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SOME NEW MULTIDIMENSIONAL HARDY-TYPE INEQUALITIES WITH KERNELS VIA CONVEXITY

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ABSTRACT. We prove some new multidimensional Hardy-type inequalities involving general Hardy type operators with positive kernels for functions ϕ which may not necessarily be convex but satisfy the condition $A\psi(\boldsymbol{x}) \leqslant \phi(\boldsymbol{x}) \leqslant B\psi(\boldsymbol{x})$, where ψ is convex. Our approach is mainly the use of convexity argument and the results obtained are new even for the one-dimensional case and also unify and extend several inequalities of Hardy type known in the literature.

1. Introduction

In a note published in 1920, Hardy [**3**] in a surprising way discovered and announced (without proof) (see also [**4**, **13**, **14**]) the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \quad p>1, \tag{1.1}$$

in an attempt to simplify the proof of Hilbert's double series theorem. Inequality (1.1) is today referred to as the classical Hardy's integral inequality and it has an interesting prehistory and history (see e.g., [**4**, **13**, **14**]) and the references cited therein). It is interesting to note that Hardy could hardly had forseen the profound influence this inequality and its various variants and generalizations would have on the development of many areas in analysis (see [**6**]). Owing to the usefulness of inequality (1.1), it has been investigated and generalized in several directions, e.g., one chapter in the book [**16**] is devoted to this subject. In addition, there are three

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books [**12**,**13**,**23**] completely devoted to this subject, however, none of these books treated Hardy's inequality via convexity approach. We also refer interested readers to the references in these books and the classical book [**5**] for some complementary historical remarks connected to the development of this inequality.

Nowadays, a well-known simple fact is that (1.1) can equivalently (via the substitution $f(x) = h(x^{1-1/p})x^{-1/p}$), be rewritten in the form

$$\int_0^\infty \left(\frac{1}{x}\int_0^x h(t)\,dt\right)^p \frac{dx}{x} \leqslant \int_0^\infty h^p(x)\frac{dx}{x} \tag{1.2}$$

for which equality even holds when $p = 1$. In this form we see that Hardy's inequality is a simple consequence of Jensen's inequality but this was not discovered in the dramatic period when Hardy discovered and finally proved inequality (1.1) in his famous paper [**4**] from 1925 (see [**13**, **14**, **20**]). In 1965, Godunova [**1**] while studying inequalities with convex functions, pioneered a simple direct way of obtaining Hardy's inequality via a convexity argument. However, the result of Godunova seems to be fairly little referred to and almost unknown in the western literature (see e.g., [**20**]). It was rediscovered independently by Imoru [**7**] and Kaijser et al. [**10**] in 1978 and 2002 respectively. This is the starting point of a new development of Hardy-type inequalities and most of the results reported in this paper are influenced by the work of Kaijser et al. [**10**]. Obviously, the prehistory and history of Hardy's inequality (1.1) would have been completely changed if Hardy (or some collaborators in the dramatic period 1915–1925) had discovered that (1.1) can be rewritten in the form (1.2) and that this inequality follows directly from Jensen's inequality and Fubini's theorem (see e.g., [**24**]). Even though, Jensen's inequality from 1906 (see [**8**] was of course known to Hardy).

Again, the first author to obtain the multidimensional Hardy type inequality with a general kernel was Godunova [**1**] (see also [**2**]), while Kaijser et al. [**9**] obtained some new integral inequalities with general integral operators (without additional restrictions on the kernel). The corresponding results for the case $0 < p < 1$ and $p < 0$ was recently obtained by Oguntuase et al. [**18**] as a consequences of a more general inequalities for convex and concave functions (see [**17**] for further details). Furthermore, Oguntuase et al. [**21**] obtained a new class of general multidimensional strengthened Hardy type inequalities with power weights, related to all possible choices of parameter $p \in \mathbb{R} \setminus \{0\}$ and to arbitrary almost everywhere positive convex (or concave) function ϕ, such that $Ax^p \leqslant \phi(x) \leqslant Bx^p$ holds on $(0, \infty)$ with some positive real constants $A \leqslant B$. The multidimensional version of these results were recently obtained in [**22**] (see e.g., [**11**, **20**] and the references given there). In particular, Oguntuase et al. [**22**] obtained the following new multidimensional Hardy-type inequality

$$\left(\int_0^{b_1}\!\!\cdots\int_0^{b_n} \phi(A_K f(\boldsymbol{x}))^q u(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}} \leqslant C\left(\int_0^{b_1}\!\!\cdots\int_0^{b_n} \phi^p(f(\boldsymbol{x}))v(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{p}}$$

involving arithmetic mean operator

$$A_K f(\boldsymbol{x}) := \frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\!\!\cdots\int_0^{x_n} k(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y} \tag{1.3}$$

with general positive kernel

$$K(\boldsymbol{x}) := \int_0^{x_1} \cdots \int_0^{x_n} k(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y}. \tag{1.4}$$

Our aim is to derive some new multidimensional Hardy-type inequalities involving general Hardy type operators with positive kernels (see [**15**]). Our approach is to use mainly a convexity argument to prove our results. This idea was first introduced by Oguntuase and Persson in [**19**] and further developed in [**21**] for convex and concave functions. Our aim is to further extend this idea to functions ϕ, which may not necessarily be convex but for which $A\psi(\boldsymbol{x}) \leqslant \phi(\boldsymbol{x}) \leqslant A\psi(\boldsymbol{x})$ holds on $\mathbb{R}^n_+$ for some constants $A \leqslant B$ and convex function ψ.

Notations and Conventions. Throughout this paper we use bold letters to denote n-tuples of real numbers, e.g., $\boldsymbol{x} = (x_1, \dots, x_n t)$, or $\boldsymbol{y} = (y_1, \dots, y_n)$. Also, we set $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. Furthermore, the relations $<$, $\leqslant$, $>$, and $\geqslant$ are, as usual, defined componentwise. For example, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we write $\boldsymbol{x} < \boldsymbol{y}$ if $x_i < y_i$, $i = 1, \dots, n$. Moreover, $\mathbf{0} < \boldsymbol{b} \leqslant \boldsymbol{\infty}$ means that $0 < b_i \leqslant \infty$, $i = 1, \dots, n$. In addition, we introduce a notation for n-cells, that is, axis-parallel to rectangular blocks in $\mathbb{R}^n$. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, $\boldsymbol{a} < \boldsymbol{b}$, let

$$(\boldsymbol{a}, \boldsymbol{b}) = (a_1, b_1) \times \cdots \times (a_n, b_n) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a} < \boldsymbol{x} < \boldsymbol{b}\},$$
$$(\boldsymbol{a}, \boldsymbol{b}] = (a_1, b_1] \times \cdots \times (a_n, b_n] = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a} < \boldsymbol{x} \leqslant \boldsymbol{b}\},$$

and analogously also for $[\boldsymbol{a}, \boldsymbol{b})$ and $[\boldsymbol{a}, \boldsymbol{b}]$. In particular, we have $\mathbb{R}^n_+ = (\mathbf{0}, \boldsymbol{\infty})$,

$$(\mathbf{0}, \boldsymbol{\infty}] = \{\boldsymbol{x} : \mathbf{0} < \boldsymbol{x} \leqslant \boldsymbol{\infty}\}, \text{ and } [\mathbf{0}, \boldsymbol{\infty}) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{x} \geqslant \mathbf{0}\}.$$

Furthermore, all functions are assumed to be measurable, and expressions of the form $\mathbf{0} \cdot \boldsymbol{\infty}$, $\frac{\boldsymbol{\infty}}{\boldsymbol{\infty}}$, and $\frac{\mathbf{0}}{\mathbf{0}}$ are taken to be equal to zero. Moreover, $u(\boldsymbol{x})$ denotes a weight function, i.e., a nonnegative and measurable function on $\mathbb{R}^n$, and we define $d\boldsymbol{x}$ by $d\boldsymbol{x} = dx_1 \cdots dx_n$.

2. Main Results

In order to prove our results we need the following multidimensional Minkowski type inequality proved in [**22**].

Lemma 2.1. [**22**] *Let $p > 1$, $-\infty \leqslant a_i < b_i \leqslant \infty$, $k = k(\boldsymbol{x}, \boldsymbol{y})$ be a locally integrable kernel and ϕ and ψ be positive and measurable functions. Then*

$$\left(\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left(\int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} k(\boldsymbol{x}, \boldsymbol{y})\, \psi(\boldsymbol{y})\, d\boldsymbol{y} \right)^p \phi(\boldsymbol{x})\, d\boldsymbol{x} \right)^{\frac{1}{p}}$$
$$\leqslant \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \left(\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \phi(\boldsymbol{x}) k^p(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{x} \right)^{\frac{1}{p}} \psi(\boldsymbol{y})\, d\boldsymbol{y}.$$

Proof. Follows from the fact that we can have equality in Hölder's inequality and application of Fubini's theorem (see [**22**, Lemma 2.1 and Proposition 2.3] for details). □

Our first result which generalizes Theorem 2.1 in [**22**] reads:

Theorem 2.1. *Let $1 < p \leqslant q < \infty$, $0 < b_i \leqslant \infty$, $s_1, \ldots, s_n \in (1, p)$, $i = 1, 2, \ldots, n$ and let ϕ be a nonnegative increasing function on (a, c), $-\infty \leqslant a < c \leqslant \infty$ for which there exists a convex function ψ on (a, c) such that*

$$A\psi(\boldsymbol{x}) \leqslant \phi(\boldsymbol{x}) \leqslant B\psi(\boldsymbol{x}) \tag{2.1}$$

holds for constants $0 < A \leqslant B < \infty$. Let A_K be the general Hardy operator defined by (1.3) *and let $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ be weight functions, where $v(\boldsymbol{x})$ is of product type i.e., $v(\boldsymbol{x}) = v(x_1)\, v(x_2) \cdots v(x_n)$. Then the inequality*

$$\left(\int_0^{b_1} \cdots \int_0^{b_n} [\phi(A_K f(\boldsymbol{x}))]^q u(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}} \leqslant C\left(\int_0^{b_1} \cdots \int_0^{b_n} \phi(f(\boldsymbol{x}))\, v(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{p}} \tag{2.2}$$

holds for all functions f such that $a < f(\boldsymbol{x}) < c$, if

$$A(\boldsymbol{s}) := \sup_{0<\boldsymbol{y}<\boldsymbol{b}} \left(\int_{t_1}^{b_1} \cdots \int_{t_n}^{b_n} \left[\frac{k(\boldsymbol{x}, \boldsymbol{y})}{K(\boldsymbol{x})}\right]^q u(\boldsymbol{x}) V_1^{\frac{q(p-s_1)}{p}}(x_1) \cdots V_n^{\frac{q(p-s_n)}{p}}(x_n) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}} \times V_1^{\frac{s_1-1}{p}}(t_1) \cdots V_n^{\frac{s_n-1}{p}}(t_n) < \infty$$

holds, where $V_i(x_i) = \int_0^{x_i} v_i^{1-p'}\, dt_i$, $i = 1, 2, \ldots, n$, and $p' = \frac{p}{p-1}$.

Furthermore, if C is the best possible constant in (2.2)*, then*

$$C \leqslant \frac{B}{A} \inf_{1<\boldsymbol{s}<p} \left(\frac{p-1}{p-s_1}\right)^{1/p'} \cdots \left(\frac{p-1}{p-s_n}\right)^{1/p'} A(\boldsymbol{s}).$$

Remark 2.1. For the special case $A = B = 1$, Theorem 2.1 coincides with Theorem 2.1 in [**22**].

Proof. By applying condition (2.1) followed by Jensen's inequality to the left-hand side of (2.2) we obtain that

$$\begin{aligned}
&\left(\int_0^{b_1} \cdots \int_0^{b_n} [\phi(A_K f(\boldsymbol{x}))]^q\, u(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}} \\
&\leqslant B\left(\int_0^{b_1} \cdots \int_0^{b_n} \left[\psi\left(\frac{1}{K(\boldsymbol{x})} \int_0^{x_1} \cdots \int_0^{x_n} k(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y})\, d\boldsymbol{y}\right)\right]^q u(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}} \\
&\leqslant B\left(\int_0^{b_1} \cdots \int_0^{b_n} \left[\frac{1}{K(\boldsymbol{x})} \int_0^{x_1} \cdots \int_0^{x_n} k(\boldsymbol{x}, \boldsymbol{y})\, \psi(f(\boldsymbol{y}))\, d\boldsymbol{y}\right]^q u(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.3}$$

Define a function g so that $\psi^p(f(\boldsymbol{x})) \frac{v(\boldsymbol{x})}{x_1 \cdots x_n} = \psi(g(\boldsymbol{x}))$ and substitute it into inequality (2.3) to obtain that

$$\begin{aligned}
&\left(\int_0^{b_1} \cdots \int_0^{b_n} [\phi(A_K f(\boldsymbol{x}))]^q u(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}} \\
&\leqslant B\left(\int_0^{b_1} \cdots \int_0^{b_n} \left[\frac{1}{K(\boldsymbol{x})} \int_0^{x_1} \cdots \int_0^{x_n} k(\boldsymbol{x}, \boldsymbol{y})\, \psi^{\frac{1}{p}}(g(\boldsymbol{y}))\right.\right.
\end{aligned} \tag{2.4}$$

$$\times\, v_1^{-\frac{1}{p}}(y_1)\cdots v_n^{-\frac{1}{p}}(y_n)\, y_1^{\frac{1}{p}}\cdots y_n^{\frac{1}{p}}\, d\boldsymbol{y}\Big]^q u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\Bigg)^{\frac{1}{q}}$$

Now apply Hölder's inequality, Lemma 2.1 with p replaced by $\frac{q}{p}$ on the the right-hand side of (2.4) and using the fact that

$$\frac{dV_i(x_i)}{dx_i} = v_i^{1-p'}(x_{i)} = v^{\frac{-p'}{p}}(x_i), \quad v(\boldsymbol{x}) = v_1(x_1)\cdots v_n(x_n), \quad \frac{-p'}{p} = 1-p',$$

we obtain

$$\begin{aligned}
&\bigg(\int_0^{b_1}\!\!\cdots\int_0^{b_n}\Big[\phi(A_K f(\boldsymbol{x}))\Big]^q u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\bigg)^{\frac{1}{q}}\\
&\leqslant B\bigg(\int_0^{b_1}\!\!\cdots\int_0^{b_n}\bigg[\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\!\!\cdots\int_0^{x_n} k(\boldsymbol{x},\boldsymbol{y})\,\psi^{\frac{1}{p}}(g(\boldsymbol{y}))V_1^{\frac{s_1-1}{p}}(y_1)\cdots V_n^{\frac{s_n-1}{p}}(y_n)\\
&\quad\times V_1^{\frac{-(s_1-1)}{p}}(y_1)\cdots V_n^{\frac{-(s_n-1)}{p}}(y_n)\, v_1^{-\frac{1}{p}}(y_1)\cdots v_n^{-\frac{1}{p}}(y_n)\, y_1^{\frac{1}{p}}\cdots y_n^{\frac{1}{p}}\, d\boldsymbol{y}\bigg]^q u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\bigg)^{\frac{1}{q}}\\
&\leqslant B\bigg(\int_0^{b_1}\!\!\cdots\int_0^{b_n}\bigg[\int_0^{x_1}\!\!\cdots\int_0^{x_n} k^p(\boldsymbol{x},\boldsymbol{y})\,\psi(g(\boldsymbol{y}))V_1^{s_1-1}(y_1)\cdots V_n^{s_n-1}(y_n)\, d\boldsymbol{y}\bigg]^{\frac{q}{p}}\\
&\quad\times\bigg[\int_0^{x_1}\!\!\cdots\int_0^{x_n} V_1^{\frac{-p'(s_1-1)}{p}}(y_1)\cdots V_n^{\frac{-p'(s_n-1)}{p}}(y_n)\, v_1^{-\frac{p'}{p}}(y_1)\cdots v_n^{-\frac{p'}{p}}(y_n)\, y_1^{\frac{p'}{p}}\cdots y_n^{\frac{p'}{p}}\bigg]^{\frac{q}{p'}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\frac{u(\boldsymbol{x})}{K^q(\boldsymbol{x})}\frac{d\boldsymbol{x}}{x_1\cdots x_n}\bigg)^{\frac{1}{q}}\\
&= B\bigg(\frac{p-1}{p-s_1}\bigg)^{\frac{1}{p'}}\cdots\bigg(\frac{p-1}{p-s_n}\bigg)^{\frac{1}{p'}}\bigg(\int_0^{b_1}\!\!\cdots\int_0^{b_n}\bigg[\int_0^{x_1}\!\!\cdots\int_0^{x_n} k^p(\boldsymbol{x},\boldsymbol{y})\,\psi(g(\boldsymbol{y}))\\
&\quad\times V_1^{s_1-1}(y_1)\cdots V_n^{s_n-1}(y_n)\, d\boldsymbol{y}\bigg]^{\frac{q}{p}} V_1^{\frac{q(p-s_1)}{p}}(x_1)\cdots V_n^{\frac{q(p-s_n)}{p}}(x_n)\frac{u(\boldsymbol{x})}{K^q(\boldsymbol{x})}\frac{d\boldsymbol{x}}{x_1\cdots x_n}\bigg)^{\frac{1}{q}}\\
&\leqslant B\bigg(\frac{p-1}{p-s_1}\bigg)^{\frac{1}{p'}}\cdots\bigg(\frac{p-1}{p-s_n}\bigg)^{\frac{1}{p'}}\bigg(\int_0^{b_1}\!\!\cdots\int_0^{b_n}\psi(g(\boldsymbol{y}))V_1^{s_1-1}(y_1)\cdots V_n^{s_n-1}(y_n)\\
&\quad\times\bigg[\int_{y_1}^{b_1}\!\!\cdots\int_{y_n}^{b_n} V_1^{\frac{q(p-s_1)}{p}}(x_1)\cdots V_n^{\frac{q(p-s_n)}{p}}(x_n)\, u(\boldsymbol{x})\bigg(\frac{k(\boldsymbol{x},\boldsymbol{y})}{K(\boldsymbol{x})}\bigg)^q\frac{d\boldsymbol{x}}{x_1\cdots x_n}\bigg]^{\frac{p}{q}} d\boldsymbol{y}\bigg)^{\frac{1}{p}}\\
&\leqslant B\bigg(\frac{p-1}{p-s_1}\bigg)^{\frac{1}{p'}}\cdots\bigg(\frac{p-1}{p-s_n}\bigg)^{\frac{1}{p'}} A(\boldsymbol{s})\bigg(\int_0^{b_1}\!\!\cdots\int_0^{b_n}\psi(g(\boldsymbol{y}))\, d\boldsymbol{y}\bigg)^{\frac{1}{p}}\\
&\leqslant \frac{B}{A}\bigg(\frac{p-1}{p-s_1}\bigg)^{\frac{1}{p'}}\cdots\bigg(\frac{p-1}{p-s_n}\bigg)^{\frac{1}{p'}} A(\boldsymbol{s})\bigg(\int_0^{b_1}\!\!\cdots\int_0^{b_n}\phi(g(\boldsymbol{y}))d\boldsymbol{y}\bigg)^{\frac{1}{p}},
\end{aligned}$$

and the proof is complete. □

REMARK 2.2. Note that by taking $A = B$ in Theorem 2.1, condition (2.1) reduces to $\phi(\boldsymbol{x}) = A\psi(\boldsymbol{x})$ and so Theorem 2.1 yields a result that is more general than Theorem 2.1 in [22].

COROLLARY 2.1. *Let $1 < p \leqslant q < \infty$, $0 < b_i \leqslant \infty$, $s_1, \dots, s_n \in (1, p)$, $i = 1, 2, \dots, n$ and let ϕ be a nonnegative increasing function on (a, c), $-\infty \leqslant a < c \leqslant \infty$ for which there exists a convex function ψ on (a, c) such that $\phi(\boldsymbol{x}) \leqslant A\psi(\boldsymbol{x})$ holds for a constant $A > 0$. Let A_K be the general Hardy operator defined by* (1.3) *and let $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ be weight functions, where $v(\boldsymbol{x})$ is of product type i.e., $v(\boldsymbol{x}) = v(x_1)\, v(x_2) \cdots v(x_n)$. Then the inequality*

$$\left(\int_0^{b_1} \cdots \int_0^{b_n} [\phi(A_K f(\boldsymbol{x}))]^q u(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}} \leqslant C\left(\int_0^{b_1} \cdots \int_0^{b_n} \phi(f(\boldsymbol{x}))\, v(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{p}}$$

holds for all functions f such that $a < f(\boldsymbol{x}) < c$, if

$$A(\boldsymbol{s}) := \sup_{0<\boldsymbol{y}<\boldsymbol{b}} \left(\int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \left[\frac{k(\boldsymbol{x}, \boldsymbol{y})}{K(\boldsymbol{x})}\right]^q u(\boldsymbol{x}) V_1^{\frac{q(p-s_1)}{p}}(x_1) \cdots V_n^{\frac{q(p-s_n)}{p}}(x_n) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}} \times V_1^{\frac{s_1-1}{p}}(t_1) \cdots V_n^{\frac{s_n-1}{p}}(t_n) < \infty$$

holds, where $V_i(x_i) = \int_0^{x_i} v_i^{1-p'} dt_i$, $i = 1, 2, \dots, n$. Furthermore, if C is the best possible constant in (2.2)*, then*

$$C \leqslant A \inf_{1<\boldsymbol{s}<p} \left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}} \cdots \left(\frac{p-1}{p-s_n}\right)^{\frac{1}{p'}} A(\boldsymbol{s}).$$

REMARK 2.3. For the case $A = 1$, Corollary 2.1 reduces to Theorem 2.1 in [**22**]. Observe also that the case $n = 1$ yields a result which is more general than Theorem 4.4 in [**9**].

Our next result reads:

THEOREM 2.2. *Let $1 < p \leqslant q < \infty$, $0 < b_i \leqslant \infty$, $s_1, \dots, s_n \in (1, p)$, $i = 1, 2, \dots, n$ ($n \in \mathbb{Z}_+$) and let ϕ be a nonnegative increasing function on (a, c) satisfying* (2.1). *Let A_K be a general Hardy type operator defined by* (1.3) *and let $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ be weight functions, where $v(\boldsymbol{x})$ is of product type i.e., $v(\boldsymbol{x}) = v(x_1)\, v(x_2) \cdots v(x_n)$. Then the inequality*

$$\left(\int_0^{b_1} \cdots \int_0^{b_n} \phi^{\frac{q}{p}}(A_K f(\boldsymbol{x}))\, u(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}} \leqslant C\left(\int_0^{b_1} \cdots \int_0^{b_n} \phi(f(\boldsymbol{x}))\, v(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{p}} \tag{2.5}$$

holds for all functions f such that $a < f(\boldsymbol{x}) < c$, if

$$A(\boldsymbol{s}) := \left(\sup_{0<\boldsymbol{y}<\boldsymbol{b}} \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \left[\frac{k(\boldsymbol{x}, \boldsymbol{y})}{K(\boldsymbol{x})}\right]^{\frac{q}{p}} u(\boldsymbol{x}) V_1^{q(p-s_1)}(x_1) \cdots V_n^{q(p-s_n)}(x_n) \frac{d\boldsymbol{x}}{x_1 \cdots x_n}\right)^{\frac{1}{q}} \times V_1^{\frac{s_1-1}{p}}(y_1) \cdots V_n^{\frac{s_n-1}{p}}(y_n) < \infty$$

holds, where $V_i(x_i) = \int_0^{x_i} v_i^{1-p'} dt_i$, $i = 1, 2, \dots, n$.

Furthermore, if C *is the best possible constant in* (2.5)*, then*

$$C \leqslant \left(\frac{B}{A}\right)^{\frac{1}{p}} \inf_{1<\boldsymbol{s}<\boldsymbol{p}} \left(\frac{p-1}{p-s_1}\right)^{\frac{p}{p'}} \cdots \left(\frac{p-1}{p-s_n}\right)^{\frac{p}{p'}} A(\boldsymbol{s}).$$

REMARK 2.4. If $A = B = 1$, then Theorem 2.2 coincides with Theorem 3.1 in [**9**].

PROOF. By applying condition (2.1) followed by Jensen's inequality to the left-hand side of (2.5) we obtain that

$$\begin{aligned}
&\left(\int_0^{b_1}\cdots\int_0^{b_n}\phi^{\frac{q}{p}}(A_K f(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}} \qquad (2.6)\\
&= \left(\int_0^{b_1}\cdots\int_0^{b_n}\left[\phi\left(\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\cdots\int_0^{x_n}k(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}\right)\right]^{\frac{q}{p}}u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}}\\
&\leqslant B^{\frac{1}{p}}\left(\int_0^{b_1}\cdots\int_0^{b_n}\left[\psi\left(\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\cdots\int_0^{x_n}k(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}\right)\right]^{\frac{q}{p}}u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}}\\
&\leqslant B^{\frac{1}{p}}\left(\int_0^{b_1}\cdots\int_0^{b_n}\left[\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\cdots\int_0^{x_n}k(\boldsymbol{x},\boldsymbol{y})\,\psi(f(\boldsymbol{y}))\,d\boldsymbol{y}\right]^{\frac{q}{p}}u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}}.
\end{aligned}$$

Define g such that $\psi^{\frac{1}{p}}(f(\boldsymbol{x}))\frac{v(\boldsymbol{x})}{x_1\cdots x_n} = \psi(g(x_1,\dots,x_n))$ and substitute in (2.6) to find that

$$\begin{aligned}
&\left(\int_0^{b_1}\cdots\int_0^{b_n}\phi^{\frac{q}{p}}(A_K f(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}}\\
&\leqslant B^{\frac{1}{p}}\left(\int_0^{b_1}\cdots\int_0^{b_n}\left[\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\cdots\int_0^{x_n}k(\boldsymbol{x},\boldsymbol{y})\,\psi^p(g(\boldsymbol{y}))\,v_1^{-p}(y_1)\cdots v_n^{-p}(y_n)\,y_1^p\cdots y_n^p d\boldsymbol{y}\right]^{\frac{q}{p}}\right.\\
&\qquad\qquad\left.\times\, u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}}.
\end{aligned}$$

By applying Hölder's inequality, Lemma 2.1 and using the fact that

$$v(\boldsymbol{x}) = v_1(x_1)\cdots v_n(x_n),$$

we have that

$$\begin{aligned}
&\left(\int_0^{b_1}\cdots\int_0^{b_n}\phi^{\frac{q}{p}}(A_K f(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}}\\
&\leqslant B^{\frac{1}{p}}\left(\int_0^{b_1}\cdots\int_0^{b_n}\left[\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\cdots\int_0^{x_n}k(\boldsymbol{x},\boldsymbol{y})\,\psi^p(g(\boldsymbol{y}))V_1^{(s_1-1)p}(y_1)\cdots V_n^{(s_n-1)p}(y_n)\right.\right.\\
&\qquad\left.\left.\times\, V_1^{-(s_1-1)p}(y_1)\cdots V_n^{-(s_n-1)p}(y_n)\,v_1^{-p}(y_1)\cdots v_n^{-p}(y_n)\,y_1^p\cdots y_n^p d\boldsymbol{y}\right]^{\frac{q}{p}}u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}}\\
&\leqslant B^{\frac{1}{p}}\left(\int_0^{b_1}\cdots\int_0^{b_n}\left[\int_0^{x_1}\cdots\int_0^{x_n}k^{\frac{1}{p}}(\boldsymbol{x},\boldsymbol{y})\,\psi(g(\boldsymbol{y}))V_1^{s_1-1}(\boldsymbol{y}_1)\cdots V_n^{s_n-1}(\boldsymbol{y}_n)\,d\boldsymbol{y}\right]^{q}\right.
\end{aligned}$$

$$\times \left[\int_0^{x_1}\cdots\int_0^{x_n} V_1^{\frac{-p'(s_1-1)}{p}}(y_1)\cdots V_n^{\frac{-p'(s_n-1)}{p}}(y_n)\, v_1^{-\frac{p'}{p}}(y_1)\cdots v_n^{-\frac{p'}{p}}(y_n)\, y_1^{\frac{p'}{p}}\cdots y_n^{\frac{p'}{p}}\, d\boldsymbol{y}\right]^{\frac{qp}{p'}}$$

$$\times \left.\frac{u(\boldsymbol{x})d\boldsymbol{x}}{K^{\frac{q}{p}}(\boldsymbol{x})x_1\cdots x_n}\right)^{\frac{1}{q}}$$

$$\leqslant B^{\frac{1}{p}}\left(\frac{p-1}{p-s_1}\right)^{\frac{p}{p'}}\cdots\left(\frac{p-1}{p-s_n}\right)^{\frac{p}{p'}}\left(\int_0^{b_1}\cdots\int_0^{b_n}\psi(g(\boldsymbol{y}))V_1^{s_1-1}(\boldsymbol{y}_1)\cdots V_n^{s_n-1}(\boldsymbol{y}_n)\right.$$

$$\times\left.\left[\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n} V_1^{(p-s_1)q}(x_1)\cdots V_n^{(p-s_n)q}(x_n)u(\boldsymbol{x})\left(\frac{k(\boldsymbol{x},\boldsymbol{y})}{K(\boldsymbol{x})}\right)^{\frac{q}{p}}\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right]d\boldsymbol{y}\right)^{\frac{1}{p}}$$

$$\leqslant B^{\frac{1}{p}}\left(\frac{p-1}{p-s_1}\right)^{\frac{p}{p'}}\cdots\left(\frac{p-1}{p-s_n}\right)^{\frac{p}{p'}}A(\boldsymbol{s})\left(\int_0^{b_1}\cdots\int_0^{b_n}\psi(g(\boldsymbol{y}))\,d\boldsymbol{y}\right)^{\frac{1}{p}}$$

$$\leqslant \left(\frac{B}{A}\right)^{\frac{1}{p}}\left(\frac{p-1}{p-s_1}\right)^{\frac{p}{p'}}\cdots\left(\frac{p-1}{p-s_n}\right)^{\frac{p}{p'}}A(\boldsymbol{s})\left(\int_0^{b_1}\cdots\int_0^{b_n}\phi(g(\boldsymbol{y}))\,d\boldsymbol{y}\right)^{\frac{1}{p}}$$

and the proof is complete. □

Next, we give the following more general result:

THEOREM 2.3. *Let $t\in[1,\infty)$, $0<b_i\leqslant\infty$, $i=1,2,\dots,n$ $(n\in\mathbb{Z}_+)$ and let ϕ be a nonnegative increasing function on (a,c), $-\infty\leqslant a<c\leqslant\infty$ for which there exist a convex function ψ on (a,c) such that*

$$A\psi(\boldsymbol{x})\leqslant\phi(\boldsymbol{x})\leqslant B\psi(\boldsymbol{x}) \tag{2.7}$$

holds for constants $0<A\leqslant B<\infty$. Let A_K be a general Hardy type operator defined by (1.3), $K(\boldsymbol{x})$ be as defined in (1.4) and $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ be weight functions on $(\boldsymbol{0},\boldsymbol{b})$. Then the inequality

$$\int_0^{b_1}\cdots\int_0^{b_n}\phi^t(A_Kf(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\leqslant\left(\frac{B}{A}\right)^t\left(\int_0^{b_1}\cdots\int_0^{b_n}\phi(f(\boldsymbol{x}))\,v(\boldsymbol{x})\,d\boldsymbol{x}\right)^t \tag{2.8}$$

holds for all functions f such that $a<f(\boldsymbol{x})<c$, where

$$v(\boldsymbol{y}):=\left(\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\left[\frac{k(\boldsymbol{x},\boldsymbol{y})}{K(\boldsymbol{x})}\right]^t u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{t}}.$$

REMARK 2.5. In Theorem 2.3, if we set $t=1$, then we obtain that

$$\int_0^{b_1}\cdots\int_0^{b_n}\phi(A_Kf(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\leqslant\frac{B}{A}\int_0^{b_1}\cdots\int_0^{b_n}\phi(f(\boldsymbol{x}))\,v(\boldsymbol{x})\,d\boldsymbol{x}$$

where

$$v(\boldsymbol{y}):=\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\frac{k(\boldsymbol{x},\boldsymbol{y})}{K(\boldsymbol{x})}\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}.$$

Proof. By using assumption (2.7), Jensen's inequality, Lemma 2.1 and the monotonicity of the function $\alpha \to \alpha^t$, to the left-hand side of inequality (2.8) we obtain that

$$\begin{aligned}
&\int_0^{b_1}\cdots\int_0^{b_n}\phi^t(A_K f(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\\
&=\int_0^{b_1}\cdots\int_0^{b_n}\left[\phi\left(\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\cdots\int_0^{x_n}k(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}\right)\right]^t u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\\
&\leqslant B^t\int_0^{b_1}\cdots\int_0^{b_n}\left[\psi\left(\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\cdots\int_0^{x_n}k(\boldsymbol{x},\boldsymbol{y})f(\boldsymbol{y})\,d\boldsymbol{y}\right)\right]^t u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\\
&\leqslant B^t\int_0^{b_1}\cdots\int_0^{b_n}\left(\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\cdots\int_0^{x_n}k(\boldsymbol{x},\boldsymbol{y})\,\psi(f(\boldsymbol{y}))\,d\boldsymbol{y}\right)^t u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\\
&\leqslant B^t\left\{\left(\int_0^{b_1}\cdots\int_0^{b_n}\left(\frac{1}{K(\boldsymbol{x})}\int_0^{x_1}\cdots\int_0^{x_n}k(\boldsymbol{x},\boldsymbol{y})\,\psi(f(\boldsymbol{y}))\,d\boldsymbol{y}\right)^t u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{t}}\right\}^t\\
&\leqslant B^t\left(\int_0^{b_1}\cdots\int_0^{b_n}\psi(f(\boldsymbol{y}))\left(\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}k^t(\boldsymbol{x},\boldsymbol{y})\frac{u(\boldsymbol{x})}{K^t(\boldsymbol{x})}\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{t}}d\boldsymbol{y}\right)^t\\
&\leqslant B^t\left(\int_0^{b_1}\cdots\int_0^{b_n}\psi(f(\boldsymbol{y}))v(y)\,d\boldsymbol{y}\right)^t\leqslant\left(\frac{B}{A}\right)^t\left(\int_0^{b_1}\cdots\int_0^{b_n}\phi(f(\boldsymbol{y}))\,v(y)\,d\boldsymbol{y}\right)^t,
\end{aligned}$$

and the proof is complete. □

3. Further results and examples

By using our results in special cases, we obtain some multidimensional Hardy-type inequalities which are also new even for the case $n = 1$. Specifically, we obtain the following results:

Remark 3.1. For the case $p = q = 1$ in Theorem 2.2, then inequality (2.5) reduces to

$$\int_0^{b_1}\cdots\int_0^{b_n}\phi(A_K f(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\leqslant C\int_0^{b_1}\cdots\int_0^{b_n}\phi(f(\boldsymbol{x}))\,v(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n} \tag{3.1}$$

where

$$C\leqslant\left(\frac{B}{A}\right)\inf_{1<\boldsymbol{s}<\boldsymbol{p}}\left(\frac{p-1}{p-s_1}\right)^{\frac{p}{p'}}\cdots\left(\frac{p-1}{p-s_n}\right)^{\frac{p}{p'}}A(\boldsymbol{s}).$$

Observe that (3.1) is new for the case $n = 1$.

In Theorem 2.3 if $t = \frac{q}{p}$, then we obtain:

Example 3.1. Let $t = \frac{q}{p} \geqslant 1$, $0 < b_i \leqslant \infty$, $i = 1, 2, \ldots, n$ ($n \in \mathbb{Z}_+$) and let ϕ be a nonnegative increasing function on (a, c), $-\infty \leqslant a < c \leqslant \infty$ for which there exist a convex function ψ on (a, c) such that $A\psi(\boldsymbol{x}) \leqslant \phi(\boldsymbol{x}) \leqslant B\psi(\boldsymbol{x})$ holds for constants $0 < A \leqslant B < \infty$. Let A_K be a general Hardy type operator defined by (1.3), $K(\boldsymbol{x})$

be as defined in (1.4) and $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ be weight functions on $(\boldsymbol{0},\boldsymbol{b})$. Then

$$\int_0^{b_1}\cdots\int_0^{b_n}\phi^{\frac{q}{p}}(A_K f(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\leqslant\left(\frac{B}{A}\right)^{\frac{q}{p}}\left(\int_0^{b_1}\cdots\int_0^{b_n}\phi(f(\boldsymbol{x}))\,v(\boldsymbol{x})\,d\boldsymbol{x}\right)^{\frac{q}{p}}, \tag{3.2}$$

where

$$v(\boldsymbol{y}):=\left(\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\left[\frac{k(\boldsymbol{x},\boldsymbol{y})}{K(\boldsymbol{x})}\right]^{\frac{q}{p}}u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{p}{q}},$$

and

$$C\leqslant\left(\frac{B}{A}\right)^{\frac{1}{p}}\inf_{1<\boldsymbol{s}<\boldsymbol{p}}\left(\frac{p-1}{p-s_1}\right)^{\frac{p}{p'}}\cdots\left(\frac{p-1}{p-s_n}\right)^{\frac{p}{p'}}A(\boldsymbol{s}).$$

REMARK 3.2. Note that in the case $A=B=1$ in Example 3.1, we see that $\phi(x)=A\psi(x)$ is convex, and coupled with the fact that $\frac{q}{p}\geqslant 1$, then (3.2) yields

$$\int_0^{b_1}\cdots\int_0^{b_n}\phi^{\frac{q}{p}}(A_K f(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\leqslant\left(\int_0^{b_1}\cdots\int_0^{b_n}\phi(f(\boldsymbol{x}))\,v(\boldsymbol{x})\,d\boldsymbol{x}\right)^{\frac{q}{p}} \tag{3.3}$$

and the inequality sign is reversed if ϕ is concave

REMARK 3.3. For the special case $p=q=1$ in (3.3) we obtain that

$$\int_0^{b_1}\cdots\int_0^{b_n}\phi(A_K f(\boldsymbol{x}))\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\leqslant\int_0^{b_1}\cdots\int_0^{b_n}\phi(f(\boldsymbol{x}))\,v(\boldsymbol{x})\,d\boldsymbol{x}.$$

which coincides with [**9**, Theorem 4.1] when $n=1$.

In Theorem 2.2, if we set $\psi(x)=x^p$, then we obtain the following result:

EXAMPLE 3.2. Let $1<p\leqslant q<\infty$, $0<b_i\leqslant\infty$, $s_1,\dots,s_n\in(1,p)$, $i=1,2,\dots,n$ $(n\in\mathbb{Z}_+)$ and let ϕ be a nonnegative increasing function on (a,c) such that $A\boldsymbol{x}^p\leqslant\phi(\boldsymbol{x})\leqslant B\boldsymbol{x}^p$. Let A_K be a general Hardy type operator defined by (1.3) and let $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ be weight functions, where $v(\boldsymbol{x})$ is of product type i.e., $v(\boldsymbol{x})=v(x_1)\,v(x_2)\cdots v(x_n)$; then

$$\left(\int_0^{b_1}\cdots\int_0^{b_n}[\phi(A_K f(\boldsymbol{x}))]^q\,u(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}}\leqslant C\left(\int_0^{b_1}\cdots\int_0^{b_n}\phi^p(f(\boldsymbol{x}))\,v(\boldsymbol{x})\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{p}}$$

holds for all functions f such that $a<f(\boldsymbol{x})<c$, if

$$A(\boldsymbol{s}):=\left(\sup_{0<\boldsymbol{y}<\boldsymbol{b}}\int_{y_1}^{b_1}\cdots\int_{y_n}^{b_n}\left[\frac{k(\boldsymbol{x},\boldsymbol{y})}{K(\boldsymbol{x})}\right]^{\frac{q}{p}}u(\boldsymbol{x})V_1^{q(p-s_1)}(x_1)\cdots V_n^{q(p-s_n)}(x_n)\frac{d\boldsymbol{x}}{x_1\cdots x_n}\right)^{\frac{1}{q}}$$
$$\times V_1^{\frac{s_1-1}{p}}(y_1)\cdots V_n^{\frac{s_n-1}{p}}(y_n)<\infty$$

and

$$C\leqslant\left(\frac{B}{A}\right)^{\frac{1}{p}}\inf_{1<\boldsymbol{s}<\boldsymbol{p}}\left(\frac{p-1}{p-s_1}\right)^{\frac{p}{p'}}\cdots\left(\frac{p-1}{p-s_n}\right)^{\frac{p}{p'}}A(\boldsymbol{s}).$$

Finally, by putting $k(\boldsymbol{x},\boldsymbol{y})=1$, $\psi(x)=x^p$ and $t=1$ in Theorem 2.3, condition (2.1) reduces to $A\boldsymbol{x}^p\leqslant\phi(\boldsymbol{x})\leqslant B\boldsymbol{x}^p$ and so we obtain the following result:

EXAMPLE 3.3. Let $1 < p < \infty$, $0 < b_i \leqslant \infty$, $i = 1, 2, \dots, n$ ($n \in \mathbb{Z}_+$) and let ϕ be a nonnegative convex function on (a, c), $-\infty \leqslant a < c \leqslant \infty$ for which

$$A\boldsymbol{x}^p \leqslant \phi(\boldsymbol{x}) \leqslant B\boldsymbol{x}^p$$

holds for constants $0 < A \leqslant B < \infty$. If $u(\boldsymbol{x})$ and $v(\boldsymbol{x})$ are weight functions on $(\boldsymbol{0}, \boldsymbol{b})$, then the inequality

$$\int_0^{b_1} \cdots \int_0^{b_n} \left(\frac{1}{x_1 \cdots x_n} \int_0^{x_1} \cdots \int_0^{x_n} f(\boldsymbol{y})\, d\boldsymbol{y} \right)^p u(\boldsymbol{x}) \frac{d\boldsymbol{x}}{x_1 \cdots x_n} \leqslant \frac{B}{A} \int_0^{b_1} \cdots \int_0^{b_n} f^p(\boldsymbol{x})\, v(\boldsymbol{x})\, d\boldsymbol{x} \tag{3.4}$$

holds, where

$$v(\boldsymbol{y}) := \int_{y_1}^{b_1} \cdots \int_{y_n}^{b_n} \frac{u(\boldsymbol{x})}{x_1^2 \cdots x_n^2} d\boldsymbol{x}.$$

The sign of (3.4) is reversed if ϕ is concave.

REMARK 3.4. For the case $n = 1$, inequality (3.4) yields the following result

$$\int_0^b \left(\frac{1}{x} \int_0^x f(y)\, dy \right)^p u(x) \frac{dx}{x} \leqslant \frac{B}{A} \int_0^b f^p(x) v(x)\, dx,$$

which is also new.

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