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ASYMPTOTICALLY PERIODIC SOLUTIONS FOR A CLASS OF NONLINEAR COUPLED OSCILLATORS

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Abstract: We consider the Hamiltonian system

$$\begin{cases} u'' + u + (u^2+v^2)^\alpha\, u = 0 \;, \\ v'' + k\,v + (u^2+v^2)^\alpha\, v = 0 \;, \end{cases}$$

where k, α are real numbers, $k>1$ and $\alpha>0$. This system is a special case of the nonlinear wave equation

$$u_{tt} - \Delta u + \|u\|_{L^2}^{2\alpha}\, u = 0 \;,$$

when only two Fourier components of the solution are nonzero. We show that for sufficiently large energy, every periodic solution of the above system with $v \equiv 0$ has a nontrivial stable manifold. Thus, we obtain asymptotically periodic, and therefore nonrecurrent, solutions of this nonlinear wave equation. The same result is also true for a wider class of nonlinearities.

Résumé: On considère le système Hamiltonien

$$\begin{cases} u'' + u + (u^2+v^2)^\alpha\, u = 0 \;, \\ v'' + k\,v + (u^2+v^2)^\alpha\, v = 0 \;, \end{cases}$$

où k, α sont réels, $k>1$ et $\alpha>0$. Ce système est un cas particulier de l'équation des ondes non-linéaire

$$u_{tt} - \Delta u + \|u\|_{L^2}^{2\alpha}\, u = 0 \;,$$

lorsque seulement deux composantes de Fourier de la solution sont non nulles. Nous montrons que pour toute valeur assez grande de l'énergie, toute solution périodique du

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système Hamiltonien telle que $v \equiv 0$, possède une variété stable non-triviale. Nous obtenons donc des solutions asymptotiquement périodiques, et en particulier non-récurrentes, de l' équation des ondes ci-dessus. Nous obtenons ce même résultat pour une classe plus vaste de non-linéarités.

1 – Introduction

This paper continues our study, begun in [1, 2, 3, 4], of the asymptotic behavior of solutions to conservative nonlinear wave equations in bounded domains. For the (linear) wave equation $u_{tt} - \Delta u = 0$ on a bounded domain, all solutions are almost periodic. It is natural to wonder to what extent this property persists with the addition of a nonlinear term. A classic result of Rabinowitz [6] says that the equation

$$u_{tt} - u_{xx} + u^3 = 0 \ , \tag{1.1}$$

where $x \in [0, \pi]$ and $u(t, 0) = u(t, \pi) = 0$ for all $t \in \mathbb{R}$, has nontrivial time-periodic solutions. However, the asymptotic behavior of the general solution is not yet well understood.

In [2, 3, 4] we, along with A. Haraux, studied a modified version of (1.1), i.e.

$$u_{tt} - u_{xx} + u\Big(\int_0^\pi u(t, x)^2 \, dx\Big) = 0 \ . \tag{1.2}$$

We chose equation (1.2) in the hope that studying its solutions would provide an indication of what to expect of solutions to (1.1), and that (1.2) would prove more tractible than (1.1). Due to the special structure of the nonlinear term, equation (1.2) can be written as an infinite system of ODE's in the Fourier coefficients of $u(t, x)$. If only the first two components are present, this system becomes

$$\begin{cases} u'' + u + (u^2 + v^2)\, u = 0 \ , \\ v'' + 4v + (u^2 + v^2)\, v = 0 \ . \end{cases} \tag{1.3}$$

Among other results, it is shown in [3, Corollary 5.4] that (1.3) admits non-periodic solutions which are asymptotic to a periodic solution of (1.3) with $v \equiv 0$. In particular, these solutions are nonrecurrent, and therefore not almost periodic.

Essential to the proofs in [3] is the fact that the Hamiltonian system (1.3) admits a second conservation law. This leaves open the question as to whether or not the results themselves are due to the completely integrable character of the system (1.3). In this paper, we prove the existence of such nonrecurrent solutions for an entire class of nonlinear coupled oscillators

$$\begin{cases} u'' + u + u\,f(u^2+v^2) = 0 \;, \\ v'' + k\,v + v\,f(u^2+v^2) = 0 \;, \end{cases} \tag{1.4}$$

where $k > 1$ and the conditions on f will be specified below. As with the system (1.3), these nonrecurrent solutions are asymptotic to a periodic solution of (1.4) with $v \equiv 0$. Since the general system (1.4) does not seem to be completely integrable, we use here arguments different from those of [3].

In the same way that (1.2) and (1.3) are related, the system (1.4) is a special case of the nonlinear wave equation

$$u_{tt} - \Delta u + u\,f\Big(\|u\|_{L^2}^2\Big) = 0 \;, \tag{1.5}$$

when only two Fourier components are present. Thus, we prove the existence of nonrecurrent solutions for an entire class of nonlinear conservative wave equations.

Our results are contained in Theorems 1.1 and 1.2 below. In order to simplify the exposition, we treat the case $f(s) = s^\alpha$ in complete detail and indicate the necessary modifications for the general case.

We therefore consider the system

$$\begin{cases} u'' + u + (u^2+v^2)^\alpha\,u = 0 \;, \\ v'' + k\,v + (u^2+v^2)^\alpha\,v = 0 \;, \end{cases} \tag{1.6}$$

where k, α are real numbers, $k > 1$ and $\alpha > 0$. (1.6) is the Hamiltonian system associated with the Hamiltonian

$$E(u,v,u',v') = \frac{1}{2}\,u'^2 + \frac{1}{2}\,v'^2 + \frac{1}{2}\,u^2 + \frac{k}{2}\,v^2 + \frac{1}{2(\alpha+1)}\,(u^2+v^2)^{\alpha+1} \;. \tag{1.7}$$

Since all the terms in E are nonnegative, it is clear that all the solutions of (1.6) are global and bounded.

Theorem 1.1. *There exists $C > 0$ such that for every $E_0 \ge C$, there exists a two dimensional submanifold $\mathcal{M}$ of the (three dimensional) manifold $\{E(u,v,u',v') = E_0\}$ with the following property. If $(u_0, v_0, u_0', v_0') \in \mathcal{M}$, and if (u,v) is the solution of (1.6) with initial data (u_0, v_0, u_0', v_0'), then v and v' converge exponentially to 0 as $t \to \infty$, and there exists a solution w of the equation $w'' + w + |w|^{2\alpha}\,w = 0$ with energy $\frac{1}{2}\,w'^2 + \frac{1}{2}\,w^2 + \frac{1}{2(\alpha+1)}\,|w|^{2(\alpha+1)} = E_0$ such that $u - w$ and $u' - w'$ converge exponentially to 0 as $t \to \infty$.*

Note that the manifold $\{E(u,v,u',v') = E_0\} \cap \{v = v' = 0\}$ is one dimensional; and so, for most solutions of (1.6) with initial values in $\mathcal{M}$, we have $v \not\equiv 0$.

More generally, let $f\colon [0,\infty) \to \mathbb{R}$ satisfy the following properties.

$$f \in C([0,\infty)) \cap C^1((0,\infty)) \ , \tag{1.8}$$

$$f(0) = 0 \ , \tag{1.9}$$

$$s\, f'(s) \underset{s\downarrow 0}{\longrightarrow} 0 \ , \tag{1.10}$$

$$\inf_{s\geq 0} f(s) > -1 \ , \tag{1.11}$$

$$f(s) \underset{s\to\infty}{\longrightarrow} +\infty \ , \tag{1.12}$$

$$\frac{f(t\,x)}{f(t)} \underset{t\to\infty}{\longrightarrow} |x|^\alpha \ , \tag{1.13}$$

where α is a positive number and (1.13) holds uniformly for x in a bounded set. It follows from (1.8)–(1.10) that the map $U \mapsto f(|U|^2)\, U$ is $C^1(\mathbb{R}^2, \mathbb{R}^2)$ and in particular that the local Cauchy problem for (1.4) is well posed. Furthermore, (1.4) is the Hamiltonian system associated with the Hamiltonian

$$E(u,v,u',v') = \frac{1}{2}\left(u'^2 + v'^2 + u^2 + k\, v^2 + F(u^2+v^2)\right) \ ,$$

where $F(t) = \int_0^t f(s)\, ds$. It follows from (1.12) that $E(u,v,u',v') \to \infty$ as $u^2 + v^2 + u'^2 + v'^2 \to \infty$; and so, all solutions of (1.4) are global and bounded.

Theorem 1.2. *Assume f verifies (1.8)–(1.13). There exists $C > 0$ such that for every $E_0 \geq C$, there exists a two dimensional submanifold $\mathcal{M}$ of the (three dimensional) manifold $\{E(u,v,u',v') = E_0\}$ with the following property. If $(u_0, v_0, u_0', v_0') \in \mathcal{M}$, and if (u,v) is the solution of (1.4) with initial data (u_0, v_0, u_0', v_0'), then v and v' converge exponentially to 0 as $t \to \infty$, and there exists a solution w of the equation $w'' + w + f(w^2)\, w = 0$ with energy $\frac{1}{2}\,(w'^2 + w^2 + F(w^2)) = E_0$ such that $u - w$ and $u' - w'$ converge exponentially to 0 as $t \to \infty$.*

Note that Theorem 1.2 applies in particular to nonlinearities of the form $f(x) = x^p(\log(1+x))^q$, for $p > 0$ and $q \geq 0$. Also, the large energy requirement in Theorems 1.1 and 1.2 can not be eliminated. Indeed, if $\alpha = 1$ then all solutions with $E_0 \leq k(k-1)$ are quasi-periodic (Theorem 2.1 and Lemma 4.1 in [3]).

Furthermore, the conclusion of Theorem 1.1 is valid if we consider solutions with fixed energy and $k > 1$ sufficiently close to 1. The proof is essentially the same, except that the limiting system described in Section 3 is obtained from the linearized system of Section 2 by setting $k = 1$. (See also Remark 3.2.)

To prove Theorem 1.1, we study the Poincaré map associated to a periodic solution with $v \equiv 0$ on a given energy surface. In particular, the Poincaré map is defined on a two dimensional submanifold of this energy surface. (See Section 2.) In fact, symmetry properties of the system allow us to study a simpler map, T, defined using a half-period. (See the discussion just preceding formula (2.8).) As is well known, a periodic solution of (1.4) corresponds to a fixed point of T; and we show explicitly that for sufficiently large energy, the eigenvalues of DT at this fixed point are of the form λ and λ^{-1} with $0 < \lambda < 1$. Standard arguments [5, Chapter 5] then give the existence of the desired solution of (1.4).

The proof of Theorem 1.1 is contained in Sections 2 and 3 below. The proof of Theorem 1.2 is essentially the same as that of Theorem 1.1, except for a few modifications, which we describe in Section 4.

2 – Reduction to the linearized system

Throughout this section, $E_0 > 0$ is fixed. We define the mapping $T\colon \mathcal{U} \to \mathbb{R}^2$, where $\mathcal{U} \subset \mathbb{R}^2$ is a neighborhood of 0, as follows. Let

$$\mathcal{U} = \left\{ (a,b) \in \mathbb{R}^2;\ \frac{b^2}{2} + \frac{k\,a^2}{2} + \frac{a^{2(\alpha+1)}}{2(\alpha+1)} < E_0 \right\} ,$$

and for $(a,b) \in \mathcal{U}$ let $u_0' > 0$ be defined by

$$\frac{u_0'^2}{2} + \frac{b^2}{2} + \frac{k\,a^2}{2} + \frac{a^{2(\alpha+1)}}{2(\alpha+1)} = E_0 \ .$$

Given $(a,b) \in \mathcal{U}$, we consider the solution (u,v) of (1.6) with initial data $(u,v,u',v')(0) = (0,a,u_0',b)$. Since $u(0) = 0$ and $u'(0) > 0$, we have $u(t) > 0$ for $t > 0$ and small. On the other hand, multiplying the equation for u by $\sin \pi t$ and integrating twice by parts on $(0,\pi)$, it follows easily that u must have a zero on $(0,\pi)$. Let τ be the first positive zero of u. Note that τ itself depends on (a,b). We define

$$T(a,b) = -\Big(v(\tau), v'(\tau)\Big) \ , \tag{2.1}$$

for all $(a,b) \in \mathcal{U}$. It is clear that T is of class C^1 and that

$$T(0,0) = (0,0) \ . \tag{2.2}$$

We next define the linear operator $B \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ as follows. Let $w_0' > 0$ be defined by

$$\frac{w_0'^2}{2} = E_0 \ ,$$

i.e. $w_0' = \sqrt{2E_0}$, and let w be the solution of the equation

$$w'' + w + |w|^{2\alpha}\, w = 0 \ ,$$

with initial data $w(0) = 0$ and $w'(0) = w_0'$. Since $w(0) = 0$ and $w'(0) > 0$, we have $w(t) > 0$ for $t > 0$ and small. On the other hand, w must have a zero on $(0, \pi)$ (see above). Let ρ be the first positive zero of w. Given $(a, b) \in \mathbb{R}^2$, let z be the solution of the (linear) equation

$$z'' + k\, z + |w|^{2\alpha}\, z = 0 \ ,$$

with initial data $z(0) = a$ and $z'(0) = b$. We define $B \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ by

$$B(a, b) = -\Big(z(\rho), z'(\rho)\Big) \ , \tag{2.3}$$

for all $(a, b) \in \mathbb{R}^2$. T and B are related as follows.

Proposition 2.1. $DT(0, 0) = B$.

Proof: We know that $DT(0, 0)$ exists. To prove it equals B, it suffices to show that

$$\lim_{\varepsilon \downarrow 0} \frac{T\Big((0, 0) + \varepsilon(a, b)\Big) - T(0, 0)}{\varepsilon} = B(a, b) \ ,$$

for all $(a, b) \in \mathbb{R}^2$. In view of (2.2), we need to show that

$$\lim_{\varepsilon \downarrow 0} \frac{T\Big(\varepsilon(a, b)\Big)}{\varepsilon} = B(a, b) \ , \tag{2.4}$$

for all $(a, b) \in \mathbb{R}^2$. Fix $(a, b) \in \mathbb{R}^2$, and for $\varepsilon > 0$ small enough, let $(u_\varepsilon, v_\varepsilon)$ be the solution of (1.6) with initial data $(u_\varepsilon, v_\varepsilon, u_\varepsilon', v_\varepsilon')(0) = (0, \varepsilon a, c_\varepsilon, \varepsilon b)$, where $c_\varepsilon > 0$ is defined by

$$\frac{c_\varepsilon^2}{2} + \frac{\varepsilon^2\, b^2}{2} + \frac{k\, \varepsilon^2\, a^2}{2} + \frac{\varepsilon^{2(\alpha+1)}\, a^{2(\alpha+1)}}{2(\alpha+1)} = E_0 \ , \tag{2.5}$$

and let τ_ε be the first positive zero of u_ε (see the definition of T). Set $z_\varepsilon(t) = \dfrac{v_\varepsilon(t)}{\varepsilon}$. Then

$$\begin{cases} u_\varepsilon'' + u_\varepsilon + (u_\varepsilon^2 + \varepsilon^2\, z_\varepsilon^2)^\alpha\, u_\varepsilon = 0 \ , \\ z_\varepsilon'' + k\, z_\varepsilon + (u_\varepsilon^2 + \varepsilon^2\, z_\varepsilon^2)^\alpha\, z_\varepsilon = 0 \ , \\ u_\varepsilon(0) = 0, \quad u_\varepsilon'(0) = c_\varepsilon, \quad z_\varepsilon(0) = a, \quad z_\varepsilon'(0) = b \ . \end{cases}$$

It follows from (2.5) that $\lim_{\varepsilon \downarrow 0} c_\varepsilon = \sqrt{2E_0}$. Also,

$$T\Big(\varepsilon(a,b)\Big) = -\Big(v_\varepsilon(\tau_\varepsilon), v'_\varepsilon(\tau_\varepsilon)\Big) = -\varepsilon\Big(z_\varepsilon(\tau_\varepsilon), z'_\varepsilon(\tau_\varepsilon)\Big) \ ;$$

and so,

$$\frac{T(\varepsilon(a,b))}{\varepsilon} = -\Big(z_\varepsilon(\tau_\varepsilon), z'_\varepsilon(\tau_\varepsilon)\Big) \ .$$

By continuous dependence, it is clear that $(u_\varepsilon, z_\varepsilon)$ converges to the solution (w,z) of

$$\begin{cases} w'' + w + |w|^{2\alpha}\, w = 0 \ , \\ z'' + k\, z + |w|^{2\alpha}\, z = 0 \ , \\ w(0) = 0, \quad w'(0) = \sqrt{2E_0}, \quad z(0) = a, \quad z'(0) = b \ , \end{cases}$$

in $C^1([0,T])$ for every $T > 0$. On the other hand, $B(a,b) = -(z(\rho), z'(\rho))$, where ρ is the first positive zero of w. Therefore, to establish (2.4), it suffices to show that $\tau_\varepsilon \to \rho$, as $\varepsilon \downarrow 0$. This, however, is obvious since $w'(0) \neq 0$, $w'(\rho) \neq 0$, $w > 0$ on $(0,\rho)$ and $(u_\varepsilon, z_\varepsilon) \to (w,z)$ in $C^1([0,\rho+1])$. This concludes the proof. ∎

The main result of this section is the following.

Theorem 2.2. *For E_0 large enough, there exist $0 < \delta < 1$ and a nontrivial C^1 parametrized curve $\mathcal{C}$ in the neighborhood of $(0,0)$ which is invariant under the action of T and such that $|T(P)| \le \delta |P|$ for all $P \in \mathcal{C}$.*

The proof of Theorem 2.2 relies on the following result, which will be proved in Section 3.

Theorem 2.3. *If E_0 is sufficiently large, then the eigenvalues of B are of the form λ and λ^{-1} where $0 < \lambda < 1$.*

Proof (assuming Theorem 2.3)**:** We consider coordinates in $\mathbb{R}^2$ such that B is diagonal. Therefore, $B(x,y) = (\lambda x, \lambda^{-1} y)$. Proposition 2.1 implies that $T(x,y) = (\lambda x, \lambda^{-1} y) + F(x,y)$, for all (x,y) in a neighborhood of $(0,0)$, where F is C^1 and $F(0,0) = 0$, $DF(0,0) = 0$. The result now follows from Lemma 5.1, p. 234 of Hartman [5]. ∎

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1: Let $P = (v_0, v'_0) \in \mathcal{C}$, the curve constructed in Theorem 2.2. It follows that

$$|T^n(P)| \le \delta^n |P| \ , \tag{2.6}$$

for all integers $n \geq 1$. Let $u_0' > 0$ be defined by

$$\frac{u_0'^2}{2} + \frac{v_0'^2}{2} + \frac{k\,v_0^2}{2} + \frac{v_0^{2(\alpha+1)}}{2(\alpha+1)} = E_0 \ , \tag{2.7}$$

and let (u, v) be the solution of (1.6) with initial data $(u, v, u', v')(0) = (0, v_0, u_0', v_0')$. We denote by $(\tau_n)_{n\geq 1}$ the sequence of positive zeroes of u. It follows that $T(v_0, v_0') = -(v(\tau_1), v'(\tau_1)) \in \mathcal{C}$, and $u'(\tau_1) < 0$ is given by

$$\frac{u'(\tau_1)^2}{2} + \frac{v'(\tau_1)^2}{2} + \frac{k\,v(\tau_1)^2}{2} + \frac{v(\tau_1)^{2(\alpha+1)}}{2(\alpha+1)} = E_0 \ .$$

If $\widetilde{u}(t) = -u(\tau_1+t)$ and $\widetilde{v}(t) = -v(\tau_1+t)$, then $(\widetilde{u}, \widetilde{v})$ solves (1.6), and $\tau_2-\tau_1$ is the first positive zero of $\widetilde{u}$. It follows easily that $T(-v(\tau_1), -v'(\tau_1)) = (v(\tau_2), v'(\tau_2))$. An obvious iteration argument shows that

$$T^n(v_0, v_0') = (-1)^n \Big(v(\tau_n), v'(\tau_n)\Big) \ , \tag{2.8}$$

and that $u'(\tau_n)$ is given by $(-1)^n\, u'(\tau_n) > 0$ and

$$\frac{u'(\tau_n)^2}{2} + \frac{v'(\tau_n)^2}{2} + \frac{k\,v(\tau_n)^2}{2} + \frac{v(\tau_n)^{2(\alpha+1)}}{2(\alpha+1)} = E_0 \ . \tag{2.9}$$

Furthermore, we have (see the definition of T)

$$\tau_{n+1} - \tau_n \leq \pi \ . \tag{2.10}$$

Since $u^2 + v^2$ is a bounded function of t by (1.7), it follows that there exists a constant C, independent of n, such that

$$v(t)^2 + v'(t)^2 \leq C\Big(v(\tau_n)^2 + v'(\tau_n)^2\Big) \ , \tag{2.11}$$

for all $t \in [\tau_n, \tau_{n+1}]$. Applying now (2.6), (2.8) and (2.11), we get

$$v(t)^2 + v'(t)^2 \leq C\,\delta^n \ , \tag{2.12}$$

for all $t \in [\tau_n, \tau_{n+1}]$. Let $\varepsilon > 0$ be such that $e^{-\varepsilon\pi} = \delta$. It follows from (2.10) that $\tau_n \leq n\pi$; and so, for $t \in [\tau_n, \tau_{n+1}]$ we have

$$e^{-\varepsilon t} \geq e^{-\varepsilon\tau_{n+1}} \geq e^{-\varepsilon(n+1)\pi} = e^{-\varepsilon\pi}\,\delta^n \ .$$

Therefore, it follows from (2.12) that there exists C such that

$$v(t)^2 + v'(t)^2 \leq C\,e^{-\varepsilon t} \ , \tag{2.13}$$

for all $t \geq 0$. This implies, together with (2.9) that there exists a constant C such that

$$\left|(-1)^n u'(\tau_n) - \sqrt{2E_0}\right| \leq C\, e^{-\varepsilon \tau_n} \ . \tag{2.14}$$

Let now w be the solution of the equation

$$w'' + w + |w|^{2\alpha} w = 0 \ , \tag{2.15}$$

with the initial data $w(0) = 0$ and $w'(0) = \sqrt{2E_0}$. It follows easily from (2.13) and (2.14) that there exists C such that

$$\left\|(-1)^n u(\tau_n + \cdot) - w(\cdot)\right\|_{C^1([0,2\pi])} \leq C\, e^{-\varepsilon \tau_n} \ , \tag{2.16}$$

for all $n \geq 1$. If we denote by ρ the first positive zero of w, it follows from (2.16) that there exists C such that $|\tau_{n+1} - \tau_n - \rho| \leq Ce^{-\varepsilon \tau_n}$. Therefore, since $\tau_n - n\rho = \sum_{j=0}^{n-1}(\tau_{j+1} - \tau_j - \rho)$ and $\tau_{n+1} - \tau_n \geq \rho/2$ for n large enough, there exists $\theta \in \mathbb{R}$ such that

$$|\tau_n - n\rho - \theta| \leq C\, e^{-\varepsilon \tau_n} \ . \tag{2.17}$$

Since w is clearly ρ anti-periodic, i.e. $w(n\rho + \cdot) = (-1)^n\, w(\cdot)$, it follows from (2.17) and continuous dependence of the solutions of equation (2.15) on the initial values that

$$\left\|w(\cdot) - (-1)^n\, w(\tau_n - \theta + \cdot)\right\|_{C^1([0,2\pi]} = \left\|w(n\rho + \cdot) - w(\tau_n - \theta + \cdot)\right\|_{C^1([0,2\pi]} \leq C\, e^{-\varepsilon \tau_n} \ .$$

Therefore, by (2.16),

$$\left\|u(\tau_n + \cdot) - w(\tau_n - \theta + \cdot)\right\|_{C^1([0,2\pi])} \leq C\, e^{-\varepsilon \tau_n} \ ,$$

for all $n \geq 1$. This implies that

$$\left\|u(t + \cdot) - w(t - \theta + \cdot)\right\|_{C^1([0,2\pi])} \leq C\, e^{-\varepsilon t} \ ,$$

for all $t \geq 0$. This estimate, together with (2.13) implies that (u, v) verifies the conclusion of the theorem. The theorem now follows by setting

$$\mathcal{M} = \bigcup_{(a,b)\in\mathcal{C}} \bigcup_{t\in\mathbb{R}} \Big(u(t), v(t), u'(t), v'(t)\Big) \ ,$$

where (u, v) is the solution of (1.6) with initial data $(u, v, u', v')(0) = (0, a, u_0', b)$ and $u_0' > 0$ is determined by (2.7). ∎

3 – Analysis of the linearized system

This section is devoted to the proof of Theorem 2.3. For that purpose, we define the mapping $\mathcal{B}\colon S^1 \to S^1$, where $S^1 = \{(a,b) \in \mathbb{R}^2;\, a^2+b^2=1\}$, by

$$\mathcal{B}(a,b) = \frac{B(a,b)}{|B(a,b)|}\ . \tag{3.1}$$

Our goal is to show that $\mathcal{B}$ has a fixed point (a,b), which implies that $|B(a,b)|$ is an eigenvalue of B, and that this fixed point determines a real number $0<\lambda<1$ such that λ and $\dfrac{1}{\lambda}$ are the eigenvalues of B. These results will be established for E_0 large enough. Our methods depend on a limiting argument as $E_0 \to \infty$, which leads us naturally to consider a limiting system. We summarize in the following lemma the properties of this limiting system that we will use later on.

Lemma 3.1. *Let $\widetilde{w}$ be the solution of equation*

$$\widetilde{w}'' + |\widetilde{w}|^{2\alpha}\, \widetilde{w} = 0\ , \tag{3.2}$$

with initial values $\widetilde{w}(0)=0$ and $\widetilde{w}'(0)=1$, and let θ be the first positive zero of $\widetilde{w}$. Let $(\widetilde{z}_0, \widetilde{z}'_0) \in \mathbb{R}^2$ be such that $\widetilde{z}_0^2 + \widetilde{z}_0'^2 = 1$ and $\widetilde{z}_0, \widetilde{z}'_0 \ge 0$, and let $\widetilde{z}$ be the solution of equation

$$\widetilde{z}'' + |\widetilde{w}|^{2\alpha}\, \widetilde{z} = 0\ , \tag{3.3}$$

with initial values $\widetilde{z}(0)=\widetilde{z}_0$ and $\widetilde{z}'(0)=\widetilde{z}'_0$. Then $\widetilde{w}$ and $\widetilde{z}$ verify the following properties.

- **(i)** *Either $\widetilde{z}_0 = 0$ in which case $\widetilde{z}=\widetilde{w}$, or else $\widetilde{z}_0>0$ in which case $\widetilde{z}$ has a unique zero in $(0,\theta)$ and $\widetilde{z}(\theta)<0$.*
- **(ii)** *$\widetilde{z}'(\theta)<0$.*
- **(iii)** *$\displaystyle\int_0^\theta \widetilde{w}\,\widetilde{z} > 0$.*

Proof: It is clear that $\widetilde{w}$ is periodic, increasing on $(0,\theta/2)$ and symmetric about $\theta/2$. Therefore,

$$\widetilde{w}(t) = \widetilde{w}(\theta - t)\ , \tag{3.4}$$

for all $t\in\mathbb{R}$. We first prove property (i). It follows from uniqueness that if $\widetilde{z}_0 = 0$ (hence $\widetilde{z}'_0 = 1$), then $\widetilde{z} \equiv \widetilde{w}$. Suppose now that $\widetilde{z}_0>0$. It follows from (3.2) and (3.3) that $(\widetilde{z}\widetilde{w}' - \widetilde{z}'\widetilde{w})' = 0$; and so,

$$\widetilde{z}\,\widetilde{w}' - \widetilde{z}'\,\widetilde{w} = \widetilde{z}_0 > 0\ . \tag{3.5}$$

In particular, $\widetilde{z}(\theta)\,\widetilde{w}'(\theta)=\widetilde{z}_0$. Since $\widetilde{w}'(\theta)=-\widetilde{w}'(0)=-1$, we have $\widetilde{z}(\theta)=-\widetilde{z}_0<0$. Therefore, $\widetilde{z}$ has at least one zero in $(0,\theta)$. Let $\sigma\in(0,\theta)$ be a zero of $\widetilde{z}$. It follows from (3.5) that $-\widetilde{z}'(\sigma)\,\widetilde{w}(\sigma)=\widetilde{z}_0>0$. Therefore, $\widetilde{z}'(\sigma)<0$ if $\sigma\in(0,\theta)$ is a zero of $\widetilde{z}$. This shows that $\widetilde{z}$ has at most one zero in $(0,\theta)$. Hence (i). The second statement is clear in the case $\widetilde{z}_0=0$, and so we assume again that $\widetilde{z}_0>0$. Let $\varphi=\widetilde{w}'$. Then $\varphi''+(2\alpha+1)\,|\widetilde{w}|^{2\alpha}\,\varphi=0$ on $[0,\theta]$, and

$$(\widetilde{z}\,\varphi'-\widetilde{z}'\,\varphi)'=-2\,\alpha\,|\widetilde{w}|^{2\alpha}\,\widetilde{z}\,\varphi\ , \tag{3.6}$$

on $[0,\theta]$. Let σ be the (unique) zero of $\widetilde{z}$ in $(0,\theta)$. We first show that $\sigma>\theta/2$. Otherwise, $\varphi>0$ on $(0,\sigma)$; and it follows from (3.6) that $(\widetilde{z}\varphi'-\widetilde{z}'\varphi)(\sigma)<(\widetilde{z}\varphi'-\widetilde{z}'\varphi)(0)$, which means that $0\le-\widetilde{z}'(\sigma)\,\varphi(\sigma)<-\widetilde{z}'(0)\,\varphi(0)\le0$, which is absurd. Therefore, $\sigma>\theta/2$; and in particular, $\varphi<0$ on $[\sigma,\theta]$. It now follows from (3.6) that $(\widetilde{z}\varphi'-\widetilde{z}'\varphi)(\theta)<(\widetilde{z}\varphi'-\widetilde{z}'\varphi)(\sigma)$, which means that $\widetilde{z}'(\theta)<-\widetilde{z}'(\sigma)\,\varphi(\sigma)<0$ (since $\varphi(\theta)=-1$ and $\varphi'(\theta)=0$). This proves (ii). Finally, we show (iii). Note that if $\widetilde{z}_0=0$, then $\widetilde{z}\equiv\widetilde{w}$; and so,

$$\int_0^\theta \widetilde{w}\,\widetilde{z}=\int_0^\theta \widetilde{w}^2>0\ .$$

If $\widetilde{z}_0>0$, then (as shown above) $\sigma>\theta/2$, and in particular, $\widetilde{z}(\theta/2)>0$. Set $x(t)=\widetilde{z}(t)+\widetilde{z}(\theta-t)$. We have $x(\theta/2)=2\,\widetilde{z}(\theta/2)$, $x'(\theta/2)=0$, and x solves equation $x''+|\widetilde{w}|^{2\alpha}\,x=0$. By uniqueness, it follows that

$$x\equiv\frac{2\,\widetilde{z}(\theta/2)}{\widetilde{w}(\theta/2)}\,\widetilde{w}\ ;$$

and so, $x>0$ on $(0,\theta)$. Now, by (3.4),

$$\int_0^\theta \widetilde{w}\,\widetilde{z}=\int_0^{\theta/2}\widetilde{w}(\theta-t)\,\widetilde{z}(t)\,dt+\int_{\theta/2}^\theta \widetilde{w}(t)\,\widetilde{z}(t)\,dt=\int_{\theta/2}^\theta \widetilde{w}\,x>0\ ,$$

which shows (iii). ∎

Remark 3.2. Note that the conclusions of Lemma 3.1 holds as well for the equations $\widetilde{w}''+m\,\widetilde{w}+|\widetilde{w}|^{2\alpha}\,\widetilde{w}=0$ and $\widetilde{z}''+m\,\widetilde{z}+|\widetilde{w}|^{2\alpha}\,\widetilde{z}=0$, where m is a nonnegative real number. The proof is the same.

Corollary 3.3. *There exists $C>0$ such that if $E_0\ge C$, then (with the notation introduced in the definition of B in Section 2) for every $(a,b)\in S^1$ such that $a,b\ge0$, w and z verify the following properties.*

(i) *$z'(\rho)<0$.*

(ii) *$z(\rho)<-a$.*

(iii) $\displaystyle\int_0^\rho w\,z > 0.$

Proof: We argue by contradiction, and we assume that there exists a sequence of energies $E_0^n \to \infty$ and a sequence (a_n, b_n) such that the corresponding solutions (w_n, z_n) do not satisfy at least one of the stated conditions. With the above notation, let $\lambda_n = (2E_0^n)^{\frac{\alpha}{2(\alpha+1)}} = (w_n'(0))^{\frac{\alpha}{\alpha+1}}$ and define $C(\lambda_n) = \sqrt{a_n^2 + \lambda_n^{-2}\, b_n^2}$. Set $w_n(t) = \lambda_n^{\frac{1}{\alpha}}\, x_n(\lambda_n\, t)$ and $z_n(t) = C(\lambda_n)\, y_n(\lambda_n\, t)$. It follows that (x_n, y_n) solves the system

$$\begin{cases} x_n'' + \lambda_n^{-2}\, x_n + |x_n|^{2\alpha}\, x_n = 0 \ , \\ y_n'' + k\,\lambda_n^{-2}\, y_n + |x_n|^{2\alpha}\, y_n = 0 \ , \end{cases}$$

with the initial data

$$x_n(0) = 0, \quad x_n'(0) = 1, \quad y_n(0) = \frac{a_n}{C(\lambda_n)}, \quad y_n'(0) = \frac{b_n}{\lambda_n\, C(\lambda_n)} \ ,$$

so that $y_n(0), y_n'(0) \geq 0$ and $y_n(0)^2 + y_n'(0)^2 = 1$. Since $\lambda_n \to \infty$ as $n \to \infty$, it follows from an obvious continuous dependence argument that (x_n, y_n) converges as $n \to \infty$ to a solution of (3.2)–(3.3) as considered in Lemma 3.1. Therefore, it follows from Property (ii), respectively Property (iii), of Lemma 3.1 that Property (i), respectively Property (iii), holds for E_0 large enough. It now remains to show Property (ii) for n large. We have $(z_n w_n' - z_n' w_n)' = (k-1)\, w_n z_n$. Integrating this identity on $(0, \rho_n)$ and applying Property (iii), we obtain

$$(z_n w_n' - z_n' w_n)(\rho_n) > (z_n w_n' - z_n' w_n)(0) \ .$$

Since $w_n(\rho_n) = w_n(0) = 0$ and $w_n'(\rho_n) = -w_n'(0) = -\sqrt{2E_0^n}$, this implies that

$$z_n(\rho_n) < -a_n \ .$$

Therefore, (w_n, z_n) do indeed verify Properties (i), (ii) and (iii) for all sufficiently large n, which completes the argument by contradiction. ∎

Remark 3.4. In fact, by applying Property (i) of Lemma 3.1, one can show that, after possibly choosing a larger C, z has exactly one zero in $(0, \rho)$, but we do not need this fact here.

Proof of Theorem 2.3: Consider the mapping $\mathcal{B}\colon S^1 \to S^1$ defined by (3.1). If $\theta \in [0, \pi/2]$, then $e^{i\theta} \in \{(a, b) \in S^1;\ a, b \geq 0\}$; and so, it follows from Property (ii) of Corollary 3.3 that $\mathcal{B}(e^{i\theta}) \in \{(a, b) \in S^1;\ a > 0\}$. Therefore, for every $\theta \in [0, \pi/2]$ there exists a unique $\varphi \in (-\pi/2, \pi/2)$ such that $\mathcal{B}(e^{i\theta}) = e^{i\varphi}$. In particular, it follows from Property (i) of Corollary 3.3 that $\varphi > 0$ if $\theta = 0$. If we denote by f the mapping $\theta \mapsto \varphi$, then $f\colon [0,\pi/2] \to (-\pi/2,\pi/2)$ and $f(0) > 0$. Setting $g(\theta) = f(\theta) - \theta$, we see that $g(0) > 0$ and $g(\pi/2) = f(\pi/2) - \pi/2 < 0$. Since g is clearly continuous, there exists $\theta \in (0,\pi/2)$ such that $g(\theta) = 0$; and so, $\mathcal{B}(e^{i\theta}) = e^{i\theta}$. In other words, $Be^{i\theta} = |Be^{i\theta}|\, e^{i\theta}$ for some $\theta \in (0,\pi/2)$. Note that by Property (ii) of Corollary 3.3, it follows that $z(\rho) < -z(0)$; and so,

$$|Be^{i\theta}| = \frac{-z(\rho)}{z(0)} > 1 \ .$$

Set $\lambda = |Be^{i\theta}|^{-1} \in (0,1)$. We have $Be^{i\theta} = \lambda^{-1}\, e^{i\theta}$; and so, λ^{-1} is an eigenvalue of B. We claim that $Be^{-i\theta} = \lambda\, e^{-i\theta}$. Indeed, if $(a,b) = e^{i\theta}$, then $(z(\rho), z'(\rho)) = -\lambda^{-1}\, e^{i\theta}$. Now let $\widehat{z}(t) = -\lambda\, z(\rho - t)$. Since w is clearly symmetric about $\rho/2$, it follows that $\widehat{z}$ solves the equation $\widehat{z}'' + k\,\widehat{z} + |w|^{2\alpha}\,\widehat{z} = 0$. Moreover, we have $\widehat{z}(0) = \cos\theta$, $\widehat{z}'(0) = -\sin\theta$, $\widehat{z}(\rho) = -\lambda\cos\theta$ and $\widehat{z}'(\rho) = \lambda\sin\theta$; and so, $B(\cos\theta, -\sin\theta) = \lambda(\cos\theta, -\sin\theta)$, which proves the claim. This shows that B has the eigenvalue $\lambda \in (0,1)$, and completes the proof. ∎

Remark 3.5. It is perhaps interesting to note that while Theorem 2.3 is proved by passing to a limiting set of equations (i.e. (3.2) and (3.3)), these limiting equations do not satisfy the conclusions of Theorem 2.3. More precisely, one can define a limiting linear operator $\widetilde{B}$ analogous to B. By the argument in the proof of Corollary 3.3, it is clear that $\widetilde{B} \to B$ as $E_0 \to \infty$, and so $\det(\widetilde{B}) = \lim \det(B) = 1$. Moreover, since $\widetilde{w}$ is itself a solution of (3.3), one of the eigenvalues of $\widetilde{B}$ is equal to one; and therefore they both are.

Proof of Theorem 1.2: In this section, we describe how to adapt the proof of Theorem 1.1 to prove Theorem 1.2. Most modifications are quite obvious, except perhaps for the analogue of Corollary 3.3. The linearized system is

$$\begin{cases} w'' + w + f(w^2)\, w = 0 \ , \\ z'' + k\, z + f(w^2)\, z = 0 \ , \end{cases}$$

with the initial conditions $w(0) = 0$, $w'(0) = \sqrt{2E_0}$, $z(0) = a$ and $z'(0) = b$, where $a,b \geq 0$ and $a^2 + b^2 = 1$. Let $g\colon [0,\infty) \to \mathbb{R}$ be a positive, continuous function such that

$$g(t) = \sqrt{f(t^2)} \ , \tag{4.1}$$

for t large. It follows from assumption (1.12) that such a function exists. It now follows from assumption (1.13) and from (4.1) that

$$\frac{f(\lambda^2 x^2)}{g(\lambda)^2} = \frac{f(\lambda^2 x^2)}{f(\lambda^2)}\, \frac{f(\lambda^2)}{g(\lambda)^2} \underset{\lambda\to\infty}{\longrightarrow} |x|^{2\alpha} \ , \tag{4.2}$$

uniformly on bounded sets of $\mathbb{R}$. For every $E_0 > 0$, set $\lambda = \sup\{t > 0;\ tg(t) = \sqrt{2E_0}\}$. It follows from assumption (1.12) that λ is well defined, that

$$\lambda\, g(\lambda) = \sqrt{2E_0} = w'(0)\ , \tag{4.3}$$

and that

$$\lambda \underset{E_0\to\infty}{\longrightarrow} \infty\,, \qquad g(\lambda) \underset{E_0\to\infty}{\longrightarrow} \infty\ . \tag{4.4}$$

Finally, let

$$C(\lambda) = \sqrt{a^2 + g(\lambda)^{-2}\, b^2}\ , \tag{4.5}$$

and define x and y by $w(t) = \lambda\, x(g(\lambda)t)$ and $z(t) = C(\lambda)\, y(g(\lambda)t)$. It follows that (x, y) solves the system

$$\begin{cases} x'' + \dfrac{1}{g(\lambda)^2}\, x + \dfrac{f(\lambda^2\, x^2)}{g(\lambda)^2}\, x = 0\ , \\[2ex] y'' + \dfrac{k}{g(\lambda)^2}\, y + \dfrac{f(\lambda^2\, x^2)}{g(\lambda)^2}\, y = 0\ , \end{cases}$$

with the initial conditions $x(0) = 0$, $x'(0) = 1$ (by (4.3)), $y(0) = a\, C(\lambda)^{-1}$ and $y'(0) = b\, C(\lambda)^{-1}\, g(\lambda)^{-1}$. In particular, it follows from (4.5) that $y(0)^2 + y'(0)^2 = 1$. One concludes with the argument of the proof of Corollary 3.3, by applying (4.2) and (4.4). ∎

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