PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 98(112) (2015), 251–263 DOI: 10.2298/PIM150304024K

# SIMPLE GROUPS WITH THE SAME PRIME GRAPH AS ${}^2D_n(q)$

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Abstract. In 2006, Vasil'ev posed the problem: *Does there exist a positive integer $k$ such that there are no $k$ pairwise nonisomorphic nonabelian finite simple groups with the same graphs of primes? Conjecture: $k = 5$.* In 2013, Zvezdina, confirmed the conjecture for the case when one of the groups is alternating. We continue this work and determine all nonabelian simple groups having the same prime graphs as the nonabelian simple group ${}^2D_n(q)$.

## 1. Introduction

If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. If $G$ is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. The *spectrum* of a finite group $G$ which is denoted by $\omega(G)$ is the set of its element orders. We construct the *prime graph* of $G$, denoted by $\Gamma(G)$, as follows: the vertex set is $\pi(G)$ and two distinct primes $p$ and $q$ are joined by an edge (we write $p \sim q$) if and only if $G$ contains an element of order $pq$. Let $s(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_i(G)$, $i = 1, \dots, s(G)$, be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$ we always suppose that $2 \in \pi_1(G)$. The connected components of the prime graph of nonabelian simple groups with disconnected prime graph are listed in [**13**]. In graph theory a subset of vertices of a graph is called an independent set if its vertices are pairwise non-adjacent. Denote by $t(G)$ the maximal number of primes in $\pi(G)$ pairwise non-adjacent in $\Gamma(G)$. In other words, if $\rho(G)$ is an independent set with the maximal number of vertices in $\Gamma(G)$, then $t(G) = |\rho(G)|$. Similarly if $p \in \pi(G)$, then let $\rho(p, G)$ be an independent set with the maximal number of vertices in $\Gamma(G)$ containing $p$ and $t(p, G) = |\rho(p, G)|$. In [**11**, Tables 2-9], independent sets also independence numbers for all simple groups are listed.

Hagie [**6**] determined finite groups $G$ satisfying $\Gamma(G) = \Gamma(S)$, where $S$ is a sporadic simple group. The same problem is considered for some finite simple groups (see [**1, 2, 3, 8, 14**]).

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2010 *Mathematics Subject Classification:* 20D05; 20D60.
*Key words and phrases:* prime graph, simple group, Vasil'ev conjecture.
The first author was supported in part by a grant from IPM (93200043).
Communicated by Zoran Petrović.

Vasil'ev formulated the following problem in [**9**]:

Problem 16.26. *Does there exist a positive integer $k$ such that there are no $k$ pairwise nonisomorphic nonabelian finite simple groups with the same graphs of primes? Conjecture: $k = 5$.*

In [**16**], the problem was solved when one of the two groups is an alternating group. The conjecture is true in this case.

Here we continue this work and determine all nonabelian simple groups, with the same prime graph as ${}^2D_n(q)$.

Throughout the paper, we use the classification of finite simple groups, all groups are finite and by simple groups we mean nonabelian simple groups. All further unexplained notations are standard and refer to [**4**]. Also for a natural number $n$ and a prime number $p$, we denote by $(n)_p$, the $p$-part of $n$, i.e., $(n)_p = p^\alpha$, such that $p^\alpha \mid n$, but $p^{\alpha+1} \nmid n$.

## 2. Preliminary results

In this section, we will quote some useful facts which will be used during the proof of the main theorem.

Remark 2.1. [**10**] Let $p$ be a prime number and $(q,p) = 1$. Let $k \geqslant 1$ be the smallest positive integer such that $q^k \equiv 1 \pmod p$. Then $k$ is called *the order of $q$ with respect to $p$* and we denote it by $\mathrm{ord}_p(q)$. Obviously by Fermat's little theorem it follows that $\mathrm{ord}_p(q)|(p-1)$. Also if $q^n \equiv 1 \pmod p$, then $\mathrm{ord}_p(q)|n$. Similarly if $m > 1$ is an integer and $(q,m) = 1$, we can define $\mathrm{ord}_m(q)$. If $a$ is odd, then $\mathrm{ord}_a(q)$ is denoted by $e(a,q)$, too. If $q$ is odd, let $e(2,q) = 1$ if $q \equiv 1 \pmod 4$ and $e(2,q) = 2$ if $q \equiv -1 \pmod 4$.

Lemma 2.1. [**5**, Remark 1] *The equation $p^m - q^n = 1$, where $p$ and $q$ are primes and $m,n > 1$ has only one solution, namely $3^2 - 2^3 = 1$.*

Lemma 2.2. [**5, 7**] *Except the relations $(239)^2 - 2(13)^4 = -1$ and $(3)^5 - 2(11)^2 = 1$ every solution of the equation*

$$p^m - 2q^n = \pm 1; \quad p, q \text{ prime}; \quad m, n > 1,$$

*has exponents $m = n = 2$; i.e., it comes from a unit $p - q2^{1/2}$ of the quadratic field $Q(2^{1/2})$ for which the coefficients $p, q$ are primes.*

Lemma 2.3. (Zsigmondy Theorem) [**15**] *Let $p$ be a prime and $n$ a positive integer. Then one of the following holds:*

(i) *there is a primitive prime $p'$ for $p^n - 1$, that is, $p' \mid (p^n - 1)$ but $p' \nmid (p^m - 1)$, for every $1 \leqslant m < n$, (usually $p'$ is denoted by $r_n$)*

(ii) *$p = 2$, $n = 1$ or $6$,*

(iii) *$p$ is a Mersenne prime and $n = 2$.*

We denote by $D_n^+(q)$ the simple group $D_n(q)$, and by $D_n^-(q)$ the simple group ${}^2D_n(q)$.

TABLE 1. 2-independence numbers for group ${}^2D_n(q)$

| $G$ | conditions | $t(2,G)$ | $\rho(2,G)$ |
|---|---|---|---|
| ${}^2D_n(q)$ | $n \equiv 0 \pmod 2$, $n \geqslant 4$ | 2 | $\{2, r_{2n}\}$ |
| | $n \equiv 1 \pmod 2$, $n > 4$, $q \equiv 1 \pmod 4$ | 2 | $\{2, r_{2n}\}$ |
| | $n \equiv 1 \pmod 2$, $n > 4$, $q \equiv 7 \pmod 8$ | 2 | $\{2, r_{2(n-1)}\}$ |
| | $n \equiv 1 \pmod 2$, $n > 4$, $q \equiv 3 \pmod 8$ | 3 | $\{2, r_{2(n-1)}, r_{2n}\}$ |

LEMMA 2.4. [**12**, Proposition 2.5] *Let $G = D_n^{\varepsilon}(q)$ be a finite simple group of Lie type over a field of characteristic $p$. Define*

$$\eta(m) = \begin{cases} m & \text{if } m \text{ is odd,} \\ m/2 & \text{otherwise.} \end{cases}$$

*Let $r$ and $s$ be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and $1 \leqslant \eta(k) \leqslant \eta(l)$. Then $r$ and $s$ are non-adjacent if and only if $2\eta(k) + 2\eta(l) > 2n - (1 - \varepsilon(-1)^{k+l})$, and $l/k$ is not an odd natural number, and if $\varepsilon = +$, then the chain of equalities $n = l = 2\eta(l) = 2\eta(k) = 2k$ is not true.*

LEMMA 2.5. [**12**, Proposition 2.4] *Let $G$ be a simple group of Lie type, $B_n(q)$ or $C_n(q)$ over a field of characteristic $p$. Let $r, s$ be odd primes with $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and suppose that $1 \leqslant \eta(k) \leqslant \eta(l)$. Then $r$ and $s$ are non-adjacent if and only if $\eta(k) + \eta(l) > n$, and $l/k$ is not an odd natural number.*

LEMMA 2.6. [**11**, Proposition 2.1] *Let $G = A_{n-1}(q)$ be a finite simple group of Lie type over a field of characteristic $p$. Let $r$ and $s$ be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and suppose that $2 \leqslant k \leqslant l$. Then $r$ and $s$ are non-adjacent if and only if $k + l > n$, and $k$ does not divide $l$.*

LEMMA 2.7. [**11**, Proposition 2.2] *Let $G = {}^2A_{n'-1}(q)$ be a finite simple group of Lie type over a field of characteristic $p$. Define*

$$\nu(m) = \begin{cases} m & \text{if } m \equiv 0 \pmod 4; \\ m/2 & \text{if } m \equiv 2 \pmod 4; \\ 2m & \text{if } m \equiv 1 \pmod 4. \end{cases}$$

*Let $r$ and $s$ be odd primes and $r, s \in \pi(G) \setminus \{p\}$. Put $k = e(r,q)$ and $l = e(s,q)$, and suppose that $2 \leqslant \nu(k) \leqslant \nu(l)$. Then $r$ and $s$ are non-adjacent if and only if $\nu(k) + \nu(l) > n$, and $\nu(k)$ does not divide $\nu(l)$.*

## 3. Prime graph of simple classical Lie type groups

In this section, we denote by $r_i$, a primitive prime divisor of $q^i - 1$ and we consider $R_i(q)$ as the set of all primitive prime divisors of $q^i - 1$.

REMARK 3.1. Let $G =^2 D_n(q)$, where $q = p^{\alpha}$ and $n \geqslant 4$. Using [**11**, Table 6], we give the 2-independence number for the simple group ${}^2D_n(q)$ in Table 1. Let $n$ be odd. By [**11**, Proposition 3.1], we have $\rho(p, G) = \{p, r_{2(n-1)}, r_{2n}\}$. By Lemma 2.4, we know that $\rho(r_1, G) = \{r_1, r_{2n}\}$ and also $\rho(r_2, G) = \{r_2, r_{2(n-1)}\}$.

Let $r_k \nsim r_i$, where $k \geqslant 3$ is a fixed odd number. Hence $2k + 2\eta(i) > 2n - (1 + (-1)^{i+k})$. Therefore, $i \in A \cup B$, where $A = \{2n, 2(n-1), \dots, 2(n-k+1)\}$ and $B = \{n-2, n-4, \dots, n-k+1\}$. Since $k \geqslant 3$, so $r_k$ is not adjacent to $r_{2n}$, $r_{2(n-1)}$ and $r_{2(n-2)}$. Moreover, $\{r_{2(n-2)}, r_{2(n-1)}, r_{2n}\}$ is an independent set. So $\{r_k, r_{2(n-2)}, r_{2(n-1)}, r_{2n}\} \subseteq \rho(r_k, G)$. Therefore, $t(r_k, G) \geqslant 4$. Let $r_k \nsim r_i$, where $k \geqslant 4$ is a fixed even number. Hence $k + 2\eta(i) > 2n - (1 + (-1)^{i+k})$. Define $A = \{2n, 2(n-1), \dots, 2(n-k/2)\}$ and $B = \{n-2, n-4, \dots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 2$ and otherwise, $a = 1$. Therefore, $i \in A \cup B$. Let $k = 4$. If $n \equiv 1 \pmod 4$, then $\rho(r_4, G) = \{r_4, r_{2(n-2)}, r_{2(n-1)}, r_{2n}\}$, otherwise $\rho(r_4, G) = \{r_4, r_{2(n-2)}, r_{2n}\}$. If $k \geqslant 6$, then $t(r_k, G) \geqslant 5$.

Let $n$ be even. We know that $\rho(p, G) = \{p, r_{n-1}, r_{2(n-1)}, r_{2n}\}$, by [**11**, Proposition 3.1]. By Lemma 2.4, we know that $\rho(r_1, G) = \{r_1, r_{2n}\}$ and $\rho(r_2, G) = \{r_2, r_{2n}\}$.

Let $r_k \nsim r_i$, where $k \geqslant 3$ is a fixed odd number. Hence $2k + 2\eta(i) > 2n - (1 + (-1)^{i+k})$. Suppose $A = \{2n, 2(n-1), \dots, 2(n-k+1)\}$ and $B = \{n-1, n-3, \dots, n-k\}$. Therefore, $i \in A \cup B$. Since $k \geqslant 3$, so $t(r_k, G) \geqslant 5$. Let $r_k \nsim r_i$, where $k \geqslant 4$ is a fixed even number. Hence $k + 2\eta(i) > 2n - (1 + (-1)^{i+k})$. Define $A = \{2n, 2(n-1), \dots, 2(n-k/2)\}$ and $B = \{n-1, n-3, \dots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 1$, otherwise $a = 2$. Therefore, $i \in A \cup B$. Let $k = 4$. If $n \equiv 0 \pmod 4$, then $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-1)}, r_{2n}\}$, otherwise $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-2)}, r_{2(n-1)}\}$. Also if $k \geqslant 6$, then $t(r_k, G) \geqslant 5$.

Remark 3.2. Let $G = D_n(q)$, where $q = p^\alpha$ and $n > 4$. Let $n$ be odd. By [**11**, Proposition 3.1], we have $\rho(p, G) = \{p, r_n, r_{2(n-1)}\}$. By Lemma 2.4, we know that $\rho(r_1, G) = \{r_1, r_{2(n-1)}\}$ and $\rho(r_2, G) = \{r_2, r_n\}$. Let $r_k \nsim r_i$, where $k \geqslant 3$ is a fixed odd number. Hence $2k + 2\eta(i) > 2n - (1 - (-1)^{i+k})$. Suppose $A = \{2(n-1), 2(n-2), \dots, 2(n-k)\}$ and $B = \{n, n-2, \dots, n-k+1\}$. Therefore, $i \in A \cup B$. Since $k \geqslant 3$, so $t(r_k, G) \geqslant 4$. Let $r_k \nsim r_i$, where $k \geqslant 4$ is a fixed even number. Hence $k + 2\eta(i) > 2n - (1 - (-1)^{i+k})$. Define $A = \{2(n-1), 2(n-2), \dots, 2(n-k/2+1)\}$ and $B = \{n, n-2, \dots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 0$, otherwise $a = 1$. Therefore, $i \in A \cup B$. Let $k = 4$, if $n \equiv 1 \pmod 4$, then $\rho(r_4, G) = \{r_4, r_{n-2}, r_n, r_{2(n-1)}\}$, otherwise $\rho(r_4, G) = \{r_4, r_{n-2}, r_n\}$. If $k \geqslant 6$, then $t(r_k, G) \geqslant 4$.

Let $n$ be even. By [**11**, Proposition 3.1], we have $\rho(p, G) = \{p, r_{n-1}, r_{2(n-1)}\}$. By Lemma 2.4, we know that $\rho(r_1, G) = \{r_1, r_{2(n-1)}\}$ and $\rho(r_2, G) = \{r_2, r_{n-1}\}$. Let $r_k \nsim r_i$, where $k \geqslant 3$ is a fixed odd number. Hence $2k + 2\eta(i) > 2n - (1 - (-1)^{i+k})$. Therefore, $i \in A \cup B$, where $A = \{2(n-1), 2(n-2), \dots, 2(n-k)\}$ and $B = \{n-1, n-3, \dots, n-k+2\}$. Since $k \geqslant 3$, if $n \neq 4$, then $t(r_k, G) \geqslant 4$. Let $r_k \nsim r_i$, where $k \geqslant 4$ is a fixed even number. Hence $k + 2\eta(i) > 2n - (1 - (-1)^{i+k})$. Define $A = \{2(n-1), 2(n-2), \dots, 2(n-k/2+1)\}$ and $B = \{n-1, n-3, \dots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 1$, otherwise $a = 0$. Therefore, $i \in A \cup B$. Similarly to the above, if $k = 4$, then $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-1)}\}$ otherwise, $t(r_k, G) \geqslant 4$.

Remark 3.3. Let $G = C_n(q)$ or $G = B_n(q)$, where $q = p^\alpha$ and $n > 3$. Let $n$ be odd. By [**11**, Proposition 3.1], we have $\rho(p, G) = \{p, r_n, r_{2n}\}$. By Lemma 2.5, we know that $\rho(r_1, G) = \{r_1, r_{2n}\}$ and $\rho(r_2, G) = \{r_2, r_n\}$. Let $r_k \nsim r_i$, where $k \geqslant 3$ is a fixed odd number. Hence $k+\eta(i) > n$. Suppose $A = \{2n, 2(n-1), \dots, 2(n-k+1)\}$ and $B = \{n, n-2, \dots, n-k+1\}$. Therefore, $i \in A \cup B$. Since $k \geqslant 3$, so $t(r_k, G) \geqslant 4$. Let $r_k \nsim r_i$, where $k \geqslant 4$ is a fixed even number. Hence $k/2 + \eta(i) > n$. Define $A = \{2n, 2(n-1), 2(n-2), \dots, 2(n-k/2+1)\}$ and $B = \{n, n-2, \dots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 2$, otherwise $a = 1$. Therefore, $i \in A \cup B$. If $k \neq 4$, then $t(r_k, G) \geqslant 4$. Let $k = 4$, if $n \equiv 1 \pmod 4$, then $t(r_4, G) = 4$, otherwise $\rho(r_4, G) = \{r_4, r_n, r_{2n}\}$.

Let $n$ be even. By [**11**, Proposition 3.1], we have $\rho(p, G) = \{p, r_{2n}\}$. By Lemma 2.5, we know that $\rho(r_1, G) = \{r_1, r_{2n}\}$ and $\rho(r_2, G) = \{r_2, r_{2n}\}$. Let $r_k \nsim r_i$, where $k \geqslant 3$ is a fixed odd number. Hence $k + \eta(i) > n$. Suppose $A = \{2n, 2(n-1), \dots, 2(n-k+1)\}$ and $B = \{n-1, n-3, \dots, n-k+2\}$. Therefore, $i \in A \cup B$. Since $k \geqslant 3$, so $t(r_k, G) \geqslant 4$. Let $r_k \nsim r_i$, where $k \geqslant 4$ is a fixed even number. Hence $k/2 + \eta(i) > n$. Define $A = \{2n, 2(n-1), \dots, 2(n-k/2+1)\}$ and $B = \{n-1, n-3, \dots, n-k/2+a\}$, where if $k \equiv 0 \pmod 4$, then $a = 1$, otherwise, $a = 2$. Therefore, $i \in A \cup B$. Let $k = 4$, if $n \equiv 0 \pmod 4$, then $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-1)}, r_{2n}\}$, otherwise $\rho(r_4, G) = \{r_4, r_{n-1}, r_{2(n-1)}\}$. Let $k \geqslant 6$, so $t(r_k, G) \geqslant 4$.

Remark 3.4. Let $G = A_{n-1}(q)$. By [**11**, Proposition 3.1], we have $t(p, G) = 3$ and also by [**11**, Proposition 4.1], we know that $2 \leqslant t(r_1, G) \leqslant 3$. Let $r_k \nsim r_i$, where $k \neq 1$ is a fixed number, hence $i \in \{n, n-1, \dots, n-k+1\}$. Therefore, by Lemma 2.6, we have $t(r_2, G) = 2$ and $t(r_3, G) = 3$. Let $k \geqslant 4$, so $t(r_k, G) \geqslant 4$.

Remark 3.5. Let $G = {}^2A_{n-1}(q)$. By [**11**, Proposition 3.1], we have $t(p, G) = 3$ and also by [**11**, Proposition 4.2], we know that $2 \leqslant t(r_2, G) \leqslant 3$. Let $r_k \nsim r_i$, where $k \neq 2$ is a fixed number, hence $i \in \{n, n-1, \dots, n-k+1\}$. Therefore, by Lemma 2.7, we have $t(r_1, G) = 2$ and $t(r_3, G) = 3$. Let $k \geqslant 4$, so $t(r_k, G) \geqslant 4$.

## 4. Main results

In the sequel, we denote by $r_i$ and $u_i$, a primitive prime divisor of $q^i - 1$ and $q'^i - 1$, respectively. Also we consider $R_i(q)$ and $U_i(q')$ as the set of all primitive prime divisors of $q^i - 1$ and $q'^i - 1$, respectively.

Theorem 4.1. *Let $G = {}^2D_n(q)$, where $n \geqslant 4$ and $q = p^\alpha$, and also $S$ be a classical simple group of Lie type over the field* $\mathrm{GF}(q')$*, where $q' = p'^\beta$. Then $\Gamma(S) = \Gamma(G)$ if and only if one of the following holds:*

(1) *$S = G$.*
(2) *$S = {}^2D_n(q')$, where $q' = p'^\beta$, $4 \mid n$, $p' \neq p$, $p' \equiv 1 \pmod 4$, $p \equiv 1 \pmod 4$, $\pi(q^2 - 1) = \pi(q'^2 - 1)$, $R_{2n}(q) = U_{2n}(q')$ and $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$.*
(3) *$S = B_n(q')$ or $S = C_n(q')$, where $q' = p'^\beta$, $4 \mid n$, $p' \neq p$, $R_1(q) \cup R_2(q) = \{p'\} \cup U_1(q') \cup U_2(q')$, $R_{2n}(q) = U_{2n}(q')$ and either $\{p\} \cup R_4(q) = U_3(q') \cup U_4(q') \cup U_6(q')$ or $\{p\} \cup R_4(q) = U_4(q')$.*

Proof. We know that $\Gamma(S) = \Gamma(G)$, therefore $t(S) = t(G)$, $t(2, S) = t(2, G)$ and for every $r \in \pi(G)$, we have $t(r, G) = t(r, S)$. We know that $t(p, G) \geqslant 3$

and $t(r_1, G) = t(r_2, G) = 2$ and for every $r_i \in \pi(G)$, where $i \notin \{1, 2\}$, we have $t(r_i, G) > 2$, by Remark 3.1. Now we consider each possibility for $S$ by [**13**, Tables 1a–1c].

**Case 1**. Let $S = {}^2D_{n'}(q')$, where $q' = p'^{\beta}$. We have $t(S) = t(G)$ so $[(3n' + 4)/4] = [(3n + 4)/4]$. Therefore, $n = n'$, $n + 1 = n'$ or $n' + 1 = n$. Also $t(p, S) \geqslant 3$, $t(u_1, S) = t(u_2, S) = 2$ and for every $u_i \in \pi(S)$, where $i \notin \{1, 2\}$, we have $t(u_i, S) > 2$, by Remark 3.1. Therefore, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$. Now we consider the following subcases:

**1.1.** Let $n$ be odd.

1.1.1 Let $n + 1 = n'$. It is clear that $p' \neq 2$ otherwise, $t(2, S) = 4$, which is a contradiction, since $t(2, G) \leqslant 3$, by [**11**, Tables 4, 6]. Therefore, $\rho(2, S) = \{2, u_{2n'}\}$, hence $t(2, G) = 2$. Consequently, $p \neq 2$ otherwise, $t(2, G) = 3$, which is a contradiction. Now we consider the following two cases:

1.1.1.1. Let $\rho(2, G) = \{2, r_{2n}\}$. Therefore, $R_{2n}(q) = U_{2n'}(q')$. We know that $r_1 \nsim r_{2n} \sim r_2$, and $u_1 \nsim u_{2n'} \nsim u_2$, which is a contradiction.

1.1.1.2. Let $\rho(2, G) = \{2, r_{2(n-1)}\}$. Therefore, $R_{2(n-1)}(q) = U_{2n'}(q')$. We know that $r_1 \sim r_{2(n-1)} \nsim r_2$, and $u_1 \nsim u_{2n'} \nsim u_2$, which is a contradiction.

Similarly, if $n' + 1 = n$, then we get a contradiction.

1.1.2. Let $n = n'$, now we consider the following subcases:

1.1.2.1. Let $q \equiv 1 \pmod 4$, hence $\rho(2, G) = \{2, r_{2n}\}$. Therefore, $t(2, S) = 2$, and so $q' \not\equiv 3 \pmod 8$. If $q' \equiv 1 \pmod 4$, then $\rho(2, S) = \{2, u_{2n}\}$ and so $R_{2n}(q) = U_{2n}(q')$. We know that $r_1 \nsim r_{2n} \sim r_2$, and $u_1 \nsim u_{2n} \sim u_2$. Consequently, $R_1(q) = U_1(q')$ and $R_2(q) = U_2(q')$. Moreover, we know that $u_2$ is adjacent to all vertices except $u_{2(n-1)}$ and also $r_2$ is adjacent to all vertices except $r_{2(n-1)}$, which implies that $R_{2(n-1)}(q) = U_{2(n-1)}(q')$. Consequently, $R_{2(n-1)}(q) \cup R_{2n}(q) = U_{2(n-1)}(q') \cup U_{2n}(q')$. Therefore, every $r_{2n}$ and $r_{2(n-1)}$ can be regarded as $u_{2n}$ and $u_{2(n-1)}$. For convenience in the sequel we write $\{r_{2n}, r_{2(n-1)}\} \approx \{u_{2n}, u_{2(n-1)}\}$ to illustrate the above statement. By Remark 3.1, we know that $p$ is the only vertex in $\Gamma(G)$, which is adjacent to all vertices except $r_{2(n-1)}$ and $r_{2n}$, and similarly $p'$ is the only vertex in $\Gamma(S)$, which is adjacent to all vertices except $u_{2(n-1)}$ and $u_{2n}$. Consequently, $p = p'$. Since $\pi(S) = \pi(G)$, so $\alpha = \beta$, by Lemma 2.3, which implies that $S = G$. If $q' \equiv 7 \pmod 8$, then $R_{2n}(q) = U_{2(n-1)}(q')$. Similarly to the above, by the above notation $\{r_{2(n-1)}, r_{2n}\} \approx \{u_{2(n-1)}, u_{2n}\}$, and by Remark 3.1, $p = p'$ and so $q = q'$, which is a contradiction, since $q \equiv 1 \pmod 4$.

1.1.2.2. Let $q \equiv 7 \pmod 8$, hence $\rho(2, G) = \{2, r_{2(n-1)}\}$, completely similar to the above case we get that $S = G$.

1.1.2.3. Let $q \equiv 3 \pmod 8$, hence $\rho(2, G) = \{2, r_{2(n-1)}, r_{2n}\}$ so $t(2, S) = 3$. It follows that $\rho(2, S) = \{2, u_{2(n-1)}, u_{2n}\}$, by [**11**, Tables 4, 6]. Therefore, $R_{2(n-1)}(q) \cup R_{2n}(q) = U_{2(n-1)}(q') \cup U_{2n}(q')$. Therefore, $\{r_{2(n-1)}, r_{2n}\} \approx \{u_{2(n-1)}, u_{2n}\}$. We know that $p$ and 2 are the only vertices which are adjacent to all vertices in $\Gamma(G)$ except $r_{2n}$ and $r_{2(n-1)}$. Also we know that $p'$ and 2 are the only vertices which are adjacent to all vertices in $\Gamma(S)$ except $u_{2n}$ and $u_{2(n-1)}$. Consequently, $\{2, p\} = \{2, p'\}$. Since $q \equiv 3 \pmod 8$, so $p \neq 2$. Therefore, $p = p'$. Since $\pi(S) = \pi(G)$, so $\alpha = \beta$, by Lemma 2.3, which implies that $S = G$.

1.1.2.4. Let $q = 2^{\alpha}$, hence $\rho(2, G) = \{2, r_{2(n-1)}, r_{2n}\}$ so $t(2, S) = 3$. Consequently, either $q' \equiv 3 \pmod 8$ or $q' = 2^\beta$, so $\rho(2, S) = \{2, u_{2(n-1)}, u_{2n}\}$. Therefore, similarly to the above $\{r_{2(n-1)}, r_{2n}\} \approx \{u_{2(n-1)}, u_{2n}\}$. If $q' \equiv 3 \pmod 8$, then $p'$ and 2 are the only vertices, which are adjacent to all vertices except $u_{2n}$ and $u_{2(n-1)}$, by [**11**, Tables 4, 6] and Remark 3.1. On the other hand, $p$ is the only vertex, which is adjacent to all vertices except $r_{2n}$ and $r_{2(n-1)}$. Consequently, $\{p\} = \{p', 2\}$, which is a contradiction. It follows that $p' = 2 = p$. Since $\pi(G) = \pi(S)$, so $\alpha = \beta$, by Lemma 2.3. Therefore, $S = G$.

**1.2** Let $n$ be even.

1.2.1. Let $n + 1 = n'$. It is clear that $p \neq 2$, since $t(2, S) \neq t(2, G)$. Therefore, $\rho(2, G) = \{2, r_{2n}\}$ and we know that $t(2, S) = 2$. Now we consider the following two cases:

1.2.1.1. Let $\rho(2, S) = \{2, u_{2n'}\}$. Therefore, $R_{2n}(q) = U_{2n'}(q')$. We know that $r_1 \nsim r_{2n} \nsim r_2$, and $u_1 \nsim u_{2n'} \sim u_2$, which is a contradiction.

1.2.1.2 Let $\rho(2, S) = \{2, u_{2(n'-1)}\}$. Therefore, $R_{2n}(q) = U_{2(n'-1)}(q')$. We know that $r_1 \nsim r_{2n} \nsim r_2$, and $u_1 \sim u_{2(n'-1)} \nsim u_2$, which is a contradiction.

Similarly, if $n' + 1 = n$, then we get a contradiction.

1.2.2. Let $n = n'$. If $p = 2$, then $t(2, G) = 4$. It follows that $p' = 2$, by [**11**, Tables 4, 6]. Consequently, $p = p'$ and so similarly to the above we have $G = S$. If $p \neq 2$, then $t(2, G) = 2$. Hence $p' \neq 2$, since otherwise, $t(2, S) = 4$. Consequently, $p \neq 2$ and $p' \neq 2$. Since $n$ is even, so $\rho(2, G) = \{2, r_{2n}\}$ and $\rho(2, S) = \{2, u_{2n}\}$. Therefore, $R_{2n}(q) = U_{2n}(q')$. By Remark 3.1, $p$ and $r_4$ are the only vertices in $\Gamma(G)$ such that their independence numbers are 4. Also $p'$ and $u_4$ are the only vertices in $\Gamma(S)$ such that their independence numbers are equal to 4. Therefore, $R_4(q) \cup \{p\} = U_4(q') \cup \{p'\}$. Now we consider the following two cases:

1.2.2.1. Let $n \equiv 2 \pmod 4$. If $p = p'$, then similarly to the above we have $S = G$. Otherwise, there exists $u_4$ such that $p = u_4$. We know that $\{p, r_{n-1}, r_{2(n-1)}, r_{2n}\}$ is the unique maximal independent set in $\Gamma(G)$ which contains $p$. Also we know that $\{u_4, u_{n-1}, u_{2(n-2)}, u_{2(n-1)}\}$ is the unique maximal independent set in $\Gamma(S)$ which contains $u_4$. So $R_{n-1}(q) \cup R_{2(n-1)}(q) \cup R_{2n}(q) = U_{n-1}(q') \cup U_{2(n-2)}(q') \cup U_{2(n-1)}(q')$, which is a contradiction, since $R_{2n}(q) = U_{2n}(q')$.

1.2.2.2. Let $n \equiv 0 \pmod 4$. Thus $\{r_{n-1}, r_{2(n-1)}, r_{2n}\}$ is equal to $\rho(r_4, G) \setminus \{r_4\}$ and $\rho(p, G) \setminus \{p\}$. Similarly we can consider $\rho(u_4, S) \setminus \{u_4\} = \rho(p', S) \setminus \{p'\}$. If $p = p'$, then similarly to the above we have $S = G$. Otherwise, there exist $r_4$ and $u_4$ such that $p = u_4$ and $p' = r_4$. Consequently, $S = {}^2D_n(r_4^\beta)$.

**Case 2**. Let $S = D_{n'}(q')$, where $q' = p'^\beta$.

If $n' = 4$, then $t(S) = 3$, and so $t(G) = 3$. Therefore, $n = 3$, which is a contradiction. Consequently $n' > 4$. We know that $t(p, S) = 3$ and $t(u_1, S) = t(u_2, S) = 2$ and for every $u_i \in \pi(S)$, where $i > 2$, we have $t(u_i, S) > 2$, by Remark 3.2. Therefore, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$.

Let $n$ be even. By Remark 3.1, there is no vertex in $\Gamma(G)$, whose independence number is 3, while $p' \in \pi(S)$ and $t(p', S) = 3$, which is a contradiction.

Therefore $n$ is odd. We know that $t(S) = t(G)$.

**2.1**. If $t(S) = [(3n' + 1)/4]$, then $n' = n$, $n' = n + 1$ or $n' = n + 2$.

2.1.1. Let $n = n'$. We know that $t(2, S) = 2$ or 3. We consider the following two

cases:
2.1.1.1. Let $t(2,S)=3$, hence $\rho(2,S)=\{2,u_n,u_{2(n-1)}\}$, by [**11**, Tables 4, 6]. So $t(2,G)=3$ and $\rho(2,G)=\{2,r_{2(n-1)},r_{2n}\}$, by [**11**, Tables 4, 6]. Therefore, similarly to the above, $R_{2n}(q)\cup R_{2(n-1)}(q)=U_n(q')\cup U_{2(n-1)}(q')$ and consequently $\{r_{2(n-1)},r_{2n}\}\approx\{u_n,u_{2(n-1)}\}$. By Remarks 3.1 and 3.2, $\{p,2\}=\{p',2\}$. Therefore, $p=p'$. Since $\pi(S)=\pi(G)$, so $2n\alpha=2(n-1)\beta$, by Lemma 2.3, so $(\alpha)_2>(\beta)_2$. Let $x$ be a primitive prime divisor of $p^{2n\alpha}-1$, so $x$ is a primitive prime divisor of $q^{2n}-1$. By assumption, $x\in U_n(q')\cup U_{2(n-1)}(q')$. If $x\in U_n(q')$, then $x\mid(p^{n\beta}-1)$, which implies that $2n\alpha\leqslant n\beta$, and this is a contradiction. Therefore, $R_{2n}(q)=U_{2(n-1)}(q')$ and $R_{2(n-1)}(q)=U_n(q')$. Let $x$ be a primitive prime divisor of $p^{2(n-1)\alpha}-1$. Then similarly to the above $2(n-1)\alpha\leqslant n\beta$. If $y$ is a primitive prime divisor of $p^{n\beta}-1$, then similarly we have $n\beta\leqslant 2(n-1)\alpha$. Therefore, $n\beta=2(n-1)\alpha$. Since $n\alpha=(n-1)\beta$, so $(n-2)\alpha=\beta$, which is a contradiction, since $n$ is odd and $(\alpha)_2>(\beta)_2$.
2.2.1.2. Let $t(2,S)=2$ and so $p'\neq 2$. Let $q'\equiv 3 \pmod 4$ and so $\rho(2,S)=\{2,u_n\}$. We know that $t(2,G)=t(2,S)=2$. Now we consider the following two cases:
2.2.1.2.1. Let $q\equiv 1 \pmod 4$, so $R_{2n}(q)=U_n(q')$. We know that $r_1\nsim r_{2n}\sim r_2$ and $u_1\sim u_n\nsim u_2$. Therefore, $R_2(q)=U_1(q')$ and so $R_{2(n-1)}(q)=U_{2(n-1)}(q')$.
2.2.1.2.2. Let $q\equiv 7 \pmod 8$, so $R_{2(n-1)}(q)=U_n(q')$. We know that $r_1\sim r_{2(n-1)}\nsim r_2$ and $u_1\sim u_n\nsim u_2$. Therefore, $R_1(q)=U_1(q')$ and so $R_{2n}(q)=U_{2(n-1)}(q')$.
Consequently, $R_{2n}(q)\cup R_{2(n-1)}(q)=U_n(q')\cup U_{2(n-1)}(q')$ and similarly to the above $\{r_{2(n-1)},r_{2n}\}\approx\{u_n,u_{2(n-1)}\}$. By Remarks 3.1 and 3.2, $p=p'$. Similarly to the above we have $2n\alpha=2(n-1)\beta$ and so $(\alpha)_2>(\beta)_2$. If $R_{2n}(q)=U_{2(n-1)}(q')$, then similarly to the above we get a contradiction. Therefore, $R_{2n}(q)=U_n(q')$ and similarly $2n\alpha=n\beta$, which is a contradiction.
Let $q'\equiv 1 \pmod 4$, then similarly to the above we get a contradiction.
2.1.2. Let $n=n'+1$. If $p'=2$, then $t(2,S)=t(2,G)=3$. We know that $\rho(2,S)=\{2,u_{n'-1},u_{2(n'-1)}\}$ and $\rho(2,G)=\{2,r_{2(n-1)},r_{2n}\}$, by [**11**, Tables 4,6]. Therefore, $R_{2(n-1)}(q)\cup R_{2n}(q)=U_{n'-1}(q')\cup U_{2(n'-1)}(q')$ and so $\{r_{2n},r_{2(n-1)}\}\approx\{u_{n'-1},u_{2(n'-1)}\}$. By Remarks 3.1 and 3.2, we have $\{2,p\}=U_4(q')\cup\{p'\}$. If $p=2$, then we get a contradiction. It follows that $U_4(q')$ has one member and $p=u_4$, and so there exists a natural number $m$ such that $p^m=q'^2+1$. By Lemma 2.1, we have $m=1$. On the other hand, we know that $R_1(q)\cup R_2(q)=U_1(q')\cup U_2(q')$ or in other words $\pi(q^2-1)=\pi(q'^2-1)$. Consequently, $\pi((q'^2+1)^{2\alpha}-1)=\pi(q'^2-1)$, which is a contradiction. Therefore, $p'\neq 2$ and since $n'$ is even, so $t(2,S)=2$.
If $q'\equiv 3 \pmod 4$, so $\rho(2,S)=\{2,u_{n'-1}\}$. We know that $t(2,G)=t(2,S)=2$. Now we consider the following two cases:
2.1.2.1. Let $q\equiv 1 \pmod 4$, so $R_{2n}(q)=U_{n'-1}(q')$. We know that $r_1\nsim r_{2n}\sim r_2$ and $u_1\sim u_{n'-1}\nsim u_2$. Therefore, $R_2(q)=U_1(q')$ and so $R_{2(n-1)}(q)=U_{2(n'-1)}(q')$.
2.1.2.2. Let $q\equiv 7 \pmod 8$, so $R_{2(n-1)}(q)=U_{n'-1}(q')$. We know that $r_1\sim r_{2(n-1)}\nsim r_2$ and $u_1\sim u_{n'-1}\nsim u_2$. Therefore, $R_1(q)=U_1(q')$ and so $R_{2n}(q)=U_{2(n'-1)}(q')$. Thus, similarly to the above we have $\{r_{2(n-1)},r_{2n}\}\approx\{u_{n'-1},u_{2(n'-1)}\}$. By Remarks 3.1 and 3.2, we have $\{p\}=\{p'\}\cup U_4(q')$, which is a contradiction.
If $q'\equiv 1 \pmod 4$, then $\rho(2,S)=\{2,u_{2(n'-1)}\}$ and similarly to the above we get a

contradiction.
2.1.3. Let $n' = n+2$, so $n'$ is odd. It is clear, either $n \equiv 1 \pmod 4$ or $n' \equiv 1 \pmod 4$. If $n \equiv 1 \pmod 4$, then $p$ is the only vertex in $\Gamma(G)$, whose independence number is 3. Also independence number of $p'$ and $u_4$ are 3 in $\Gamma(S)$. Therefore, $\{p\} = \{p'\} \cup U_4(q')$, which is a contradiction. Similarly, if $n' \equiv 1 \pmod 4$, then we get a contradiction.
**2.2**. If $t(S) = (3n'+3)/4$, then $n = n'$ and similarly to the above we get a contradiction.

**Case 3**. Let $S = C_{n'}(q')$ or $S = B_{n'}(q')$, where $q' = p'^{\beta}$.

If $n' \leqslant 3$, then $t(S) \leqslant 3$, and so $t(G) \leqslant 3$. Therefore, $n \leqslant 3$, which is a contradiction. Consequently, $n' > 3$. We have $t(S) = t(G)$ so $[(3n'+5)/4] = [(3n+4)/4]$. Therefore, $n = n'+1$ or $n = n'$.
**3.1**. Let $n$ be odd.
3.1.1. Let $n = n'$. We know that $t(p,S) = 3$, $t(u_1,S) = t(u_2,S) = 2$ and for every $u_i \in \pi(S)$, where $i > 2$, we have $t(u_i,S) > 2$, by Remark 3.3. Therefore, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$. Let $p' = 2$. Then $p = 2$, since $\pi(q^2-1) = \pi(q'^2-1)$. Since $\pi(G) = \pi(S)$, so $\alpha = \beta$, by Lemma 2.3. Therefore, $S = C_n(q)$ or $S = B_n(q)$. We know that $r_n \in \pi(S) \smallsetminus \pi(G)$, which is a contradiction. Consequently, $p' \neq 2$ and so $t(2,S) = t(2,G) = 2$. Therefore $\rho(2,S) = \{2, u_{2n}\}$ or $\rho(2,S) = \{2, u_n\}$. Since the proofs are similar, for convenience we give a proof for $\rho(2,S) = \{2, u_{2n}\}$ and the proof of the other case is similar. Let $\rho(2,S) = \{2, u_{2n}\}$.
3.1.1.1. If $q \equiv 1 \pmod 4$, then $R_{2n}(q) = U_{2n}(q')$. We know that $r_1 \nsim r_{2n} \sim r_2$ and $u_1 \nsim u_{2n} \sim u_2$. Consequently, $R_2(q) = U_2(q')$ and so $R_{2(n-1)}(q) = U_n(q')$. Similarly to the above $\{r_{2(n-1)}, r_{2n}\} \approx \{u_n, u_{2n}\}$. By Remarks 3.1 and 3.3, if $n \equiv 1 \pmod 4$, then $p = p'$. Similarly to the above we can see that $S = C_n(q)$ or $S = B_n(q)$, which is a contradiction. Otherwise, $\{p\} = \{p'\} \cup U_4(q')$, which is a contradiction.
3.1.1.2. If $q \equiv 7 \pmod 8$, then $R_{2(n-1)}(q) = U_{2n}(q')$ and similarly to the above we get a contradiction.
3.1.2. Let $n = n'+1$, so $n'$ is even. Hence $\rho(2,S) = \{2, u_{2n'}\}$ and $t(p',S) = 2$, by [**11**, Tables 4, 6]. By Remarks 3.1 and 3.3, and similarly to the above, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q') \cup \{p'\}$ and $u_{2n'}$ is not adjacent to $u_1$, $u_2$ and $p'$. If $q \equiv 1 \pmod 4$, then $R_{2n}(q) = U_{2n'}(q')$. On the other hand, we know that $r_{2n} \sim r_2$, which is a contradiction. Similarly, if $q \equiv 7 \pmod 8$, then $R_{2(n-1)}(q) = U_{2n'}(q')$, while $r_{2(n-1)} \sim r_1$, which is a contradiction.
**3.2** Let $n$ be even. It is clear that $p \neq 2$, otherwise, $t(2,S) \neq t(2,G)$, which is a contradiction. Therefore, $\rho(2,G) = \{2, r_{2n}\}$.
3.2.1. Let $n = n'$, so $\rho(2,S) = \{2, u_{2n}\}$. Therefore, $R_{2n}(q) = U_{2n}(q')$. By Remarks 3.1 and 3.3, $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q') \cup \{p'\}$.
Let $n \not\equiv 0 \pmod 4$, so $t(u_4,S) = 3$. But we know that $t(x,G) \neq 3$, for every $x \in \pi(G)$, which is a contradiction. Consequently, $n \equiv 0 \pmod 4$. By Remark 3.1, we have $p$ and $r_4$ are the only vertices in $\Gamma(G)$ such that their independence numbers are equal to 4.
If $3 \mid (n-1)$, then $u_3$, $u_4$ and $u_6$ are the only vertices in $\Gamma(S)$ such that their independence numbers are equal to 4. In this case, we have $\{p\} \cup R_4(q) = U_3(q') \cup$

$U_4(q') \cup U_6(q')$. Otherwise, $3 \nmid (n-1)$ and so $\{p\} \cup R_4(q) = U_4(q')$. Similarly, we can find some relations between other vertices.
3.2.2. Let $n = n' + 1$. Since $n$ is even, so by Remark 3.1, there is not any vertex in $\Gamma(G)$ such that its independence number is 3. On the other hand, $n'$ is odd, so $t(p', S) = 3$, which is a contradiction.

**Case 4.** Let $S = A_{n'-1}(q')$ or ${}^2A_{n'-1}(q')$, where $q' = p'^{\beta}$.

Since the proofs for these groups are similar, we state the details of the proof for one of them, say $A_{n'-1}(q')$. So in the sequel let $S = A_{n'-1}(q')$, where $q' = p'^{\beta}$.

Let $n$ be even. By Remark 3.1, there is not any vertex in $\Gamma(G)$, whose independence number is 3, while $t(p', S) = 3$, which is a contradiction. Therefore, $n$ is odd. We know that by [**11**], $t(u_1, S)$ is equal to 2 or 3.
**4.1**. Let $t(u_1, S) = 3$, so by Remarks 3.1 and 3.4, $R_1(q) \cup R_2(q) = U_2(q')$. Now we consider the following cases:
4.1.1. Let $n'_2 < (q'-1)_2$, so $\rho(2, S) = \{2, u_{n'}\}$. Hence $t(2, G) = 2$. If $q \equiv 1 \pmod 4$, then $\rho(2, G) = \{2, r_{2n}\}$. Therefore, $R_{2n}(q) = U_{n'}(q')$. We know that $r_1 \nsim r_{2n} \sim r_2$, which is a contradiction, since $R_1(q) \cup R_2(q) = U_2(q')$. Consequently, $q \equiv 7 \pmod 8$, and so $R_{2(n-1)}(q) = U_{n'}(q')$. We have $r_1 \sim r_{2(n-1)} \nsim r_2$, which is a contradiction.
4.1.2. Let $n'_2 > (q'-1)_2$ or $n'_2 = (q'-1)_2 = 2$, so $\rho(2, S) = \{2, u_{n'-1}\}$. If $q \equiv 1 \pmod 4$, then $R_{2n}(q) = U_{n'-1}(q')$. We know that $r_1 \nsim r_{2n} \sim r_2$, which is a contradiction, since $R_1(q) \cup R_2(q) = U_2(q')$. Similarly, if $q \equiv 7 \pmod 8$, then $R_{2(n-1)}(q) = U_{n'-1}(q')$, while we know that $r_1 \sim r_{2(n-1)} \nsim r_2$, which is a contradiction.
4.1.3. Let $2 < n'_2 = (q'-1)_2$, so $\rho(2, S) = \{2, u_{n'-1}, u_{n'}\}$. Therefore, $q \equiv 3 \pmod 8$, hence similarly to the above we get that $\{r_{2(n-1)}, r_{2n}\} \approx \{u_{n'-1}, u_{n'}\}$. It follows that $R_{2n}(q) = U_{n'}(q')$ or $R_{2n}(q) = U_{n'-1}(q')$ and similarly to the above we get a contradiction.
**4.2**. Let $t(u_1, S) = 2$, hence $R_1(q) \cup R_2(q) = U_1(q') \cup U_2(q')$. Let $n \equiv 1 \pmod 4$. Let $t(2, S) = t(2, G) = 2$. By Remark 3.4, we know that $p'$ and $u_3$ are the only vertices in $\Gamma(S)$ such that their independence number is 3. On the other hand, $p$ is the only vertex in $\Gamma(G)$ such that its independence number is 3 so $\{p\} = \{p'\} \cup U_3(q')$, which is a contradiction. So $t(2, S) = t(2, G) = 3$. Similarly to the above we have, $\{2, p\} = \{2, p'\} \cup U_3(q')$. Also $p' = 2$ if and only if $p = 2$, since $\pi(q^2-1) = \pi(q'^2-1)$. It follows that $u_3 = 2$, which is a contradiction. Therefore, $p = p' \neq 2$ and so $\{2\} = \{2\} \cup U_3(q')$, which is a contradiction. Hence, $n \not\equiv 1 \pmod 4$.

Now we claim that $t(2, S) = t(2, G) = 3$. Otherwise, $t(2, S) = t(2, G) = 2$ and $\{p\} \cup R_4(q) = \{p'\} \cup U_3(q')$. If $p' = p$, then $\{r_{2(n-1)}, r_{2n}\} \approx \{u_{n'-1}, u_{n'}\}$.

Let $R_{2n}(q) = U_{n'}(q')$ and $R_{2(n-1)}(q) = U_{n'-1}(q')$. If $x$ is a primitive prime divisor of $p^{2n\alpha} - 1$, then $x$ is a primitive prime divisor of $q^{2n} - 1$. Therefore, $x \mid (p^{n'\beta} - 1)$, which implies that $2n\alpha \leqslant n'\beta$. Let $y$ be a primitive prime divisor of $p^{n'\beta} - 1$, so $y \mid (p^{2n\alpha} - 1)$, which implies that $n'\beta \leqslant 2n\alpha$. Consequently, $n'\beta = 2n\alpha$, similarly we have $(n'-1)\beta = 2(n-1)\alpha$. It follows that $2\alpha = \beta$. On the other hand, $t(S) = t(G)$ so $n' \in \{(3n-1)/2, (3n+1)/2, (3n+3)/2\}$. Consequently, $S = A_{n'-1}(p^{2\alpha})$, which is a contradiction, since $\pi(S) \neq \pi(G)$.

Let $R_{2n}(q) = U_{n'-1}(q')$ and $R_{2(n-1)}(q) = U_{n'}(q')$. Similarly to the above, we have $2n\alpha = (n'-1)\beta$ and $2(n-1)\alpha = n'\beta$, which is a contradiction.

Therefore, $p \neq p'$ and so $p$ is a primitive prime divisor of $q'^3 - 1$ and $p'$ is a primitive prime divisor of $q^4 - 1$. Hence we can consider $p' = r_4$ and $p = u_3$.

Since $p' = r_4$, so $\rho(r_4, G) \smallsetminus \{r_4\} = \rho(p', S) \smallsetminus \{p'\}$, hence $R_{2(n-2)}(q) \cup R_{2n}(q) = U_{n'-1}(q') \cup U_{n'}(q')$. On the other hand, $u_3$ is not adjacent to two elements of $\{u_{n'-2}, u_{n'-1}, u_{n'}\}$ in $\Gamma(S)$ and $\rho(p, G) = \{p, r_{2(n-1)}, r_{2n}\}$. Consequently, $3 \nmid (n'-2)$ and $u_3 \nsim u_{n'-2}$, since $p = u_3$. Therefore, either $3 \mid (n'-1)$ and so $R_{2(n-1)}(q) \cup R_{2n}(q) = U_{n'-2}(q') \cup U_{n'}(q')$ or $3 \mid n'$ and $R_{2(n-1)}(q) \cup R_{2n}(q) = U_{n'-2}(q') \cup U_{n'-1}(q')$. Moreover, we know that $t(S) = t(G)$, which implies $2n' \in \{3n-1, 3n+1, 3n+3\}$.

Let $3n+1 = 2n'$. It is clear that neither $3 \mid (n'-1)$ nor $3 \mid n'$, which is a contradiction.

By Remark 3.1, if $3 \nmid (n-2)$, then for every $x \in \pi(G)$, we have $t(x, G) \neq 4$, while $t(u_4, S) = 4$, which is a contradiction. So $3 \mid (n-2)$ and $t(r_3, G) = 4$. Since $u_4$ is the only vertex in $\Gamma(S)$, whose independence number is equal to 4, so $r_3 = u_4$. Therefore, $\rho(r_3, G) \smallsetminus \{r_3\} = \rho(u_4, S) \smallsetminus \{u_4\}$. It follows that $R_{2(n-2)}(q) \cup R_{2(n-1)}(q) \cup R_{2n}(q) = \rho(u_4, S) \smallsetminus \{u_4\}$. By the above discussion, we get that $R_{2(n-2)}(q) \cup R_{2(n-1)}(q) \cup R_{2n}(q) = U_{n'-2}(q') \cup U_{n'-1}(q') \cup U_{n'}(q')$. Hence $4 \mid (n'-3)$, by Lemma 2.6, and so $n'$ is odd.

If $3n+3 = 2n'$, then $3(n+1) = 2n'$, which is a contradiction, since $n \not\equiv 1 \pmod 4$. If $3n-1 = 2n'$, then $3(n-1) = 2(n'-1)$, we get a contradiction.

Hence, $t(2, S) = t(2, G) = 3$, so $\{2, p\} \cup R_4(q) = \{2, p'\} \cup U_3(q')$. If $p = 2$, then $p' = 2$, since $\pi(q^2-1) = \pi(q'^2-1)$. Therefore, $p = p'$ and similarly to the above we get a contradiction. Consequently, since $r_4 \neq 2$ and $u_3 \neq 2$ so $\{p\} \cup R_4(q) = \{p'\} \cup U_3(q')$. Now completely similar to the above we get a contradiction. $\square$

Theorem 4.2. *Let $G = {}^2D_n(q)$, where $q = p^\alpha$ and $n \geqslant 4$, and also $S$ be an exceptional group of Lie type. Then $\Gamma(S)$ and $\Gamma(G)$ are not equal.*

Proof. We consider the following cases:
(1) Let $S = E_8(q')$. Since $s(E_8(q')) \geqslant 4$ and $s(G) \leqslant 3$, by [**13**, Tables 1a–1c], so we get a contradiction. Similarly $S \neq {}^2B_2(q')$, where $q' = 2^{2m+1}$.
(2) Let $S = G_2(q')$. We know that $t(S) = t(G)$. Therefore, $[(3n+4)/4] = 3$, so $n = 3$, which is a contradiction. Similarly $S \neq {}^3D_4(q')$ and ${}^2F_4(2')$.
(3) Let $S = E_6(q')$. Since $t(S) = t(G)$, so $[(3n+4)/4] = 5$, hence $n = 6$. We know that $\Gamma(S)$ has two components so $s(G) = 2$, which is a contradiction, by [**13**, Tables 1a–1c]. Similarly $S$ is not isomorphic to ${}^2E_6(q')$, ${}^2G_2(q')$, where $q' = 3^{2m+1}$, ${}^2F_4(q')$, where $q' = 2^{2m+1}$ and $F_4(q')$, where $q' > 2$.
(4) Let $S = E_7(q')$. So $[(3n+4)/4] = 8$, hence $n = 10$. Therefore, $t(2, G) = 2$ or 4, while $t(2, S) = 3$, which is a contradiction. Similarly $S \neq F_4(2)$. $\square$

Theorem 4.3. *Let $G = {}^2D_n(q)$, where $q = p^\alpha$ and $n \geqslant 4$, and also $S$ be an alternating or sporadic group. Then $\Gamma(S)$ and $\Gamma(G)$ are not equal.*

Proof. We consider the following cases:
(1) Let $S = M_{22}$. Since $s(M_{22}) = 4$ and $s(G) \leqslant 3$, by [**13**, Tables 1a–1c], so we get

a contradiction. Similarly $S \neq J_1$, $J_4$, $ON$, $Ly$, $F'_{24}$ and $F_1$.
(2) Let $S = M_{11}$. We know that $t(S) = t(G)$. Therefore, $[(3n+4)/4] = 3$, so $n = 3$, which is a contradiction. Similarly $S$ is not isomorphic to $M_{12}$, $J_2$, $J_3$, $He$, $McL$, $HN$ and $HiS$.
(3) Let $S = F_3$. Since $t(S) = t(G)$, so $[(3n+4)/4] = 5$, hence $n = 6$. We know that $\Gamma(S)$ has three components so $s(G) = 3$, which is a contradiction, by [**13**, Tables 1a–1c]. Similarly $S \neq Fi_{23}$ and $F_2$.
(4) Let $S = M_{23}$. So $[(3n+4)/4] = 4$, hence $n = 4$ or $5$. By [**13**, Tables 1a–1c], $s(G) = 1$ while $s(S) > 1$, which is a contradiction. Similarly for the other sporadic groups, we get a contradiction.

By [**16**], it is clear that $S$ cannot be equal to an alternating group. $\square$

Corollary 4.1. (i) *If $n$ is a natural number such that $4 \nmid n$, then the Vasil'ev Conjecture is true for the nonabelian simple group ${}^2D_n(p^\alpha)$.*

(ii) *Let $4 \mid n$ and $q = p^\alpha$. If $S$ is a nonabelian simple group such that $\Gamma(S) = \Gamma({}^2D_n(q))$, then one of the following holds:*

(1) $S = G$.
(2) $S = {}^2D_n(q')$, *where* $q' = p'^\beta$, $p' \neq p$, $p' \equiv 1 \pmod 4$, $p \equiv 1 \pmod 4$, $\pi(q^2-1) = \pi(q'^2-1)$, $R_{2n}(q) = U_{2n}(q')$ *and* $\{p\} \cup R_4(q) = \{p'\} \cup U_4(q')$.
(3) $S = B_n(q')$ *or* $S = C_n(q')$, *where* $q' = p'^\beta$, $p' \neq p$, $R_1(q) \cup R_2(q) = \{p'\} \cup U_1(q') \cup U_2(q')$, $R_{2n}(q) = U_{2n}(q')$ *and either* $\{p\} \cup R_4(q) = U_3(q') \cup U_4(q') \cup U_6(q')$ *or* $\{p\} \cup R_4(q) = U_4(q')$.

Finally we state

Conjecture. Cases (2) and (3) in above corollary can not occur.

It is clear that if the conjecture is true, then Vasil'ev's conjecture will be true for ${}^2D_n(q)$, for each $n$ and prime power $q$.

## References


1. A. Babai, B. Khosravi, N. Hasani, *Quasirecognition by prime graph of ${}^2D_p(3)$ where $p = 2^n + 1 \geqslant 5$ is a prime*, Bull. Malays. Math. Sci. Soc. (2) **32**(3) (2009), 343–350.
2. A. Babai, B. Khosravi, *Recognition by prime graph of ${}^2D_{2^m+1}(3)$*, Sib. Math. J. **52**(5) (2011), 788–795.
3. ———, *Quasirecognition by prime graph of ${}^2D_n(3^\alpha)$, where $n = 4m + 1 \geqslant 21$ and $\alpha$ is odd*, Math. Notes. **95**(3) (2014), 3–13.
4. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of finite groups*, Oxford University Press, Oxford, 1985.
5. P. Crescenzo, *A diophantine equation which arises in the theory of finite groups*, Adv. Math. **17**(1) (1975), 25–29.
6. M. Hagie, *The prime graph of a sporadic simple group*, Comm. Algebra **31**(9) (2003), 4405–4424.
7. A. Khosravi, B. Khosravi, *A new characterization of some alternating and symmetric groups (II)*, Houston J. Math. **30** (2004), 465–478.
8. ———, *Quasirecognition by prime graph of the simple group ${}^2G_2(q)$*, Sib. Math. J. **48**(3) (2007), 570–577.
9. V. D. Mazurov, E. I. Khukhro (Eds.), *The Kourovka Notebook:Unsolved Problems in Group Theory*, 16th ed., Sobolev Inst. Math., Novosibirsk, 2006.

10. W. Sierpiński, *Elementary theory of numbers*, Monografie Matematyczne **42**, Panstwowe Wydawnictwo Naukowe, Warsaw, 1964.
11. A. V. Vasil'ev, E. P. Vdovin, *An adjacency criterion in the prime graph of a finite simple group*, Algebra Logic **44**(6) (2005), 381–405.
12. ______, *Cocliques of maximal size in the prime graph of a finite simple group*, arXiv:0905.1164.
13. A. V. Vasil'ev, M. A. Grechkoseeva, *On the recognition of the finite simple orthogonal groups of dimension* $2^m$*,* $2^m+1$ *and* $2^m+2$ *over a field of characteristic 2*, Sib. Math. J. **45**(3) (2004), 420–431.
14. A. V. Zavarnitsin, *On the recognition of finite groups by the prime graph*, Algebra Logic **43**(4) (2006), 220–231.
15. K. Zsigmondy, *Zur theorie der potenzreste*, Monatsh. Math. Phys. **3** (1892), 265–284.
16. M. A. Zvezdina, *On nonabelian simple groups having the same prime graph as an alternating group*, Sib. Math. J. **54**(1) (2013), 47–55.

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(Received 23 04 2014)

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