PORTUGALIAE MATHEMATICA
Vol. 51 Fasc. 2 – 1994

PROPER LEFT TYPE-A COVERS

John Fountain and Gracinda M.S. Gomes[1]

Introduction

Left type-A monoids form a special class of left abundant monoids. Interest in the latter arose originally from the study of monoids by means of their associated S-sets. A left abundant monoid is a monoid with the property that all principal left ideals are projective. All regular monoids are left abundant and so are many other types of monoid including right cancellative monoids. A left abundant monoid S is said to be left type-A if the set $E(S)$ of idempotents of S is a commutative submonoid of S and S also satisfies the condition that for any elements e in $E(S)$ and a in S we have $eS \cap aS = eaS$. In fact, [see 2] left type-A monoids are precisely those monoids which are isomorphic to certain submonoids of symmetric inverse monoids, namely those submonoids S of $\mathcal{I}(X)$ which satisfy the condition that if α is in S, then $\alpha\alpha^{-1}$ is in S. Thus all inverse monoids are left type-A but there are many left type-A monoids which are not inverse, for example, right cancellative monoids which are not groups. We see from the characterization just given that for a topological space X, the submonoid of $\mathcal{I}(X)$ consisting of continuous one-one partial maps is left type-A. In general, of course, this example is not inverse. A significant body of structure theory has been developed for left type-A monoids, much of it inspired by corresponding theory for inverse monoids. In particular, it is shown in [2] that for the study of general left type-A monoids the subclass of proper left type-A monoids plays a special role.

This paper is the last of a series of three devoted to studying proper left type-A monoids via categories. The ideas and techniques are inspired by those which Margolis and Pin introduced [5] in their study of E-dense and inverse monoids. The first paper [3] of the series showed that the work of Margolis and

Received: November 29, 1991; *Revised*: November 12, 1993.
[1] This research was supported by "Plano de Reforço da Capacidade Científica do Departamento de Matemática", Fundação Calouste Gulbenkian.

Pin for E-dense monoids could be strengthened in the case of left type-A E-dense monoids to give generalizations of results on inverse monoids. This paper and the second [4] of the series are concerned with extending the techniques to apply to left type-A monoids in general. The concept of a left type-A monoid is essentially a one-sided notion and this is reflected in the fact that it is possible to generalize the methods in two ways. In [4] we considered right actions on categories and were led to new results on left type-A monoids.

In the present paper we study left type-A monoids by means of left actions on categories. This forces us to change both the nature of the categories considered and the definition of the action.

In Section 1 we use our new techniques to obtain a new proof of a theorem of Palmer [6] which characterizes proper left type-A monoids in terms of M-systems. Palmer's result is a variation of a characterization obtained in [2]. The other main result of [2] is that every left type-A monoid has a proper left type-A cover. In [1] the categorical methods of Margolis and Pin were used to show that every E-dense monoid has an E-unitary dense cover. This result was relativized in [3] to the case of left type-A E-dense monoids showing that the cover constructed is proper and respects the relation $\mathcal{R}^*$. In Section 2 of the present paper we adapt the techniques of [1] to obtain a new proof of the covering theorem of [2]. That is, we prove that every left type-A monoid has a left type-A $^+$-cover. It is not difficult to see that this is, in fact, the dual of Theorem 3.3 of [2].

1 – Preliminaries

We start by recalling some of the definitions and results, presented in [3], for both left type-A monoids and categories.

On left type-A monoids

Let S be a monoid, with set of idempotents $E(S)$. On S, we define a binary relation $\mathcal{R}^*$, which contains the Green's relation $\mathcal{R}$, as follows: for all $a, b \in S$,

$$(a,b) \in \mathcal{R}^* \quad \Leftrightarrow \quad [(\forall\, s,t \in S)\ sa = ta \ \Leftrightarrow\ sb = tb]\ .$$

The monoid S is said to be *left abundant* if each $\mathcal{R}^*$-class, R^*_a, contains an idempotent. When $E(S)$ is a semilattice, such idempotent is unique and it is denoted by a^+. If, in addition, S satisfies the *type-A condition*: for all $a \in S$ and $e \in E(S)$,

$$a\,e = (a\,e)^+\,a\ ,$$

we say that S is a *left type-A monoid*. It is shown in [2] that this definition is equivalent to those given in the Introduction.

We remind the reader of the following basic properties of left type-A monoids which we use frequently and without further mention:

1) For every $a, b, c \in S$, $a\,\mathcal{R}^*\,b$ implies $c\,a\,\mathcal{R}^*\,c\,b$;

2) For every $a \in S$, $a = a^+\,a$;

3) For every $e \in E(S)$ and $a \in S$, $(e\,a)^+ = e\,a^+$.

On a left type-A monoid S, the *least right cancellative monoid congruence*, σ, is defined by: for all $a, b \in S$,

$$(a,b) \in \sigma \quad \Leftrightarrow \quad (\exists\, e \in E(S)) \;\; e\,a = e\,b \ ;$$

and we say that S is *proper* if

$$\sigma \cap \mathcal{R}^* = \iota \ ,$$

where ι is the identity relation [2].

As usual by an *E-unitary semigroup*, we mean a semigroup S such that, for all $a \in S$ and $e \in E(S)$,

$$a\,e \in E(S) \;\text{ or }\; e\,a \in E(S) \quad \Rightarrow \quad a \in E(S) \ .$$

In [2], it is shown that every proper left type-A monoid is E- unitary but, however, the converse is not true.

On left type-A categories

Let $\mathcal{C}$ be a (small) category. We denote the set of *objects* of $\mathcal{C}$ by $\mathrm{Obj}\,\mathcal{C}$ and the set of *morphisms* by $\mathrm{Mor}\,\mathcal{C}$. For any object u of $\mathcal{C}$, $\mathrm{Mor}(u,-)$ stands for the set of morphisms of $\mathcal{C}$ with *domain* u and $\mathrm{Mor}(-,u)$ for the set of morphisms of $\mathcal{C}$ with *codomain* u; we denote the *identity morphism* at the object u by O_u.

As in [5], we adopt an additive notation for the composition of morphisms. A morphism p is said to be an *idempotent* if $p = p+p$. Clearly, if p is an idempotent then $p \in \mathrm{Mor}(u,u)$, for some $u \in \mathrm{Obj}\,\mathcal{C}$.

On the partial groupoid $\mathrm{Mor}\,\mathcal{C}$, we define the $\mathcal{R}^*$-relation as for a monoid.

A category $\mathcal{C}$ is said to be *E-left type-A* if, for all $u \in \mathrm{Obj}\,\mathcal{C}$, $E(\mathrm{Mor}(u,u))$ is a semilattice, every $\mathcal{R}^*$-class R^*_p of $\mathrm{Mor}\,\mathcal{C}$ contains an idempotent p^+ (necessarily unique) and $\mathcal{C}$ satisfies the *type-A condition*, i.e. for all $u, v \in \mathrm{Obj}\,\mathcal{C}$, $p \in \mathrm{Mor}(u,v)$ and $f \in E(\mathrm{Mor}(v,v))$,

$$p + f = (p+f)^+ + p \ .$$

Let $\mathcal{C}^0$ be an E-left type-A category *with a distinguished* object u_0 such that $\mathrm{Mor}(u_0,u_0)$ is a semilattice. We say that $\mathcal{C}^0$ is (left) *u_0-connected* if, for all

$v \in \operatorname{Obj}\mathcal{C}^0$, $\operatorname{Mor}(u_0, v) \neq \emptyset$. Also, $\mathcal{C}^0$ is called (left) u_0-*proper* if, for all $v \in \operatorname{Obj}\mathcal{C}^0$ and $p, q \in \operatorname{Mor}(u_0, v)$,

$$p^+ = q^+ \quad \Rightarrow \quad p = q \ ,$$

i.e. each $\mathcal{R}^*$-class has at most an element of $\operatorname{Mor}(u_0, v)$.

To simplify the terminology, we say that an E-left type-A, u_0-connected and u_0-proper category $\mathcal{C}^0$, with distinguished element u_0 is a u_0-*proper left category*.

2 – u_0-proper left categories

In this section, we begin by considering left actions of right cancellative monoids on E-left type-A categories. In particular, we introduce the ideas of a downwards action and a u_0-closed action. We show that given a right cancellative monoid acting in this way on a u_0-proper left category we can form a proper left type-A monoid and that any proper left type-A monoid arises in this way. We then use this result to recover a theorem of Palmer which states that every proper left type-A monoid is isomorphic to an M-monoid.

Definition 2.1. Let $\mathcal{C}$ be an E-left type-A category and T a right cancellative monoid. We say that T *acts* (on the left) on $\mathcal{C}$ (by $\mathcal{R}^*$-endomorphisms) if, for all $u \in \operatorname{Obj}\mathcal{C}$ and $t \in T$, there exists a unique $tu \in \operatorname{Obj}\mathcal{C}$, and, for all $u, v \in \operatorname{Obj}\mathcal{C}$, $p \in \operatorname{Mor}(u, v)$, there is a unique $tp \in \operatorname{Mor}(tu, tv)$ such that, for all $u, v, w \in \operatorname{Obj}\mathcal{C}$, $p \in \operatorname{Mor}(u, v)$, $q \in \operatorname{Mor}(v, w)$ and $t, t_1, t_2 \in T$,

- $t(p + q) = tp + tq$,
- $(t_1\, t_2)\, p = t_1(t_2\, p)$,
- $t\, O_v = O_{tv}$,
- $1\, p = p$,
- $(t\, p)^+ = t\, p^+$.

It is not difficult to check that

Lemma 2.2. *Let $\mathcal{C}^0$ be a u_0-proper left category and T a right cancellative monoid acting on $\mathcal{C}^0$. Then*

$$\mathcal{C}_{u_0} = \Big\{(p, t)\colon\ t \in T,\ p \in \operatorname{Mor}(u_0, tu_0)\Big\} \ ,$$

with multiplication given by

$$(p, t)\, (q, s) = (p + tq, ts)$$

is a proper left type-A monoid such that $E(\mathcal{C}_{u_0}) \simeq \mathrm{Mor}(u_0, u_0)$. ∎

Definition 2.3. Let $\mathcal{C}^0$ be an E-left type-A category, with a distinguished object u_0, and T a right cancellative monoid acting on $\mathcal{C}^0$. We say that the action of T on $\mathcal{C}^0$ is *downwards* if, for all $u \in \mathrm{Obj}\,\mathcal{C}^0$ and $t \in T$,

$$\mathrm{Mor}(tv, -) = t\,\mathrm{Mor}(v, -)\ .$$

On the other side, if the action of T over u_0 satisfies the following properties:

- $\mathrm{Obj}\,\mathcal{C}^0 = Tu_0$,
- for all $v \in \mathrm{Obj}\,\mathcal{C}^0$, if $\mathrm{Mor}(v, u_0) \neq \emptyset$ then $v = gu_0$, for some unit $g \in T$,

we say that the action is u_0-*closed*.

Lemma 2.4. *Let $\mathcal{C}^0$ be a u_0-proper left category and T a right cancellative monoid acting on $\mathcal{C}^0$. If, for all $v \in \mathrm{Obj}\,\mathcal{C}^0$,*

$$\mathrm{Mor}(v, u_0) \neq \emptyset \quad\Rightarrow\quad v = g\,u_0, \qquad \textit{for some unit}\ \ g \in T\ ,$$

then, for all $p, q \in \mathrm{Mor}(v, u_0)$,

$$p^+ = q^+ \quad\Rightarrow\quad p = q\ .$$

Proof: Let $p, q \in \mathrm{Mor}(v, u_0)$ be such that $p^+ = q^+$. As $\mathrm{Mor}(v, u_0) \neq \emptyset$, there exists a unit $g \in T$ such that $v = gu_0$. Now, as the action respects the operation $^+$, we have

$$(g^{-1}p)^+ = g^{-1}\,p^+ = g^{-1}\,q^+ = (g^{-1}q)^+\ ,$$

where $g^{-1}p, g^{-1}q \in \mathrm{Mor}(u_0, g^{-1}u_0)$. Whence, $\mathcal{C}^0$ being u_0-proper, $g^{-1}p = g^{-1}q$ and, so $p = q$. ∎

Let M be a proper left type-A monoid and $T = M/\sigma$. We define the *derived category* $\mathcal{D}^0$ (of the natural morphism $M \to M/\sigma$) as in [3]: $\mathrm{Obj}\,\mathcal{D}^0 = T$ and, for all $t_1, t_2 \in T$,

$$\mathrm{Mor}(t_1, t_2) = \Big\{(t_1, m, t_2)\colon\ m \in M,\ t_1(m\,\sigma) = t_2\Big\}\ ,$$

with composition given by

$$(t_1, m, t_2)\,(t_2, n, t_3) = (t_1, mn, t_3)\ .$$

The distinguished object of $\mathcal{D}^0$ is 1, the identity of T. The action of T over $\mathcal{D}^0$ is given by: for all $u \in \mathrm{Obj}\,\mathcal{D}^0$ and $t \in T$, tu is the result of the multiplication of t by u in T and for all $(u, m, v) \in \mathrm{Mor}(u, v)$,

$$t(u, m, v) = (tu, m, tv)\ .$$

Lemma 2.5. *Let M be a proper left type-A monoid. Then the derived category $\mathcal{D}^0$ is a 1-proper left category and the action of T on $\mathcal{D}^0$ is downwards and 1-closed.*

Proof: First, notice that if M is a proper left type-A monoid then M is E-unitary and, so $1 = E(M)$. Then, following [3, 4], we have that $\mathcal{D}^0$ is an E-left type-A category where, for all $(t_1, m, t_2) \in \operatorname{Mor} \mathcal{D}^0$,

$$(t_1, m, t_2)^+ = (t_1, m^+, t_1)$$

and

$$E(\operatorname{Mor}(t,t)) = \Big\{(t, e, t)\colon\ e \in E(M)\Big\} \simeq E(M)\ .$$

In particular,

$$\operatorname{Mor}(1,1) = E(\operatorname{Mor}(1,1)) \simeq E(M)\ .$$

The category $\mathcal{D}^0$ is 1-connected since, for all $m\sigma \in M/\sigma = T$,

$$(1, m, m\sigma) \in \operatorname{Mor}(1, m\sigma)\ .$$

On the other hand, $\mathcal{D}^0$ is 1-proper, since M is proper, i.e. $\mathcal{R}^* \cap \sigma = \iota$.

It is a routine matter to verify that T acts on $\mathcal{D}^0$ in such a way that $\operatorname{Obj} \mathcal{D}^0 = T1$. To prove that T acts downwards, let $t \in T$, $u \in \operatorname{Obj} \mathcal{D}^0$ and $p \in \operatorname{Mor}(tu, -)$. Then, there exists $m \in M$ such that

$$p = (tu, m, tu.m\sigma)\ ,$$

and, so

$$p = t(u, m, u.m\sigma) \in t \operatorname{Mor}(u, -)\ .$$

It is obvious that $t \operatorname{Mor}(u,-) \subseteq \operatorname{Mor}(tu,-)$, hence $t \operatorname{Mor}(u,-) = \operatorname{Mor}(tu,-)$.

Finally, let $p \in \operatorname{Mor}(v, 1)$. Then, $p = (v, m, 1)$ for some $m \in M$ and $v.m\sigma = 1$. As T is right cancellative, $v.m\sigma = 1 = m\sigma.v$ and $v = v.1$ is a unit of T, as required. ∎

Theorem 2.6. *Let M be a monoid. Then, M is proper and left type-A if and only if $M \simeq \mathcal{C}_{u_0}$, where u_0 is the distinguished object of a u_0-proper left category $\mathcal{C}^0$ on which a right cancellative monoid T acts via an action which is downwards and u_0-closed.*

Proof: In view of Lemma 2.2, under the above conditions, if $M \simeq \mathcal{C}_{u_0}$, then M is a proper left type-A monoid.

Conversely, let M be a proper left type-A monoid. Then, by Lemma 2.5, the derived category $\mathcal{D}^0$ of M is a 1-proper left category and $T = M/\sigma$ is a right cancellative monoid which acts on $\mathcal{D}^0$ with an action which is downwards and 1-closed. Now, we consider the map

$$\begin{aligned}\psi\colon M \to C_1 = \Big\{(p,t)\colon\ t\in T,\ p\in \mathrm{Mor}(1,t)\Big\}\\ m \mapsto ((1,m,m\phi),m\phi)\ ,\end{aligned}$$

which is easily seen to be an isomorphism and the result follows. ∎

Let $\mathcal{C}$ be an E-left type-A category. On $\mathrm{Mor}\,\mathcal{C}$, we define a relation $\preceq$ as follows: for all $p,q\in \mathrm{Mor}\,\mathcal{C}$,

$$p\preceq q \quad\Leftrightarrow\quad (\exists\, a\in \mathrm{Mor}\,\mathcal{C})\ \ p^+=a^+,\ \ a+q^+=a\ .$$

In [3], we showed that $\preceq$ is a *preorder* on $\mathrm{Mor}\,\mathcal{C}$ and that the relation defined by

$$p\sim q \quad\Leftrightarrow\quad p\preceq q \ \text{ and }\ q\preceq p$$

defines an *equivalence relation* on $\mathrm{Mor}\,\mathcal{C}$ which contains $\mathcal{R}^*$. Also, on the quotient set $\mathcal{X}=\mathrm{Mor}\,\mathcal{C}/\sim$, we consider the *partial order* $\leq$ given by, for all $A_p,A_q\in\mathcal{X}$,

$$A_p\leq A_q \quad\Leftrightarrow\quad p\preceq q\ .$$

If T is a right cancellative monoid acting on $\mathcal{C}$, we define an action (on the left) of T on the partially ordered set $\mathcal{X}$ in the following way: for all $A_p\in\mathcal{X}$ and $t\in T$,

$$t\,A_p=A_{tp}\ .$$

Lemma 2.7. *Let $\mathcal{C}^0$ be a u_0-proper left category and T a right cancellative monoid acting on $\mathcal{C}^0$. If the action is such that, for all $v\in \mathrm{Obj}\,\mathcal{C}^0$,*

$$(*)\qquad \mathrm{Mor}(v,u_0)\neq\emptyset \quad\Rightarrow\quad v=g\,u_0,\ \ \textit{for some unit } g\in T\ ,$$

then the action of T over $\mathcal{X}$ respects the relations $\preceq$, $\sim$ and $\leq$.

Moreover, for all $t,t'\in T$, $p\in \mathrm{Mor}(u_0,tu_0)$ and $q\in\mathrm{Mor}(u_0,t'\,u_0)$,

$$A_p\wedge A_{tq}=A_{p+tq}\ .$$

Proof: By bearing in mind condition $(*)$ and Lemma 2.4, the proof is similar to the proof of Lemma 3.12 of [3]. Notice that here we need $\mathcal{C}^0$ to be u_0-proper. ∎

Lemma 2.8. *Under the conditions of Lemma 2.7, let*

$$\mathcal{Y}=\Big\{A\in\mathcal{X}\colon\ A\cap \mathrm{Mor}(u_0,u_0)\neq\emptyset\Big\}\ .$$

Then

a) *$\mathcal{Y}$ is a semilattice of $\mathcal{X}$ with greatest element $F = A_{O_{u_0}}$;*

b) $\mathcal{Y} = \Big\{A \in \mathcal{X}\colon\ (\exists\, v \in \operatorname{Obj}\mathcal{C}^0)\ A \cap \operatorname{Mor}(u_0, v) \neq \emptyset\Big\}$*;*

c) *$(\forall\, t \in T)\ (\forall\, B \in \mathcal{Y})\ B \leq tF \Leftrightarrow B \cap \operatorname{Mor}(u_0, tu_0) \neq \emptyset$;*

d) *$(\forall\, t \in T)\ (\exists\, B \in \mathcal{Y})\ B \leq tF$.*

Proof: Since $\mathcal{C}^0$ is a u_0-proper left category, $\operatorname{Mor}(u_0, u_0)$ is a semilattice and condition **a)** follows from the previous lemma.

On any E-left type-A category $\mathcal{C}$, for all $u, v \in \operatorname{Obj}\mathcal{C}$ and $p \in \operatorname{Mor}(u_0, v)$, we must have $p^+ \in \operatorname{Mor}(u_0, u_0)$. Since the equivalence $\sim$ contains $\mathcal{R}^*$, condition **b)** must hold.

c) Let $t \in T$ then $tF = A_{O_{tu_0}}$. Let $B = A_q \in \mathcal{Y}$, with $q \in \operatorname{Mor}(u_0, u_0)$. Suppose that $B \leq tF$. Then, $q \preceq O_{tu_0}$. Thus, there exists $r \in \operatorname{Mor}(u_0, tu_0)$ such that $q^+ = r^+$ and, so

$$r \in A_q \cap \operatorname{Mor}(u_0, tu_0)\ .$$

Conversely, suppose that there exists $r \in A_q \cap \operatorname{Mor}(u_0, tu_0)$. Then, $r + O_{tu_0} = r$. Hence, $r \preceq O_{tu_0}$ and $A_r = B \leq tF$.

d) Let $t \in T$. Since $\mathcal{C}^0$ is u_0-connected, there exists $a \in \operatorname{Mor}(u_0, tu_0)$. Thus, $A_a \in \mathcal{Y}$, by condition b), and $a \preceq O_{tu_0}$. ∎

Next, we make the connection between the characterization of a proper left type-A monoid M as an M-monoid [6] and the characterization of M, via categories, as a $\mathcal{C}_{u_0}$ monoid. We start by describing an M-monoid.

Definition 2.9 [6]**.** Let X be a partially ordered set and Y a subsemilattice of X with greatest element f. Let T be a right cancellative monoid acting (on the left) on X, in such a way that

- $(\forall\, a \in X)\ 1\, a = a$;
- $(\forall\, a, b \in X)\ (\forall\, t \in T)$, $a \leq b \Rightarrow ta \leq tb$;
- $X = TY$;
- $(\forall\, t \in T)\ (\exists\, b \in Y)\ b \leq tf$;
- $(\forall\, a, b \in Y)\ (\forall\, t \in T)\ a \leq tf \Rightarrow a \wedge tb \in Y$;
- $(\forall\, a, b, c \in Y)\ (\forall\, t, t' \in T)$, $a \leq tf$, $b \leq t'f \Rightarrow (a \wedge tb) \wedge tt'c = a \wedge t(b \wedge t'c)$.

Then, we define

$$M(T, X, Y) = \Big\{(a, t) \in Y \times T\colon\ a \leq tf\Big\}\ ,$$

with multiplication given by

$$(a,t)\,(b,t') = (a \wedge tb, tt')\ ,$$

and obtain a monoid which we call an *M-monoid*.

Theorem 2.10 [6]**.** *Every proper left type-A monoid M is isomorphic to an M-monoid $M(T,X,Y)$. Also, in $M(T,X,Y)$, for all (a,t), (b,t'):*

- $(a,t)\,\mathcal{R}^*\,(b,t') \Leftrightarrow a=b$;
- $(a,t)\,\sigma\,(b,t') \Leftrightarrow t=t'$;

and so $T \simeq M(T,X,Y)/\sigma$. ∎

Lemma 2.11. *Let $\mathcal{C}^0$ be a u_0-proper left category and T be a right cancellative monoid acting downwards on $\mathcal{C}^0$. If this action is u_0-closed, then $M(T,\mathcal{X},\mathcal{Y})$ is an M-monoid.*

Proof: By Lemma 2.8, $\mathcal{Y}$ is a subsemilattice, with greatest element $F = A_{O_{u_0}}$, of the partially ordered set $\mathcal{X}$. Now, we verify that $(T,\mathcal{X},\mathcal{Y})$ satisfies the properties of Definition 2.9. Let $A_p, A_q \in \mathcal{X}$ and $t \in T$. Clearly, $1A_p = A_{1p} = A_p$ and, by Lemma 2.7,

$$A_p \le A_q \ \Rightarrow\ p \preceq q \ \Rightarrow\ tp \preceq tq \ \Rightarrow\ tA_p \le tA_q\ .$$

Now, let $A_p \in \mathcal{X}$ with $p \in \mathrm{Mor}(v,v)$. As the action of T on $\mathcal{C}^0$ is u_0-closed, $v = tu_0$, for some $t \in T$. Thus, $p^+ \in \mathrm{Mor}(tu_0,tu_0)$ and, as T acts downwards on $\mathcal{C}^0$, there exists $r \in \mathrm{Mor}(u_0,u_0)$ such that $p^+ = t\,r$. Whence, $A_r \in \mathcal{Y}$ and

$$A_p = A_{p^+} = A_{tr} = t\,A_r \ \in\ \mathcal{Y}\ .$$

Next, let $t \in T$. By Lemma 2.8 d), there exists $A_a \in \mathcal{Y}$ such that

$$A_a \preceq t\,F\ .$$

To prove the fifth condition suppose that $A_a, A_b \in \mathcal{Y}$, with $a,b \in \mathrm{Mor}(u_0,u_0)$, and let $t \in T$ be such that $A_a \le tA_{O_{u_0}}$. By Lemma 2.8 c), $A_a = A_r$, for some $r \in \mathrm{Mor}(u_0,tu_0)$. Hence, by Lemma 2.7, there exists

$$A_a \wedge tA_b = A_r \wedge A_{tb} = A_{r+tb} = A_{(r+tb)^+} \in \mathcal{Y}\ .$$

Finally, let $A_a, A_b, A_c \in \mathcal{Y}$ with $a,b,c \in \mathrm{Mor}(u_0,u_0)$ and $t,t' \in T$. Suppose that $A_a \le tF$ and $A_b \le t'\,F$. Then, as before, there exist $r \in \mathrm{Mor}(u_0,tu_0) \cap A_a$ and $r' \in \mathrm{Mor}(u_0,t'\,u_0) \cap A_b$. Now, by Lemma 2.7,

$$A_a \wedge tA_b = A_r \wedge tA_{r'} = A_{r+tr'}$$

and

$$A_b \wedge t' A_c = A_{r'+t'c} \ .$$

Again, by Lemma 2.7,

$$\begin{aligned}(A_a \wedge tA_b) \wedge t\,t' A_c &= A_{r+tr'} \wedge t\,t' A_c \\ &= A_{r+tr'+tt'c}\end{aligned}$$

and

$$\begin{aligned}A_a \wedge t(A_b \wedge t' A_c) &= A_r \wedge tA_{r'+t'c} = A_{r+t(r'+t'c)} \\ &= A_{r+tr'+t'c} \ .\end{aligned}$$

Therefore $M(T, \mathcal{X}, \mathcal{Y})$ is an M-monoid, as required. ∎

By Theorem 2.10, we know that every proper left type-A monoid M is isomorphic to an M-monoid $\mathcal{M}$. The above results allow us to obtain a clearer construction of such an $\mathcal{M}$ and a new proof of the theorem.

Theorem 2.12. *Let M be a proper left type-A monoid, $T = M/\sigma$ and $\mathcal{D}^0$ its derived category. Then, $M \simeq M(T, \mathcal{X}, \mathcal{Y})$, where $\mathcal{X} = \operatorname{Mor} \mathcal{D}^0/\sim$ and $\mathcal{Y} = \{A \in \mathcal{X}\colon\ A \cap \operatorname{Mor}(1,1) \neq \emptyset\}$.*

Proof: In view of Theorem 2.6 and Lemma 2.11, it only remains to prove that $C_1 \simeq M(T, \mathcal{X}, \mathcal{Y})$. Consider the map

$$\begin{aligned}\theta\colon C_1 &\to M(T, \mathcal{X}, \mathcal{Y}) \\ (p,t) &\mapsto (A_p, t) \ .\end{aligned}$$

It follows from Lemma 2.8 c) that θ is well defined. By Lemma 2.7, θ is a morphism. Again, by Lemma 2.8 c), θ is onto. To see that θ is injective, let $q, p \in \operatorname{Mor}(1,t)$, for some $t \in T$, be such that $A_p = A_q$, i.e. $p \sim q$. Thus, there exists $a \in \operatorname{Mor} \mathcal{D}^0$ such that $p^+ = a^+$, $a + q^+ = a$. Hence $a \in \operatorname{Mor}(1,1)$ and $a = a^+$. Thus $p^+ = a^+ = a^+ + q^+ = p^+ + q^+$. Similarly, $q^+ = q^+ + p^+$. As $\operatorname{Mor}(1,1)$ is a semilattice, $p^+ = q^+$. Finally, $\mathcal{D}^0$ being 1-proper, it follows that $p = q$, as required. ∎

3 – Proper left type-A covers of left type-A monoids

In this section we are concerned to show that for each left type-A monoid M there is a proper left type-A monoid P and an idempotent separating homomorphism $\theta\colon\ P \to M$ from P onto M such that $a^+\theta = (a\,\theta)^+$. We express this result by saying that M has a proper left type-A $^+$-cover. It (or rather its dual) was originally proved in [2] although it is stated somewhat differently there. For the alternative proof which we present here we use the theory developed in Section 2 and a modification of the method of [1].

Before embarking on the proof we illustrate the notion of proper left type-A $^+$-cover by the following example. Let X be a topological space. We denote by $G(X)$ the monoid of all continuous bijections from X to itself under composition. Certainly $G(X)$ is cancellative but it is not a group in general. We let $\mathcal{I}_c(X)$ denote the monoid of all continuous one-one partial maps from X to itself under composition of partial functions. Finally, $\mathcal{P}(X)$ denotes the power set of X regarded as a semilattice under the operation of intersection. We define a left action of $G(X)$ on $\mathcal{P}(X)$ by the rule that $\sigma\, Y = Y\, \sigma^{-1}$ for all σ in $G(X)$ and all subsets Y of X. It is then easy to verify that the multiplication

$$(Y,\sigma)(Z,\tau) = (Y \cap \sigma Z, \sigma\tau)$$

makes the set $\mathcal{P}(X) \times G(X)$ into a monoid $\mathcal{P}(X) * G(X)$ (a semidirect product of $\mathcal{P}(X)$ and $G(X)$). It is also readily checked that this monoid is proper left type-A with semilattice of idempotents $\{(Y,1)\colon\ Y \in \mathcal{P}(X)\}$ and $(Y,\sigma)^+ = (Y,1)$. Indeed, $\mathcal{P}(X) * G(X)$ is nothing other than $M(G(X), \mathcal{P}(X), \mathcal{P}(X))$. We claim that it is a left type-A $^+$-cover of $\mathcal{I}_c(X)$. To see this consider the surjective function $\theta\colon\ \mathcal{P}(X) * G(X) \to \mathcal{I}_c(X)$ defined by

$$(Y,\sigma)\,\theta = \sigma_Y \ ,$$

where σ_Y denotes the partial map with domain Y obtained by restricting σ. It is routine to show that θ is an idempotent separating homomorphism and that $((Y,\sigma)^+)\,\theta = ((Y,\sigma)\,\theta)^+$. Of course, this example is very familiar when X has the discrete topology and we have an E-unitary cover of the symmetric inverse monoid on X.

We now start our proof with a technical lemma on left type-A monoids.

Lemma 3.1. *Let M be a left type-A monoid and let $s \in S$. If $s = e_0 x_1 e_1 \cdots e_{n-1} x_n e_n$, for some $n \in \mathbb{N}$, $x_i \in M$ $(i = 1, ..., n)$ and $e_j \in E(M)$ $(j = 0, ..., n)$, then*

$$s = s^+ (x_1 \cdots x_n) \ .$$

Proof: Suppose that $n = 0$, then $s = e_0$ and $s = s^+$. Now, let us assume that the result is true for n. Suppose that

$$s = e_0 x_1 \cdots x_n\, e_n x_{n+1} e_{n+1} \ .$$

Then,

$$s = r\, x_{n+1} e_{n+1} \ ,$$

where $r = e_0\, x_1 e_1 \cdots x_n e_n$. Hence, by the induction hypothesis, $r = r^+(x_1 \cdots x_n)$ and so

$$s = r^+(x_1 \cdots x_n) \cdot x_{n+1} e_{n+1} \ .$$

Thus

$$\begin{aligned} s &= r^+(x_1 \cdots x_{n+1} e_{n+1})^+\, x_1 \cdots x_{n+1} \\ &= (r^+\, x_1 \cdots x_{n+1} e_{n+1})^+\, x_1 \cdots x_{n+1} \\ &= s^+\, x_1 \cdots x_{n+1} \ , \end{aligned}$$

as required. ∎

Let M be a left type-A monoid with set of idempotents E. Put $X = M \backslash \{1\}$. We start by considering X^*, the free monoid on X with identity 1. We write the non-identity elements as sequences $(x_1, ..., x_n)$, where $n \geq 1$ and $x_i \in X$ $(i = 1, ..., n)$. To each word $w \in X^*$ we associate a subset M_w of M, in the following way:

$$M_w = \begin{cases} E & \text{if } w = 1, \\ E x_1 E x_2 E \cdots x_{n-1} E x_n E & \text{if } w = (x_1, ..., x_n)\ . \end{cases}$$

It is clear that, for all $v, w \in X^*$, we have

$$M_{vw} = M_v\, M_w \ .$$

Now, define a category $\mathcal{C}^0$ as follows:

$$\operatorname{Obj} \mathcal{C}^0 = X^*$$

and, for all $v, w \in X^*$,

$$\operatorname{Mor}(v, w) = \begin{cases} \{(v, s, w) \colon\ s \in M_{w_1}\} & \text{if } w = v w_1, \text{ for some } w_1 \in X^*, \\ \emptyset, & \text{otherwise}\ . \end{cases}$$

The composition law is given by

$$(v, s, w) + (w, t, u) = (v, st, u) \ .$$

Clearly, the composition is well defined and associative. Also, for any object v,

$$\operatorname{Mor}(v, v) = \Big\{ (v, e, v) \colon\ e \in E \Big\}$$

and $(v, 1_M, v)$ is the identity on $\operatorname{Mor}(v, v)$, where 1_M denotes the identity of M. Thus, $\mathcal{C}^0$ is indeed a category.

Next, we consider a (left) action of the (right) cancellative monoid X^* on the category $\mathcal{C}^0$: the action of X^* on $\operatorname{Obj}\mathcal{C}^0$ is given by the multiplication on X^* and, for all $u \in X^*$ and $(v,s,w) \in \operatorname{Mor}\mathcal{C}^0$,

$$u(v,s,w) = (uv,s,uw) \ .$$

It is easy to verify that this action is well defined.

We choose 1 to be the distinguished object of $\mathcal{C}^0$.

Lemma 3.2. *Let M be a left type-A monoid. Then $\mathcal{C}^0$ is a left proper category with distinguished object 1. Also, the right cancellative monoid X^* acts (on the left) downwards on $\mathcal{C}^0$. The action is 1-closed.*

Proof: Most of the required properties of $\mathcal{C}^0$ and of the action of X^* over $\mathcal{C}^0$ are easy to prove, once we notice that:

– For all $u \in \operatorname{Obj}\mathcal{C}^0$, $\operatorname{Mor}(u,u) = \{(u,e,u)\colon\ e \in E\} \simeq E$;

– For all $(u,s,v) \in \operatorname{Mor}(u,v)$, $(u,s,v)^+ = (u,s^+,u)$;

– The unique unit of X^* is the empty word 1.

Here, we only prove that $\mathcal{C}^0$ is 1-proper. Let $v \in X^*$ and $(1,s,v),(1,t,v) \in \operatorname{Mor}(1,v)$ be such that $(1,s,v)^+ = (1,t,v)^+$. Then, $s^+ = t^+$ and $s,t \in M_v$. If $v = 1$, then $M_v = E$ and we have $s = s^+ = t^+ = t$. Whence $(1,s,v) = (1,t,v)$. If $v \neq 1$, let $v = (x_1,...,x_n)$, where $n > 0$ and $x_i \in X$ $(i = 1,...,n)$. Thus, there exist $e_1,...,e_n,f_1,...,f_n \in E$ such that

$$s = e_1x_1e_2\cdots e_nx_ne_{n+1}$$

and

$$t = f_1x_1f_2\cdots f_nx_nf_{n+1} \ .$$

By Lemma 3.1,

$$s = s^+(x_1\cdots x_n) \quad \text{ and } \quad t = t^+(x_1\cdots x_n) \ .$$

Hence, as $s^+ = t^+$, we have $s = t$. Therefore

$$(1,s,v) = (1,t,v)$$

and $\mathcal{C}^0$ is 1-proper, as required. ∎

Definition 3.3. Let M and N be left type-A monoids we say that N is a ${}^+$*-cover* of M if there exists an idempotent separating monoid morphism θ from N onto M that respects the operation ${}^+$, that is, for all $a \in N$, $a^+\theta = (a\theta)^+$.

Theorem 3.4. *Every left type-A monoid has a proper left type-A $^+$-cover.*

Proof: Suppose that M is a left type-A monoid. Let $\mathcal{C}^0$ be the category defined before. We have

$$C_1 = \Big\{((1,s,u),u)\colon\ u\in X^*,\ \ s\in M_u\Big\}$$

and the multiplication on C_1 is given by

$$((1,s,u),u)\,((1,t,v),v) = ((1,st,uv),uv)\ .$$

The identity of C_1 is $((1,1_M,1),1)$. By Lemmas 3.2 and 2.2, C_1 is a proper left type-A monoid. Now, let us consider the map

$$\begin{aligned}\theta\colon C_1 &\longrightarrow M\\ ((1,s,u),u) &\mapsto s\ .\end{aligned}$$

Clearly, θ is monoid morphism and is, in fact, a $^+$-morphism. Because

$$((1,s,u),u)^+\,\theta = ((1,s^+,1),1)\,\theta = s^+ = (((1,s,u),u)\,\theta)^+\ .$$

That θ is onto follows from the fact that, for all $a\in M\backslash\{1\}=X$,

$$a = ((1,a,(a)),(a))\,\theta\ .$$

Finally, as

$$E(C_1) = \Big\{((1,e,1),1)\colon\ e\in E\Big\}\ ,$$

we have that $\theta|_{E(C_1)}$ is an isomorphism from $E(C_1)$ into E. Therefore, C_1 is a proper left type-A $^+$-cover of M, as required. ∎

REFERENCES

[1] Fountain, J. – E-unitary dense covers of E-dense monoids, *Bull. London Math. Soc.*, 22 (1990), 353–358.

[2] Fountain, J. – A class of right PP monoids, *Quart. J. Math. Oxford,* 28(2) (1977), 285–300.

[3] Fountain, J. and Gomes, G.M.S. – Left proper E-dense monoids, *J. Pure and Applied Algebra*, 80 (1992), 1–27.

[4] Fountain, J. and Gomes, G.M.S. – Proper left type-A monoids revisited, *Glasgow Math. J.*, 35 (1993), 293–306.

[5] Margolis, S.W. and Pin, J.-E. – Inverse semigroups and extensions of groups by semilattices, *J. Algebra*, 110 (1987), 277– 297.

[6] Palmer, A. – *Proper right type-A semigroups*, M. Phil. Thesis, York, 1982.

John Fountain,
Dept. Mathematics, University of York,
Heslington, York, YO15DD – ENGLAND

and

Gracinda M.S. Gomes,
Dep. Matemática, Universidade de Lisboa,
Rua Ernesto de Vasconcelos, C1, 1700 Lisboa – PORTUGAL