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ON THE ARITHMETICAL FUNCTIONS $d_k(n)$ AND $d_k^*(n)$

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1 – Introduction

1. Let $\varphi(n)$, $\sigma(n)$, $d(n)$, $\omega(n)$, $\Omega(n)$ denote as usual the Euler totient, the sum of divisors of n, the number of divisors of n, the number of distinct prime factors of n, and the total number of prime factors of n, respectively. We note that by convention $\varphi(1) = \sigma(1) = d(1) = 1$, $\omega(1) = \Omega(1) = 0$. Let $e(n) = 1$, $I_k(n) = n^k$, $I(n) = I_1(n)$ ($n = 1, 2, ..., k \geq 0$), and μ denote the Möbius function. In terms of Dirichlet convolution, denoted by $\cdot$, we have ([1], [6], [7])

$$\varphi(n) = (I \cdot \mu)(n)\,, \quad \sigma(n) = (I \cdot e)(n)\,, \quad d(n) = (e \cdot e)(n)\;, \tag{1}$$

Similarly, for the Jordan's generalization $\varphi_k(n)$, of $\varphi(n)$; and for the sum $\sigma_k(n)$, of k-th powers of divisors of n, we have

$$\varphi_k(n) = (I_k \cdot \mu)(n)\,, \quad \sigma_k(n) = (I_k \cdot e)(n)\;. \tag{2}$$

Clearly

$$\varphi_1 \equiv \varphi\,, \quad \sigma_1 \equiv \sigma\,, \quad \sigma_0 \equiv d\;.$$

Let $d_k(n)$ denote the Piltz divisor function counting the number of distinct solutions of the equation $x_1\,x_2 \cdots x_k = n$ (where $x_1, x_2, ..., x_k$ run independently through the set of positive integers). Then $d_2 \equiv d$ and $d_1 \equiv e$. It is easy to see that ([9], [11])

$$d_k(n) = (d_{k-1} \cdot e)(n)\,, \quad k \geq 2\;. \tag{3}$$

The arithmetical functions φ, σ, d, φ_k, σ_k, d_k are all multiplicative, while ω and Ω are additive. For many properties of these classical functions, see e.g. [1], [2], [6], [9].

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2. A divisor i of n is called unitary, if $(i, \frac{n}{i}) = 1$. The unitary convolution of the arithmetical functions f and g is defined by ([3])

$$(f \oplus g)(n) = \sum_{i \| n} f(i)\, g\left(\frac{n}{i}\right) , \tag{4}$$

where $i \| n$ means that i runs through the unitary divisors of n.

The unitary analogue μ^*, of μ, is given by ([3], [4])

$$\mu^*(n) = (-1)^{\omega(n)} \tag{5}$$

and the unitary analogue of φ_k is given by

$$\varphi_k^*(n) = (I_k \oplus \mu^*)(n) . \tag{6}$$

The unitary analogues of d and σ_k are d^* and σ_k^*, counting the number, and the sum of powers, of unitary divisors of n, respectively. We have ([4], [8]):

$$d^*(n) = (e \oplus e)(n) = 2^{\omega(n)} , \tag{7}$$

$$\sigma_k^*(n) = (I_k \oplus e)(n) . \tag{8}$$

For more properties of σ_k^*, see [8]. It is known that the unitary convolution of multiplicative functions is also multiplicative, so the functions φ_k^*, d^*, σ_k^* are all multiplicative, too.

Given a prime p and a positive integer $m \geq 1$, the following formulae are valid:

$$\varphi_k(p^m) = p^{km}\left(1-\frac{1}{p}\right), \quad \sigma_k(p^m) = \frac{p^{k(m+1)}-1}{p^k-1} \quad (k \geq 1), \quad d(p^m) = m+1 , \tag{9}$$

and

$$\varphi_k^*(p^m) = p^{km}-1, \quad \sigma_k^*(p^m) = p^{km}+1 \quad (k \geq 1), \quad d^*(p^m) = 2 . \tag{10}$$

The arithmetical function d_k $(k \geq 2)$ is also multiplicative, and

$$d_k(p^m) = \binom{k+m-1}{m} , \tag{11}$$

where $\binom{a}{b} = C_a^b$ denotes a binomial coefficient. For a review of properties of d_k, see e.g. [11]. For more theorems, see [6].

The aim of this note is to introduce and study certain properties of an unitary analogue of the function d_k, as well as to prove new relations for the above mentioned arithmetical functions.

II – The function d_k^* and normal, maximal and average orders

1. The unitary analogue d_k^*, of d_k, will be defined recurrently by

$$d_2^*(n) = d^*(n)\,, \qquad d_k^*(n) = (d_{k-1}^* \oplus e)(n), \quad k \ge 2\ . \tag{12}$$

Then, by induction on k, it follows that d_k^* is multiplicative for all $k \ge 2$ and, for a prime power, one has

$$d_k^*(p^m) = k\ . \tag{13}$$

This is a consequence of (10) and (12). Then a similar formula, as in (7), holds for $d_k^*(n)$:

$$d_k^*(n) = k^{\omega(n)}\ . \tag{14}$$

2. The formula above, attending to a well known theorem of Hardy and Ramanujan on the normal order of magnitude of the function $\omega(n)$ (see [1], [6], [2], [7], [9]), immediately gives:

The normal order of magnitude of $\log d_k^*(n)$ is

$$\log k \cdot \log\log n\ . \tag{15}$$

Indeed, let $\varepsilon > 0$. Then for almost all n one has $(1-\varepsilon)\log\log n < \omega(n) < (1+\varepsilon)\log\log n$, giving, by (14), $(1-\varepsilon)\log k \cdot \log\log n < \log d_k^*(n) < (1+\varepsilon)\log k \cdot \log\log n$, yielding (15).

3. For the maximal order of magnitude of $\log d_k^*(n)$, one can write:

$$\limsup_{n\to\infty} \frac{\log d_k^*(n)\cdot \log\log n}{\log n} = \log k\ . \tag{16}$$

This is a simple consequence of (14) and the known result

$$\limsup_{n\to\infty} \frac{\omega(n)\log\log n}{\log n} = 1 \qquad \text{(see e.g. [6])}\ .$$

4. We note that $d^*(n)$ counts also the number of squarefree divisors of n, so for the average order of $d^*(n)$ the first result was obtained by Mertens in 1874 (see [7]):

$$\sum_{n\le x} d^*(n) = Ax\log x + Bx + O(x^{1/2}\log x)\ , \tag{17}$$

where A, B are certain explicit constants. This has been rediscovered in [4]. The O-term in (17) can be much improved, for example to $O(x^{1/2})$ (see [5]).

In order to obtain the average order of $d_k^*(n)$, we can apply a result of Selberg ([16]):

$$\sum_{n\le x} z^{\omega(n)} = zF(z)\,x(\log x)^{z-1} + O\Big(x(\log x)^{\mathrm{Re}(z-2)}\Big)\ , \tag{18}$$

where $z\in\mathbb{C}$, and the O-constant is uniform for $|z|\le R$ (>0, given), and

$$F(z) = \frac{1}{\Gamma(z+1)}\cdot\prod_p\Big(1+\frac{z}{p-1}\Big)\cdot\Big(1-\frac{1}{p}\Big)^z\ .$$

For $z=k$, a fixed positive integer, one obtains

$$\sum_{n\le x} d_k^*(n) = Ax(\log x)^{k-1} + O\Big(x(\log x)^{k-2}\Big)\ , \tag{19}$$

where A is a positive constant (depending only on k, but the O-constant is not uniform for all k).

III – Inequalities

1. In the paper [11], the following inequalities are proved:

$$k^{\omega(n)} \le \prod_{i=1}^r\Big(1+\frac{k-1}{a_i}\Big)^{a_i} \le d_k(n) \le k^{\Omega(n)}\ , \tag{20}$$

where $k\ge 2$ and $n=\prod_{i=1}^r p_i^{a_i}$ (p_i primes) is the canonical representation of $n\ge 2$. In view of (14), this means that

$$d_k^*(n) \le \prod_{i=1}^r\Big(1+\frac{k-1}{a_i}\Big)^{a_i} \le d_k(n) \le (d_k^*(n))^{\Omega(n)/\omega(n)}\ , \tag{21}$$

with equality only for squarefree n (i.e. $a_i=1$ for all i).

In a recent note [14], as an application of an inequality of Klamkin, the following has been proved:

$$\frac{\varphi_{k+1}^*(n)}{\varphi^*(n)} \le \Big(\frac{k+1}{2}\Big)^{\omega(n)}\cdot\sigma_k^*(n)\ . \tag{22}$$

By using the function d_k^*, this can be rewritten as

$$\frac{\varphi_{k+1}^*(n)}{\varphi^*(n)} \le \frac{d_{k+1}^*(n)}{d^*(n)} \cdot \sigma_k^*(n) \; . \tag{23}$$

2. Since it is known that

$$\frac{\sigma_k(n)}{\sigma_k^*(n)} \le \frac{d(n)}{d^*(n)} = \frac{d_2(n)}{d_2^*(n)} \tag{24}$$

([12], [14]), and that

$$\frac{\sigma_k(n)}{\sigma_k^*(n)} > \frac{\sigma_{k+1}(n)}{\sigma_{k+1}^*(n)} \qquad (k \ge 1, \quad n \ge 2) \; , \tag{25}$$

it is natural (see [15]) the problem of monotony of the sequence (d_k/d_k^*). One has:

$$\frac{d_k(n)}{d_k^*(n)} \le \frac{d_{k+1}(n)}{d_{k+1}^*(n)} \qquad (k \ge 2, \quad n \ge 2) \; , \tag{26}$$

with equality only for squarefree n.

Since the involved functions are multiplicative, it is sufficient to prove (26) for prime powers $n = p^m$. Using (11) and (13), (26) becomes

$$\frac{\binom{k+m-1}{m}}{k} \le \frac{\binom{k+m}{m}}{k+1} \; . \tag{27}$$

By the known relation $\binom{n}{m} = \frac{n}{n-m} \cdot \binom{n-1}{m}$, a simple calculus transforms (27) into $k + m \ge k + 1$, which is true, with equality only for $m = 1$.

As a corollary of (24) and (26), we note that

$$\frac{\sigma_k(n)}{\sigma_k^*(n)} \le \frac{d_r(n)}{d_r^*(n)} \qquad \text{for all} \;\; k, r \ge 2 \; . \tag{28}$$

3. Since for $m|n$ (m divides n) we have $\omega(n) \le \omega(n)$, clearly

$$m \,|\, n \;\; \Rightarrow \;\; d_k^*(m) \le d_k^*(n) \; . \tag{29}$$

On the other hand, by (12) and (29),

$$d_k^*(n)=\sum_{i\|n} d_{k-1}^*(i)\le d_{k-1}^*(n)\cdot\sum_{i\|n}1\ ,$$

so

$$d_k^*(n)\le d_{k-1}^*(n)\,d^*(n)\qquad (k\ge 2)\ . \tag{30}$$

By successive applications of (30), one can deduce

$$d_k^*(n)\le (d^*(n))^{k-1}\ . \tag{31}$$

For $k=2$, one has equality for all n.

In fact, it is true that

$$(d_{k+1}^*(n))^{1/k}<(d_k^*(n))^{1/(k-1)}\qquad (k\ge 2)\ . \tag{32}$$

This follows from (14) and from the inequality $(k+1)^{k-1}<k^k$, or written equivalently, $(1+\frac{1}{k})^k<k+1$. Since $(1+\frac{1}{k})^k<e<k+1$ for $k\ge 2$, this holds true, proving (32).

In the same manner, by applying the inequality $(1+\frac{1}{k})^{k+2/5}<e<k+1$ for $k\ge 2$ (see e.g. [10]), one obtains:

$$(d_{k+1}^*(n))^{k-3/5}<(d_k^*(n))^{k+2/5}\qquad (k\ge 2)\ . \tag{33}$$

For example, for $k=2$ this means that

$$(d_3^*(n))^7<(d_2^*(n))^{12}=(d^*(n))^{12}\ . \tag{34}$$

4. A connection among d_m^*, φ_k^*, σ_k^* is given by

$$(d_m^*(n))^2\,\varphi_k^*(n)>\sigma_k^*(n)\qquad (m,n\ge 2;\quad k\ge 1)\ . \tag{35}$$

Indeed, for prime powers $n=p^a$, we have $m^2(p^{ka}-1)\ge 4(p^{ka}-1)>p^{ka}+1$ since $3p^{ka}\ge 6>5$. Inequality (35) follows by the multiplicativity of the involved functions.

By $(p^{ka}-1)\,m\ge 2(p^{ka}-1)\ge p^{ka}$, with equality for $p=2$, $k=a=1$, we get:

$$\varphi_k^*(n)\,d_m^*(n)\ge n^k\qquad (m\ge 2,\quad k\ge 1)\ , \tag{36}$$

with equality for $k=1$, $m=2$, $n=2$.

Now we prove that

$$\varphi_k^*(n)\,(d_m^*(n))^\lambda \le n^{2k} \quad (m \ge 2,\ \ k \ge 1)\ , \tag{37}$$

where $0 < \lambda \le \log_m 4$.

Let $n = p^a$. Then (37) becomes

$$x^2 - m^\lambda \cdot x + m^\lambda \ge 0\ , \tag{38}$$

where $x = p^{ka}$. The discriminant of this trinomial is $\Delta = m^\lambda \cdot (m^\lambda - 4) \le 0$ for $m^\lambda \le 4$, i.e. $\lambda \le \log_m^4$.

Certain particular cases are of interest to be noted: For $m = 4$, $\lambda = 1$ we have

$$\varphi_k^*(n)\,d_4^*(n) \le n^{2k}\ . \tag{39}$$

For $m = 2$, $\lambda = 2$, we get

$$\varphi_k^*(n)(d^*(n))^2 \le n^{2k}\ , \tag{40}$$

which has been considered also in [13].

For $m = 5$, $\lambda = \log_5 4$ we have

$$\varphi_k^*(n)(d_5^*(n))^{\log_5 4} \le n^{2k}\ . \tag{41}$$

Finally, we prove:

$$d_k(n)\,\varphi_m(n) \ge \frac{d_k^*(n)}{d^*(n)} \cdot n^m \quad (m \ge 1;\ k, n \ge 2)\ . \tag{42}$$

By (20), it is sufficient to show that

$$p^{ma}\left(1 - \frac{1}{p}\right) \ge \frac{p^{ma}}{2} \tag{43}$$

(where $n = p^a$). For $n \ge 2$, one has $1 - \frac{1}{p} \ge \frac{1}{2}$, so relation (43) is trivial. This finishes the proof of (42), since the considered functions are multiplicative.

5. Finally, we study the submultiplicative property of d_k and d_k^*. By $\omega(ab) \le \omega(a) + \omega(b)$ for $a, b \ge 1$, we have

$$d_k^*(ab) \le d_k^*(a)\,d_k^*(b) \quad (k \ge 2;\ \ a, b \ge 1)\ , \tag{44}$$

where equality occurs only for $(a, b) = 1$.

The submultiplicativity of d_k is more difficult to prove. Let $a = \prod p^r \cdot \prod q^s$, $b = \prod p^m \cdot \prod t^h$ be the prime factorizations of a and b, where $(p,q) = (p,t) = (q,t) = 1$ (we do not use indices for simplicity). Using (11), the inequality

$$d_k(ab) \le d_k(a)\, d_k(b) \tag{45}$$

becomes (after certain elementary computations)

$$r!\, m!\, (k-1)!\, (k+r+m-1)! \le (r+m)!\, (k+r-1)!\, (k+m-1)! \tag{46}$$

Let $k-1 = u$. Then, using the definition of a factorial, (46) is transformed into

$$(1\cdot 2\cdots u)\cdot(r+m+1)\cdots(r+m+u) \le (r+1)\cdots(r+u)\cdot(m+1)\cdots(m+u)\ . \tag{47}$$

remarking that $k\,(r+m+k) \le (r+k)\,(m+k)$ and writing $k = 1, 2, ..., u$, after term-by-term multiplication we get (47). Equality occurs in (45), when all $r = m = 0$, i.e., when a and b are coprime.

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