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NSE CHARACTERIZATION OF THE SIMPLE GROUP $L_2(3^n)$

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Abstract. Let G be a group and $\pi(G)$ be the set of primes p such that G contains an element of order p. Let $\mathrm{nse}(G)$ be the set of the numbers of elements of G of the same order. We prove that the simple group $L_2(3^n)$ is uniquely determined by $\mathrm{nse}(L_2(3^n))$, where $|\pi(L_2(3^n))| = 4$.

1. Introduction

Let G be a group and $\pi(G)$ be the set of primes p such that G contains an element of order p and $\pi_e(G)$ be the set of element orders of G. If $k \in \pi_e(G)$, then we denote by m_k, the number of elements of order k in G. Let $\mathrm{nse}(G) = \{m_k \mid k \in \pi_e(G)\}$. In 1987, Thompson posed a problem [**6**, Problem 12.37] related to algebraic number fields as follows:

Thompson Problem. Let $T(G) = \{(k, m_k) \mid k \in \pi_e(G)\}$. Suppose that $T(G) = T(H)$. If G is a finite solvable group, then is H also necessarily solvable?

Up to now, no one has been able to solve this problem completely even to give a counterexample. It is easy to see that if G and H are two finite groups with $T(G) = T(H)$, then $|G| = |H|$ and $\mathrm{nse}(G) = \mathrm{nse}(H)$. Studies on characterizations related to nse started by Shao et al. in [**9**]. They proved that if S is a finite simple group with $|\pi(S)| = 4$, then S is characterizable by $\mathrm{nse}(S)$ and $|S|$, i.e., S is uniquely determined by $\mathrm{nse}(S)$ and $|S|$. Also, in [**4**], it is proved that sporadic simple group S is characterizable by $\mathrm{nse}(S)$ and $|S|$. Moreover, there are some research on the characterization of finite simple groups by nse. For instance, in [**3, 5, 8, 10**], it is proved that the groups A_5, A_6, A_7, A_8, J_1 and $L_2(q)$, where $q \in \{7, 8, 11, 13\}$ can be uniquely determined by nse. It is worth mentioning that considering the characterization of a simple group S by $\mathrm{nse}(S)$ is much more complicated than its characterization by $\mathrm{nse}(S)$ and $|S|$. Because when G is a group with $\mathrm{nse}(G) = \mathrm{nse}(S)$, the most challenging part to show that $G \cong S$, is to prove $\pi(G) = \pi(S)$.

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We here consider this characterization motivated by the following problem, which appeared in [**3**].

Problem. Let G be a group such that $\mathrm{nse}(G) = \mathrm{nse}(L_2(q))$, where q is a prime power. Is G isomorphic to $L_2(q)$?

Our main purpose is to show that the problem has an affirmative answer for $q = 3^n$ and $|\pi(L_2(q))| = 4$. In fact, we have the following main theorem.

Main Theorem. *Let G be a group such that $\mathrm{nse}(G) = \mathrm{nse}(L_2(3^n))$, where n, $(3^n - 1)/2$ and $(3^n + 1)/4$ are odd primes. Then $G \cong L_2(3^n)$.*

2. Notation and preliminaries

For a natural number m, by $\pi(m)$, we mean the set of all prime divisors of m, so it is obvious that if G is a finite group, then $\pi(G) = \pi(|G|)$. A Sylow p-subgroup of G is denoted by G_p and by $n_p(G)$, we mean the number of Sylow p-subgroups of G. Also, the largest element order of G_p is denoted by $\exp(G_p)$. Moreover, we denote by φ, the Euler totient function and by (a, b), the greatest common divisor of integers a and b.

In the following, we bring some useful lemmas which will be used in the proof of the main theorem.

Lemma 2.1. [**2**] *Let G be a finite group and m be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G : g^m = 1\}$, then $m \mid |L_m(G)|$.*

Lemma 2.2. [**7**] *Let G be a finite group and $p \in \pi(G) - \{2\}$. Suppose that P is a Sylow p-subgroup of G and $n = p^s m$, where $(p, m) = 1$. If P is not cyclic and $s > 1$, then the number of elements of order n is either 0, or a multiple of p^s.*

Lemma 2.3. [**11**, Theorem 3] *In a group of order g, the number of elements whose orders are multiples of n is either zero, or a multiple of the greatest divisor of g that is prime to n.*

Lemma 2.4. [**10**] *Let G be a group containing more than two elements. Let $k \in \pi_e(G)$ and m_k be the number of elements of order k in G. If $s = \sup\{m_k : k \in \pi_e(G)\}$ is finite, then G is finite and $|G| \leqslant s(s^2 - 1)$.*

Lemma 2.5. [**1**] *Let p be a prime number.*

(1) *If $p \neq 3$, then $y^2 \equiv -3 \pmod p$ is solvable if and only if $p \equiv 1 \pmod 3$.*
(2) *The equation $y^2 \equiv -1 \pmod p$ is solvable if and only if $p \equiv 1 \pmod 4$.*

From now on, we assume that n, $u = (3^n - 1)/2$ and $t = (3^n + 1)/4$ are odd primes and $x = 3^n$.

Lemma 2.6. *We have*

(1) $(u, t) = 1$, $(u, 3) = 1$, $(u, 2) = 1$, $(t, 3) = 1$, $(t, 2) = 1$;
(2) $(u - 1) = 2(t - 1)$, $(u - 1, t) = 1$, $(t - 1, u) = 1$;
(3) $4 \mid 1 + xu$.

PROOF. Parts (1) and (2) are straightforward. To prove part (3), we know that $t-1$ is even and hence, $8 \mid x-3$. Now if $4 \nmid 1+xu$, then $8 \nmid x^2-x+2$ and hence $8 \nmid x^2-x+2+x-3=(x-1)(x+1)=8tu$, which is a contradiction. □

LEMMA 2.7. *Let p be a prime number which is prime to $2, 3, t, u, t-1$.*

(1) *If $p \mid x^2-5x+8$, then $p \nmid 3x+1$, $p \nmid x^2+x-4$ and $p \nmid x^2-5$.*
(2) *If $p \mid x^2-x-4$, then $p \nmid 3x+1$ and $p \nmid x^2+5x+8$.*
(3) *If $16 \mid x-3$, then $16 \nmid x^2-x+2$.*

PROOF. Let $p \mid x^2-5x+8$. If $p \mid 3x+1$, then $p \mid 9(x^2-5x+8)+(-3x+16)(3x+1) = 8.11$ and since $(p,2)=1$, we conclude that $p=11$. Now since $p \mid (3x+1)$, we conclude that $3^{n+1} \equiv -1 \pmod{p}$. Lemma 2.5 implies $4 \mid (p-1)=10$, which is a contradiction. If $p \mid x^2+x-4$, then $p \mid (x^2-5x+8)+2(x^2+x-4)=6xu$ and since $(p,2)=(p,3)=(p,u)=1$, we get a contradiction. If $p \mid x^2-5$, then $p \mid (-5x-13)(x^2-5x+8)+(5x-12)(x^2-5)=-4.11$ and since $(p,2)=1$, we conclude that $p=11$. Thus $11 \mid x^2-5-11=(x-4)(x+4)$ and hence $11 \mid (x+4)$ or $11 \mid (x-4)$. If $11 \mid (x+4)$, then $11 \mid 3x+1$. Thus $3^{n+1} \equiv -1 \pmod{11}$ and hence Lemma 2.5 implies that $4 \mid (p-1)=10$, which is a contradiction. If $11 \mid (x-4)$, then $11 \mid (x^2-5x+8)-(x^2-5)+5(x-4)=-7$, which is a contradiction.

Let $p \mid x^2-x-4$. If $p \mid 3x+1$, then $p \mid -3(x^2-x-4)+(x-12)(3x+1)=-32x$ and since $(p,3)=(p,2)=1$, we get a contradiction. If $p \mid x^2+5x+8$, then $p \mid (x^2+5x+8)+2(x^2-x-4)=12xt$ and since $(p,2)=(p,3)=(p,t)=1$, we get a contradiction.

If $16 \mid x-3$ and x^2-x+2, then $16 \mid (x^2-x+2)+(x-3)=(x^2-1)=8tu$ and since $(u,2)=(t,2)=1$, we get a contradiction. □

3. Proof of the main theorem

According to [**9**], we know that $|L_2(3^n)| = 2^2 3^n tu$ and

$$\mathrm{nse}(L_2(3^n)) = \{1, 3^n u, 8tu, (t-1)3^n u, (u-1)3^n 2t\}.$$

We prove the main theorem in a sequence of lemmas.

LEMMA 3.1. *The group G is finite and for every $i \in \pi_e(G)$, we have $\varphi(i) \mid m_i$ and $i \mid \sum_{d|i} m_d$. Moreover, if $i > 2$, then m_i is even.*

PROOF. According to Lemma 2.4, it is obvious that G is a finite group. Now, if $i \in \pi_e(G)$, then Lemma 2.1 implies that $i \mid \sum_{d|i} m_d$. We know that the number of elements of order i in a cyclic group of order i is equal to $\varphi(i)$. Thus $m_i = \varphi(i)k$, where k is the number of cyclic subgroups of order i in G and hence, $\varphi(i) \mid m_i$. Also, it is known that if $i > 2$, then $\varphi(i)$ is even and since $\varphi(i) \mid m_i$, we conclude that m_i is even as well. □

LEMMA 3.2. $2 \in \pi(G)$ *and* $\exp(G_2) \leqslant 2^5$.

PROOF. Since $3^n u$ is the only odd element of $\mathrm{nse}(G)-\{1\}$, Lemma 3.1 yields $2 \in \pi(G)$ and $m_2 = 3^n u$. If $\exp(G_2) > 2^5$, then $2^6 \in \pi_e(G)$ and hence by Lemma 3.1, we have $2^5 = \varphi(2^6) \mid m_{2^6}$. Lemma 2.6 now implies that $8 \mid t-1$. Therefore $32 \mid x-3$ and also according to $\mathrm{nse}(G)$, we have 8 divides m_4 and m_8.

Thus Lemma 3.1 implies that $8 \mid 1 + m_2$ and hence, $16 \mid x^2 - x + 2$, which is impossible according to Lemma 2.7. So $\exp(G_2) \leqslant 2^5$. □

LEMMA 3.3. *We have* $\pi(G) \neq \{2\}$.

PROOF. If $\pi(G) = \{2\}$, then $|G| = 2^k = 1 + 3^n u + k_1 8tu + k_2(t-1)3^n u + k_3(u-1)3^n 2t$, where k, k_1, k_2, k_3 are natural numbers. By Lemma 3.2, $\exp(G_2) \leqslant 2^5$ and hence $|\pi_e(G)| \leqslant 6$. Thus $k_1 + k_2 + k_3 \leqslant 4$. If $k_1 = 1$, then $3^n \mid |G|$, which is a contradiction. Thus, we have $(k_1, k_2, k_3) = (2, 1, 1)$, which implies that $u \mid |G|$, a contradiction. □

LEMMA 3.4. *We have* $\pi(G) \neq \{2, 3\}$.

PROOF. Let $\pi(G) = \{2, 3\}$. We prove the lemma in the following four steps.

Step 1. In this step, we prove that 3 is the only element of $\pi_e(G)$ such that $m_3 = 8tu$ and hence $|G| = 1 + 3^n u + 8tu + k_1(t-1)3^n u + k_2(u-1)3^n 2t$, where k_1, k_2 are natural numbers and $k_1 + k_2 \leqslant |\pi_e(G)| - 3$.

According to $\mathrm{nse}(G)$, there exists $i \in \pi_e(G)$ such that $m_i = 8tu$. Since $i \in \pi_e(G)$ and $\pi(G) = \{2, 3\}$, we have $i = 2^\alpha 3^\beta$, where $\alpha, \beta \geqslant 0$ and the case $\alpha = \beta = 0$ does not happen. By Lemma 3.1, it is obvious that if $i = 3$, then $m_3 = 8tu$. Now we are going to reach a contradiction for the other cases of α and β.

Case 1. Let $\alpha \geqslant 1$ and $\beta = 0$. Thus $m_i = m_{2^\alpha} = 8tu$. If $2^\alpha 3 \notin \pi_e(G)$, then G_3 acts fixed point freely on the set of elements of order 2^α by conjugation and hence, $|G_3| \mid m_{2^\alpha}$, which is a contradiction. Thus $2^\alpha 3 \in \pi_e(G)$ and according to Lemma 3.1, we conclude that $2^\alpha 3 \mid \sum_{d \mid 2^\alpha 3} m_d$ and $2^{\alpha-1} 3 \mid \sum_{d \mid 2^{\alpha-1} 3} m_d$. Therefore, $3 \mid m_{2^\alpha} + m_{2^\alpha 3}$. Since $3 \nmid m_{2^\alpha}$, we have $3 \nmid m_{2^\alpha 3}$ and hence $m_{2^\alpha 3} = m_{2^\alpha} = 8tu$ which implies that $3 \mid m_{2^\alpha} + m_{2^\alpha 3} = 16tu$, a contradiction.

Case 2. Let $\alpha \geqslant 1$ and $\beta = 1$. Thus $m_i = m_{2^\alpha 3} = 8tu$ and Lemma 3.1 implies that $2^\alpha 3 \mid \sum_{d \mid 2^\alpha 3} m_d$ and $2^{\alpha-1} 3 \mid \sum_{d \mid 2^{\alpha-1} 3} m_d$. Thus $3 \mid m_{2^\alpha} + m_{2^\alpha 3}$. According to Case 1, we know that $m_{2^\alpha} \neq 8tu$ and hence, $3 \mid m_{2^\alpha}$. Thus $3 \mid m_{2^\alpha 3}$, a contradiction.

Case 3. Let $\alpha \geqslant 0$ and $\beta \geqslant 2$. If G_3 is cyclic of order 3^k, then $n_3(G) = \frac{m_{3^k}}{\varphi(3^k)} = \frac{m_{3^k}}{2.3^{k-1}}$. Thus according to $\mathrm{nse}(G)$, we have $u \mid n_3(G)$ or $t \mid n_3(G)$ and since $n_3(G) \mid |G|$, we can get a contradiction. So G_3 is not cyclic and Lemma 2.2 now yields $9 \mid m_{2^\alpha 3^\beta}$, which is a contradiction.

Step 2. In this step, we prove that $|G_3| = 3^n$ and $\exp(G_3) = 3$.

According to step 1, we have

$$|G| = |G_2||G_3| = 1 + 3^n u + 8tu + k_1(t-1)3^n u + k_2(u-1)3^n 2t.$$

Since

$$3^n \mid 1 + 3^n u + 8tu + k_1(t-1)3^n u + k_2(u-1)3^n 2t,$$
$$3^{n+1} \nmid 1 + 3^n u + 8tu + k_1(t-1)3^n u + k_2(u-1)3^n 2t,$$

we conclude that $|G_3| = 3^n$. Let $\exp(G_3) = 3^k$, where $k \geqslant 2$. Then by Step 1, for every $i \geqslant 0$, $m_{2^i 3^k} \neq 8tu$. Lemma 2.3 now implies

$$|G_2| \mid \sum_{i\geqslant 0} m_{2^i 3^k} = k_1'(t-1)3^n u + k_2'(u-1)3^n 2t,$$

where $k_1', k_2' \geqslant 0$ are integers and since $3^k \in \pi_e(G)$, we have $k_1' + k_2' \geqslant 1$. Thus $|G_2| \mid k_1'(t-1)u + k_2'(u-1)2t$. On the other hand, $|G| = |G_2||G_3| = |G_2|3^n = 1 + 3^n u + 8tu + k_1(t-1)3^n u + k_2(u-1)3^n 2t$, where k_1, k_2 are natural numbers and $k_1' \leqslant k_1,\ k_2' \leqslant k_2$. Thus $|G_2| = u + 3^n + k_1(t-1)u + k_2(u-1)2t$. Therefore $u + 3^n + k_1(t-1)u + k_2(u-1)2t \leqslant k_1'(t-1)u + k_2'(u-1)2t$, which is a contradiction because $k_1' \leqslant k_1,\ k_2' \leqslant k_2$. So $\exp(G_3) = 3$.

Step 3. In this step, we prove that $6 \in \pi_e(G)$.

Let $6 \notin \pi_e(G)$. Then G_2 acts fixed point freely on the set of elements of order 3 by conjugation and hence, $|G_2| \mid 8$ and since $|\operatorname{nse}(G)| = 5$ and $\exp(G_3) = 3$, we conclude that $8 \in \pi_e(G)$. Thus G_2 is cyclic and $n_2(G) = m_8/4$. Now according to $\operatorname{nse}(G)$, we conclude that t or u divides $n_2(G)$ and since $n_2(G) \mid |G|$, we can get a contradiction.

Step 4. In this step, we prove that $\pi(G) \neq \{2, 3\}$.

By steps 1 and 2, we have $|G_3| = 3^n$ and $|G| = |G_2||G_3| = 1 + 3^n u + 8tu + k_1(t-1)3^n u + k_2(u-1)3^n 2t$, where k_1, k_2 are natural numbers. Thus $|G_2| = u + 3^n + k_1(t-1)u + k_2(u-1)2t = 1/8p(x)$, where $p(x) = Ax^2 + Bx + C$, $A = k_1 + 2k_2$, $B = 12 - 4k_1 - 4k_2$ and $C = -4 + 3k_1 - 6k_2$. By Lemma 3.2 and Step 2, $\exp(G_2) \leqslant 2^5$ and $\exp(G_3) = 3$. Thus $|\pi_e(G)| \leqslant 12$, which implies that $k_1 + k_2 \leqslant 9$. According to Lemma 2.3 we know that $|G_2| \mid \sum_{i\geqslant 0} m_{2^i 3} = 8tu + k_1'(t-1)3^n u + k_2'(u-1)3^n 2t$, where $k_1', k_2' \geqslant 0$ are integers. Thus $|G_2| \mid 1/8q(x)$, where $q(x) = A'x^3 + B'x^2 + C'x + D'$, $A' = k_1' + 2k_2'$, $B' = 8 - 4k_1' - 4k_2'$, $C' = 3k_1' - 6k_2'$ and $D' = -8$. Now by step 3, we can conclude that $k_1' + k_2' \geqslant 1$ and since $\exp(G_2) \leqslant 2^5$, we have $5 \geqslant k_1' + k_2'$. It is obvious that

$$\begin{aligned} p(x) &\mid (AA'x + AB' - A'B)p(x) - A^2 q(x) \\ &= (AB'B - A'B^2 - A^2C' + AA'C)x + (AB'C - A'BC - A^2D'). \end{aligned}$$

Thus

$$Ax^2 + Bx + C \leqslant (AB'B - A'B^2 - A^2C' + AA'C)x + (AB'C - A'BC - A^2D')$$

which implies that

$$Ax^2 + (B - AB'B + A'B^2 + A^2C' - AA'C)x + (C - AB'C + A'BC + A^2D') \leqslant 0.$$

But according to $k_1 + k_2 \leqslant 9$ and $1 \leqslant k_1' + k_2' \leqslant 5$, we conclude that $x < 201$. Thus $m = 3$ and $k_1 + k_2 = 8$ and $|G_2| = 2^{10}$. Since $\exp(G_2) \leqslant 2^5$ and $\exp(G_3) = 3$, we conclude that $\exp(G_2) = 2^5$. Hence Lemma 3.1 implies that

$$\begin{aligned} 2^{10} &= |G_2| \mid 1 + m_2 + m_4 + m_8 + m_{16} + m_{32} \\ &= 1 + 3^n u + k_1''(t-1)3^n u + k_2''(u-1)3^n 2t = 1 + 351 + k_1''2106 + k_2''4536, \end{aligned}$$

where $k_1'', k_2'' \geqslant 0$ are integers and $k_1'' + k_2'' = 4$, which is a contradiction. $\square$

Lemma 3.5. *We have* $\pi(G) \subseteq \{2, 3, t, u\}$.

Proof. Let $p \in \pi(G) - \{2, 3, t, u\}$. We prove the lemma in the following six steps.

Step 1. In this step, we prove that $m_p \neq 8tu$ and hence $(p, t-1) = 1$.

Let $m_p = 8tu$. Then $p \mid (1 + m_p) = 3^{2n}$, which is a contradiction. So $m_p \neq 8tu$ and hence $m_p \in \{(t-1)3^n u, (u-1)3^n 2t\}$. Since $(p, m_p) = 1$, we conclude that $(p, t-1) = 1$.

Step 2. In this step, we prove that $\exp(G_p) = p$.

Let $\exp(G_p) > p$. Then $p^2 \in \pi_e(G)$. Since $p(p-1) = \varphi(p^2) \mid m_{p^2}$, we conclude that p divides one of the numbers $2, 3, t, u$ or $t-1$, which is a contradiction. So $\exp(G_p) = p$.

Step 3. In this step, we prove that if $q \in \pi_e(G)$ and $(p, q) = 1$, then $qp \in \pi_e(G)$ and $p \mid m_q + m_{qp}$.

Let $q \in \pi_e(G)$ which is prime to p. If $qp \notin \pi_e(G)$, then G_p acts fixed point freely on the set of elements of order q by conjugation and hence $|G_p| \mid m_q$, which implies that p divides one of the numbers $2, 3, t, u$ or $(t-1)$, which is a contradiction. So $qp \in \pi_e(G)$. Let $q = q_1^{s_1} \cdots q_k^{s_k}$, where $q_1, \ldots, q_k$ are distinct prime numbers and $k, s_1, \ldots, s_k$ are natural numbers. We prove $p \mid m_q + m_{qp}$ by induction on $s = s_1 + \cdots + s_k$. Let $s = 1$. Then we have $p \mid 1 + m_p + m_{q_i} + m_{q_i p}$ and since $p \mid 1 + m_p$, $p \mid m_{q_i} + m_{q_i p}$. Let $s = 2$. Then there exist $1 \leqslant i < j \leqslant k$ such that $q = q_i q_j$ or $q = q_i^2$. If $q = q_i q_j$, then we have $p \mid 1 + m_p + m_{q_i} + m_{q_j} + m_{q_i p} + m_{q_j p} + m_{q_i q_j} + m_{q_i q_j p}$ and since $p \mid 1 + m_p,\ m_{q_i} + m_{q_i p},\ m_{q_j} + m_{q_j p}$, we conclude that $p \mid m_{q_i q_j} + m_{q_i q_j p}$, as desired. The case $q = q_i^2$ is similar and we omit the details for the sake of convenience. Now, assume the statement is true for the values less than s. We have

$$p \mid \sum_{d \mid qp} m_d = \sum_{\substack{d \mid qp \\ d \neq q, qp}} m_d + m_q + m_{qp}.$$

Moreover, according to the induction hypothesis, $p \mid \sum_{d \mid qp,\, d \neq q, qp} m_d$. Therefore, $p \mid m_q + m_{qp}$.

Step 4. $m_p \neq (t-1)3^n u$.

Let $m_p = (t-1)3^n u$. Then $p \mid 1 + m_p$ and hence $p \mid x^2 - 5x + 8$. On the other hand, by $\mathrm{nse}(G)$, there is $q \in \pi_e(G)$ such that $(q, p) = 1$ and m_q or $m_{qp} = (u-1)3^n 2t$. Thus by step 3, $p \mid m_q + m_{qp}$. Now, there are four cases:

Case 1. If $\{m_q, m_{qp}\} = \{(u-1)3^n 2t, 3^n u\}$, then $p \mid x^2 - 5$, which is a contradiction by Lemma 2.7(1).

Case 2. If $\{m_q, m_{qp}\} = \{(u-1)3^n 2t, 8tu\}$, then $p \mid x^2 + x - 4$, which is a contradiction by Lemma 2.7(1).

Case 3. If $\{m_q, m_{qp}\} = \{(u-1)3^n 2t, (t-1)3^n u\}$, then $p \mid 3x + 1$, which is a contradiction by Lemma 2.7(1).

Case 4. If $\{m_q, m_{qp}\} = \{(u-1)3^n 2t\}$, then $p \mid (u-1)3^n 4t$, which is a contradiction according to Step 1 and Lemma 2.6(2).

Step 5. $m_p \neq (u-1)3^n 2t$.

Let $m_p = (u-1)3^n 2t$. Then $p \mid 1 + m_p$ and hence $p \mid x^2 - x - 4$. On the other hand, by $\mathrm{nse}(G)$, there is $q \in \pi_e(G)$ such that $(q, p) = 1$ and m_q or $m_{qp} = (t-1)3^n u$. Thus by Step 3, $p \mid m_q + m_{qp}$. Now there are four cases:

Case 1. If $\{m_q, m_{qp}\} = \{(t-1)3^n u, 3^n u\}$, then $p \mid 3^n tu$, which is a contradiction.

Case 2. If $\{m_q, m_{qp}\} = \{(t-1)3^n u, 8tu\}$, then $p \mid x^2 + 5x + 8$, which is a contradiction by Lemma 2.7(2).

Case 3. If $\{m_q, m_{qp}\} = \{(t-1)3^n u\}$, then $p \mid 2(t-1)3^n u$, which is a contradiction according to Step 1.

Case 4. If $\{m_q, m_{qp}\} = \{(t-1)3^n u, (u-1)3^n 2t\}$, then $p \mid 3x + 1$, which is a contradiction by Lemma 2.7(2).

Step 6. $p \notin \pi(G)$. According to steps 1, 4 and 5, $m_p \notin \mathrm{nse}(G)$ and hence, $p \notin \pi(G)$. □

LEMMA 3.6. *If $t \in \pi(G)$, then $u \in \pi(G)$.*

PROOF. If $t \in \pi(G)$, then according to Lemma 3.1, we have m_t is even and $(m_t, t) = 1$ and hence, according to $\mathrm{nse}(G)$, it is obvious that $m_t = (t-1)3^n u$. Now we claim that $t^2 \notin \pi_e(G)$. Suppose, contraty to our claim, that $t^2 \in \pi_e(G)$. Lemma 3.1 implies that $t(t-1) = \varphi(t^2) \mid m_{t^2}$ and hence according to Lemma 2.6, we have $m_{t^2} = (u-1)3^n 2t$. On the other hand, Lemma 3.1 implies that $t^2 \mid 1+m_t+m_{t^2}$ and hence $(x+1)^2 \mid (x+1)^2(6x-28)+44(x+1)$. Thus $(x+1) \mid 44$. So we conclude that $t = 11$, which is a contradiction because $4t = 3^n + 1$. Therefore, $t^2 \notin \pi_e(G)$. Now we are going to show that $|G_t| = t$. Since $2 \in \pi_e(G)$, we can assume that q is the largest element of $\pi_e(G)$ satisfies $(q, t) = 1$. Thus $\{q\} \subseteq \{s \in \pi_e(G) : s \text{ is multiple of } q\} \subseteq \{q, qt\}$. Now Lemma 2.3 implies that $|G_t| \mid m_q$ or $m_q + m_{qt}$. Hence by $\mathrm{nse}(G)$ and Lemma 2.6, we conclude that $|G_t| = t$. So $n_t(G) = \frac{m_t}{\varphi(t)} = 3^n u$ and since $n_t(G) \mid |G|$, we have $u \in \pi(G)$. □

LEMMA 3.7. *We have $\pi(G) = \{2, 3, t, u\}$.*

PROOF. We first show that $u^2 \notin \pi_e(G)$. If $u^2 \in \pi_e(G)$, then by Lemma 3.1, $u(u-1) = \varphi(u^2) \mid m_{u^2}$. But according to Lemma 2.6 and $\mathrm{nse}(G)$, we can easily see that there is no choice for m_{u^2}. Therefore $u^2 \notin \pi_e(G)$.

If $u \in \pi(G)$, then according to $\mathrm{nse}(G)$ and Lemma 3.1, we can easily conclude that $m_u = (u-1)3^n 2t$. Since $u^2 \notin \pi_e(G)$, Lemma 2.1 implies that $|G_u| \mid 1 + m_u$. If $u^2 \mid 1 + m_u$, then $(x-1)^2 \mid (x-1)^2 x - 4(x-1)$ and hence $(x-1) \mid 4$, which is a contradiction. Thus $|G_u| = u$ and we have $n_u(G) = \frac{m_u}{\varphi(u)} = 3^n 2t$. Now according to Lemmas 3.2-3.6, we have $u \in \pi(G)$ and since $n_u(G) = 3^n 2t$ and $n_u(G) \mid |G|$, we conclude that $\{2, 3, t, u\} \subseteq \pi(G)$. On the other hand, by Lemma 3.5, there is no $p \in \pi(G)$ such that $p \neq 2, 3, t, u$. Therefore, $\pi(G) = \{2, 3, t, u\}$. □

LEMMA 3.8. *We have $2u, 3u, tu, 3t, 4t \notin \pi_e(G)$ but $2t \in \pi_e(G)$ and $m_{2t} = m_t$. Moreover if $6 \in \pi_e(G)$, then $m_6 \neq 8tu$*

PROOF. If $2u \in \pi_e(G)$, then $m_{2u} = \varphi(2u)n_u(G)k$, where k is the number of cyclic subgroups of order 2 in $C_G(G_u)$. Actually, this follows from the fact that all centralizers of Sylow u-subgroups of G in G are conjugate in G. Therefore we have $(u-1)3^n2t \mid m_{2u}$. Thus $m_{2u} = m_u$. On the other hand, $2u \mid 1+m_2+m_u+m_{2u}$ and $u \mid 1+m_2+m_u$ which implies that $u \mid m_{2u}$, a contradiction. Similarly, we can prove that $3u, tu, 3t, 4t \notin \pi_e(G)$. If $2t \notin \pi_e(G)$, then G_t acts fixed point freely on the set of elements of order 2 by conjugation and hence, $|G_t| \mid m_2$, which is a contradiction. So $2t \in \pi_e(G)$. Similarly, we can prove that $m_{2t} = m_t$. Let $6 \in \pi_e(G)$ such that $m_6 = 8tu$. Since $3 \mid 1 + m_2 + m_3$ and $6 \mid 1 + m_2 + m_3 + m_6$, we conclude that $3 \mid m_6$, which is a contradiction. □

Lemma 3.9. *We have* $\exp(G_3) = 3$.

Proof. If $\exp(G_3) > 3$, then $9 \in \pi_e(G)$. Since $3t, 3u \notin \pi_e(G)$, we conclude that $9t, 9u \notin \pi_e(G)$. Thus G_t and G_u act fixed point freely on the set of elements of order 9 by conjugation and hence, $|G_t| \mid m_9$ and $|G_u| \mid m_9$. So we have $tu \mid m_9$, which implies that $m_9 = 8tu$, a contradiction, because $6 = \varphi(9) \mid m_9$. □

Lemma 3.10. *We have* $\exp(G_2) = 2$.

Proof. If $\exp(G_2) > 2$, then $4 \in \pi_e(G)$. Thus Lemma 3.8 implies that $4t, 4u \notin \pi_e(G)$. Thus G_t and G_u act fixed point freely on the set of elements of order 4 by conjugation and hence, $|G_t| \mid m_4$ and $|G_u| \mid m_4$. So $tu \mid m_4$, which implies that $m_4 = 8tu$. On the other hand, $12 \in \pi_e(G)$ because otherwise G_3 acts fixed point freely on the set of elements of order 4 by conjugation and hence, $3 \mid m_4$, a contradiction. So $12 \in \pi_e(G)$. Now since $6 \mid 1 + m_2 + m_3 + m_6$ and $12 \mid 1 + m_2 + m_3 + m_4 + m_6 + m_{12}$, we conclude that $3 \mid m_4 + m_{12}$. Thus $3 \nmid m_{12}$, which implies that $m_{12} = 8tu$. But we have $3 \mid m_4 + m_{12} = 16tu$, a contradiction. □

Lemma 3.11. *We have* $|G| = 1 + 3^nu + 8tu + k_1(t-1)3^nu + k_2(u-1)3^n2t$, *where* k_1, k_2 *are natural and* $3 \leqslant k_1 + k_2 \leqslant 4$.

Proof. According to Lemmas 3.8–3.10, we have $\{1, 2, 3, t, 2t, u\} \subseteq \pi_e(G) \subseteq \{1, 2, 3, 6, t, 2t, u\}$ and $m_1 = 1$, $m_2 = 3^nu$, $m_3 = 8tu$, $m_t = m_{2t} = (t-1)3^nu$, $m_u = (u-1)3^n2t$ and $m_6 \in \{(t-1)3^nu, (u-1)3^n2t\}$. So we conclude that $|G| = 1 + 3^nu + 8tu + k_1(t-1)3^nu + k_2(u-1)3^n2t$, where k_1, k_2 are natural numbers and $3 \leqslant k_1 + k_2 \leqslant 4$. □

Lemma 3.12. *We have* $|G_3| = 3^n$.

Proof. By Lemma 3.11, $|G| = 1+3^nu+8tu+k_1(t-1)3^nu+k_2(u-1)3^n2t$, where k_1, k_2 are natural. Now since $3^n \mid 1 + 3^nu + 8tu + k_1(t-1)3^nu + k_2(u-1)3^n2t$ and $3^{n+1} \nmid 1+3^nu+8tu+k_1(t-1)3^nu+k_2(u-1)3^n2t$, we conclude that $|G_3| = 3^n$. □

Lemma 3.13. *We have* $|G_2| = 4$.

Proof. If $|G_2| = 2$, then G_2 is cyclic and $n_2(G) = 3^nu$. Since $|G_3| = 3^n$ and $n_2(G) = |G : N_G(G_2)|$, we conclude that $3 \nmid |N_G(G_2)|$. Thus $6 \notin \pi_e(G)$ and by Lemma 3.11, we have $|G| = 1 + 3^nu + 8tu + 2(t-1)3^nu + (u-1)3^n2t$. Lemma 2.6 now yields $4 \mid |G|$, which is a contradiction. Therefore, $|G_2| \geqslant 4$. If $|G_2| \geqslant 8$, then by Lemmas 2.1 and 3.10, $8 \mid 1 + m_2$. Thus $16 \mid x^2 - x + 2$. On the other hand, by Lemma 3.8, $t, 2t$ are only elements of $\pi_e(G)$ which are multiple, of t and $m_t = m_{2t}$.

Thus Lemma 2.3 implies that $|G_2||G_3||G_u| \mid m_t + m_{2t} = 2(t-1)3^n u$. Therefore $4 \mid (t-1)$. So $16 \mid (x-3)$, which is a contradiction according to Lemma 2.7. □

LEMMA 3.14. *We have* $G \cong L_2(3^n)$.

PROOF. Since $|G_u| = u$, $|G_t| = t$, $|G_3| = 3^n$ and $|G_2| = 4$, we conclude that $|G| = |L_2(3^n)| = 2^2 3^n tu$. Now according to [**9**] we have $G \cong L_2(3^n)$. □

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References

1. W. W. Adams, L. J. Goldstein, *Introduction to Number Theory*, Prentice-Hall Englewood Cliffs, New Jersey, 1976.
2. G. Frobenius, *Verallgemeinerung des sylowschen Satze*, Berliner Sitz (1895), 981–993.
3. M. Khatami, B. Khosravi, Z. Akhlaghi, *A new characterization for some linear groups*, Monatsh. Math. **163** (2011), 39–50.
4. A. Khalili Asboei, S. S. Salehi Amiri, A. Iranmanesh, A. Tehranian, *A characterization of sporadic simple groups by nse and order*, J. Algebra Appl. **12** (2013), 1250158.
5. A. Khalili Asboei, S. S. Salehi Amiri, A. Iranmanesh, A. Tehranian, *A new characterization of A_7, A_8*, An. Ştiinţ. Univ. ŞOvidiusŤ Constanţa, Ser. Mat. **21** (2013), 43–50.
6. V. D. Mazurov, E. I. Khukhro, *Unsolved Problems in Group Theory*, The Kourovka Notebook, 16 ed. Inst. Mat. Sibirsk. Otdel. Akad. Novosibirsk (2006).
7. G. Miller, *Addition to a theorem due to Frobenius*, Bull. Am. Math. Soc. **11** (1904), 6–7.
8. S. S. Salehi Amiri, A. Khalili Asboei, A. Tehranian, *A new characterization of Janko group*, Aust. J. Basic Appl. Sci. **6** (2012), 130–132.
9. C. Shao, W. Shi, Q. Jiang, *Characterization of simple K_4-groups*, Front. Math. China **3** (2008), 355–370.
10. R. Shen, C. Shao, Q. Jiang, W. Shi, V. Mazurov, *A new characterization of A_5*, Monatsh. Math. **160** (2010), 337–341.
11. L. Weisner, *On the number of elements of a group which have a power in a given conjugate set*, Bull. Amer. Math. Soc. **31** (1925), 492–496.

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