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UNIFORM DISTRIBUTION MODULO 1 AND THE UNIVERSALITY OF ZETA-FUNCTIONS OF CERTAIN CUSP FORMS

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Abstract. An universality theorem on the approximation of analytic functions by shifts $\zeta(s+i\tau, F)$ of zeta-functions of normalized Hecke-eigen forms F, where τ takes values from the set $\{k^\alpha h : k = 0, 1, 2, \dots\}$ with fixed $0 < \alpha < 1$ and $h > 0$, is obtained.

1. Introduction

Denote by $\mathrm{SL}(2, \mathbb{Z})$ the full modular group, i.e.,

$$\mathrm{SL}(2,\mathbb{Z}) = \Big\{\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) : a, b, c, d \in \mathbb{Z},\ ad - bc = 1\Big\}.$$

The function $F(z)$ is called a holomorphic cusp form of weight κ for $\mathrm{SL}(2, \mathbb{Z})$ if $F(z)$ is holomorphic in the half-plane $\mathrm{Im} z > 0$, for all $\big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) \in \mathrm{SL}(2, \mathbb{Z})$ satisfies the functional equation

$$F\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^\kappa F(z),$$

and at infinity has the Fourier series expansion $F(z) = \sum_{m=1}^{\infty} c(m) e^{2\pi i m z}$. Assume additionally that $F(z)$ is a normalized Hecke-eigen form, i.e., is an eigen form of all Hecke operators

$$T_m F(z) = m^{\kappa-1} \sum_{\substack{a,d>0\\ ad=m}} \frac{1}{d^\kappa} \sum_{b (\mathrm{mod}\, d)} F\Big(\frac{az+b}{d}\Big), \quad m \in \mathbb{N},$$

and $c(1) = 1$.

The associated zeta-function $\zeta(s, F)$, $s = \sigma + it$, is defined, for $\sigma > \frac{\kappa+1}{2}$, by the Dirichlet series

$$\zeta(s, F) = \sum_{m=1}^{\infty} \frac{c(m)}{m^s},$$

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and can be analytically continued to an entire function. Moreover, the function $\zeta(s,F)$ can be written, for $\sigma > \frac{\kappa+1}{2}$, as a product over primes

$$\zeta(s,F) = \prod_p \Big(1 - \frac{\alpha(p)}{p^s}\Big)^{-1} \Big(1 - \frac{\beta(p)}{p^s}\Big)^{-1},$$

where $\alpha(p)$ and $\beta(p)$ are conjugate complex numbers satisfying $\alpha(p) + \beta(p) = c(p)$.

The zeta-function $\zeta(s,F)$, as the Riemann zeta-function, Dirichlet L-functions, and some other zeta and L-functions, is universal in that sense that a wide class of analytic functions can be approximated by shifts $\zeta(s+i\tau,F)$ with some real τ. This was obtained in [**6**] by using the probabilistic approach and positive density method. Let $D = D_F = \big\{s \in \mathbb{C} : \frac{\kappa}{2} < \sigma < \frac{\kappa+1}{2}\big\}$. Denote by $\mathcal{K} = \mathcal{K}_F$ the class of compact subsets of the strip D with connected complements, and by $H_0(K)$, $K \in \mathcal{K}$, the class of continuous non-vanishing functions on K which are analytic in the interior of K. Let meas$A$ stand for the Lebesgue measure of a measurable set $A \subset \mathbb{R}$. Then, in [**7**], the following statement was proved.

Theorem 1.1. *Suppose that $K \in \mathcal{K}$ and $f(s) \in H_0(K)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{\tau \in [0,T] : \sup_{s\in K} |\zeta(s+i\tau,F) - f(s)| < \varepsilon\Big\} > 0.$$

Investigations of universality of zeta-functions of cusp forms were continued in [**8**] and [**6**], where the analogues of Theorem 1.1 were obtained for zeta-functions attached to new forms and for zeta-functions of primitive normalized Hecke-eigen forms for the Hecke subgroup with character, respectively.

Theorem 1.1 and its generalizations in [**8**], [**6**] are of continuous type because the shifts τ in $\zeta(s+i\tau,F)$ can take arbitrary real values. Also, the discrete universality of zeta-functions is considered. In this case, τ takes values from some discrete sets. The discrete analogue of Theorem 1.1 was begun to study in [**9**], and a general result was obtained in [**11**]. Denote by $\#A$ the cardinality of the set A.

Theorem 1.2. *Suppose that $K \in \mathcal{K}$, $f(s) \in H_0(K)$ and $h > 0$ is an arbitrary fixed number. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ikh,F) - f(s)| < \varepsilon\Big\} > 0.$$

In Theorem 1.2, the shift τ in $\zeta(s+i\tau,F)$ takes values from the arithmetical progression $\{0,h,2h,\dots\}$ with difference h. It is an interesting problem to prove Theorem 1.2 when τ takes values from a more complicated discrete set, and the present paper is devoted to the case of the set $\{k^\alpha h : k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\}$, where $h > 0$ and $0 < \alpha < 1$ are arbitrary fixed numbers.

Theorem 1.3. *Suppose that $K \in \mathcal{K}$, $f(s) \in H_0(K)$, and $h > 0$ and $0 < \alpha < 1$ are arbitrary fixed numbers. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\zeta(s+ik^\alpha h,F) - f(s)| < \varepsilon\Big\} > 0.$$

Let $H(G)$ be the space of analytic functions on the region $G \subset \mathbb{C}$ endowed with the topology of uniform convergence on compacta. In [**10**], Theorem 1.1 was generalized to composite functions $\Phi(\zeta(s,F))$ for some classes of operators $\Phi : H(D) \to H(D)$. Similarly, discrete analogues of Theorem 1.2 for $\Phi(\zeta(s,F))$ were obtained in [**11**]. Theorem 1.3 also can be rewritten for composite functions. We give only one example. For $a_1, \dots, a_r \in \mathbf{C}$ and $\Phi : H(D) \to H(D)$, define

$$H_{\Phi;a_1,\dots,a_r}(D) = \{g \in H(D) : g(s) \neq a_j,\ j = 1, \dots, r\} \cup \{\Phi(0)\},$$
$$S = \{g \in H(D) : g(s) \neq 0 \text{ or } g(s) \equiv 0\}.$$

Theorem 1.4. *Suppose that $\Phi : H(D) \to H(D)$ is a continuous operator such that $\Phi(S) \supset H_{\Phi;a_1,\dots,a_r}(D)$, and $h > 0$ and $0 < \alpha < 1$ are arbitrary fixed numbers. If $r = 1$, let $K \in \mathcal{K}$, $f(s) \in H(K)$ and $f(s) \neq a_1$ on K. Then, for every $\varepsilon > 0$,*

$$\liminf_{N\to\infty} \frac{1}{N+1} \#\Big\{0 \leqslant k \leqslant N : \sup_{s\in K} |\Phi(\zeta(s + ik^\alpha h, F)) - f(s)| < \varepsilon\Big\} > 0. \tag{1.1}$$

Let $K \subset D$ be an arbitrary compact subset, and $f(s) \in H_{\Phi;a_1,\dots,a_r}(D)$. Then inequality (1.1) *holds for any $\varepsilon > 0$.*

For example, Theorem 1.4 implies the discrete universality for the functions $e^{\zeta(s,F)}$, $\sin(\zeta(s,F))$, $\cos(\zeta(s,F))$, etc.

2. Probabilistic limit theorems

For the proof of Theorem 1.3, we need the weak convergence for

$$P_N(A) = \frac{1}{N+1} \#\big\{0 \leqslant k \leqslant N : \zeta(s + ik^\alpha h, F) \in A\big\}, \quad A \in \mathcal{B}(H(D)),$$

with explicitly given limit measure. Here the sequel, $\mathcal{B}(X)$ denotes the Borel σ-field of the space X.

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on the complex plane, and $\mathbb{P}$ be the set of all prime numbers. Define $\Omega = \prod_{p\in\mathbb{P}} \gamma_p$ where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$. It is well known that the torus Ω, with the product topology and pointwise multiplication, is a compact topological Abelian group. Thus, on a measurable space $(\Omega, \mathcal{B}(\Omega))$, the probability Haar measure m_H can be defined, and we have the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of an element $\omega \in \Omega$ to the circle γ_p, $p \in \mathbb{P}$. Then we have that $\{\omega(p) : p \in \mathbb{P}\}$ is a sequence of independent random variables defined on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. On the latter space, define the $H(D)$-valued random element $\zeta(s, \omega, F)$ by

$$\zeta(s, \omega, F) = \prod_{p\in\mathbb{P}} \Big(1 - \frac{\alpha(p)\omega(p)}{p^s}\Big)^{-1} \Big(1 - \frac{\beta(p)\omega(p)}{p^s}\Big)^{-1},$$

and denote by P_ζ the distribution of $\zeta(s, \omega, F)$, i.e., for $A \in \mathcal{B}(H(D))$,

$$P_\zeta(A) = m_H(\omega \in \Omega : \zeta(s, \omega, F) \in A).$$

Theorem 2.1. *The measure P_N converges weakly to P_ζ as $N \to \infty$. Moreover, the support of P_ζ is the set S.*

The proof of Theorem 2.1 is based on individual properties of the sequence $\{k^\alpha : k \in \mathbb{N}_0\}$. We recall that a sequence $\{x_k\} \subset \mathbb{R}$ is uniformly distributed modulo 1 if, for each interval $I = [a, b) \subset [0, 1)$ of length $|I|$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_I(\{x_k\}) = |I|,$$

where $\{u\}$ denotes the fractional part of $u \in \mathbb{R}$, and χ_I is the indicator function of I.

LEMMA 2.1. *For an arbitrary fixed $a \neq 0$ and $0 < \alpha < 1$, the sequence $\{k^\alpha a\}$ is uniformly distributed modulo* 1.

The lemma is Exercise 3.10 of [**4**].

LEMMA 2.2. *Suppose that a sequence $\{x_k\} \subset \mathbb{R}$ is such that, for every $a \neq 0$, the sequence $\{x_k a\}$ is uniformly distributed modulo* 1. *Then the measure Q_N, defined, for $h > 0$, by*

$$Q_N(A) = \frac{1}{N+1} \#\big\{0 \leqslant k \leqslant N : (p^{-ix_k h} : p \in \mathbb{P}) \in A\big\}, \quad A \in \mathcal{B}(\Omega),$$

converges weakly to the Haar measure m_H as $N \to \infty$.

PROOF. Let $g_N(\underline{k})$, $\underline{k} = (k_p : p \in \mathbb{P})$ denote the Fourier transform of Q_N, i.e.,

$$g_N(\underline{k}) = \int_\Omega \prod_p \omega^{k_p}(p) \mathrm{d}Q_N,$$

where only a finite number of integers k_p are distinct from zero. By the definition of Q_N, we find that

$$g_N(\underline{k}) = \frac{1}{N+1} \sum_{k=0}^{N} \prod_p p^{-ix_k h k_p} = \frac{1}{N+1} \sum_{k=0}^{N} \exp\Big\{ -ix_k h \sum_p k_p \log p\Big\}. \tag{2.1}$$

It is well known that the set $\{\log p : p \in \mathbb{P}\}$ is linearly independent over the field of rational numbers $\mathbb{Q}$. Therefore, the equality $\sum_p k_p \log p = 0$ holds if and only if $\underline{k} = \underline{0}$. Clearly,

$$g_N(\underline{0}) = 1. \tag{2.2}$$

In the case $\underline{k} \neq \underline{0}$, we have that $h \sum_p k_p \log p \neq 0$. Therefore, by the hypothesis on the sequence $\{x_k\}$, the sequence

$$\left\{ \frac{x_k h}{2\pi} \sum_p k_p \log p \right\}$$

is uniformly distributed modulo 1. Hence, an application of the Weyl criterion together with (2.1) shows that $\lim_{N\to\infty} g_N(\underline{k}) = 0$ for $\underline{k} \neq \underline{0}$. This and (2.2) yield that

$$\lim_{N\to\infty} g_N(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases} \tag{2.3}$$

Since

$$g(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}, \end{cases}$$

is the Fourier transform of the Haar measure m_H, a continuity theorem for probability measures on compact groups, see, for example, [**3**], and (2.3) prove the lemma. □

Lemma 2.2 for the sequence $\{k^\alpha\}$ with $\alpha > 0$ and $\alpha \notin \mathbb{N}$ was proved in [**2**].

For each $\omega \in \Omega$, extend the function $\omega(p)$ from the set $\mathbb{P}$ to the set $\mathbb{N}$ by

$$\omega(m) = \prod_{\substack{p^l \mid m \\ p^{l+1} \nmid m}} \omega^l(p), \quad m \in \mathbb{N}.$$

Further, we consider two functions

$$\zeta_n(s, F) = \sum_{m=1}^{\infty} \frac{c(m) v_n(m)}{m^s} \quad \text{and} \quad \zeta_n(s, \omega, F) = \sum_{m=1}^{\infty} \frac{c(m)\omega(m) v_n(m)}{m^s},$$

where, for a fixed number $\sigma_0 > \frac{1}{2}$ and $m, n \in \mathbb{N}$,

$$v_n(m) = \exp\big\{ -(m/n)^{\sigma_0} \big\}.$$

Then the series for $\zeta_n(s, F)$ and $\zeta_n(s, \omega, F)$ are absolutely convergent for $\sigma > \frac{\kappa}{2}$.

For $A \in \mathcal{B}(H(D))$, define

$$P_{N,n}(A) = \frac{1}{N+1} \#\big\{ 0 \leqslant k \leqslant N : \zeta_n(s + i x_k h, F) \in A \big\}.$$

Moreover, let the function $u_n : \Omega \to H(D)$ be given by $u_n(\omega) = \zeta_n(s, \omega, F)$, and let the probability measure $\hat{P}_n$ be defined by $\hat{P}_n = m_H u_n^{-1}$, i.e., for $A \in \mathcal{B}(H(D))$,

$$\hat{P}_n(A) = m_H u_n^{-1}(A) = m_H(u_n^{-1} A).$$

Lemma 2.3. *Under hypotheses of Lemma* 2.2*,* $P_{N,n}$ *converges weakly to* $\hat{P}_n$ *as* $N \to \infty$*.*

Proof. Since the series for $\zeta_n(s, \omega, F)$ is absolutely convergent for $\sigma > \frac{\kappa}{2}$, we have that the function u_n is a continuous one. Moreover,

$$u_n(p^{-i x_k h} : p \in \mathbb{P}) = \zeta_n(s + i x_k h, F).$$

Therefore, $P_{N,n} = Q_N u_n^{-1}$. This, Lemma 2.2, and Theorem 5.1 of [**1**] prove the lemma. □

For the proof of Theorem 1.1, a limit theorem for

$$\hat{P}_T(A) \stackrel{\text{def}}{=} \frac{1}{T} \operatorname{meas}\big\{ \tau \in [0, T] : \zeta(s + i\tau, F) \in A \big\}, \quad A \in \mathcal{B}(H(D)),$$

as $T \to \infty$ was applied. For our propose, we need some facts from the proof of the above limit theorem.

Lemma 2.4. *The measure $\hat{Q}_T$ defined by*

$$\hat{Q}_T(A) = \frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0,T] : (p^{-i\tau} : p \in \mathbb{P}) \in A\bigr\}, \quad A \in \mathcal{B}(\Omega),$$

converges weakly to the Haar measure m_H as $T \to \infty$.

Proof. We use the method of Fourier transform and the linear independence over the field of rational numbers $\mathbb{Q}$ for the set $\{\log p : p \in \mathbb{P}\}$. □

Lemma 2.5. *The measure $\hat{P}_{T,n}$ defined by*

$$\hat{P}_{T,n}(A) = \frac{1}{T} \operatorname{meas}\bigl\{\tau \in [0,T] : \zeta_n(s+i\tau, F) \in A\bigr\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $\hat{P}_n$ as $T \to \infty$, where $\hat{P}_n$ is defined in Lemma 2.3.

Proof. We use Lemma 2.4 and repeat the proof of Lemma 2.3. □

Lemma 2.6. *$\hat{P}_T$ converges weakly to P_ζ, and the support of P_ζ is the set S. Moreover, P_ζ coincides with the limit measure P of $\hat{P}_n$ as $n \to \infty$.*

Proof. We apply Lemma 2.5, the approximation of $\zeta(s,F)$ and $\zeta(s,\omega,F)$ by $\zeta_n(s,F)$ and $\zeta_n(s,\omega,F)$, respectively, and the classical Birkhoff-Khintchine ergodic theorem. For the investigation of the support, the positive density method is applied, see [**7**]. □

Our next aim is to show that the measure P_N, as $N \to \infty$, also converges weakly to the limit measure P of $\hat{P}_n$ as $n \to \infty$, i.e., that P_N converges weakly to P_ζ.

First we need a discrete version of approximation $\zeta(s,F)$ by $\zeta_n(s,F)$. Let $\{K_l : l \in \mathbb{N}\} \subset D$ be a sequence of compact subsets such that $D = \bigcup_{l=1}^{\infty} K_l$, $K_l \subset K_{l+1}$ for all $l \in \mathbb{N}$, and if $K \subset D$ is a compact subset, then $K \subset K_l$ for some $l \in \mathbb{N}$. For $g_1, g_2 \in H(D)$, set

$$\rho(g_1,g_2) = \sum_{l=1}^{\infty} 2^{-l} \frac{\sup_{s \in K_l} |g_1(s) - g_2(s)|}{1 + \sup_{s \in K_l} |g_1(s) - g_2(s)|}.$$

Then ρ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta.

We also recall the Gallagher lemma which relates continuous and discrete mean values of certain functions.

Lemma 2.7. *Let T_0 and $T \geqslant \delta > 0$ be real numbers, and $\mathcal{T}$ be a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Define*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1,$$

and let $S(x)$ be a complex-valued continuous function on $[T_0, T+T_0]$ having a continuous derivative on $(T_0, T+T_0)$. Then

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leqslant \frac{1}{\delta}\int_{T_0}^{T_0+T} |S(x)|^2\,dx + \Big(\int_{T_0}^{T_0+T} |S(x)|^2\,dx \int_{T_0}^{T_0+T} |S'(x)|^2\,dx\Big)^{\frac{1}{2}}.$$

Proof of the lemma can be found in [**13**], Lemma 1.4.

Lemma 2.8. *Suppose that $\alpha \in (0,1)$ and $h > 0$ are fixed numbers. Then*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\rho(\zeta(s+ik^\alpha h, F), \zeta_n(s+ik^\alpha h, F)) = 0.$$

Proof. It is known that, for fixed $\sigma \in (\frac{\kappa}{2}, \frac{\kappa+1}{2})$,

$$\int_0^T |\zeta(\sigma+it, F)|^2 \mathrm{d}t = O(T). \tag{2.4}$$

This together with the Cauchy integral formula implies, for the same σ, the estimate

$$\int_0^T |\zeta'(\sigma+it, F)|^2 \mathrm{d}t = O(T). \tag{2.5}$$

Further, we will apply Lemma 2.7. For $2 \leqslant k \leqslant N$ and sufficiently large N, we have that

$$\begin{aligned}(k+1)^\alpha - k^\alpha &= k^\alpha\Big(1+\frac{1}{k}\Big)^\alpha - k^\alpha = k^\alpha\Big(1+\frac{\alpha}{k}+\frac{\alpha(\alpha-1)}{2k^2}+\cdots\Big) - k^\alpha \\ &= \frac{\alpha}{k^{1-\alpha}} + \frac{\alpha(\alpha-1)}{2k^{2-\alpha}} + \cdots > \frac{\alpha}{2N^{1-\alpha}}.\end{aligned}$$

We take $\delta = \frac{\alpha h}{2N^{1-\alpha}}$ in Lemma 2.7. Then estimates (2.4), (2.5) and Lemma 2.7, for $\sigma \in (\frac{\kappa}{2}, \frac{\kappa+1}{2})$, yield

$$\begin{aligned}\sum_{k=0}^{N} |\zeta(\sigma+ik^\alpha h, F)|^2 &\ll N^{1-\alpha}\int_0^{N^\alpha h} |\zeta(\sigma+it, F|^2 \mathrm{d}t \\ &+ \Big(\int_0^{N^\alpha h} |\zeta(\sigma+it, F)|^2\mathrm{d}t\Big)^{\frac{1}{2}}\Big(\int_0^{N^\alpha h_j} |\zeta'(\sigma+it, F)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \ll N.\end{aligned} \tag{2.6}$$

Let K be a compact subset of the strip D. Then, using (2.6) and contour integration, we find similarly to the proof of Theorem 4.1 from [**5**] that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K}(\zeta(s+ik^\alpha h, F) - \zeta_n(s+ik^\alpha h, F)) = 0.$$

This and the definition of the metric ρ prove the lemma. ∎

Proof of Theorem 2.1. In view of Lemma 2.6, it suffices to show that P_N converges weakly to P as $N \to \infty$, where P is the limit measure of $\hat{P}_n$ as $n \to \infty$.

Let θ_N be a random variable defined on a certain probability space $(\Omega_0, \mathcal{A}, \mu)$, and having the distribution

$$\mu(\theta_N = k^\alpha h) = \frac{1}{N+1}, \quad k = 0, 1, \dots, N.$$

Define the $H(D)$-valued random element $X_{N,n}$ by

$$X_{N,n} = X_{N,n}(s) = \zeta_n(s + i\theta_N, F).$$

Then, by Lemmas 2.1 and 2.3, we have that $X_{N,n}$ converges in distribution to $\hat{X}_n$

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} \hat{X}_n, \tag{2.7}$$

where $\hat{X}_n$ is the $H(D)$-valued random element with the distribution $\hat{P}_n$, and $\hat{P}_n$ is the limit measure in Lemma 2.3. Since the series for $\zeta_n(s, F)$ is absolutely convergent for $\sigma > \frac{\kappa}{2}$, by a standard method it is easy to show that the family of probability measures $\{\hat{P}_n : n \in \mathbb{N}\}$ is tight, i.e., for every $\varepsilon > 0$, there exists a compact subset $K = K_F(\varepsilon) \subset D$ such that $\hat{P}_n(K) > 1 - \varepsilon$ for all $n \in \mathbb{N}$. Hence, by the Prokhorov theorem, see Theorem 6.1 in [**1**], the family $\{\hat{P}_n\}$ is relatively compact. Thus, there exists a sequence $\{\hat{P}_{n_r}\} \subset \{\hat{P}_n\}$ such that $\hat{P}_{n_r}$ converges weakly to a certain probability measure $\hat{P}$ on $(H(D), \mathcal{B}(H(D)))$ as $r \to \infty$, i.e., using a mixed notation of [**1**],

$$\hat{X}_{n_r} \xrightarrow[r\to\infty]{\mathcal{D}} \hat{P}. \tag{2.8}$$

On $(\Omega_0, \mathcal{A}, \mu)$, define one more $H(D)$-valued random element

$$X_N = X_N(s) = \zeta(s + i\theta_N, F).$$

Then, by Lemma 2.8, we find that, for every $\varepsilon > 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \limsup_{N\to\infty} \mu\big(\underline{\rho}(X_N, X_{N,n}) \geqslant \varepsilon\big) \\
&\quad = \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \#\big\{0 \leqslant k \leqslant N : \rho\big(\zeta(s + ik^\alpha h, F), \zeta_n(s + ik^\alpha h, F)\big) \geqslant \varepsilon\big\} \\
&\quad \leqslant \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{k=0}^{N} \rho\big(\zeta(s + ik^\alpha h, F), \zeta_n(s + ik^\alpha h, F)\big) = 0.
\end{aligned}$$

This and relations (2.7) and (2.8) show that all hypotheses of Theorem 4.2 of [**1**] are satisfied. Therefore,

$$X_N \xrightarrow[N\to\infty]{\mathcal{D}} \hat{P},$$

or equivalently, P_N converges weakly to $\hat{P}$ as $N \to \infty$. Moreover, the latter relation shows that the measure $\hat{P}$ is independent of the sequence $\{\hat{P}_{n_r}\}$. Therefore,

$$\hat{X}_n \xrightarrow[n\to\infty]{\mathcal{D}} \hat{P},$$

i.e., $\hat{P}_n$ converges weakly to $\hat{P}$ as $n \to \infty$, thus $\hat{P} = P$. Thus, we obtain that P_N converges weakly to the limit measure P of $\hat{P}_n$ as $n \to \infty$, and by Lemma 2.6, P coincides with P_ζ. The theorem is proved. □

3. Proof of universality theorems

Proof of Theorem 1.3. By the Mergelyan theorem on the approximation of analytic functions by polynomials [**12**], there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} \left| f(s) - e^{p(s)} \right| < \varepsilon/2. \tag{3.1}$$

Define the set

$$G = \Big\{ g \in H(D) : \sup_{s \in K} \left| g(s) - e^{p(s)} \right| < \varepsilon/2 \Big\}.$$

Then, by Theorem 2.1, G is an open neighbourhood of the element $e^{p(s)}$ of the support of the measure P_ζ. Hence, $P_\zeta(G) > 0$. This, Theorem 2.1 and an equivalent of the weak convergence of probability measures in terms of open sets show that

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \big\{ 0 \leqslant k \leqslant N : \zeta(s + ik^\alpha h, F) \in G \big\} \geqslant P_\zeta(G) > 0,$$

or, by the definition of G, we have that

$$\liminf_{N \to \infty} \frac{1}{N+1} \# \Big\{ 0 \leqslant k \leqslant N : \sup_{s \in K} \left| \zeta(s + ik^\alpha h, F) - e^{p(s)} \right| < \varepsilon/2 \Big\} > 0.$$

Combining this with (3.1) proves the theorem. □

Proof of Theorem 1.4. It follows from Theorem 2.1, the continuity of the operator Φ and Theorem 5.1 of [**1**] that the measure

$$\frac{1}{N+1} \# \big\{ 0 \leqslant k \leqslant N : \Phi(\zeta(s + ik^\alpha h, F)) \in A \big\}, \quad A \in \mathcal{B}(H(D)), \tag{3.2}$$

converges weakly to $P_\zeta \Phi^{-1}$ as $N \to \infty$. Moreover, repeating the proof of Lemma 17 from [**10**], we obtain that the support of $P_\zeta \Phi^{-1}$ includes the closure of the set $H_{\Phi; a_1, \ldots, a_r}(D)$.

First suppose that $f(s) \in H_{\Phi; a_1, \ldots, a_r}(D)$. Then, by the above remark, $f(s)$ is an element of the support of $P_\zeta \Phi^{-1}$. Therefore, putting

$$G_1 = \Big\{ g \in H(D) : \sup_{s \in K} \left| g(s) - f(s) \right| < \varepsilon \Big\},$$

we have that $P_\zeta \Phi^{-1}(G_1) > 0$. This and the weak convergence of measure (3.2) prove the theorem in this case.

Now let $r = 1$. Then, by the Mergelyan theorem, there exists a polynomial $p(s)$ such that

$$\sup_{s \in K} \left| f(s) - p(s) \right| < \varepsilon/4. \tag{3.3}$$

Since $f(s) \neq a_1$ on K, by the Mergelyan theorem again, we can find a polynomial $q(s)$ such that

$$\sup_{s\in K}\bigl|p(s)-f_1(s)\bigr| < \varepsilon/4, \tag{3.4}$$

where $f_1(s) = a_1 + e^{q(s)}$. By the above remark, $f_1(s)$ is an element of the support of the measure $P_\zeta\Phi^{-1}$. Therefore, if

$$G_2 = \Bigl\{g \in H(D): \sup_{s\in K}\bigl|g(s)-f_1(s)\bigr| < \varepsilon/2\Bigr\},$$

then $P_\zeta\Phi^{-1}(G_2) > 0$. Therefore, by the weak convergence of (3.2) to $P_\zeta\Phi^{-1}$ as $N \to \infty$, we find that

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Bigl\{0 \leqslant k \leqslant N : \sup_{s\in K}\bigl|\Psi(\zeta(s+ik^\alpha h, F)) - f_1(s)\bigr| < \varepsilon/2\Bigr\} > 0.$$

This together with (3.3) and (3.4) prove the theorem in the case $r = 1$. □

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