PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 93 (107) (2013), 173–186 DOI: 10.2298/PIM1307173H

BLOW UP RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS AND SYSTEMS

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Communicated by Stevan Pilipović

Abstract. The aim of this research paper is to establish sufficient conditions for the nonexistence of global solutions for the following nonlinear fractional differential equation

$$\mathbf{D}^{\alpha}_{0|t}u + (-\Delta)^{\beta/2}|u|^{m-1}u + a(x)\cdot\nabla|u|^{q-1}u = h(x,t)|u|^{p}, \quad (t,x)\in Q,$$
$$u(0,x) = u_0(x), \quad x\in\mathbb{R}^N$$

where $(-\Delta)^{\beta/2}$, $0<\beta<2$ is the fractional power of $-\Delta$, and $\mathbf{D}^{\alpha}_{0|t}$, $(0<\alpha<1)$ denotes the time-derivative of arbitrary $\alpha\in(0;1)$ in the sense of Caputo. The results are shown by the use of test function theory and extended to systems of the same type.

1. Introduction

In his article [3], Fujita considered the Cauchy problem

$$\begin{aligned} u_t &= \Delta u + u^{1+\tilde{p}} \text{ in } Q = \mathbb{R}^n\times\mathbb{R}_+ \\ u(0,x) &= a(x) \text{ in } \mathbb{R}^n \end{aligned} \tag{1.1}$$

where $\tilde{p}>0$. If $p_c = \frac{2}{n}$, he proved that:

1. If $0\leqslant\tilde{p}\leqslant p_c$ and $a(x_0)>0$ for some x_0, then any solution to (1.1) blows-up in a finite time.

2. If $\tilde{p}>p_c$, then there exist a solution on Q.

The critical case $\tilde{p}=p_c$ was decided later by Hayakawa [6] for $N=1,2$ and by Kobayashi, Sirao and Tanaka [9] for $n\geqslant 3$.

2010 *Mathematics Subject Classification:* 58J45, 26A33, 35B44.
Key words and phrases: blow-up; fractional derivatives; critical exponent.
This research was partly supported by MESRS-ALGERIA (CNEPRU B02120120054).

In a more recent article, Guedda and Kirane [5] extended the previous results to the equations

$$u_t = (-\Delta)^{\beta/2}u + h(x,t)u^{1+p} \text{ in } Q = \mathbb{R}^N \times \mathbb{R}_+$$
$$u(0,x) = a(x) \text{ in } \mathbb{R}^N$$

where $h(x,t) = O(t^\sigma |x|^\rho)$ for $|x|$ large.

Finally, Kirane and Qafsaoui [8] treated the more general equation

$$u_t + (-\Delta)^{\beta/2}(u^m) + a(x,t) \cdot \nabla u^q = h(x,t)u^{1+\tilde{p}}, \text{ in } Q.$$

The technique we use has been introduced by Mitidieri and Pohozaev [10], [11], Pohozaev and Tesei [12], Pohozaev and Veron [14] and used by Hakem and Berbiche [1].

Let us consider the following nonlinear fractional differential equation

$$\begin{aligned} &\mathbf{D}^\alpha_{0|t}u + (-\Delta)^{\frac{\beta}{2}}(|u|^{m-1}u) + a(x)\cdot\nabla(|u|^{q-1}u) = h(x,t)|u|^p \\ &u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \end{aligned} \tag{1.2}$$

where $\mathbf{D}^\alpha_{0|t}$ denotes the time-derivative of an arbitrary order $\alpha \in (0,\alpha)$ in the sense of Caputo [14], $(-\Delta)^{\beta/2}$, $\beta \in [1,2]$, is the $(\frac{\beta}{2})$-fractional power of the Laplacian $-\Delta_x$ in the x variable; $a(x) := (a_1(x),\dots,a_N(x))$ and $h(x,t)$ are given functions, $a(x)\cdot\nabla(|u|^{q-1}u)$ is the scalar product of $a(x)$ and $\nabla(|u|^{q-1}u)$ and the exponents $p>1$, $q \geqslant 1$ and $m \geqslant 1$ are positive constants. The nonlocal operator $(-\Delta)^{\frac{\beta}{2}}$ is defined by $(-\Delta)^{\frac{\beta}{2}}v(x) = \mathcal{F}^{-1}\big(|\xi|^\beta\mathcal{F}(v)(\xi)\big)(x)$ for every $v \in D((-\Delta)^{\frac{\beta}{2}}) = H^\beta(\mathbb{R}^N)$, where $H^\beta(\mathbb{R}^N)$ is the homogeneous Sobolev space of order β defined by

$$H^\beta(\mathbb{R}^N) = \big\{u \in \mathcal{S}';\ \ (-\Delta)^{\frac{\beta}{2}}u \in \mathbb{L}^2(\mathbb{R}^N)\big\} \quad \text{if } \ \beta \notin \mathbb{N},$$
$$H^\beta(\mathbb{R}^N) = \big\{u \in \mathbb{L}^2(\mathbb{R}^N);\ \ (-\Delta)^{\frac{\beta}{2}}u \in \mathbb{L}^2(\mathbb{R}^N)\big\} \quad \text{if } \ \beta \in \mathbb{N},$$

where $\mathcal{S}'$ is the space of Schwartz distributions; $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ its inverse. The fractional Laplacian $(-\Delta)^{\frac{\beta}{2}}$ is related to Lévy flights in physics. Many observations and experiments related to Lévy flights (superdiffusion), e.g., collective slip diffusion on solid surfaces, quantum optics or Richardson turbulent diffusion, have been recently performed. The symmetric β-stable processes ($\beta \in (0,2)$) are the basic characteristics for a class of jumping Lévy's processes. Compared with the continuous Brownian motion ($\beta = 2$), symmetric β-stable processes have infinite jumps in an arbitrary time interval. The large jumps of these processes make their variances and expectations infinite according to $\beta \in (0,2)$ or $\beta \in (0,1]$, respectively. It is worth mentioning that when $\beta = \frac{3}{2}$, the symmetric β-stable processes appear in the study of stellar dynamics. The time fractional derivative has been found to be very effective means to describe the anomalous attenuation behaviors. We here recall some definitions of fractional derivative.

The left-handed derivative and the right-handed derivative in the Riemann–Liouville sense for $\Psi \in L^1(0,T)$, $0<\alpha<1$ are defined as follows:

$$(D^\alpha_{0|t}\Psi)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\Psi(\sigma)}{(t-\sigma)^\alpha}\,d\sigma,$$

where the symbol Γ stands for the usual Euler gamma function, and

$$(D^\alpha_{t|T}\Psi)(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{\Psi(\sigma)}{(\sigma-t)^\alpha}\,d\sigma,$$

respectively. The Caputo derivative

$$(\mathbf{D}^\alpha_{0|t}\Psi)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\Psi'(\sigma)}{(t-\sigma)^\alpha}\,d\sigma,$$

requires $\Psi' \in L^1(0,T)$.

Clearly we have

$$(D^\alpha_{0|t}g)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{g(0)}{t^\alpha} + \int_0^t \frac{g'(\sigma)}{(t-\sigma)^\alpha}\,d\sigma\right],$$

$$(D^\alpha_{t|T}f)(t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{f(T)}{(T-t)^\alpha} - \int_t^T \frac{f'(\sigma)}{(\sigma-t)^\alpha}\,d\sigma\right]. \tag{1.3}$$

Therefore the Caputo derivative is related to the Riemann–Liouville derivative by

$$\mathbf{D}^\alpha_{0|t}\Psi(t) = D^\alpha_{0|t}[\Psi(t)-\Psi(0)].$$

We will use the formula of integration by parts [13, p. 46]

$$\int_0^T f(t)(D^\alpha_{0|t}g)(t)\,dt = \int_0^T g(t)(D^\alpha_{t|T}f)(t)\,dt.$$

Solutions to problem (1.2) are meant in the following sense.

DEFINITION 1.1. A function $u \in L^p_{\mathrm{loc}}(Q_T)$, where $Q_T := \mathbb{R}^N \times (0,T)$, is a local weak solution to (1.2) defined on Q_T, if $uh^{1/p} \in L^1_{\mathrm{loc}}(Q_T, dx\,dt)$ such that

$$\begin{aligned}\int_{Q_T} h(x,t)\xi|u|^p dx\,dt + \int_{Q_T} u_0 D^\alpha_{t|T}\xi\,dx\,dt \\ = \int_{Q_T} uD^\alpha_{t|T}\xi\,dx\,dt + \int_{Q_T}|u|^{m-1}u(-\Delta)^{\frac{\beta}{2}}\xi\,dx\,dt \\ - \sum_{i=1}^N\int_{Q_T}|u|^{q-1}u\xi\frac{\partial a_i}{\partial x_i}\,dx\,dt - \int_{Q_T}|u|^{q-1}ua\cdot\nabla\xi\,dx\,dt\end{aligned} \tag{1.4}$$

for any test function $\xi \in C^{2,1}_{x,t}(Q_T)$, such that $\xi(x,T)=0$.

The integrals in the definition are supposed to be convergent. If in the above definition, $T=+\infty$ the solution is called global.

To begin, we set some hypotheses. For the function h, we require the condition

$$h(yR,\tau T^{\beta/\alpha}) \geqslant C_h R^\sigma T^{\rho\beta/\alpha}, \quad C_h>0, \tag{$\mathcal{H}_h$}$$

for some $\sigma, \rho > 0$ to be determined later, R, T large and $\tau \geqslant 0$, y in a bounded domain. It can easily be seen that there is no conditions imposed on σ. The vector $a(x) = (a_1(x), \ldots, a_N(x))$ is required to satisfy

$$(\mathcal{H}_a) \qquad |a_i(x)| \sim c|x|^{\delta_i} \text{ for } x \text{ large and } \delta_i > 2.$$

For later use, we define $\delta = \max(\delta_i)$.

1.1. The Results.

Now, we may state our first result.

THEOREM 1.1. *Let $N \geqslant 1$, $p > \max(m, q) \geqslant 1$. The exponent ρ satisfies*

$$(\rho + 1) > \max\Big\{\frac{p}{m}, (1-\alpha)p, \frac{p}{q}\Big\}.$$

Assume that $(\mathcal{H}_h)$ and $(\mathcal{H}_a)$ are satisfied and $u_0(x)$ satisfies $u_0(x) \geqslant 0$. If

$$p \leqslant \min\left(1 + \frac{\alpha(\sigma+\beta) + \beta\rho}{\alpha N + \beta(1-\alpha)}, \frac{((\alpha N + \beta) + (\alpha\sigma + \beta\rho))q}{((\delta-1)\alpha + (N\alpha + \beta))}\right),$$

then problem (1.2) admits no global weak solutions other than the trivial one.

PROOF. The proof proceeds by contradiction. Suppose that u is a nontrivial solution which exists globally in time. That is exists in $(0, T^*)$ for any arbitrary $T^* > 0$. Let T and R be two positive real numbers such that $0 < TR^{\beta/\alpha} < T^*$. For later use, let Φ be a smooth nonincreasing function such that

$$\Phi(z) = \begin{cases} 1, & \text{if } z \leqslant 1, \\ 0, & \text{if } z \geqslant 2. \end{cases}$$

and $0 \leqslant \Phi \leqslant 1$. The test function ξ is chosen so that

$$\int_{Q_T} (h\xi)^{-m/(p-m)} \big|(-\Delta)^{\beta/2}\xi\big|^{p/(p-m)} dx\,dt < \infty, \quad \int_{Q_T} (h\xi)^{-1/(p-1)} |D^{\alpha}_{t|T}\xi|^{p/(p-1)} dx\,dt < \infty,$$

$$\int_{Q_T} h^{-q/(p-q)}\xi \bigg|\sum_{i=1}^{N} \frac{\partial a}{\partial x_i}\bigg|^{p/(p-q)} dx\,dt < \infty, \quad \int_{Q_T} (h\xi)^{-q/(p-q)} |a\cdot\nabla\xi|^{p/(p-q)} dx\,dt < \infty.$$

To estimate the right-hand side of (1.2) on $Q_{TR^{2/\theta}}$, we write

$$\int_{Q_{TR^{2/\theta}}} |u|^{m-1}u(-\Delta)^{\beta/2}\xi dx\,dt = \int_{Q_{TR^{2/\theta}}} |u|^{m-1}u(h\xi)^{m/p}(h\xi)^{-m/p}(-\Delta)^{\beta/2}\xi dx\,dt.$$

Using the ε-Young inequality

$$XY \leqslant \varepsilon X^p + C(\varepsilon)Y^{p'}, \quad p + p' = pp', \quad X, Y \geqslant 0,$$

we have the estimate

$$\int_{Q_{TR^{2/\theta}}} |u|^{m-1}u(-\Delta)^{\frac{\beta}{2}}\xi dx\,dt$$
$$\leqslant \varepsilon \int_{Q_{TR^{2/\theta}}} h\xi|u|^p dx\,dt + C(\varepsilon)\int_{Q_{TR^{2/\theta}}} (h\xi)^{\frac{-m}{p-m}}|(-\Delta)^{\frac{\beta}{2}}\xi|^{\frac{p}{p-m}} dx\,dt.$$

Similarly,

$$\int_{Q_{TR^{2/\theta}}} uD^{\alpha}_{t|TR^{2/\theta}}\xi dx\,dt = \int_{Q_{TR^{2/\theta}}} u(h\xi)^{\frac{1}{p}}(h\xi)^{\frac{-1}{p}} D^{\alpha}_{t|TR^{2/\theta}}\xi dx\,dt$$
$$\leqslant \varepsilon \int_{Q_{TR^{2/\theta}}} h\xi|u|^p dx\,dt + C(\varepsilon)\int_{Q_{TR^{2/\theta}}} (h\xi)^{\frac{-1}{p-1}}|D^{\alpha}_{t|TR^{2/\theta}}\xi|^{\frac{p}{p-1}}dx\,dt.$$

Integrating by parts, we get

$$\int_{Q_{TR^{2/\theta}}} a\cdot\nabla(|u|^{q-1}u)\xi dx\,dt$$
$$= -\int_{Q_{TR^{2/\theta}}} |u|^{q-1}ua\cdot\nabla\xi dx\,dt - \int_{Q_{TR^{2/\theta}}} \sum_{i=1}^{N}|u|^{q-1}u\xi\frac{\partial a}{\partial x_i}dx\,dt.$$

Now writing

$$\int_{Q_{TR^{2/\theta}}} |u|^{q-1}u\xi\sum_{i=1}^{N}\frac{\partial a}{\partial x_i}dx\,dt = \int_{Q_{TR^{2/\theta}}} |u|^{q-1}u(h\xi)^{\frac{q}{p}}h^{\frac{-q}{p}}\xi^{\frac{(p-q)}{p}}\sum_{i=1}^{N}\frac{\partial a}{\partial x_i}dx\,dt,$$

and using the ε-Young inequality, we get

$$\int_{Q_{TR^{2/\theta}}} |u|^{q-1}u\xi\sum_{i=1}^{N}\frac{\partial a}{\partial x_i}dx\,dt$$
$$\leqslant \varepsilon\int h\xi|u|^p dx\,dt + C(\varepsilon)\int_{Q_{TR^{2/\theta}}} h^{\frac{-q}{p-q}}\xi\left|\sum_{i=1}^{N}\frac{\partial a}{\partial x_i}\right|^{\frac{p}{p-q}}dx\,dt.$$

Similarly, we have

$$\int_{Q_{TR^{2/\theta}}} |u|^{q-1}u(a\cdot\nabla\xi)\,dx\,dt$$
$$\leqslant \varepsilon\int_{Q_{TR^{2/\theta}}} h\xi|u|^p dx\,dt + C(\varepsilon)\int_{Q_{TR^{2/\theta}}} (h\xi)^{\frac{-q}{p-q}}|a\cdot\nabla\xi|^{\frac{p}{p-q}}dx\,dt.$$

Combining the above estimates with (1.4) and taking ε small enough, we infer that

(1.5)

$$\int_{Q_{TR^{2/\theta}}} u_0 D^{\alpha}_{t|TR^{2/\theta}}\xi\,dx\,dt + \int_{Q_{TR^{2/\theta}}} |u|^p\xi h\,dx\,dt$$
$$\leqslant C(\varepsilon)\Big(\int_{Q_{TR^{2/\theta}}} (h\xi)^{\frac{-1}{p-1}}\left|D^{\alpha}_{t|TR^{2/\theta}}\xi\right|^{\frac{p}{p-1}}dx\,dt + \int_{Q_{TR^{2/\theta}}} h^{\frac{-q}{p-q}}\xi\left|\sum_{i=1}^{N}\frac{\partial a}{\partial x_i}\right|^{\frac{p}{p-q}}dx\,dt$$
$$+\int_{Q_{TR^{2/\theta}}} (h\xi)^{\frac{-q}{p-q}}|a\cdot\nabla\xi|^{\frac{p}{p-q}}dx\,dt + \int_{Q_{TR^{2/\theta}}} (h\xi)^{-\frac{m}{p-m}}\left|(-\Delta)^{\frac{\beta}{2}}\xi\right|^{\frac{m}{p-m}}dx\,dt\Big).$$

At this stage, we set

$$\xi(x,t) := \Phi\Big(\frac{|x|^2+t^{\theta}}{R^2}\Big),$$

where R and θ are positive real numbers to be determined latter.

We note that $\xi(x, TR^{2/\theta}) = 0$ for $T^\theta \geqslant 2$, then by (1.3) we have

$$\int_{Q_{TR^{2/\theta}}} u_0 D^\alpha_{t|TR^{2/\theta}} \xi \, dx\, dt \geqslant 0.$$

Let us perform the change of variables $\tau = t/R^{2/\theta}$, $y = x/R$, and set

$$\Omega := \big\{(y,\tau) \in \mathbb{R}^N \times \mathbb{R}^+,\ |y|^2 + \tau^\theta < 2\big\},\quad \mu(y,\tau) := |y|^2 + \tau^\theta.$$

We have the estimates

$$\begin{aligned}\int_{Q_{TR^{2/\theta}}} &|D^\alpha_{t|TR^{2/\theta}} \xi|^{p/(p-1)} (h\xi)^{-1/(p-1)} dx\, dt\\ &\leqslant R^{-\frac{2}{\theta}\alpha p/(p-1) + N + \frac{2}{\theta} - \frac{1}{p-1}(\sigma + \frac{2\rho}{\theta})} \int_\Omega |D^\alpha_{\tau|T} \Phi o \mu|^{\frac{p}{p-1}} (C_h |y|^\sigma \tau^\rho \Phi o\mu)^{-\frac{1}{p-1}} dy\, d\tau,\end{aligned}$$

$$\begin{aligned}\int_{Q_{TR^{2/\theta}}} &|(-\Delta)^{\frac{\beta}{2}} \xi|^{\frac{p}{p-m}} (h\xi)^{\frac{-m}{p-m}} dx\, dt\\ &\leqslant R^{\frac{-\beta p}{p-m} + N + \frac{2}{\theta} - \frac{m}{p-m}(\sigma + \frac{2\rho}{\theta})} \int_\Omega |(-\Delta)^{\frac{\beta}{2}} \Phi o\mu|^{\frac{p}{p-m}} (C_h |y|^\sigma \tau^\rho \Phi o\mu)^{\frac{-m}{p-m}} dy\, d\tau,\end{aligned}$$

$$\begin{aligned}\int_{Q_{TR^{2/\theta}}} &h^{\frac{-q}{p-q}} \Big| \sum_{i=1}^N \frac{\partial a}{\partial x_i} \Big|^{\frac{p}{p-q}} \xi dx\, dt\\ &\leqslant R^{\frac{-q}{p-q}(\sigma + \frac{2}{\theta}\rho) + \frac{p}{p-q}(\delta - 2) + N + \frac{2}{\theta}} \int_\Omega (C_h |y|^\sigma \tau^\rho)^{\frac{-q}{p-q}} \Big| \sum_{i=1}^N \frac{\partial a_i}{\partial y_i} \Big|^{\frac{p}{p-q}} \Phi o\mu dy\, d\tau,\end{aligned}$$

and

$$\begin{aligned}\int_{Q_{TR^{2/\theta}}} &(h\xi)^{-q/(p-q)} |a\cdot\nabla\xi|^{p/(p-q)} dx\, dt\\ &\leqslant R^{\frac{-q}{p-q}(\sigma + \frac{2}{\theta}\rho) + (\delta-1)\frac{p}{p-q} + N + \frac{2}{\theta}} \int_\Omega (C_h |y|^\sigma \tau^\rho \Phi o\mu)^{\frac{-q}{p-q}} |a\cdot\nabla\xi|^{\frac{p}{p-q}} d\tau\, dy.\end{aligned}$$

Now, we choose θ such that

$$\begin{aligned}&-\beta p\theta + (p-m)\theta N + 2(p-m) - m(\theta\sigma + 2\rho)\\ &\qquad\leqslant -2\alpha p + (p-1)\theta N + 2(p-1) - (\theta\sigma + 2\rho),\end{aligned}$$

then it is sufficient to take $\theta = \frac{2\alpha}{\beta}$. We then have the estimate

$$\int_{Q_{TR^{\beta/\alpha}}} h|u|^p \xi\, dx\, dt \leqslant C(\varepsilon)(R^{s_1} + R^{s_2} + R^{s_3}), \tag{1.6}$$

where

$$\begin{aligned}(p-1)\theta s_1 &= -2\alpha p + (p-1)\theta N + 2(p-1) - (\theta\sigma + 2\rho)\\ (p-m)\theta s_2 &= -\beta p\theta + (p-m)\theta N + 2(p-m) - m(\theta\sigma + 2\rho)\\ (p-q)\theta s_3 &= (\delta-1)p\theta + (N\theta + 2)(p-q) - q(\theta\sigma + 2\rho)\end{aligned}$$

and $C(\varepsilon)$ is a generic positive constant depending on ε. Now if we choose $\max(s_1, s_2, s_3) < 0$, that is

$$p < \min\left(1 + \frac{\alpha(\sigma+\beta)+\beta\rho}{\alpha N + \beta(1-\alpha)}, \frac{((\alpha N+\beta)+(\alpha\sigma+\beta\rho))q}{((\delta-1)\alpha+(N\alpha+\beta))}\right)$$

and let $R \to \infty$ in (1.6); we obtain

$$\int_{\mathbb{R}^N \times \mathbb{R}^+} h|u|^p dx\, dt \leqslant 0.$$

This implies that $u = 0$ a.e., which is a contradiction. If $p = p_c$ (i.e., $\max(s_1, s_2, s_3) = 0$) the critical case, we have from (1.6)

$$\int_{\mathbb{R}^n \times \mathbb{R}^+} h|u|^p dx\, dt \leqslant C. \tag{1.7}$$

We modify the test function ξ by introducing a new fixed constant S $(0 < S < R)$, such that

$$\xi(x,t) := \Phi\left(\frac{|x|^2}{R^2} + \frac{t^\theta}{(SR)^2}\right).$$

We set

$$C_{R,S} = \Big\{(x,t) \in \mathbb{R}^n \times \mathbb{R}^+ : R^2 \leqslant |x|^2 + \frac{t^\theta}{S^2} \leqslant 2R^2\Big\}.$$

See that because of the convergence of the integral in (1.7), then

$$\lim_{R\to\infty} \int_{C_{R,S}} h|u|^p \xi\, dx\, dt = 0. \tag{1.8}$$

By using the Hölder inequality, we get

$$\int_{Q_{T(SR)^{2/\theta}}} |u|^{q-1} u a \cdot \nabla\xi\, dx\, dt = \int_{C_{R,S}} |u|^{q-1} u a \cdot \nabla\xi\, dx\, dt$$
$$\leqslant \left(\int_{C_{R,S}} |u|^p h\xi\, dx\, dt\right)^{\frac{q}{p}} \left(\int_{C_{R,S}} (h\xi)^{-\frac{q}{p-q}} |a \cdot \nabla\xi|^{\frac{p}{p-q}} dx\, dt\right)^{\frac{p-q}{p}},$$

where we have used that the support of $a \cdot \nabla\xi$ is $C_{R,S}$. Taking into account of the scaled variables: $t = (RS)^{\frac{2}{\theta}}\tau$, $x = Ry$ $\xi(x,t) = \xi(Ry, (RS)^{\frac{2}{\theta}}\tau) = \chi(y,\tau)$ and the fact that $p = p_c$ then instead of estimate (1.5), we get

$$(1-3\varepsilon)\int_{Q_{T(SR)^{2/\theta}}} h|u|^p \xi\, dx\, dt \tag{1.9}$$
$$\leqslant \left(\int_{C_{R,S}} |u|^p h\xi\, dx\, dt\right)^{\frac{q}{p}} \left(\int_{C_{R,S}} (h\xi)^{-\frac{q}{p-q}} |a \cdot \nabla\xi|^{\frac{p}{p-q}} dx\, dt\right)^{\frac{p-q}{p}}$$
$$+ C(\varepsilon)\big(L_1 S^{-\frac{1}{p-1}(\frac{2\rho}{\theta}) - \frac{2}{\theta}\alpha\frac{p}{p-1} + \frac{2}{\theta}} + L_2 S^{-\frac{m}{p-m}(\frac{2\rho}{\theta}) + \frac{2}{\theta}} + L_3 S^{\frac{-q}{p-q}(\frac{2}{\theta}\rho) + \frac{2}{\theta}}\big),$$

where

$$L_1 := \int_\Omega \chi^{\frac{-1}{p-1}} \left|D^\alpha_{t|T}\chi\right|^{\frac{p}{p-1}} dy\, d\tau,$$

$$L_2 := \int_\Omega \chi^{-\frac{m}{p-m}} \left|(-\Delta)^{\frac{\beta}{2}}\chi\right|^{\frac{m}{p-m}} dy\, d\tau,$$

$$L_3 := \int_\Omega \left(\chi \left|\sum_{i=1}^{n} \frac{\partial a}{\partial y_i}\right|^{\frac{p}{p-q}} dy\, d\tau\right).$$

Using (1.9), we obtain via (1.8), after passing to the limit as $R \to \infty$

(1.10)
$$\int_{\mathbb{R}^n \times \mathbb{R}^+} h|u|^p dx\, dt \leqslant C\Big(S^{-\frac{1}{p-1}(\frac{2\rho}{\theta})-\frac{2}{\theta}\alpha\frac{p}{p-1}+\frac{2}{\theta}} + S^{-\frac{m}{p-m}(\frac{2\rho}{\theta})+\frac{2}{\theta}} + S^{\frac{-q}{p-q}(\frac{2}{\theta}\rho)+\frac{2}{\theta}}\Big).$$

Finally, we realize that the left-hand side of (1.10) is independent of S, then by passing to the limit when S goes to infinity, we obtain $u = 0$, which is contradiction and this completes the proof. □

Remark 1.1. When the vector $a = 0$ and $q = m = 1$, we recover the case studied by Kirane-Tatar [6]. When $a = 0$, $q = m = 1$, $\sigma = \rho = 0$, $\alpha = 1$ and $\beta = 2$, the critical exponent coincides with the well known Fujita exponent [2].

2. System of Fractional Differential Equations

This section is devoted to the following system of reaction-diffusion equations

$$\text{(FDS)} \qquad \begin{aligned} D^{\alpha}_{0|t}(u-u_0) + (-\Delta)^{\frac{\beta}{2}}(|u|^{m-1}u) &= h(t,x)|v|^p + |g(t,x)||u|^r \quad \text{in } Q \\ D^{\delta}_{0|t}(v-v_0) + (-\Delta)^{\frac{\gamma}{2}}(|v|^{m-1}v) &= k(t,x)|u|^q + |l(t,x)||v|^s \quad \text{in } Q \end{aligned}$$

subject to the initial conditions $u(x,0) = u_0(x) \geqslant 0$, $v(x,0) = v_0(x) \geqslant 0$, $x \in \mathbb{R}^N$, where $0 < \alpha, \delta < 1$ and $0 \leqslant \gamma, \beta \leqslant 2$.

The functions h, g, k, l are assumed to satisfy the conditions

$$h(t,x) \geqslant C_1 t^{\omega_1}|x|^{d_1}, \quad g(t,x) \sim t^{\omega_2}|x|^{d_2} \quad \text{when } |x| \text{ large}$$
$$k(t,x) \geqslant C_3 t^{\omega_3}|x|^{d_3}, \quad l(t,x) \sim t^{\omega_4}|x|^{d_4} \quad \text{when } |x| \text{ large},$$

for $t > 0$, $x \gg 1$, $\omega_1 \geqslant 0$, $\omega_2 \geqslant 0$, $\omega_3 \geqslant 0$, $\omega_4 \geqslant 0$, $d_1 \geqslant 0$, $d_2 \geqslant 0$, $d_3 \geqslant 0$ and $d_4 \geqslant 0$. We set $\lambda_i = \omega_i + \frac{\alpha}{\beta}d_i$ and $\eta_i = \omega_i + \frac{\delta}{\gamma}d_i$ for $i = 1, \ldots, 4$. For the system (FDS) we have

Theorem 2.1. *Let $p, q > 1, m, r,$ and $s \geqslant 1$ and assume that*

(1) $pq > \max(m^2, sm, sr, mr)$.

(2) $\frac{(\eta_3+1)}{(\eta_2+1)} \geqslant \frac{q}{r} > 1$, $\frac{(\lambda_1+1)}{(\lambda_4+1)} > \frac{p}{s} > 1$, $\frac{(\lambda_3+1)}{(\lambda_4+1)} > \frac{q^2}{ms} > 1$.

If

$$N \leqslant \max(\theta_1, \theta_2) \tag{2.1}$$

where $\theta_1 = \min_{1\leqslant j\leqslant 7}\theta_{1j}$, $\theta_2 = \min_{1\leqslant j\leqslant 7}\theta_{2j}$

$$\theta_{11} = \frac{q\eta_1 + p^2\lambda_3 + pq(\delta - (1-\frac{1}{p})) + p^2q(\alpha - (1-\frac{1}{q}))}{(\frac{(p-1)q\delta}{\gamma} + \frac{(q-1)p^2\alpha}{\beta})},$$

$$\theta_{12} = \frac{mq\eta_1 + p^2\lambda_3 + pq(\delta - (1-\frac{m}{p})) + p^2q(\alpha - (1-\frac{1}{q}))}{(\frac{(p-m)q\delta}{\gamma} + \frac{(q-1)p^2\alpha}{\beta})}$$

$$\theta_{13}=\frac{sq(\lambda_1+1)-pq(\lambda_4+1)+p^2\lambda_3+p^2q(\alpha-(1-\frac{1}{q}))}{((p-s)q+(q-1)p^2)\frac{\alpha}{\beta}},$$

$$\theta_{14}=\frac{qm\eta_1+p^2m\lambda_3+pqm(\delta-(1-\frac{1}{p}))+p^2q(\alpha-(1-\frac{m}{q}))}{(\frac{\delta(p-1)mq}{\gamma}+\frac{\alpha p^2(q-m)}{\beta})},$$

$$\theta_{15}=\frac{sp\lambda_3+sq^2(\lambda_1+1)-pq^2(\lambda_4+1)+pqs(\alpha-(1-\frac{1}{q}))}{((q-1)sp+(p-s)q^2)\frac{\alpha}{\beta}},$$

$$\theta_{16}=\frac{rq\eta_1-p^2q(\eta_2+1)+rp^2(\eta_3+1)+rpq(\delta-1+\frac{1}{p})}{((p-1)rq+(q-r)p^2)\frac{\delta}{\gamma}},$$

$$\theta_{17}=\frac{qrs(\lambda_1+1)-pqr(\lambda_4+1)+p^2r(\lambda_3+1)-p^2q(\lambda_2+1)}{((p-s)rq+(q-r)p^2)\frac{\alpha}{\beta}}.$$

$$\theta_{21}=\frac{q^2\eta_1+pq(\alpha-(1-\frac{1}{q}))+p\lambda_3+pq^2(\delta-(1-\frac{1}{p}))}{\left[p(q-1)\frac{\alpha}{\beta}+(p-1)q^2\frac{\delta}{\gamma}\right]},$$

$$\theta_{22}=\frac{(\eta_3+1)rp-(\eta_2+1)pq+q^2\eta_1+pq^2(\delta-1+\frac{1}{p})}{((q-r)p+(p-1)q^2)\frac{\delta}{\gamma}},$$

$$\theta_{23}=\frac{mp\lambda_3+mq^2\eta_1+mpq(\alpha-(1-\frac{1}{q}))+pq^2(\delta-(1-\frac{m}{p}))}{(\frac{mp(q-1)\alpha}{\beta}+\frac{(p-m)q^2\delta}{\gamma})},$$

$$\theta_{24}=\frac{prm(\eta_3+1)-mpq(\eta_2+1)+mq^2\eta_1+pq^2(\delta-(1-\frac{m}{p}))}{((q-r)mp+(p-m)q^2)\frac{\delta}{\gamma}},$$

$$\theta_{25}=\frac{smq(\lambda_1+1)-mpq(\lambda_4+1)+p^2m\lambda_3+p^2q(\alpha-(1-\frac{m}{q}))}{((p-s)mq+p^2(q-m))\frac{\alpha}{\beta}},$$

$$\theta_{26}=\frac{msp(\lambda_3+1)+sq^2\lambda_1-pq^2(\lambda_4+1)+pqs(\alpha-(1-\frac{q}{p}))}{((q-m)sp+(p-s)q^2)\frac{\alpha}{\beta}},$$

$$\theta_{27}=\frac{rsp(\lambda_3+1)-spq(\lambda_2+1)+sq^2(\lambda_1+1)-pq^2(\lambda_4+1)}{((q-r)sp+(p-s)q^2)\frac{\alpha}{\beta}},$$

then the system (FDS) *(with the initial data) does not admit nontrivial global weak solutions.*

PROOF. Here again the proof proceeds by contradiction. Let

$$\xi_j(x,t)=\Phi\Big(\frac{t^2+|x|^{2\mu_j}}{R^2}\Big),\quad j=1,2$$

where $R>0$, $\mu_1=\beta/\alpha$ and $\mu_2=\gamma/\delta$.

The weak formulation of solutions to (FDS) reads

$$\int_{Q_{TR}}h|v|^p\xi_1+\int_{Q_{TR}}u_0D^{\alpha}_{t|T}\xi_1$$

$$= \int_{Q_{TR}} u D^{\alpha}_{t|T}\xi_1 + \int_{Q_{TR}} (|u|^{m-1}u)(-\Delta)^{\frac{\beta}{2}}\xi_1 - \int_{Q_{TR}} g|u|^r\xi_1,$$

$$\int_{Q_{TR}} k\xi_2|u|^q + \int_{Q_{TR}} v_0 D^{\delta}_{t|TR}\xi_2$$
$$= \int_{Q_{TR}} v D^{\delta}_{t|T}\xi_2 + \int_{Q_{TR}} (|v|^{m-1}v)(-\Delta)^{\frac{\gamma}{2}}\xi_2 - \int_{Q_{TR}} l\xi_2|v|^s.$$

Using the Hölder inequality, we may write

$$\int_{Q_{TR}} u D^{\alpha}_{t|T}\xi_1 \leqslant \left(\int_{Q_{TR}} k\xi_2|u|^q\right)^{\frac{1}{q}} \left(\int_{Q_{TR}} (k\xi_2)^{-\frac{1}{q-1}}|D^{\alpha}_{t|T}\xi_1|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}},$$

$$\int_{Q_{TR}} (|u|^{m-1}u)(-\Delta)^{\frac{\beta}{2}}\xi_1 \leqslant \left(\int_{Q_{TR}} k\xi_2|u|^q\right)^{\frac{m}{q}}\left(\int_{Q_{TR}} |(-\Delta)^{\frac{\beta}{2}}\xi_1|^{\frac{-q}{m-q}}(k\xi_2)^{\frac{m}{m-q}}\right)^{-\frac{m-q}{q}},$$

$$\int_{Q_{TR}} g|u|^r\xi_1 \leqslant \left(\int_{Q_{TR}} k\xi_2|u|^q\right)^{\frac{r}{q}} \left(\int_{Q_{TR}} (k\xi_2)^{-\frac{r}{q-r}}(g\xi_1)^{\frac{q}{q-r}}\right)^{\frac{q-r}{q}}.$$

Consequently

$$\int_{Q_{TR}} h|v|^p\xi_1 \leqslant \left(\int_{Q_{TR}} k\xi_2|u|^q\right)^{\frac{1}{q}}\cdot\mathcal{A} + \left(\int_{Q_{TR}} k\xi_2|u|^q\right)^{\frac{m}{q}}\cdot\mathcal{B} + \left(\int_{Q_{TR}} k\xi_2|u|^q\right)^{\frac{r}{q}}\cdot\mathcal{C}$$

where

$$\mathcal{A} = \left(\int_{Q_{TR}} (k\xi_2)^{-\frac{1}{q-1}}|D^{\alpha}_{t|T}\xi_1|^{\frac{q}{q-1}}\right)^{\frac{q-1}{q}},$$

$$\mathcal{B} = \left(\int_{Q_{TR}} |(-\Delta)^{\frac{\beta}{2}}\xi_1|^{\frac{-q}{m-q}}(k\xi_2)^{\frac{m}{m-q}}\right)^{-\frac{m-q}{q}},$$

$$\mathcal{C} = \left(\int_{Q_{TR}} (k\xi_2)^{-\frac{r}{q-r}}(g\xi_1)^{\frac{q}{q-r}}\right)^{\frac{q-r}{q}}.$$

Similarly we obtain the estimates

$$\int_{Q_{TR}} v D^{\delta}_{t|T}\xi_2 \leqslant \left(\int_{Q_{TR}} |v|^p(h\xi_1)\right)^{\frac{1}{p}} \left(\int_{Q_{TR}} (h\xi_1)^{-\frac{1}{p-1}}|D^{\delta}_{t|T}\xi_2|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}},$$

$$\int_{Q_{TR}} (|v|^{m-1}v)(-\Delta)^{\frac{\gamma}{2}}\xi_2 \leqslant \left(\int_{Q_{TR}} |v|^p h\xi_1\right)^{\frac{m}{p}}\left(\int_{Q_{TR}} (h\xi_1)^{\frac{m}{m-p}}|(-\Delta)^{\frac{\gamma}{2}}\xi_2|^{\frac{-p}{m-p}}\right)^{-\frac{m-p}{p}},$$

$$\int_{Q_{TR}} l\xi_2|v|^s \leqslant \left(\int_{Q_{TR}} |v|^p h\xi_1\right)^{\frac{s}{p}} \left(\int_{Q_{TR}} (h\xi_1)^{\frac{s}{s-p}}(l\xi_2)^{-\frac{p}{s-p}}\right)^{-\frac{s-p}{p}}.$$

So we get

$$\int_{Q_{TR}} k\xi_2|u|^q \leqslant \left(\int_{Q_{TR}} |v|^p(h\xi_1)\right)^{\frac{1}{p}}\cdot\mathcal{D} + \left(\int_{Q_{TR}} |v|^p h\xi_1\right)^{\frac{m}{p}}\cdot\mathcal{E} + \left(\int_{Q_{TR}} |v|^p h\xi_1\right)^{\frac{s}{p}}\cdot\mathcal{F},$$

with

$$\mathcal{D} := \left(\int_{Q_{TR}} (h\xi_1)^{-\frac{1}{p-1}} |D^{\delta}_{t|T}\xi_2|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}},$$

$$\mathcal{E} := \left(\int_{Q_{TR}} (h\xi_1)^{\frac{m}{m-p}} |(-\Delta)^{\frac{\gamma}{2}}\xi_2|^{\frac{-p}{m-p}}\right)^{-\frac{m-p}{p}},$$

$$\mathcal{F} := \left(\int_{Q_{TR}} (h\xi_1)^{\frac{s}{s-p}} (l\xi_2)^{-\frac{p}{s-p}}\right)^{-\frac{s-p}{p}}.$$

If we set

$$\mathcal{Y} := \int_{Q_{TR}} h|v|^p\xi_1, \quad \mathcal{Z} := \int_{Q_{TR}} k\xi_2|u|^q,$$

then we have

$$\mathcal{Y}^q \leqslant \mathcal{Z}\cdot\mathcal{A} + \mathcal{Z}^m\cdot\mathcal{B} + \mathcal{Z}^r\cdot\mathcal{C},$$
$$\mathcal{Z}^p \leqslant \mathcal{Y}\cdot\mathcal{D} + \mathcal{Y}^m\cdot\mathcal{E} + \mathcal{Y}^s\cdot\mathcal{F}$$

which yields

$$\mathcal{Y}^{pq} \leqslant 3^{p-1}(\mathcal{Z}^p\cdot\mathcal{A}^p + \mathcal{Z}^{pm}\cdot\mathcal{B}^p + \mathcal{Z}^{pr}\cdot\mathcal{C}^p), \tag{2.2}$$

$$\mathcal{Z}^{pq} \leqslant 3^{p-1}(\mathcal{Y}^q\cdot\mathcal{D}^q + \mathcal{Y}^{qm}\cdot\mathcal{E}^q + \mathcal{Y}^{qs}\cdot\mathcal{F}^q). \tag{2.3}$$

We have used in (2.2) and (2.3) the inequality

$$(a+b+c)^p \leqslant 3^{p-1}(a^p+b^p+c^p), \quad p\geqslant 1, \ \ a,b,c\geqslant 0.$$

It then follows from (2.2), (2.3) that

$$\begin{aligned}\mathcal{Y}^{pq} \leqslant c(p,m,r)\big(&(\mathcal{Y}\cdot\mathcal{D} + \mathcal{Y}^m\cdot\mathcal{E} + \mathcal{Y}^s\cdot\mathcal{F})\cdot\mathcal{A}^p\\ &+(\mathcal{Y}^m\cdot\mathcal{D}^m + \mathcal{Y}^{m^2}\cdot\mathcal{E}^m + \mathcal{Y}^{sm}\cdot\mathcal{F}^m)\cdot\mathcal{B}^p\\ &+(\mathcal{Y}^r\cdot\mathcal{D}^r + \mathcal{Y}^{mr}\cdot\mathcal{E}^r + \mathcal{Y}^{sr}\cdot\mathcal{F}^r)\cdot\mathcal{C}^p\big),\end{aligned}$$

where $c(p,m,r)$ is a positive constant depending on p,m and r. Using ε-Young inequality, we get

$$\begin{aligned}\mathcal{Y}^{pq} \leqslant c(p,m,r,\varepsilon)\Big(&(\mathcal{D}\mathcal{A}^p)^{\frac{pq}{pq-1}} + (\mathcal{E}\mathcal{A}^p)^{\frac{pq}{pq-m}} + (\mathcal{F}\mathcal{A}^p)^{\frac{pq}{pq-s}}\\ &+ (\mathcal{D}^m\mathcal{B}^p)^{\frac{pq}{pq-m}} + (\mathcal{E}^m\mathcal{B}^p)^{\frac{pq}{pq-m^2}} + (\mathcal{F}^m\mathcal{B}^p)^{\frac{pq}{pq-sm}}\\ &+ (\mathcal{D}^r\mathcal{C}^p)^{\frac{pq}{pq-r}} + (\mathcal{E}^r\mathcal{C}^p)^{\frac{pq}{pq-mr}} + (\mathcal{F}^r\mathcal{C}^p)^{\frac{pq}{pq-sr}}\Big).\end{aligned} \tag{2.4}$$

Now, using the scaled variables (y,τ) defined by $t=R\tau$ and $x=R^{\frac{\alpha}{\beta}}y$, in $\mathcal{A},\mathcal{B},\mathcal{F}$ while in $\mathcal{D},\mathcal{E},\mathcal{C}$ we use the variables (y,τ) defined by $t=R\tau$ and $x=R^{\frac{\delta}{\gamma}}y$, we obtain

$$\mathcal{Y}^{pq} \leqslant c\big(R^{l_1}+R^{l_2}+R^{l_3}+R^{l_4}+R^{l_5}+R^{l_6}+R^{l_7}+R^{l_8}+R^{l_9}\big) \tag{2.5}$$

where

$$(pq-1)l_1 := N - \frac{q\eta_1 + p^2\lambda_3 + pq(\delta - (1-\frac{1}{p})) + p^2q(\alpha - 1 + \frac{1}{q})}{(\frac{(p-1)q\delta}{\gamma} + \frac{(q-1)p^2\alpha}{\beta})},$$

$$(pq-m)l_2 := N - \frac{mq\eta_1 + p^2\lambda_3 + pq(\delta - (1-\frac{m}{p})) + p^2q(\alpha - (1 - \frac{1}{q}))}{(\frac{(p-m)q\delta}{\gamma} + \frac{(q-1)p^2\alpha}{\beta})},$$

$$(pq-s)l_3 := N - \frac{sq(\lambda_1+1) - pq(\lambda_4+1) + p^2\lambda_3 + p^2q(\alpha - (1-\frac{1}{q}))}{((p-s)q + (q-1)p^2)\frac{\alpha}{\beta}},$$

$$(pq-m)l_4 := N - \frac{qm\eta_1 + p^2m\lambda_3 + pqm(\delta - (1-\frac{1}{p})) + p^2q(\alpha - (1-\frac{m}{q}))}{(\frac{\delta(p-1)mq}{\gamma} + \frac{\alpha p^2(q-m)}{\beta})},$$

$$(pq-m^2)l_5 := N - \frac{m^2q\eta_1 + p^2m\lambda_3 + pqm(\delta - (1-\frac{m}{p})) + p^2q(\alpha - 1 + \frac{m}{q})}{(\frac{\delta m(p-m)q}{\gamma} + \frac{p^2(q-m)\alpha}{\beta})},$$

$$(pq-sm)l_6 := N - \frac{smq(\lambda_1+1) - mpq(\lambda_4+1) + p^2m\lambda_3 + p^2q(\alpha - (1-\frac{m}{q}))}{((p-s)mq + p^2(q-m))\frac{\alpha}{\beta}},$$

$$(pq-r)l_7 := N - \frac{rq\eta_1 - p^2q(\eta_2+1) + rp^2(\eta_3+1) + rpq(\delta - 1 + \frac{1}{p})}{((p-1)rq + (q-r)p^2)\frac{\delta}{\gamma}},$$

$$(pq-mr)l_8 := N - \frac{mrq\eta_1 + (\eta_3+1)p^2r - p^2q(\eta_2+1) + rpq(\delta - (1-\frac{m}{p}))}{((p-m)rq + (q-r)p^2)\frac{\delta}{\gamma}},$$

$$(pq-sr)l_9 := N - \frac{qrs(\lambda_1+1) - pqr(\lambda_4+1) + p^2r(\lambda_3+1) - p^2q(\lambda_2+1)}{((p-s)rq + (q-r)p^2)\frac{\alpha}{\beta}}.$$

In the same way we find

$$\begin{aligned}\mathcal{Z}^{pq} \leqslant c(\varepsilon)\Big(&(\mathcal{A}\mathcal{D}^q)^{\frac{pq}{pq-1}} + (\mathcal{B}\mathcal{D}^q)^{\frac{pq}{pq-m}} + (\mathcal{C}\mathcal{D}^q)^{\frac{pq}{pq-r}} \\ &+ (\mathcal{A}^m\mathcal{E}^q)^{\frac{pq}{pq-m}} + (\mathcal{B}^m\mathcal{E}^q)^{\frac{pq}{pq-m^2}} + (\mathcal{C}^m\mathcal{E}^q)^{\frac{pq}{pq-mr}} \\ &+ (\mathcal{A}^s\mathcal{F}^q)^{\frac{pq}{pq-s}} + (\mathcal{B}^s\mathcal{F}^q)^{\frac{pq}{pq-ms}} + (\mathcal{C}^s\mathcal{F}^q)^{\frac{pq}{pq-rs}}\Big). \end{aligned} \tag{2.6}$$

Similarly, we have for $\mathcal{Z}$

$$\mathcal{Z}^{pq} \leqslant c\big(R^{j_1} + R^{j_2} + R^{j_3} + R^{j_4} + R^{j_5} + R^{j_6} + R^{j_7} + R^{j_8} + R^{j_9}\big), \tag{2.7}$$

where

$$(pq-1)j_1 := N - \frac{q^2\eta_1 + pq(\alpha - (1-\frac{1}{q})) + p\lambda_3 + pq^2(\delta - (1-\frac{1}{p}))}{\big[p(q-1)\frac{\alpha}{\beta} + (p-1)q^2\frac{\delta}{\gamma}\big]},$$

$$(pq-m)j_2 := N - \frac{\big[mp\lambda_3 + q^2\eta_1 + pq(\alpha - (1-\frac{m}{q})) + pq^2(\delta - (1-\frac{1}{p}))\big]}{(\frac{p(q-m)\alpha}{\beta} + \frac{(p-1)q^2\delta}{\gamma})},$$

$$(pq-r)j_3 := N - \frac{(\eta_3+1)rp - (\eta_2+1)pq + q^2\eta_1 + pq^2(\delta - 1 + \frac{1}{p})}{((q-r)p + (p-1)q^2)\frac{\delta}{\gamma}},$$

$$(pq-m)j_4 := N - \frac{mp\lambda_3 + mq^2\eta_1 + mpq(\alpha - (1-\frac{1}{q})) + pq^2(\delta - (1-\frac{m}{p}))}{(\frac{(q-1)pm\alpha}{\beta} + \frac{(p-m)q^2\delta}{\gamma})},$$

$$(pq-m^2)j_5 := N - \frac{m^2p\lambda_3 + mq^2\eta_1 + pqm(\alpha - (1-\frac{m}{q})) + pq^2(\delta - (1-\frac{m}{p}))}{(\frac{\alpha(q-m)mp}{\beta} + \frac{\delta(p-m)q^2}{\gamma})},$$

$$(pq-mr)j_6 := N - \frac{prm(\eta_3+1) - mpq(\eta_2+1) + mq^2\eta_1 + pq^2(\delta - (1-\frac{m}{p}))}{((q-r)mp + (p-m)q^2)\frac{\delta}{\gamma}},$$

$$(pq-s)j_7 := N - \frac{sp\lambda_3 + sq^2(\lambda_1+1) - pq^2(\lambda_4+1) + pqs(\alpha - (1-\frac{1}{q}))}{((q-1)sp + (p-s)q^2)\frac{\alpha}{\beta}},$$

$$(pq-ms)j_8 := N - \frac{msp(\lambda_3+1) + sq^2\lambda_1 - pq^2(\lambda_4+1) + pqs(\alpha - (1-\frac{q}{p}))}{\frac{((q-m)sp + (p-s)q^2)\alpha}{\beta}},$$

$$(pq-rs)j_9 := N - \frac{rsp(\lambda_3+1) - spq(\lambda_2+1) + sq^2(\lambda_1+1) - pq^2(\lambda_4+1)}{((q-r)sp + (p-s)q^2)\frac{\alpha}{\beta}}.$$

Condition (2.1) leads to either $\max_{1\leqslant i\leqslant 9} l_i \leqslant 0$ or $\max_{1\leqslant i\leqslant 9} j_i \leqslant 0$. In both cases, the proof follows from the arguments presented above. □

Remark 2.1. When $\alpha = \delta = 1$, $\beta = \gamma = 2$, $h = k = 1$ and $g = l = 0$, we found the case studied by Escobedo and Herrero [1], however we impose the constraint $p > 1$, $q > 1$ while Escobedo and Herrero require $pq > 1$.

Acknowledgments. We would like to express our gratitude to the referees and editors for their helpful comments. Another major debt to Dr N. Guerroudj for correcting linguistic mistakes.

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(Received 10 12 2009)
(Revised 26 11 2011)

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