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2-NORMED ALGEBRAS-II

Neeraj Srivastava, S. Bhattacharya, and S. N. Lal

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ABSTRACT. In the first part of the paper [**5**], we gave a new definition of real or complex 2-normed algebras and 2-Banach algebras. Here we give two examples which establish that not all 2-normed algebras are normable and a 2-Banach algebra need not be a 2-Banach space. We conclude by deriving a new and interesting spectral radius formula for 1-Banach algebras from the basic properties of 2-Banach algebras and thus vindicating our definitions of 2-normed and 2-Banach algebras given in [**5**].

1. Introduction

This paper being the sequel to our earlier paper, for notations and definitions, we refer to the said paper [**5**].

In the next section we give two examples. The first example establishes that not all 2-normed algebras are normable and the other shows that a 2-Banach algebra need not be a 2-Banach space. In Section 3, some basic properties of a 2-Banach algebra are derived. As it turns out, these properties as well as their proofs go almost parallel to the case of an 1-Banach algebra. In Section 4, we derive, from the results obtained in Section 3, a new and interesting spectral radius formula for an 1-Banach algebra. The results in Sections 2 and 4 vindicate our definitions of a 2-normed and 2-Banach spaces given in [**5**].

2. Examples

THEOREM 2.1. *There exist* 2*-normed algebras (with or without unity) which are not normable.*

PROOF. Let $(E, \|.,.\|)$ be a 2-normed space which is not normable (for the existence of such a space, see Gähler [**1**]). We define for $x, y \in E$, $xy = 0$ and E becomes an algebra. Let a_1, a_2 be any two linearly independent elements of E ($\dim E \geqslant 2$). Then, $\|xy, a_i\| = 0\|x, a_i\|\,\|y, a_i\|$ for $i = 1, 2$ and for all $x, y \in E$ and $(E, \|.,.\|)$ becomes a 2-normed algebra with respect to a_1, a_2 without unity

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and $(E, \|.,.\|)$ is not normable. Let $(E', \|.,.\|)$ be the algebra after augmentation of unity. Then as we have observed in [**5**], $(E', \|.,.\|)$ is a 2-normed algebra with respect to a_1, a_2 with unity and as $(E, \|.,.\|)$ is not normable, $(E', \|.,.\|)$ is also not normable and we have the theorem. $\square$

We conclude this section by giving an example which shows that a 2-Banach algebra need not be a 2-Banach space. Let $I = [0,1]$,

$$A_\infty = \mathbb{Q} \cap I = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \dots\right\} = \{r_i : i \in \mathbb{N}\},$$

$A_e = \{r_{2i} : i \in \mathbb{N}\}$, $A_n = \{r_1, \dots, r_n\}$ and define sequence of functions $\{a_n\}$ and $\{\phi_n\}$ on I by

$$a_n(x) = \begin{cases} 1, & \text{if } x = r_n \\ 0, & \text{otherwise,} \end{cases} \qquad \phi_n(x) = \begin{cases} 1, & \text{if } x \notin (A_e \cup A_n) \\ 0, & \text{otherwise} \end{cases}.$$

Let $\Im(I)$ = the set of all bounded $\mathbb{K}$-valued functions on I having at most countably many points of discontinuity in I. Then the sequences $\{a_n\}$ and $\{\phi_n\}$ are in $\Im(I)$. In $\Im(I)$, let $\|.\|$ be the sup 1-norm and $\|.,.\|$ be the 2-norm defined by, for $f, g \in \Im(I)$, $\|f, g\| = \sup_{x,y \in I} |f(x)g(y) - f(y)g(x)|$. The space $\Im(I)$ is an algebra over $\mathbb{K}$ with unity with pointwise addition and multiplication. We also have for each $n \in \mathbb{N}$ and for each $f \in \Im(I)$, $fa_n = a_n f = f(r_n)a_n$.

We prove the following lemmas.

Lemma 2.1. *The 2-normed space $(\Im(I), \|.,.\|)$ is a 2-normed algebra with respect to a_1, a_2 (or any pair of distinct elements in $\{a_n\}$).*

Proof. For $f \in \Im(I)$ we have for $n \in \mathbb{N}$, $\|f, a_n\| = \sup_{x \in I,\, x \neq r_n} |f(x)|$. Therefore, for each $f, g \in \Im(I)$, and $n \in \mathbb{N}$,

$$\|fg, a_n\| = \sup_{x \neq r_n,\, x \in I} |f(x)g(x)| \leqslant \Big(\sup_{x \neq r_n,\, x \in I} |f(x)|\Big)\Big(\sup_{x \neq r_n,\, x \in I} |g(x)|\Big) = \|f, a_n\|\,\|g, a_n\|$$

and the lemma is proved. $\square$

Lemma 2.2. *The 1-normed space $(\Im(I), \|.\|)$ is an 1-Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $(\Im(I), \|.\|)$. Then for each $x \in I$, $\{f_n(x)\}$ converges to some $f(x)$ in $\mathbb{K}$, and hence $\{f_n\}$ converges to f uniformly in I. To prove the lemma it is required to show that $f \in \Im(I)$. Let F_n be the set of all points of discontinuity of f_n in I and F be the set of all points of discontinuity of f in I. We prove that $F \subseteq \bigcup F_n$ and the lemma will be established. If $x_0 \in F$, then there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists an $x_\delta \in I$ such that

$$(*) \qquad |x_0 - x_\delta| < \delta \text{ and } |f(x_0) - f(x_\delta)| \geqslant \varepsilon$$

As $\{f_n\}$ converges to f uniformly in I, there exists an $N \in \mathbb{N}$ such that

$$\text{(i)} \qquad |f_n(x) - f(x)| < \varepsilon/3 \text{ for all } n \geqslant N, \text{ for all } x \in I.$$

If possible, let $x_0 \notin \bigcup F_n$. Then $x_0 \notin F_N$ and so, as f_N is continuous at x_0, we have a $\delta_0 > 0$ such that for $|x - x_0| < \delta_0$, $x \in I$,

$$\text{(ii)} \qquad |f_N(x) - f_N(x_0)| < \varepsilon/3.$$

Then, for $|x - x_{\delta_0}| < \delta_0$,

$$|f(x_0) - f(x_{\delta_0})| \leqslant |f(x_0) - f_N(x_0)| + |f_N(x_0) - f_N(x_{\delta_0})| + |f_N(x_{\delta_0}) - f(x_{\delta_0})| < \varepsilon$$

by (i) and (ii) and this contradicts (*). Hence $x_0 \in \bigcup F_n$ and the lemma follows. □

LEMMA 2.3. *A sequence $\{f_n\}$ is Cauchy in $(\Im(I), \|.,.\|)$ with respect to a_1, a_2 (or any pair of distinct elements in $\{a_n\}$) (that is, $\lim_{m,n\to\infty} \|f_m - f_n, a_i\| = 0$, for $i = 1, 2$) if and only if $\{f_n\}$ is Cauchy in $(\Im(I), \|.\|)$.*

PROOF. As for each $f \in \Im(I)$, $\|f, a_n\| = \sup_{x\in I,\, x\neq r_n} |f(x)|$, we have, for $i = 1, 2$

$$\lim_{m,n\to\infty} \|f_m - f_n, a_i\| = 0 \Leftrightarrow \lim_{m,n\to\infty} \Big(\sup_{x\in I,\, x\neq r_n} |f_m(x) - f_n(x)| \Big) = 0$$

$$\Leftrightarrow \lim_{m,n\to\infty} \Big(\sup_{x\in I,\, x\neq r_i} |f_m(x) - f_n(x)| \Big) = 0 \Leftrightarrow \lim_{m,n\to\infty} \|f_m - f_n\| = 0$$

and the lemma is proved. □

LEMMA 2.4. *A sequence $\{f_n\}$ in $\Im(I)$ is convergent to an f in $(\Im(I), \|.\|)$ iff $\{f_n\}$ is convergent to f in $(\Im(I), \|.,.\|)$ with respect to a_1, a_2 (or any pair of distinct element in $\{a_n\}$) (that is, $\lim_{n\to\infty} \|f_n - f, a_i\| = 0$ for $i = 1, 2$).*

PROOF. Follows as in Lemma 2.3. □

LEMMA 2.5. *The 2-normed space $(\Im(I), \|.,.\|)$ is a 2-Banach algebra with respect to a_1, a_2 (or any pair of distinct elements in $\{a_n\}$).*

PROOF. Follows from Lemmas 2.1, 2.2, 2.3 and 2.4. □

LEMMA 2.6. *The sequence $\{\phi_n\}$ is Cauchy in $(\Im(I), \|.,.\|)$.*

PROOF. Define functions b_1, b_2 on I by

$$b_1(x) = \begin{cases} 1, & \text{if } x \in A_e \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad b_2(x) = \begin{cases} 1, & \text{if } x \in (A_e \setminus \{1/2\}) \\ 0, & \text{otherwise.} \end{cases}$$

Now for $f \in \Im(I)$, we have,

$$\|f, b_1\| = \max\Big\{ \sup_{k\in\mathbb{N}} |f(r_{2k})|, \sup_{k,l\in\mathbb{N}} |f(r_{2k}) - f(r_{2l})| \Big\},$$

$$\|f, b_2\| = \max\Big\{ \sup_{k\in\mathbb{N},\, k\geqslant 2} |f(r_{2k})|, \sup_{k,l\in\mathbb{N},\, k,l\geqslant 2\mathbb{N}} |f(r_{2k}) - f(r_{2l})| \Big\}.$$

Now for each $m, n \in \mathbb{N}$ and for $i = 1, 2$ we have $\|\phi_m - \phi_n, b_i\| = 0$ and hence $\{\phi_n\}$ is Cauchy in $(\Im(I), \|.,.\|)$, and the lemma is proved. □

LEMMA 2.7. *The sequence $\{\phi_n\}$ is not Cauchy in $(\Im(I), \|.\|)$.*

PROOF. We have, for $m, n \in \mathbb{N}$, $m < n$, $A_m \subseteq A_n$, and for $x \in I$,

$$|\phi_m(x) - \phi_n(x)| = \begin{cases} 1, & \text{if } x \notin (A_m \cup A_e),\ x \in (A_n \cup A_e) \\ 0, & \text{otherwise.} \end{cases}$$

As for each pair of $m, n \in \mathbb{N}$, $m < n$ and n sufficiently large, there exists an $x \in I$ such that $x \notin (A_m \cup A_e)$ but $x \in (A_n \cup A_e)$, we have $\|\phi_m - \phi_n\| = 1$ and the lemma is proved. □

LEMMA 2.8. *The* 2*-normed space* $(\Im(I), \|.,.\|)$ *is not a* 2*-Banach space.*

PROOF. If $(\Im(I), \|.,.\|)$ is a 2-Banach space, then by Lemma 2.6 as the sequence $\{\phi_n\}$ is Cauchy in $(\Im(I), \|.,.\|)$, there is a ϕ in $\Im(I)$ so that $\lim_{n\to\infty} \|\phi_n - \phi, f\| = 0$ for each $f \in \Im(I)$ and hence, in particular, $\lim_{n\to\infty} \|\phi_n - \phi, a_i\| = 0$ for $i = 1, 2$. But then by Lemma 2.4, $\{\phi_n\}$ is convergent to ϕ in $(\Im(I), \|.\|)$ contradicting Lemma 2.7 and the proof is complete. □

The Lemmas 2.5 and 2.8 imply that the 2-normed space $(\Im(I), \|.,.\|)$ is a 2-Banach algebra with respect to a_1, a_2 (or any pair of distinct elements in $\{a_n\}$) though the 2-normed space $(\Im(I), \|.,.\|)$ is not a Banach space, and we have the following.

THEOREM 2.2. *A* 2*-Banach algebra need not be a* 2*-Banach space.*

3. 2-Banach algebras: Some basic properties

Let $(E, \|.,.\|)$ be a 2-Banach algebra with respect to a_1, a_2 over $\mathbb{K}$ with unity (If E is without unity we augment unity as in [**5**]) $a_1, a_2 \in A$, where A is an algebra with unity over $\mathbb{K}$, E is a subalgebra of A and $(A, \|.,.\|)$ is a 2-normed space. As we have seen in [**5**] $(E, \|.,.\|)$ is a topological vector space, the topology being induced by the 2-norm $\|.,.\|$ in E. In this section the topological concepts like closed/open sets, continuity etc. in E, are all meant for the topological vector space $(E, \|.,.\|)$. In this context, the following proposition is useful.

PROPOSITION 3.1. *Let* $(E, \|.,.\|)$ *be a* 2*-normed linear space over* $\mathbb{K}$*,* X *be a nonempty subset of* E*. Then* X *is open if and only if for each* $a_0 \in X$*, there exists* $\varepsilon_{a_0} > 0$ *and* $b \in E$ *such that for each* $c \in E$ *with* $\rho_b(c) = \|b, c\| < \varepsilon_{a_0}$ *implies* $a_0 + c \in X$*.*

PROOF. For a proof see [**4**]. □

Before we proceed further, let us agree with the following notations. For a 2-Banach algebra $(E, \|.,.\|)$ with unity e with respect to a_1, a_2 over $\mathbb{K}$, $G(E)$ denotes the group of all invertible elements of E. For $a \in E$, $\sigma(a)$, ωa and $r(a)$ denote the spectrum, resolvent and spectral radius of a respectively.

THEOREM 3.1. *Let* $(E, \|.,.\|)$ *be a* 2*-Banach algebra with unity* e *over* $\mathbb{K}$ *with respect to* a_1, a_2*.*

(i) *If* $a \in E$ *is such that* $\|a, a_i\| < 1$ *for* $i = 1, 2$*, then* $e - a \in G(E)$ *and if* ϕ *be a nontrivial* $\mathbb{K}$*-homomorphism on* E*,* $|\phi(a)| < 1$*.*
(ii) *The group* $G(E)$ *is open in* $(E, \|.,.\|)$*, and the mapping* $f : G(E) \to G(E)$ *defined by* $f(a) = a^{-1}$*,* $a \in G(E)$ *is a homeomorphism on* $G(E)$*.*

Proof. (i) For each $a \in E$, associate a sequence $\{s_n(a)\}$ in E defined by $s_n(a) = e + a + a^2 + \cdots + a^n$. Now if $\|a, a_i\| < 1$, $i = 1, 2$ we have for $n \in \mathbb{N}$, $n \geqslant 2$, $i = 1, 2$:

$$\|a^n, a_i\| \leqslant \|a^{n-1}, a_i\|\,\|a, a_i\| \leqslant \|a^{n-2}, a_i\|\,\|a, a_i\|^2 \leqslant \cdots \leqslant \|a, a_i\|^n \to 0 \text{ as } n \to \infty.$$

So,

$$\begin{aligned}\|s_{n+p}(a) - s_n(a), a_i\| &\leqslant \|a^{n+1}, a_i\| + \cdots + \|a^{n+p}, a_i\| \\ &\leqslant \|a, a_i\|^{n+1} + \cdots + \|a, a_i\|^{n+p} \\ &\leqslant \frac{\|a, a_i\|^{n+1}}{1 - \|a, a_i\|} \text{ for } i = 1, 2;\ n, p \in \mathbb{N},\ n \geqslant 2,\ \to 0 \text{ as } n, p \to \infty.\end{aligned}$$

But $(E, \|.,.\|)$ being 2-Banach algebra with respect to a_1, a_2 there exists an $s(a)$ in E such that $\lim_{n\to\infty} \|s_n(a) - s(a), a_i\| = 0$.

Now, $s_n(a)(e - a) = e - a^{n+1} = (e - a)s_n(a)$ for all $n \in \mathbb{N}$ and therefore for $i = 1, 2$,

$$\begin{aligned}\|s(a)(e - a) - e, a_i\| &= \|s(a)(e - a) - s_n(a)(e - a) - a^{n+1}, a_i\| \\ &\leqslant \|(s(a) - s_n(a))(e - a), a_i\| + \|a^{n+1}, a_i\| \\ &\leqslant \|s(a) - s_n(a), a_i\|\,\|e - a, a_i\| + \|a^{n+1}, a_i\|\end{aligned}$$

becomes zero as $n \to \infty$. Hence $\|s(a)(e - a) - e, a_i\| = 0$ for $i = 1, 2$ and so $s(a)(e - a) = e$. Similarly, $(e - a)s(a) = e$ and therefore, $s(a) = (e - a)^{-1}$ and $e - a \in G(E)$. (We call the series $s(a) = e + a + a^2 + \cdots$, the associate series of a).

To prove the second part of (i), let, if possible, ϕ be a nontrivial $\mathbb{K}$-homomorphism on E, $|\phi(a)| \geqslant 1$. Let $\lambda \in \mathbb{K}$ be such that $\phi(a) = \lambda$. Then $|\lambda| \geqslant 1$ and $\phi(\lambda^{-1}a) = 1$ and so $\phi(e - \lambda^{-1}a) = 0$ as ϕ being nontrivial, $\phi(e) = 1$. Let $b = e - \lambda^{-1}a$. Then $\phi(b) = 0$. But as $|\lambda| \geqslant 1$ and $\|a, a_i\| < 1$ for $i = 1, 2$, we have $\|\lambda^{-1}a, a_i\| < 1$ for $i = 1, 2$ and hence $b = e - \lambda^{-1}a \in G(E)$. But then $\phi(b) \neq 0$ and we have a contradiction and (i) is completely proved.

(ii) To see that $G(E)$ is open, let $a \in G(E)$. Note that, as a_1, a_2 are linearly independent, for $a \in E$, $\|a, a_i\| = 0$ for $i = 1, 2$ if and only if $a = 0$, see [**5**]. Take $\varepsilon_a = \frac{1}{2}(\max_{i=1,2}\{\|a^{-1}, a_i\|\})^{-1}$. Then for $b \in E$ with $\|b, a_i\| < \varepsilon_a$, $i = 1, 2$, we have $\|-a^{-1}b, a_i\| \leqslant \|a^{-1}, a_i\|\,\|b, a_i\| < \frac{1}{2}$, which implies, $e + a^{-1}b \in G(E)$ by (i), and as $a + b = a(e + a^{-1}b)$, we have $a + b \in G(E)$ which using Proposition 3.1 proves that $G(E)$ is open.

To prove that f is a homeomorphism, let $b \in E$, $a \in G(E)$ and $\|b - a, a_i\| < \varepsilon_a$, for $i = 1, 2$; then, as $a + (b - a) = b$ and $G(E)$ is open, $b \in G(E)$. Write $c = a^{-1}(b - a)$. Then for $i = 1, 2$, $\|c, a_i\| \leqslant \|a^{-1}, a_i\|\,\|b - a, a_i\| \leqslant \frac{1}{2}$ and hence by (i), $e + c \in G(E)$. We also have, for each $n \in \mathbb{N}$, $s_n(-c)(e + c) = e - (-1)^{n+1}c^{n+1}$ and $s(-c) = (e + c)^{-1} \in G(E)$ and for $i = 1, 2$,

$$\begin{aligned}\|s_n(-c) - e, a_i\| &\leqslant \|c, a_i\| + \|c, a_i\|^2 + \cdots + \|c, a_i\|^n = \frac{\|c, a_i\|(1 - \|c, a_i\|^n)}{1 - \|c, a_i\|} \\ &\leqslant 2\|c, a_i\|(1 - \|c, a_i\|^n) \qquad (\text{as } \|c, a_i\| \leqslant 1/2)\end{aligned}$$

and so $\|s(-c)-e, a_i\| \leqslant 2\|c, a_i\|$ for $i = 1,2$. Since $b^{-1}-a^{-1} = [(e+c)^{-1} - e]a^{-1} = [s(-c) - e]a^{-1}$, we have for $i = 1,2$:

$$\begin{aligned}\|f(b) - f(a), a_i\| &= \|b^{-1} - a^{-1}, a_i\| = \|(s(-c) - e)a^{-1}, a_i\| \\ &\leqslant \|s(-c) - e, a_i\| \, \|a^{-1}, a_i\| \leqslant 2\|a^{-1}, a_i\| \, \|b - a, a_i\| \, \|a^{-1}, a_i\| \\ &= 2\|a^{-1}, a_i\|^2 \, \|b - a, a_i\|.\end{aligned}$$

Hence, $\|f(b) - f(a), a_i\| \leqslant 2\|a^{-1}, a_i\|^2\|b - a, a_i\|$ for $i = 1,2$; whenever $b \in E$ with $\|b - a, a_i\| < \varepsilon_a$. But this proves that f is continuous on $G(E)$. As f is one to one on $G(E)$, $f^{-1} = f$. The mapping f is a homeomorphism on $G(E)$ and (ii) is proved. This completes the proof of the theorem. □

Theorem 3.2. *Let $(E, \|.,.\|)$ be a 2-Banach algebra with unity e over $\mathbb{K}$ with respect to a_1, a_2, and $a \in E$. Then,*

- (i) *$\sigma(a)$ is closed in $\mathbb{K}$,*
- (ii) *$r(a) \leqslant \max_{i=1,2}\{\|a, a_i\|\}$,*
- (iii) *$\sigma(a)$ is compact in $\mathbb{K}$,*
- (iv) *$\sigma(a)$ is nonempty if $\mathbb{K} = \mathbb{C}$, and*
- (v) *$r(a) = \lim_{n\to\infty}\big[\max_{i=1,2}\{\|a^n, a_i\|\}\big]^{1/n}$.*

Proof. (i) For $a \in E$ define $f_a : \mathbb{K} \to E$ by $f_a(\lambda) = \lambda e - a$ for $\lambda \in \mathbb{K}$. Then f_a is continuous on $\mathbb{K}$, and so $f_a^{-1}(G(E))$ is open in $\mathbb{K}$ as $G(E)$ is open in $(E, \|.,.\|)$ by Theorem 3.1. We claim that $\Omega_a = f_a^{-1}(G(E))$. To prove the claim, let $\lambda \in \Omega_a$; then $\lambda e - a \in G(E)$ and so $f_a(\lambda) \in G(E)$ which implies $\lambda \in f_a^{-1}(G(E))$. Conversely, let $\lambda \in f_a^{-1}(G(E))$. Then $f_a(\lambda) = \lambda e - a \in G(E)$ and so $\lambda \notin \sigma(a)$ and the claim is proved. This proves that $\sigma(a)$ is closed.

(ii) Write $k = \max_{i=1,2} \|a, a_i\|$ and let, if possible, $r(a) > k$. Then there exists $\lambda \in \sigma(a)$ such that $|\lambda| > k$, and therefore $\|\lambda^{-1}a, a_i\| < 1$ for $i = 1,2$. But this implies by Theorem 3.1, $e - \lambda^{-1}a \in G(E)$ and hence $\lambda \notin \sigma(a)$, and (ii) is proved.

(iii) Combining (i) and (ii), we get (iii).

(iv) For $a \in E$, define a mapping $R_a : \Omega_a \to G(E)$ by $R_a(\lambda) = (\lambda e - a)^{-1}$, $\lambda \in \Omega_a$. Let $\lambda \in \Omega_a$, $\delta = [\max_{i=1,2} \|R_a(\lambda), a_i\|]^{-1}$. Let $\mu \in \Omega_a$ be such that

$$|\lambda - \mu| < \frac{1}{2}\,\delta, \tag{3.1}$$

and $b = (\mu - \lambda)R_a(\lambda)$. Then, as $\|b, a_i\| < 1$ for $i = 1,2$, $e - b \in G(E)$, by Theorem 3.1. For $i = 1,2$ we have

$$\|s_n(b) - e - b, a_i\| \leqslant \|b, a_i\|^2 + \cdots + \|b, a_i\|^n = \frac{\|b, a_i\|^2(1 - \|b, a_i\|^n)}{1 - \|b, a_i\|}$$

(see the proof of Theorem 3.1) and so, for $i = 1,2$,

$$\|(e - b)^{-1} - e - b, a_i\| = \|s(b) - e - b, a_i\| \leqslant \frac{\|b, a_i\|^2}{1 - \|b, a_i\|}, \qquad i = 1,2. \tag{3.2}$$

Now,

$$\begin{aligned} &R_a(\mu) - R_a(\lambda) + (\mu-\lambda)(R_a(\lambda))^2 \\ &\quad = [(\mu e - a)^{-1}(\lambda e - a) - e + (\mu-\lambda)R_a(\lambda)]R_a(\lambda) \\ &\quad = [[(\lambda e - a)^{-1}\{(\mu-\lambda)e + (\lambda e - a)\}]^{-1} - e + (\mu-\lambda)R_a(\lambda)]R_a(\lambda) \\ &\quad = [\{e + (\mu-\lambda)R_a(\lambda)\}^{-1} - e + (\mu-\lambda)R_a(\lambda)]R_a(\lambda). \end{aligned}$$

Therefore, for $i = 1, 2$,

$$\begin{aligned} &\|R_a(\mu) - R_a(\lambda) + (\mu-\lambda)(R_a(\lambda))^2, a_i\| \\ &\quad \leqslant \|\{e + (\mu-\lambda)R_a(\lambda)\}^{-1} - e + (\mu-\lambda)R_a(\lambda), a_i\| \, \|R_a(\lambda), a_i\| \\ &\quad \leqslant \frac{|(\lambda-\mu)| \, \|R_a(\lambda), a_i\|^2}{1 - |(\mu-\lambda)| \, \|R_a(\lambda), a_i\|} \|R_a(\lambda), a_i\| && \text{(by (3.2))} \\ &\quad \leqslant 2|\mu-\lambda|^2 \|R_a(\lambda), a_i\|^3 \leqslant \frac{1}{2}\,\delta \|R_a(\lambda), a_i\| && \text{(by (3.1))} \end{aligned}$$

as $\|b, a_i\| = |\mu-\lambda| \, \|R_a(\lambda), a_i\| \leqslant \frac{1}{2}$.

Therefore, for $i = 1, 2$, for $\mu \in \Omega_a$, $\mu \neq \lambda$ and $|\mu - \lambda| < \frac{1}{2}\delta$,

$$\left\| \frac{R_a(\mu) - R_a(\lambda)}{\mu - \lambda} + (R_a(\lambda))^2, a_i \right\| \leqslant 2|\mu-\lambda| \, \|R_a(\lambda), a_i\|^3.$$

So, $\lim_{\mu\to\lambda} \frac{R_a(\mu)-R_a(\lambda)}{\mu-\lambda}$ exists in the topological linear space $(E, \|.,.\|)$ and equals to $-(R_a(\lambda))^2$ for $\lambda \in \Omega_a$ and we conclude that R_a is analytic in Ω_a.

Now, if possible, let $\sigma(a)$ be empty. Then $\Omega_a = \mathbb{K} = \mathbb{C}$ and R_a is an entire function. Let $\lambda \in \mathbb{C}$ be such that $k < |\lambda|$, that is, $\|\lambda^{-1}a, a_i\| < 1$ for $i = 1, 2$; k be as in (ii). Then by Theorem 3.1, $e - \lambda^{-1}a \in G(E)$ and as $s(\lambda^{-1}a) = (e - \lambda^{-1}a)^{-1} = e + (\lambda^{-1}a) + (\lambda^{-1}a)^2 + \cdots$, we have

$$R_a(\lambda) = \lambda^{-1}(e - \lambda^{-1}a)^{-1} = \lambda^{-1}e + \lambda^{-2}a + \lambda^{-3}a^2 + \cdots \tag{3.3}$$

Let Γ_r be the circle on the complex plane with center at origin and radius r, where $k < r$. Then the series on the right-hand side of (3.3) converges uniformly on Γ_r and so term by term integration over Γ_r is allowed to the right hand-side of the series in (3.3), and we conclude that for $n = 0, 1, 2, \ldots$ and for $r > k$,

$$a^n = \frac{1}{2\pi i} \int_{\Gamma_r} \lambda^n R_a(\lambda) \, d\lambda \tag{3.4}$$

and in particular

$$e = \frac{1}{2\pi i} \int_{\Gamma_r} R_a(\lambda) \, d\lambda. \tag{3.5}$$

But as R_a is entire, by Cauchy theorem, the integral on the right-hand side of (3.5) is zero, which is a contradiction and the proof of (iv) is complete.

(v) Let a $\in E$. Then for $r > r(a)$ also (3.4) holds. The continuity of R_a in Γ_r implies that for $r > r(a)$, $B(r) = \max_{j=1,2,\, \theta\in[0,2\pi]} \|R_a(re^{i\theta}), a_j\|$ is finite. Hence by (3.4) we have $\max_{j=1,2}\{\|a^n, a_j\|\} \leqslant r^{n+1}B(r)$ for all $n \in \mathbb{N}$, which then implies that

$$\lim_{n\to\infty} \sup \left[\max_{j=1,2}\{\|a^n, a_j\|\}\right]^{1/n} \leqslant r(a) \tag{3.6}$$

Again, for $\lambda \in \sigma(a)$, as we have, for all $n \in \mathbb{N}$,

$$\lambda^n e - a^n = (\lambda e - a)(\lambda^{n-1}e + \lambda^{n-2}a + \cdots + \lambda a^{n-2} + a^{n-1}),$$

we see that $\lambda^n e - a^n \notin G(E)$ and hence $\lambda^n \in \sigma(a^n)$ for all $n \in \mathbb{N}$. Then by (ii) we have for all $n \in \mathbb{N}$, $|\lambda^n| \leqslant r(a^n) \leqslant [\max_{j=1,2}\{\|a^n, a_j\|\}]$. Hence, for all $\lambda \in \sigma(a)$, and for all $n \in \mathbb{N}$, $|\lambda| \leqslant [\max_{j=1,2}\{\|a^n, a_j\|\}]^{1/n}$ implying,

$$r(a) \leqslant \lim_{n\to\infty} \inf \left[\max_{j=1,2}\{\|a^n, a_j\|\}\right]^{1/n}. \tag{3.7}$$

Now (3.6) and (3.7) implies that $\lim_{n\to\infty} [\max_{j=1,2}\{\|a^n, a_j\|\}]^{1/n}$ exists and equals to $r(a)$. This establishes (v) and the proof of the theorem is complete. □

4. Spectral radius formula for 1-Banach algebras

The following theorem contains a new spectral radius formula for 1-Banach algebras.

THEOREM 4.1. *Let $(E, \|.\|)$ be an 1-Banach algebra with unity e over $\mathbb{C}$, $\dim E \geqslant 2$ such that a nontrivial $\mathbb{C}$-homomorphism on E exists. Then there exists an 1-Banach algebra $(B, \|.\|_1)$ of which $(E, \|.\|)$ is a closed subalgebra, the 1-norm $\|.\|_1$ on B when restricted on E becomes the 1-norm $\|.\|$ on E and $a_1, a_2 \in B$ such that for all $a \in E$,*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \left[\max_{i=1,2}\left\{\sup_{\substack{\phi,\Psi\in B^* \\ \|\phi\|=\|\Psi\|=1}} |\phi(a^n)\Psi(a_i) - \Psi(a^n)\phi(a_i)|\right\}\right]^{1/n} \tag{4.1}$$

PROOF. Follows from Lemma 5.4, Theorem 5.1 of [**5**] and Theorem 3.2. □

We conclude this section by stating the following 2-norm version of the Gelfand–Mazur theorem [**2, 3**].

THEOREM 4.2. *There does not exist a 2-Banach division algebra over $\mathbb{C}$.*

PROOF. If possible, let $(E, \|.,.\|)$ be a 2-Banach division algebra on $\mathbb{C}$ with respect to a_1, a_2. Then $\dim E \geqslant 2$. For each $0 \neq a \in E$, we claim that $\sigma(a)$ is a singleton. To prove this claim, we observe that $\sigma(a)$ is nonempty by Theorem 3.2 and if $\lambda_1, \lambda_2 \in \sigma(a)$, $\lambda_1 \neq \lambda_2$, then as $\lambda_1 e - a$ and $\lambda_2 e - a$ both are noninvertible, we have $\lambda_1 e - a = \lambda_2 e - a = 0$ as E is a division algebra. So, $\lambda_1 = \lambda_2$, and our claim is proved. Now for each a in E, let $\sigma(a) = \{\lambda(a)\}$ and by definition of $\sigma(a)$, $a = \lambda(a)e$, that is, E is generated by e and hence $\dim E = 1$ and the theorem follows. □

References

1. S. Gähler, *Lineare 2-normierte Räume*, Math. Nachr. 28 (1965), 1–43.
2. I. M. Gel'fand, *Normierte Ringe*, Mat. Sb., Nov. Ser. 9(51) (1941), 3–21.
3. I. M. Gel'fand and M. A. Neumar, *On the imbedding of normed rings into the ring of operators in Hilbert space*, Mat. Sb., Nov. Ser. 12(54) (1943), 197–213.
4. S. N. Lal, S. Bhattacharya, and E. Keshava Reddi, 2-*norm on an algebra*, Prog. Math., Varanasi 35 (2001), 27–32.
5. Neeraj Srivastava, S. Bhattacharya, and S. N. Lal, 2-*normed algebras–I*, Publ. Inst. Math., Nouv. Sér. 88(102) (2010), 111–121.

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National Academy for Scientific Development
Education and Research "Shashi Villa"
Mahamanapuri Colony
B.H.U. Post Office
Varanasi 221 005
India
neeraj_bhuin@yahoo.co.in

Department of Mathematics
Faculty of Science
Banaras Hindu University
Varanasi 221 005
India
sbhatta@bhu.ac.in

National Academy for Scientific Development,
Education and Research "Shashi Villa"
Mahamanapuri Colony
B.H.U. Post Office
Varanasi 221 005
India
snlmath@yahoo.com