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CONVERGENCE OF APPROXIMATION PROCESSES ON CONVEX CONES

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Abstract: The purpose of this paper is to establish convergence results for sequences of convex conic operators on $C(X;\mathcal{C})$ which are regular, i.e., sequences $\{T_n\}_{n\geq 1}$ such that for some positive linear operator S_n on $C(X;\mathbb{R})$ we have $T_n(g\otimes K)=S_n(g)\otimes K$, for every continuous real valued function g and every element K of the convex cone $\mathcal{C}$.

1 – Introduction

We start by reviewing some of the properties of convex cones.

Definition 1. An (abstract) *convex cone* is a non-empty set $\mathcal{C}$ such that to every pair of elements, K and L, of $\mathcal{C}$, there corresponds an element $K+L$, called the *sum* of K and L, in such a way that addition is commutative and associative, and there exists in $\mathcal{C}$ a unique element 0, called the *vertex* of $\mathcal{C}$, such that $K+0=K$, for every $K\in\mathcal{C}$. Moreover, to every pair, λ and K, where $\lambda\geq 0$ is a non-negative real number and $K\in\mathcal{C}$, there corresponds an element λK, called the *product* of λ and K, in such a way that multiplication is associative: $\lambda(\mu K)=(\lambda\mu)\,K$; $1.K=K$ and $0.K=0$ for every $K\in\mathcal{C}$; and the distributive laws are verified: $\lambda(K+L)=\lambda K+\lambda L$, $(\lambda+\mu)\,K=\lambda K+\mu K$, for every $K,L\in\mathcal{C}$ and $\lambda\geq 0$, $\mu\geq 0$.

Definition 2. Let $\mathcal{C}$ be an (abstract) convex cone and let d be a metric on $\mathcal{C}$. We say that the pair $(\mathcal{C},d)$ is a *metric convex cone* if the following properties are valid:

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a) $d\Big(\sum_{i=1}^{m} K_i, \sum_{i=1}^{m} L_i\Big) \le \sum_{i=1}^{m} d(K_i, L_i)$,

b) $d(\lambda K, \lambda L) = \lambda d(K, L)$,

for every K_i, L_i $(i = 1, ..., m)$, K, L in $\mathcal{C}$ and every $\lambda \ge 0$.

Let $(\mathcal{C}, d)$ be a metric convex cone. Then:

c) $d(\lambda K, \mu L) \le |\lambda - \mu|\, d(K, 0) + \mu\, d(K, L)$,

for every K and L in $\mathcal{C}$ and every $\lambda \ge 0$ and $\mu \ge 0$.

Definition 3. A non-empty subset $\mathcal{K}$ of an (abstract) convex cone $\mathcal{C}$ is called a *convex subcone* if $K, L \in \mathcal{K}$ and $\lambda \ge 0$ imply $K + L \in \mathcal{K}$ and $\lambda K \in \mathcal{K}$. When equipped with the induced operations, a convex subcone $\mathcal{K} \subset \mathcal{C}$ becomes a convex cone.

Example 1: If E is a vector space over the reals then the set $\mathcal{C} = \mathrm{Conv}(E)$ of all convex non-empty subsets of E is a convex cone with the operations defined by: if $K, L \in \mathrm{Conv}(E)$ and $\lambda \ge 0$

$$K + L = \{u + v;\ u \in K,\ v \in L\}\ ,$$

$$\lambda K = \{\lambda u;\ u \in K\}\ ,$$

$$0 = \{\theta\}, \quad \text{where } \theta \text{ is the origin of } E\ .$$

When E is a normed vector space, the set $\mathcal{K}$ consisting of those elements of $\mathrm{Conv}(E)$ that are bounded sets is a convex subcone of $\mathrm{Conv}(E)$.

Definition 4. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two convex cones. An operator $T\colon \mathcal{C}_1 \to \mathcal{C}_2$ is called a *convex conic operator*, if

$$T(F + G) = T(F) + T(G)$$

$$T(\lambda.F)\lambda.T(F)$$

for every pair $F, G \in \mathcal{C}_1$ and every $\lambda \ge 0$.

2 – Spaces of continuous functions

Let X be a compact Hausdorff space. Let $(\mathcal{C}, d)$ be a metric convex cone. We denote by $C(X; \mathcal{C})$ the convex cone consisting of all continuous mappings $F\colon X \to \mathcal{C}$. In $C(X; \mathcal{C})$ we consider the topology of uniform convergence over X, determined by the metric defined by

$$d(F,G) = \sup\Big\{d(F(x),G(x));\ x \in X\Big\}$$

for every pair F, G of elements of $C(X;\mathcal{C})$. Hence $F_n \to F$ in $C(X;\mathcal{C})$ if, and only if, $d(F_n, F) \to 0$.

When $(\mathcal{C}, d)$ is $\mathbb{R}$ equipped with the usual distance $d(x,y) = |x-y|$, then $C(X,\mathcal{C})$ is the classical Banach space $C(X)$ of all continuous real-valued functions $f\colon X \to \mathbb{R}$, equipped with the sup-norm $\|f\| = \sup\{|f(x)|;\ x \in X\}$.

Assume that $(X, \widetilde{d})$ is a metric compact space. We say that $F\colon X \to \mathcal{C}$ is a *Lipschitz function* if there exists a positive constant M_F such that

$$d(F(x),F(y)) \le M_F\, \widetilde{d}(x,y)$$

for all $x, y \in X$. The subset of $C(X;\mathcal{C})$ of such functions is denoted by $\mathrm{Lip}(X;\mathcal{C})$. When $(\mathcal{C}, d)$ is $\mathbb{R}$ equipped with usual distance $d(x,y) = |x-y|$ we denote $\mathrm{Lip}(X;\mathbb{R}) = \mathrm{Lip}(X)$ and $\mathrm{Lip}^+(X) = \{f \in \mathrm{Lip}(X);\ f \ge 0\}$. Notice that $\mathrm{Lip}(X;\mathcal{C})$ is a convex subcone of $C(X;\mathcal{C})$.

For each $K \in \mathcal{C}$, we denote by K^* the element of $C(X;\mathcal{C})$ defined by $K^*(t) = K$, for all $t \in X$.

For each $f \in C^+(X)$ and $K \in \mathcal{C}$ we denote by $f \otimes K$ the function of $C(X;\mathcal{C})$ defined by $(f \otimes K)(x) = f(x).K$, for all $x \in X$. The convex subcone of $C(X;\mathcal{C})$ generated by the functions $f \otimes K$, where $f \in \mathrm{Lip}^+(X)$ and $K \in \mathcal{C}$, is denoted by $\mathrm{Lip}^+(X) \otimes \mathcal{C}$.

Definition 5. Let $\mathcal{K}$ be a convex subcone of a convex cone $\mathcal{C}$. Let $T : C(X;\mathcal{C}) \to C(X;\mathcal{C})$ be a convex conic operator. We say that T is *regular over* $\mathcal{K}$ if there exists a linear operator $\widehat{T}\colon C(X;\mathbb{R}) \to C(X;\mathbb{R})$ such that

$$T(f \otimes K) = \widehat{T}(f) \otimes K$$

for all $f \in C^+(X)$ and $K \in \mathcal{K}$.

When $\mathcal{K} = \mathcal{C}$ and T is regular over $\mathcal{K}$, we say simply that T is *regular*.

Definition 6. Let $T\colon C(X;\mathcal{C}) \to C(X;\mathcal{C})$ be a convex conic operator. We say that T is *monotonically regular* if there exists a monotone linear operator $\widehat{T}\colon C(X;\mathbb{R}) \to C(X;\mathbb{R})$ such that

$$T(f \otimes K) = \widehat{T}(f) \otimes K$$

for all $f \in C^+(X)$ and $K \in \mathcal{C}$.

We recall that an operator S on $C(X;\mathbb{R})$ is called *monotone* if $S(f) \le S(g)$, whenever $f \le g$. For linear operators, to be monotone is equivalent to be positive, i.e., $S(f) \ge 0$, for all $f \ge 0$.

Remark 1. Notice that if T is regular and $\widehat{T}$ preserves the constant functions, i.e., $\widehat{T}(e_0) = e_0$, where e_0 denotes the real function $e_0(t) = 1$, for all $t \in X$, then T also preserves the constant functions, since $T(K^*) = T(e_0 \otimes K) = \widehat{T}(e_0) \otimes K = e_0 \otimes K = K^*$, for every $K \in \mathcal{C}$.

Definition 7. Let T be a regular operator on the convex cone $C(X;\mathcal{C})$. Define

$$\alpha(x) = \Big(\widehat{T}(\widetilde{d}_x), x\Big)$$

for all $x \in X$, where $\widetilde{d}_x$ is defined by $\widetilde{d}_x(y) = d(x,y)$, for all $y \in X$.

Lemma 1. *Let $(X,\widetilde{d})$ be a metric compact space and $(\mathcal{C},d)$ be a metric convex cone. Then:*

a) *If $F \in \mathrm{Lip}^+(X) \otimes \mathcal{C}$, then $F \in \mathrm{Lip}(X;\mathcal{C})$.*

b) *If $g \in \mathrm{Lip}^+(X)$ and $F \in \mathrm{Lip}^+(X) \otimes \mathcal{C}$, then the function $x \mapsto g(x)\,F(x)$, $x \in X$, belongs to $\mathrm{Lip}^+(X) \otimes \mathcal{C}$.*

Proof: **a**) Let $F \in \mathrm{Lip}^+(X) \otimes \mathcal{C}$ be given. There exist $g_i \in \mathrm{Lip}^+(X)$ and $K_i \in \mathcal{C}$, for $i = 1, ..., m$, such that $F = \sum_{i=1}^m g_i \otimes K_i$. Let $M_i > 0$ be the Lipschitz constant for g_i, $i = 1, ..., m$. Then

$$\begin{aligned} d(F(x),F(y)) &= d\Big(\sum_{i=1}^m g_i(x)\,K_i,\ \sum_{i=1}^m g_i(y)\,K_i\Big) \le \\ &\le \sum_{i=1}^m d\Big(g_i(x)\,K_i,\ g_i(y)\,K_i\Big) \le \sum_{i=1}^m |g_i(x) - g_i(y)| \cdot d(K_i,0) \le \\ &\le \sum_{i=1}^m M_i\,\widetilde{d}(x,y)\,d(K_i,0) = \Big(\sum_{i=1}^m M_i\,d(K_i,0)\Big)\,\widetilde{d}(x,y) \end{aligned}$$

for all $x, y \in X$. Hence $F \in \mathrm{Lip}(X;\mathcal{C})$.

b) Let $g \in \mathrm{Lip}^+(X)$ and $F \in \mathrm{Lip}^+(X) \otimes \mathcal{C}$ be given. Put $\|F\| = \sup\{d(F(x),0);\ x \in X\}$. Since $F \in C(X;\mathcal{C})$ it follows that $\|F\| < \infty$. Let M_g and M_F be the positive constants such that

$$|g(x) - g(y)| \le M_g\,\widetilde{d}(x,y) \quad \text{and} \quad d(F(x),F(y)) \le M_F\,\widetilde{d}(x,y)\ ,$$

for all $x, y \in X$. Then

$$\begin{aligned} d\Big(g(x)\,F(x),\; g(y)\,F(y)\Big) &\leq |g(x) - g(y)|\, d(F(x), 0) + g(y)\, d(F(x), F(y)) \\ &\leq M_g\, \widetilde{d}(x, y)\, \|F\| + \|g\|\, M_F\, \widetilde{d}(x, y) \\ &= \Big(\|F\|\, M_g + \|g\|\, M_F\Big)\, \widetilde{d}(x, y) \end{aligned}$$

for all $x, y \in X$. Hence $gF \in \mathrm{Lip}(X; \mathcal{C})$.

Now, if $g \in \mathrm{Lip}^+(X)$ and $F = \sum_{i=1}^m g_i \otimes K_i$, where $g_i \in \mathrm{Lip}^+(X)$ and $K_i \in \mathcal{C}$, then $gF = \sum_{i=1}^m h_i \otimes K_i$ where $h_i = g \cdot g_i \in \mathrm{Lip}^+(X)$. It follows that gF belongs to $\mathrm{Lip}^+(X) \otimes \mathcal{C}$. ∎

Lemma 2. *Let $(X, \widetilde{d})$ and $(\mathcal{C}, d)$ be as in Lemma 1. Then $\mathrm{Lip}^+(X) \otimes \mathcal{C}$ is dense in $C(X; \mathcal{C})$. Consequently, $\mathrm{Lip}(X; \mathcal{C})$ is dense in $C(X; \mathcal{C})$.*

Proof: Let $x, y \in X$, $x \neq y$ be given. Let $g : X \to \mathbb{R}$ be defined by $g(z) = \widetilde{d}(x, z)$, for all $z \in X$. Since $|g(z) - g(t)| = |\widetilde{d}(x, z) - \widetilde{d}(x, t)| \leq \widetilde{d}(z, t)$, for all $z, t \in X$, it follows that $g \in \mathrm{Lip}^+(X)$. Therefore $h = g/\|g\|$ belongs to $\mathrm{Lip}(X; [0, 1])$. Moreover, $h(y) > 0 = h(x)$, i.e., h separates x and y. By Lemma 1, if $F, G \in \mathrm{Lip}^+(X) \otimes \mathcal{C}$ then $hF + (1 - h)\, G$ belongs to $\mathrm{Lip}^+(X) \otimes \mathcal{C}$. Since $\mathrm{Lip}^+(X) \otimes \mathcal{C}$ contains the constant functions, the result follows from Corollary 3, Prolla [3]. ∎

Lemma 3 (Andrica and Mustata [1])**.** *Let $(X, \widetilde{d})$ be a metric compact space and let $S\colon C(X; \mathbb{R}) \to C(X; \mathbb{R})$ be a positive linear operator. If $f \in \mathrm{Lip}(X)$ then there exists a positive constant M_f such that*

$$\Big|(Sf, x) - f(x)\,(Se_0, x)\Big| \leq M_f\, \alpha(x)$$

for all $x \in X$.

Proof: Let $f \in \mathrm{Lip}(X)$ and let $M_f > 0$ be a Lipschitz constant for f, i.e.,

$$|f(x) - f(y)| \leq M_f\, \widetilde{d}(x, y)$$

for all $x, y \in X$. It follows that

$$-M_f\, \widetilde{d}(x, \cdot) \leq f(\cdot) - f(x)\, e_0 \leq M_f\, \widetilde{d}(x, \cdot)$$

for all $x \in X$. Since S is linear and positive we have

$$-M_f (S(\widetilde{d}_x), x) \leq (Sf, x) - f(x)\,(Se_0, x) \leq M_f (S(\widetilde{d}_x), x)$$

for all $x \in X$. Therefore

$$\left|(Sf,x) - f(x)\,(Se_0,x)\right| \le M_f(S(\widetilde{d}_x),x)$$

for all $x \in X$. ∎

Corollary 1. *Let $(X,\widetilde{d})$ and S be as in Lemma 3. Assume that $Se_0 = e_0$. If $f \in \mathrm{Lip}(X)$ then there exists a positive constant M_f such that*

$$\left|(Sf,x) - f(x)\right| \le M_f\,\alpha(x)$$

for all $x \in X$.

Proof: It follows immediately from Lemma 3 since $(Se_0,x) = 1$, for all $x \in X$. ∎

Remark 2. A positive linear operator S on $C(X;\mathbb{R})$ such that $Se_0 = e_0$, i.e., S preserves the constant functions, is called a Markov operator on $C(X;\mathbb{R})$. Andrica and Mustata [1] proved Lemma 3 assuming that S is a Markov operator.

Proposition 1. *Let $(X,\widetilde{d})$ be a metric compact space and $(\mathcal{C},d)$ be a metric convex cone. Let T be a monotonically regular operator on $C(X;\mathcal{C})$ and let $F \in \mathrm{Lip}^+(X)\otimes\mathcal{C}$ be given. There exist positive constants M_F and A_F such that*

$$d\Big((TF,x),F(x)\Big) \le M_F\,\alpha(x) + A_F\,|(\widehat{T}e_0,x) - 1|$$

for all $x \in X$.

Proof: Let $F = \sum_{i=1}^m g_i \otimes K_i$ be given, where $g_i \in \mathrm{Lip}^+(X)$ and $K_i \in \mathcal{C}$, for $i = 1,...,m$. Since T is convex conic and regular, we have

$$(TF,x) = \Big(\sum_{i=1}^m T(g_i \otimes K_i),x\Big) = \sum_{i=1}^m (\widehat{T}(g_i),x)\,K_i$$

for all $x \in X$.

For each $i = 1,...,m$, by Lemma 3, there exists a constant $M_i > 0$ such that

$$\left|(\widehat{T}(g_i),x) - g_i(x)\,(\widehat{T}e_0,x)\right| \le M_i\,\alpha(x)$$

for all $x \in X$. Let M_F and A_F be the positive constants defined by

$$M_F = \sum_{i=1}^m M_i\,d(K_i,0) \quad \text{and} \quad A_F = \sum_{i=1}^m \|g_i\|\,d(K_i,0)\ .$$

Then,

$$d\Big((TF,x),F(x)\Big) \le \sum_{i=1}^{m} d\Big((\widehat{T}(g_i),x)K_i,\ g_i(x)K_i\Big) \le \sum_{i=1}^{m} \Big|(\widehat{T}(g_i),x) - g_i(x)\Big|\, d(K_i,0) \le$$

$$\le \sum_{i=1}^{m} \Big[M_i\,\alpha(x) + \|g_i\|\cdot|(\widehat{T}e_0,x)-1|\Big]\, d(K_i,0) \le M_F\,\alpha(x) + A_F|(\widehat{T}e_0,x)-1|$$

for all $x \in X$. ∎

Corollary 2. *Let $(X,\widetilde{d})$, $(\mathcal{C},d)$ and T be as in Proposition 1. Assume that $\widehat{T}$ preserves the constant functions. If $F \in \mathrm{Lip}^+(x)\otimes\mathcal{C}$ then there exists a positive constant M_F such that*

$$d\Big((TF,x),F(x)\Big) \le M_F\,\alpha(x)$$

for all $x \in X$.

Proof: The result follows from Proposition 1 since $\widehat{T}(e_0) = e_0$. ∎

Definition 8. Let $\{T_n\}_{n\ge1}$ be a sequence of operators on $C(X;\mathcal{C})$. We say that $\{T_n\}_{n\ge1}$ is *uniformly equicontinuous* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $d(F,G) < \delta$ implies $d(T_nF,T_nG) < \varepsilon$, for all $n = 1,2,3,\ldots$.

Let $\{T_n\}_{n\ge1}$ be a sequence of regular operators on $C(X;\mathcal{C})$. For each $n \ge 1$ we denote by α_n the function defined by

$$\alpha_n(x) = (\widehat{T}_n(\widetilde{d}_x),x)$$

for all $x \in X$.

Theorem 1. *Let $(X,\widetilde{d})$ be a metric compact space and $(\mathcal{C},d)$ be a metric convex cone. Let $\{T_n\}_{n\ge1}$ be a sequence of monotonically regular operators on $C(X;\mathcal{C})$. Assume that $\{T_n\}_{n\ge1}$ is uniformly equicontinuous. If $\widehat{T}_ne_0 \to e_0$ and $\{\alpha_n(x)\}_{n\ge1}$ converges to zero, uniformly in $x \in X$, then $T_nF \to F$, for every $F \in C(X;\mathcal{C})$.*

Proof: Let $G \in \mathrm{Lip}^+(X)\otimes\mathcal{C}$ be given. By Proposition 1, there exist positive constants M_G and A_G such that, for each $n \ge 1$,

$$d\Big((T_nG,x),G(x)\Big) \le M_G\,\alpha_n(x) + A_G|(\widehat{T}_ne_0,x)-1|$$

for all $x \in X$. Since $\alpha_n(x) \to 0$, uniformly in $x \in X$ and $\widehat{T}_ne_0 \to e_0$ it follows that $d(T_nG,G) \to 0$. Hence $T_nG \to G$, for each G in $\mathrm{Lip}^+(X)\otimes\mathcal{C}$.

Let $F \in C(X;\mathcal{C})$ and $\varepsilon > 0$ be given. By the uniform equicontinuity of the sequence $\{T_n\}_{n\geq 1}$, there is some $\delta > 0$, which we may assume to verify $\delta < \varepsilon/3$, such that $d(F,H) < \delta$ implies $d(T_nF, T_nH) < \varepsilon/3$, for all $n \geq 1$. By Lemma 2, there exists G in $\mathrm{Lip}^+(X)\otimes\mathcal{C}$ such that $d(F,G) < \delta$. Since $T_nG \to G$ as proved above, there is n_0 such that $n \geq n_0$ implies $d(T_nG, G) < \varepsilon/3$. It follows that, for $n \geq n_0$

$$\begin{aligned} d\Big((T_nF,x),F(x)\Big) &\leq d\Big((T_nF,x),(T_nG,x)\Big) + d\Big((T_nG,x),G(x)\Big) + d\Big(G(x),F(x)\Big) \\ &\leq d(T_nF,T_nG) + d(T_nG,G) + d(G,F) < \varepsilon \end{aligned}$$

for all $x \in X$. Hence $T_nF \to F$. ∎

Remark 3. If each $\widehat{T}_n$ preserves the constant functions, then the proof of Theorem 1 implies that

$$d(T_nF,F) \leq M_F\,\|\alpha_n\|$$

for all $n \geq 1$ and all $F \in \mathrm{Lip}^+(X)\otimes\mathcal{C}$, where $\|\alpha_n\| = \sup\{|\alpha_n(x)|;\, x \in X\}$.

If we define $\beta_n(x) = (\widehat{T}_n(\widetilde{d}_x)^2, x)$, for all $x \in X$, then we have that $\|\alpha_n\| \leq \|\beta_n\|^{\frac{1}{2}}$, for all $n \in \mathbb{N}$, and the following result holds:

Corollary 3. *Let $\{T_n\}_{n\geq 1}$ be as in Theorem 1. Assume that each $\widehat{T}_n$ preserves the constant functions. If $\{\beta_n(x)\}_{n\geq 1}$ converges to zero, uniformly in $x \in X$, then $T_nF \to F$, for every $F \in C(X;\mathcal{C})$. Furthermore, if $F \in \mathrm{Lip}^+(X)\otimes\mathcal{C}$ then there exists a constant $M_F > 0$ such that*

$$d(T_nF,F) \leq M_F\,\|\beta_n\|^{\frac{1}{2}}$$

for all $n \geq 1$.

Proof: Apply Theorem 1 and Remark 3. ∎

Example 2: Let J be a finite set, and for each $k \in J$, let $t_k \in X$ and $\psi_k \in C^+(X)$ be given. The convex conic operator T defined on $C(X;\mathcal{C})$ by

$$(TF,x) = \sum_{k\in J} \psi_k(x)\,F(t_k)$$

for all $F \in C(X;\mathcal{C})$ and $x \in X$ is called an *operator of interpolation type*. If $F = f\otimes K$, where $f \in C^+(X)$ and $K \in \mathcal{C}$, then

$$(TF,x) = \sum_{k\in J} \psi_k(x)\,[f(t_k)K] = \Big[\sum_{k\in J} \psi_k(x)\,f(t_k)\Big]K\ .$$

Hence, T is regular and $T(f \otimes K) = \widehat{T}(f) \otimes K$ where, for each $f \in C(X; \mathbb{R})$,

$$(\widehat{T} f, x) = \sum_{k \in J} \psi_k(x)\, f(t_k) \ .$$

Let us assume that, for every $x \in X$,

$$\sum_{k \in J} \psi_k(x) = 1 \ .$$

It follows that $\widehat{T} e_0 = e_0$. The operators of Bernstein and of Hermite–Fejér type are examples of operators satisfying such condition.

Remark 4. If $(\mathcal{C}, d)$ is a convex cone and T is a regular operator on $C(X; \mathcal{C})$ then $TK^* = T(e_0 \otimes K) = \widehat{T}(e_0) \otimes K$, for every $K \in \mathcal{C}$, and we have

$$\begin{aligned} d\Big((TK^*, x), K^*(x)\Big) &= d\Big((\widehat{T} e_0, x) K,\ e_0(x).K\Big) \\ &\le |(\widehat{T} e_0, x) - 1|\, d(K, 0) \end{aligned}$$

for all $x \in X$. It follows that if $\{T_n\}_{n \ge 1}$ is a sequence of regular operators on $C(X; \mathcal{C})$ such that $\widehat{T}_n e_0 \to e_0$, then $T_n K^* \to K^*$, for every $K \in \mathcal{C}$.

Lemma 4. *Let $(X, \widetilde{d})$ be a metric compact space and $(\mathcal{C}, d)$ be a convex cone. Let $\{T_n\}_{n \ge 1}$ be a sequence of regular convex conic operator on $C(X; \mathcal{C})$. Assume that $\widehat{T}_n e_0 \to e_0$. If $F \in C(X; \mathcal{C})$ then $(T_n[F(x)]^*, x) \to F(x)$, uniformly in $x \in X$.*

Proof: Let $F \in C(X; \mathcal{C})$ and $\varepsilon > 0$ be given. Since $\widehat{T}_n e_0 \to e_0$ there is n_0 such that $n \ge n_0$ implies

$$\Big|(\widehat{T}_n(e_0), x) - 1\Big| < \frac{\varepsilon}{2\, \|F\|}$$

for all $x \in X$, where $\|F\| = \sup\{d(F(x), 0);\, x \in X\}$. It follows that, for $n \ge n_0$

$$\begin{aligned} d\Big((T_n[F(x)]^*, x), f(x)\Big) &\le \Big|(\widehat{T}_n(e_0), x) - 1\Big|\, d(F(x), 0) \\ &\le \left(\frac{\varepsilon}{2\, \|F\|}\right) \cdot \|F\| < \varepsilon \end{aligned}$$

for all $x \in X$. Therefore, $(T_n[F(x)]^*, x) \to F(x)$, uniformly in $x \in X$. ∎

3 – Hausdorff convex cones

Definition 9. An *ordered convex cone* is a pair $(\mathcal{C}, \leq)$, where $\mathcal{C}$ is an (abstract) convex cone and $\leq$ is an ordering of its elements, i.e., $\leq$ is a reflexive, transitive and antisymmetric relation on $\mathcal{C}$, in such a way that

a) $K \leq L$ implies $K + M \leq L + M$, for every $M \in \mathcal{C}$,

b) $K \leq L$, $\lambda \geq 0$ implies $\lambda K \leq \lambda L$,

c) $\lambda \leq \mu$ implies $\lambda K \leq \mu K$, for every $K \geq 0$.

Definition 10. Let $(\mathcal{C}, \leq)$ be an ordered convex cone and let d_H be a semi-metric on $\mathcal{C}$. We say that d_H is a *Hausdorff semi-metric* on $\mathcal{C}$ if there exists an element $B \geq 0$ on $\mathcal{C}$ such that:

a) For every pair $K, L \in \mathcal{C}$ and $\lambda \geq 0$, the following is true: $d_H(K, L) \leq \lambda$ if, and only if, $K \leq L + \lambda B$ and $L \leq K + \lambda B$,

b) $\lambda B \leq \mu B$ implies $\lambda \leq \mu$.

If d_H is a Hausdorff semi-metric on $\mathcal{C}$, we say that $(\mathcal{C}, d_H)$ is a *Hausdorff convex cone*.

Example 3: If $\mathcal{C} = \mathbb{R}$ with the usual operations and ordering, the usual distance $d_H(x, y) = |x - y|$ is a Hausdorff metric on $\mathbb{R}$, with $B = 1$.

Example 4: Let $\mathcal{C}$ be the convex subcone of $\mathrm{Conv}(E)$ of all elements of $\mathrm{Conv}(E)$ that are bounded sets and let B be the closed unit ball of E. Define on $\mathcal{C}$ the usual Hausdorff semi-metric d_H by setting

$$d_H(K, L) = \inf\Big\{\lambda > 0;\ K \subset L + \lambda B,\ L \subset K + \lambda B\Big\}$$

for every pair $K, L \in \mathcal{C}$. Then $(\mathcal{C}, d_H)$ is a Hausdorff convex cone.

Let $(X, \widetilde{d})$ be a metric compact space and $(\mathcal{C}, d_H)$ be a Hausdorff convex cone. In $C(X; \mathcal{C})$ we consider the topology determined by the metric defined by

$$d(F, G) = \sup\Big\{d_H(F(x), G(x));\ x \in X\Big\}$$

for every pair F, G in $C(X; \mathcal{C})$.

Remark 5. If $(\mathcal{C}, d_H)$ is a Hausdorff convex cone and $\{T_n\}_{n \geq 1}$ is a sequence of regular operators on $C(X; \mathcal{C})$ then $T_n B^* \to B^*$ implies $\widehat{T}_n e_0 \to e_0$. Indeed, let

$\varepsilon > 0$ be given. Since $B^* = e_0 \otimes B$ and $T_n(e_0 \otimes B) \to e_0 \otimes B$, it follows that there is n_0 such that $n \geq n_0$ implies

$$d_H\Big((\widehat{T}_n(e_0), x)\, B,\; e_0(x)\, B\Big) < \varepsilon$$

for all $x \in X$. By the definition of d_H we have

$$\Big(\widehat{T}_n(e_0), x\Big)\, B \leq B + \varepsilon B = (1+\varepsilon)\, B$$

and

$$B \leq (\widehat{T}_n(e_0), x)\, B + \varepsilon B$$

for all $x \in X$. By condition b) of Definition (10) we have $(\widehat{T}_n(e_0), x) < 1+\varepsilon$ and $1-\varepsilon < (\widehat{T}_n(e_0), x)$, for all $x \in X$. Hence $|(\widehat{T}_n(e_0), x) - 1| < \varepsilon$, for all $x \in X$ and so $\widehat{T}_n e_0 \to e_0$.

We recall that an operator T on $C(X;\mathcal{C})$ is called *monotone*, if $F \leq G$ implies $TF \leq TG$ for every pair F, G in $C(X;\mathcal{C})$.

Remark 6. If $(\mathcal{C}, d_H)$ is a Hausdorff convex cone and T is a regular operator on $C(X;\mathcal{C})$ that is monotone then $\widehat{T}$ is also monotone. Indeed, for $f, g \in C(X;\mathbb{R})$ such that $f \leq g$ we have $f \otimes B \leq g \otimes B$. It follows that $T(f \otimes B) \leq T(g \otimes B)$, and since T is regular, we get $(\widehat{T}(f), x)B \leq (\widehat{T}(g), x)B$, for all $x \in X$. Therefore $\widehat{T}f \leq \widehat{T}g$.

Theorem 2. *Let $(X, \widetilde{d})$ be a metric compact space and $(\mathcal{C}, d_H)$ be a Hausdorff convex cone. Let $\{T_n\}_{n\geq 1}$ be a sequence of regular continuous operators on $C(X;\mathcal{C})$. Assume that each T_n is monotone and $T_n B^* \to B^*$. If $\{\alpha_n(x)\}_{n\geq 1}$ converges to zero, uniformly in $x \in X$, then $T_n F \to F$, for every $F \in C(X;\mathcal{C})$.*

Proof: By Theorem 1 it suffices to show that the sequence $\{T_n\}_{n\geq 1}$ is uniformly equicontinuous. Let $\varepsilon > 0$ be given. Choose $\delta_0 > 0$ such that $\delta_0(1+\delta_0) < \varepsilon$. Since $T_n B^* \to B^*$, there is n_0 so that $n > n_0$ implies $d_H((T_n B, x), B) < \delta_0$, for all $x \in X$. It follows from the definition of d_H that

$$(T_n B^*, x) \leq B + \delta_0\, B = (1+\delta_0)\, B$$

and

$$B \leq (T_n B^*, x) + \delta_0\, B$$

for all $x \in X$, and $n > n_0$.

Let $F, G \in C(X;\mathcal{C})$ be such that $d(F,G) < \delta_0$. We claim that $d(T_nF, T_nG) < \varepsilon$, for all $n > n_0$. Indeed, since $d_H(F(x), G(x)) < \delta_0$, for all $x \in X$, it follows that $F \leq G + \delta_0\, B^*$ and $G \leq F + \delta_0\, B^*$.

Since each T_n is convex conic and monotone, we have, for each $n \geq 1$, $T_nF \leq T_nG + \delta_0\, T_nB^*$ and $T_nG \leq T_nF + \delta_0\, T_nB^*$. Therefore, for each $n \geq 1$, $(T_nF, x) \leq (T_nG, x) + \delta_0 (T_nB^*, x)$, for all $x \in X$. It follows that, for $n > n_0$

$$(T_nF, x) \leq (T_nG, x) + \delta_0(1+\delta_0)\, B < (T_nG, x) + \varepsilon\, B$$

for all $x \in X$. Similarly, for $n > n_0$

$$(T_nG, x) < (T_nF, x) + \varepsilon\, B$$

for all $x \in X$. Hence, for all $n > n_0$

$$d_H\Big((T_nF, x),\, (T_nG, x)\Big) < \varepsilon$$

for all $x \in X$. It follows that, for all $n > n_0$

$$d(T_nF, T_nG) < \varepsilon\ .$$

On the other hand, since each T_n is continuous, there exist $\delta_1, ..., \delta_{n_0}$ such that $d(F,G) < \delta_k$ implies $d(T_kF, T_kG) < \varepsilon$, for $k = 1, 2, ..., n_0$. Let $\delta = \min\{\delta_0, \delta_1, ..., \delta_{n_0}\}$. Clearly $d(F,G) < \delta$ implies $d(T_nF, T_nG) < \varepsilon$, for all $n = 1, 2, 3, ...$. ∎

Corollary 4. *Let $(X, \widetilde{d})$, $(\mathcal{C}, d_H)$ and $\{T_n\}_{n\geq 1}$ be as in Theorem 2. Assume that T_n preserves the constant functions. If $\{\beta_n(x)\}_{n\geq 1}$ converges to zero, uniformly in $x \in X$, then $T_nF \to F$, for every $F \in C(X;\mathcal{C})$. Furthermore, if $F \in \mathrm{Lip}^+(X) \otimes \mathcal{C}$ then there exists a constant $M_F > 0$ such that*

$$d(T_nF, F) \leq M_F\, \|\beta_n\|^{\frac{1}{2}}$$

for all $n = 1, 2, 3, ...$, where $\beta_n(x) = (\widehat{T}_n(\widetilde{d}_x)^2, x)$, for all $x \in X$.

Let us recall that the modulus of continuity of $F \in C(X;\mathcal{C})$ is defined as

$$w(F,\delta) = \sup\Big\{d(F(x), F(t));\ x, t \in X,\ \widetilde{d}(x,t) \leq \delta\Big\}$$

for every $\delta > 0$. By uniform continuity of F, we have $w(F,\delta) \to 0$ as $\delta \to 0$.

Let us consider the following condition:

(*) There exists a constant p with $0 < p \leq 1$ such that $w(F, \lambda\delta) \leq [1 + \lambda^{\frac{1}{p}}]\, w(F, \delta)$, for all $F \in C(X; \mathcal{C})$ and all δ, $\lambda > 0$.

If X is a compact convex subset of a q-normed linear space with $0 < q \leq 1$, then (*) holds for $p = q$.

The following result is proved in [4]:

Lemma 5. *Assume that (*) holds. Let $F \in C(X; \mathcal{C})$ and $\delta > 0$ be given. Then*

$$d_H(F(x), F(t)) \leq \left[1 + \left(\frac{\widetilde{d}(x,t)}{\delta}\right)^{\frac{1}{p}}\right] w(F, \delta)$$

for every pair, x and t, of elements of X.

If $\{T_n\}_{n \geq 1}$ is a sequence of convex conic operators on $C(X; \mathcal{C})$ that are regular, let

$$a_n(x) = \left(\widehat{T}_n((\widetilde{d}_x)^{\frac{1}{p}}), x\right)$$

for all $x \in X$, where p is given by condition (*).

Proposition 2. *Assume that (*) holds. Let $\{T_n\}_{n \geq 1}$ be a sequence of convex conic operators on $C(X; \mathcal{C})$ such that each T_n is monotone and regular. Then*

$$d_H\Big((T_n F, x), F(x)\Big) \leq \left[(\widehat{T}_n(e_0), x) + \frac{1}{\delta^{\frac{1}{p}}}\, a_n(x)\right] w(F, \delta) + d_H\Big((T_n[F(x)]^*, x), F(x)\Big)$$

for every $F \in C(X; \mathcal{C})$, $x \in X$ and $\delta > 0$.

Proof: Let $F \in C(X; \mathcal{C})$ and $\delta > 0$ be given. By Lemma 5, for $t, x \in X$

$$\begin{aligned} F(t) &\leq F(x) + \left[1 + \left(\frac{\widetilde{d}(x,t)}{\delta}\right)^{\frac{1}{p}}\right] w(F, \delta)\, B \\ &= F(x) + w(F, \delta)\left[B + \frac{1}{\delta^{\frac{1}{p}}}\, (\widetilde{d}(x,t))^{\frac{1}{p}}\, B\right] \end{aligned}$$

Hence,

$$F \leq [F(x)]^* + w(F, \delta)\left[B^* + \frac{1}{\delta^{\frac{1}{p}}} (\widetilde{d}_x)^{\frac{1}{p}} \otimes B\right].$$

Since each T_n is monotone and regular we have

$$(T_n F, x) \leq \Big(T_n[F(x)]^*, x\Big) + w(F, \delta)\left[(\widehat{T}_n(e_0), x) + \frac{1}{\delta^{\frac{1}{p}}}\, a_n(x)\right] B$$

for all $x \in X$. Similarly,

$$\Big(T_n[F(x)]^*, x\Big) \leq (T_n F, x) + w(F,\delta)\left[(\widehat{T}_n(e_0), x) + \frac{1}{\delta^{\frac{1}{p}}}\, a_n(x)\right] B$$

for all $x \in X$. Therefore

$$d_H\Big((T_n F, x), (T_n[F(x)]^*, x)\Big) \leq w(F,\delta)\left[(\widehat{T}_n(e_0), x) + \frac{1}{\delta^{\frac{1}{p}}}\, a_n(x)\right].$$

for all $x \in X$. ∎

Theorem 3. *Let $(X, \widetilde{d})$ be a compact metric space and $(\mathcal{C}, d_H)$ be a Hausdorff convex cone. Let $\{T_n\}_{n\geq 1}$ be a sequence of convex conic operators on $C(X;\mathcal{C})$ such that each T_n is monotone and regular. Assume that (*) holds and that*

i) $T_n B^* \to B^*$,

ii) $a_n(x) = 0(\frac{1}{n})$, *uniformly in $x \in X$.*

Then $T_n F \to F$, for every $F \in C(X;\mathcal{C})$.

Proof: Let $F \in C(X;\mathcal{C})$ and $\varepsilon > 0$ be given. By i), Remark 5 and Lemma 4 choose n_1 so that $n \geq n_1$ implies

(**1**) $(\widehat{T}_n(e_0), x) < 1 + \varepsilon/2$,

(**2**) $d_H((T_n[F(x)]^*, x), F(x)) < \varepsilon/2$,

for all $x \in X$. By ii) there is some constant $k > 0$ such that

(**3**) $n\, a_n(x) \leq k$,

for $n = 1, 2, ...$ and all $x \in X$. Since $w(F,\delta) \to 0$ as $\delta \to 0$, we can choose n_2 such that $n \geq n_2$ implies

(**4**) $w(F, n^{-p}) < (\varepsilon/2)\,(1 + k + \varepsilon/2)^{-1}$.

By Proposition 2 and (1)–(4), it follows that for $n \geq n_0 = \max\{n_1, n_2\}$

$$\begin{aligned} d_H\Big((T_n F, x), F(x)\Big) &\leq \left[(\widehat{T}_n(e_0), x) + \frac{1}{\delta^{\frac{1}{p}}}\, a_n(x)\right] w(F,\delta) + d_H\Big((T_n[F(x)]^*, x), F(x)\Big) \\ &= \Big[(\widehat{T}_n(e_0), x) + n\, a_n(x)\Big]\, w(F, n^{-p}) + d_H\Big((T_n[F(x)]^*, x), F(x)\Big) \\ &< (1 + k + \varepsilon/2)\, w(F, n^{-p}) + \varepsilon/2 < \varepsilon \end{aligned}$$

for all $x \in X$. Hence $T_n F \to F$. ∎

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