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SYMBOLIC DYNAMICS OF BIMODAL MAPS

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Abstract: We introduce a matrix that represents the homological behavior of 0-chains and that summarizes the dynamics of bimodal maps of the interval. Using the characteristic polynomial of this matrix we deduce the kneading determinants of bimodal maps.

1 – Introduction

Our goal in this paper is to introduce an alternative technique to the ones utilized in symbolic dynamics which allows for the simple deduction of formulae for the calculation of the topological entropy. This technique is based on a commutative diagram derived from the study of the homological configurations of graphs associated to bimodal maps of the interval. From the partition of the interval that corresponds to the itinerary of the critical points we introduce 0-1-chains of complexes that translate the dynamical properties in relations of homological type. In [11] it was introduced the kneading-determinant $D(t)$ as a formal power series. On the other hand, in [5] using homological properties we proved a precise relation between the kneading-determinant and the characteristic polynomial of a matrix A associated to the action on 1-chains of the interval with one critical point. In this paper we extend this result to the maps of the interval with two critical points. The advantage of this method is that the formula introduced in proposition 2 allows to write explicitly the characteristic polynomial which is valid for all pairs of sequences of symbols. The study of bimodal maps has found interesting applications in physics and biology [1, 3, 10, 12, 14]. Their mathematical study has drawn the attention of [4, 6, 8, 9, 11, 13]. The introduction of a homological language into these studies was used before by [2].

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2 – Kneading theory for bimodal maps

Considering a two parameter family $f_{(a,b)}$ of bimodal maps on the interval $I=[-1,1]$ and denoting by c_1 and c_2 the two critical points of $f_{(a,b)}$, we obtain for each value of the pair (a,b) the orbits

$$O_{(a,b)}(c_i) = \left\{x_j^{(i)}\colon\, x_j^{(i)} = f^j_{(a,b)}(c_i),\ j \in N\right\} ,$$

where $i=1,2$. After a reordering of the elements $x_j^{(i)}$ of these orbits we get a partition $\{I_k=[z_k,z_{k+1}]\}$ of the interval $I=[x_1^{(2)},x_1^{(1)}]$. With the aim of studying the topological properties of these orbits we associate to each orbit $O_{(a,b)}(c_i)$ a sequence of symbols $S=S_1S_2\ldots S_j\ldots$ where $S_j=L$ if $f^j_{(a,b)}(c_i)<c_1$, $S_j=A$ if $f^j_{(a,b)}(c_i)=c_1$, $S_j=M$ if $c_1<f^j_{(a,b)}(c_i)<c_2$, $S_j=B$ if $f^j_{(a,b)}(c_i)=c_2$ and $S_j=R$ if $f^j_{(a,b)}(c_i)>c_2$. If we denote by n_M the frequency of the symbol M in a finite subsequence of S we can define the M-parity of this subsequence according to whether n_M is even or odd. In what follows we define an order relation in $\Sigma=\{L,A,M,B,R\}^N$ that depends on the M-parity. So, for two of such sequences, P and Q in Σ, let i be such that $P_i\neq Q_i$ and $P_j=Q_j$ for $j<i$. If the M-parity of the block $P_1\ldots P_{i-1}=Q_1\ldots Q_{i-1}$ is even we say that $P<Q$ if $P_i=L$ and $Q_i\in\{A,M,B,R\}$ or $P_i=A$ and $Q_i\in\{M,B,R\}$ or $P_i=M$ and $Q_i\in\{B,R\}$ or $P_i=B$ and $Q_i=R$. If the M-parity of the same block is odd, we say that $P<Q$ if $P_i=A$ and $Q_i=L$ or $P_i=M$ and $Q_i\in\{L,A\}$ or $P_i=B$ and $Q_i\in\{L,A,M\}$ or $P_i=R$ and $Q_i\in\{L,A,M,B\}$. If no such index i exists, then $P=Q$. When $O_{(a,b)}(c_i)$ is a k-periodic orbit we get a sequence of symbols that can be characterized by a block of length k, $S^{(k)}=S_1\ldots S_{k-1}C_i$. In what follows, we restrict our study to the case where the two critical points are periodic $O_{(a,b)}(c_1)$ is p-periodic and $O_{(a,b)}(c_2)$ is q-periodic. Note that $O_{(a,b)}(c_1)$ is realizable iff the block $P=AP_1\ldots P_{p-1}$ is maximal, that is, $\sigma^i(P)<\sigma(P)$, where $1<i\leq p$ and $\sigma(P)=P_1\ldots P_{p-1}A$ is the usual shift operator. On the other hand, $O_{(a,b)}(c_2)$ is realizable iff the block $Q=BQ_1\ldots Q_{q-1}$ is minimal, that is, $\sigma^j(Q)>\sigma(Q)$, $1<j\leq q$. Finally, note that the pair of sequences that are realizable satisfies the following conditions $\sigma^i(P)>\sigma(Q)$, $1<i\leq p$ and $\sigma^j(Q)<\sigma(P)$, $1<j\leq q$. In what follows we denote the set of such pair of sequences by $\Sigma_{(A,B)}$. A kneading sequence $(P^{(p)},Q^{(q)})$ is a pair of sequences such that $P^{(p)}=\sigma(P)=P_1\ldots P_{p-1}A$ and $Q^{(q)}=\sigma(Q)=Q_1\ldots Q_{q-1}B$ for some pair $(P,Q)\in\Sigma_{(A,B)}$. Denoting by $\{u_i=\sigma^{i-1}(P^{(p)})\colon\, i=1,\ldots,p-1$, and $u_0=\sigma^{(p-1)}(P^{(p)})\}$, $\{v_j=\sigma^{j-1}(Q^{(q)})\colon\, j=1,\ldots,q-1$, and $v_0=\sigma^{(q-1)}(Q^{(q)})\}$ two sets of sequences and by $\{w_k\colon\, 1\leq k\leq p+q\}$ the union of the previous sets

with the order that corresponds to the elements z_i, we can define a permutation matrix π that maps the set $(y_1, \ldots, y_{p+q}) = (x_0^{(1)} \ldots, x_{p-1}^{(1)}, x_0^{(2)} \ldots, x_{p-1}^{(2)})$ into the set $(z_1, \ldots, z_{p+q})$.

Example. To illustrate the previous definition consider the pair of sequences $(RRLMA, LLRB)$. Then we have $u_1 = RRLMA$, $u_2 = RLMAR$, $u_3 = LMARR$, $u_4 = MARRL$, $u_0 = ARRLM$, $v_1 = LLRB$, $v_2 = LRBL$, $v_3 = RBLL$, $v_0 = BLLR$ and so we get $w_1 = v_1$, $w_2 = u_3$, $w_3 = v_2$, $w_4 = u_0$, $w_5 = u_4$, $w_6 = v_0$, $w_7 = u_2$, $w_8 = v_3$ and $w_9 = u_1$, (see fig. 1).

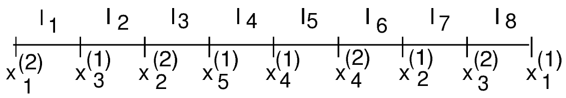

Fig. 1 – Graph G_1.

In this way, the permutation matrix π (such that, $z = \pi y$) is given by:

$$\begin{bmatrix} x_1^{(2)} \\ x_3^{(1)} \\ x_2^{(2)} \\ x_0^{(1)} \\ x_4^{(1)} \\ x_0^{(2)} \\ x_2^{(1)} \\ x_3^{(2)} \\ x_1^{(1)} \end{bmatrix} = \begin{bmatrix} 0&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&1&0&0&0&0&0&0&0 \end{bmatrix} . \begin{bmatrix} x_0^{(1)} \\ x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \\ x_0^{(2)} \\ x_1^{(2)} \\ x_2^{(2} \\ x_3^{(2)} \end{bmatrix}$$

In [11], Milnor–Thurston, introduced the concept of kneading-matrix and kneading increments. These are power series that measure the discontinuity evaluated at the turning points. For the case of bimodal maps we have two kneading-increments defined by:

$$(*) \qquad \nu_i(t) = \theta_{c_i^+}(t) - \theta_{c_i^-}(t)$$

where $\theta(x)$ is the invariant coordinate of the sequence $S_0 S_1 \ldots S_k \ldots$ associated to the itinerary of the point x. Note that the invariant coordinate is defined by:

$$\theta_x(t) = \sum_{k=0}^{\infty} \tau_k \, t^k \, S_k$$

where $\tau_k = \prod_{i=0}^{k-1} \varepsilon(S_i)$, $k > 0$, $\tau_0 = 1$, when $k = 0$,

$$\varepsilon(S_i) = \begin{cases} -1 & \text{if } S_i = M \\ 0 & \text{if } S_i = A \text{ or } S_i = B \\ 1 & \text{if } S_i = L \text{ or } S_i = R \end{cases}$$

and $\theta_{c_i^\pm}(t) = \lim_{x \to c_i^\pm} \theta_x(t)$. After separating the terms associated to the different letters in $(*)$ we get:

$$\nu_i(t) = N_{i1}(t)\, L + N_{i2}(t)\, M + N_{i3}(t)\, R$$

and from these we can define the kneading-matrix of 2×3 elements by:

$$N(t) = \begin{bmatrix} N_{11}(t) & N_{12}(t) & N_{13}(t) \\ N_{21}(t) & N_{22}(t) & N_{23}(t) \end{bmatrix} .$$

The kneading-determinant [11] is defined from the kneading-matrix according to the following formula:

$$D(t) = \frac{D_1(t)}{1-t} = -\frac{D_2(t)}{1+t} = \frac{D_3(t)}{1-t}$$

where $D_1(t) = N_{12}(t)\, N_{23}(t) - N_{22}(t)\, N_{13}(t)$, $D_2(t) = N_{11}(t)\, N_{23}(t) - N_{21}(t)\, N_{13}(t)$, $D_3(t) = N_{11}(t)\, N_{22}(t) - N_{21}(t)\, N_{12}(t)$. Finally, we define $d(t)$ by:

$$d(t) = D(t)\, (1 - t^p)\, (1 - t^q) .$$

Example. Let's return to the kneading sequence $(RRLMA, LLRB)$ considered before. The symbolic sequences that correspond to the orbits of the points c_1^+ and c_1^-, (see [11]), are the following:

$$c_1^+ \longrightarrow M(RRLMM)^\infty ,$$
$$c_1^- \longrightarrow L(RRLMM)^\infty .$$

Note that the block $RRLMM$ corresponds to the sequence $RRLMA$ where the symbol A is replaced by M because the parity of the block $RRLM$ is odd. So, we get:

$$\begin{aligned} \nu_1(t) &= M - L - 2tR - 2t^2R - 2t^3L - 2t^4M + 2t^5M - 2t^6R - \dots \\ &= M - L + \frac{-2tR - 2t^2R - 2t^3L - 2t^4M + 2t^5M}{1-t^5} \\ &= \frac{M - Mt^5 - L + Lt^5 - 2tR - 2t^2R - 2t^3L - 2t^4M + 2t^5M}{1-t^5} . \end{aligned}$$

In a similar way the symbolical sequences that correspond to the orbits of the points c_2^+ and c_2^- are given by $R(LLRR)^\infty$ and $M(LLRR)^\infty$, respectively. In this way, we get:

$$\begin{aligned}\nu_2(t) &= R - M + 2tL + 2t^2L + 2t^3R + 2t^4R + 2t^5L + \ldots\\ &= R - M + \frac{2tL + 2t^2L + 2t^3R + 2t^4R}{1-t^4}\\ &= \frac{R - Rt^4 - M + Mt^4 + 2tL + 2t^2L + 2t^3R + 2t^4R}{1-t^4}\end{aligned}$$

and from the previous definitions, we have:

$$N(t) = \begin{bmatrix} \dfrac{-1-2t^3+t^5}{1-t^5} & \dfrac{1-2t^4+t^5}{1-t^5} & \dfrac{-2t-2t^2}{1-t^5} \\ \dfrac{2t+2t^2}{1-t^4} & \dfrac{-1+t^4}{1-t^4} & \dfrac{1+2t^3+t^4}{1-t^4} \end{bmatrix},$$

$$D(t) = \frac{D_1(t)}{1-t} = \frac{1-2t-2t^2+2t^3-t^4+3t^5+2t^6-4t^7+t^9}{(1-t^4)\,(1-t^5)\,(1-t)},$$

$$\begin{aligned}d(t) &= \frac{1-2t-2t^2+2t^3-t^4+3t^5+2t^6-4t^7+t^9}{1-t}\\ &= 1-t-3t^2-t^3-2t^4+t^5+3t^6-t^7-t^8\ .\end{aligned}$$

3 – Homological configurations

In what follows we denote by G_1 the graph where the nodes $\{w_i\}$, $i = 1,\ldots,p+q$, are obtained from the permutation-matrix π associated to the kneading sequence and the edges are defined by the pairs (w_i, w_{i+1}), (see fig. 1). Let C_0 and C_1 be the vector spaces of 0-chains and 1-chains spanned by $\{u_k\}\cup\{v_j\}$, $k = 0,\ldots,p-1$, $j = 0,\ldots,q-1$ and by $\{I_i\}$, $i = 1,\ldots,p+q-1$, respectively. In what follows we use the same symbol for the linear map and their representation matrices. The border of 1-chain is defined from a map $\partial\colon C_1 \to B_0$ where $\partial I_i = w_{i+1}-w_i$. The incidence matrix of the graph G_1 is given by $\mu = [\mu_{ij}]$ where $\mu_{ij} = \delta_{i+1,j} - \delta_{i,j}$ and $\delta_{i,j}$ is the kronecker δ-symbol. Note that with these definitions $B_0 = \partial(C_1)$ is isomorphic to $\mu\pi(C_0)$. On the other hand, the shift operator σ in $\Sigma_{(A,B)}$ takes the form of a rotation $\omega\colon C_0 \to C_0$ defined by $\omega(u_i) = u_{i+1}$ where $0 \le i < p-1$ and $\omega(u_{p-1}) = u_0$ (in a similar way, $\omega(v_j) = v_{j+1}$ where $0 \le j < q-1$ and $\omega(v_{q-1}) = v_0$). If we denote by η the product of μ by π, this rotation induces in C_1 an endomorphism α that is obtained from the commutativity of the following diagram:

$$\begin{array}{ccccc} C_0 & \xrightarrow{\eta} & B_0 & \xleftarrow[\partial]{} & C_1 \\ \omega\downarrow & & \alpha\downarrow & & \alpha\downarrow \\ C_0 & \xrightarrow{\eta} & B_0 & \xleftarrow[\partial]{} & C_1 \end{array}$$

Note that from $\alpha\,\eta = \eta\,\omega$ we could get $\alpha = \eta\,\omega\,\eta^{\mathrm{T}}(\eta\,\eta^{\mathrm{T}})^{-1}$ where η^{T} is the transpose matrix of η. Note also, that if we neglect the negative signs of the matrix α then this matrix could be obtained as the Markov adjacency matrix associated to the partition $\{I_i\}$, and to the graph G_2 (see fig. 2).

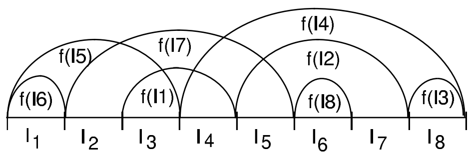

Fig. 2 – Graph G_2.

Example. If we return again to the kneading sequence ($RRLMA$, $LLRB$), we get:

$$\mu = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\omega = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\alpha = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} .$$

Lemma 1. *Suppose that $(P^{(p)}, Q^{(q)}) \in \Sigma_{(A,B)}$ and let α be the endomorphism introduced previously; then the first n_L and the last n_R rows of α are not negative. The other $n_M + 1$ rows of α are not positive.*

Proof: The result follows from the fact that the negative signs of α derive from the decreasing part of the map f. In fact, in this case the images of the intervals are obtained from the images of the boundary points $(z_i, z_{i+1}) \in [c_1, c_2]$ where $z_{i+1} > z_i$ and we have $f(z_{i+1}) < f(z_i)$. It's now quite obvious that all the other rows of α are not negative. ∎

Let's now denote by β a matrix of $(p+q-1) \times (p+q-1)$ elements defined by:

$$\beta = \begin{bmatrix} I_{n_L} & 0 & 0 \\ 0 & -I_{n_M+1} & 0 \\ 0 & 0 & I_{n_R} \end{bmatrix}$$

where I_{n_L}, I_{n_M+1} and I_{n_R} are identity matrices of rank n_L, $n_M + 1$ and n_R, respectively.

Definition 1. For each kneading sequence $(P^{(p)}, Q^{(q)})$ let $S_1 \ldots S_{p+q} = AP_1 \ldots P_{p-1} B Q_1 \ldots Q_{q-1}$, then we associate a square matrix of $(p+q)(p+q)$ elements $\gamma = [\gamma_{ij}]$ defined by:

$$\begin{aligned}
\gamma_{i1} &= -\gamma_{i,p+1} = 2 \quad \text{if } \; S_i = R \; \text{ with } i = 2, ..., p, p+2, ..., p+q \; , \\
\gamma_{i1} &= \gamma_{i,p+1} + 2 = 2 \quad \text{if } \; S_i = M \; \text{ with } i = 2, ..., p, p+2, ..., p+q \; , \\
\gamma_{i1} &= \gamma_{i,p+1} = 0 \quad \text{if } \; S_i = L \; \text{ with } i = 2, ..., p, p+2, ..., p+q \; , \\
\gamma_{p+1,1} &= \gamma_{1,p+1} + 2 = 2 \; , \\
\gamma_{i,i} &= \varepsilon(S_i), \quad \text{with } \; i = 2, ..., p, p+2, ..., p+q \; , \\
\gamma_{1,1} &= \varepsilon(S_1) + 1 = 1 \; , \\
\gamma_{p+1,p+1} &= \varepsilon(S_{p+1}) - 1 = -1
\end{aligned}$$

and all the other elements of the matrix are zeros.

Example. To the previous kneading sequence $(RRLMA, LLRB)$ we have:

$$\gamma = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 \end{bmatrix} .$$

Proposition 1. *The matrix γ introduced before satisfies the following diagram:*

$$\begin{array}{ccc} C_0 & \xrightarrow{\ \eta\ } & B_0 \\ {\scriptstyle\gamma}\downarrow & & {\scriptstyle\beta}\downarrow \\ C_0 & \xrightarrow{\ \eta\ } & B_0 \end{array}$$

Proof: Let $(y_1, \ldots, y_{p+q}) = (x_0^{(1)}, \ldots, x_{p-1}^{(1)}, x_0^{(2)}, \ldots, x_{q-1}^{(2)})$,

$$(S_1, \ldots, S_{p+q}) = (A, P_1, \ldots, P_{p-1}, B, Q_1, \ldots, Q_{q-1})$$

and denote by ρ the following permutation:

$$(y_1, \ldots, y_{p+q}) \to (y_{\rho(1)}, \ldots, y_{\rho(p+q)}) = (z_1, \ldots, z_{p+q})$$

then the matrix π defined previously is given by:

$$\pi = [\pi_{ij}]$$

where $\pi_{ij} = \delta_{\rho(i)j}$ and, in a similar way:

$$\eta = [\eta_{ik}] = [\mu_{ij}]\,[\pi_{jk}]$$

where $\mu_{ij} = \delta_{i+1,j} - \delta_{i,j}$. Thus, we have:

$$\eta_{ik} = \sum_j \mu_{ij}\,\pi_{jk} = \sum_j \Big(\delta_{i+1,j}\,\delta_{\rho(j),k} - \delta_{i,j}\,\delta_{\rho(j),k}\Big) = \delta_{\rho(i+1),k} - \delta_{\rho(i),k} \ .$$

On the other hand, the matrix β can be given by:

$$\beta_{ij} = \begin{cases} \varepsilon(S_{\rho(i)})\,\delta_{ij} & \text{if } \rho(i) \neq 1 \text{ and } \rho(i) \neq p+1, \\ -\delta_{ij} & \text{if } \rho(i) = 1 \ (i,j \in \{1,...,p+q-1\}), \\ \delta_{ij} & \text{if } \rho(i) = p+1 \ . \end{cases}$$

If we denote by $X = [X_{ik}] = [\beta_{ij}]\,[\eta_{jk}]$ so $X_{ik} = \sum_j \beta_{ij}\,\eta_{jk}$ and then we have:

$$X_{ik} = \begin{cases} \varepsilon(S_{\rho(i)})\,[\delta_{\rho(i+1)k} - \delta_{\rho(i)k}] & \text{if } \rho(i) \neq 1 \text{ and } \rho(i) \neq p+1, \\ -\delta_{1k} - \delta_{\rho(i+1)k} & \text{if } \rho(i) = 1 \ (i, j \in \{1, ..., p+q-1\}), \\ \delta_{\rho(i+1)k} - \delta_{p+1,k} & \text{if } \rho(i) = p+1\ . \end{cases}$$

In a similar way, we denote by $Y = [Y_{ik}] = [\eta_{ij}]\,[\gamma_{jk}]$. Thus

$$Y_{ik} = \sum_j \eta_{ij}\,\gamma_{jk} = \gamma_{\rho(i+1)k} - \gamma_{\rho(i)k}\ .$$

Now to achieve our goal we split our proof in several different cases:

(1) Suppose first that $k \neq \rho(i)$ and $k \neq \rho(i+1)$ then by definition of the elements X_{ik} we have $X_{ik} = 0$. Obviously, in this case, if $k \neq 1$ and $k \neq p+1$ then $\gamma_{\rho(i+1),k} = \gamma_{\rho(i),k} = 0$ (note that, according to the hypothesis these elements are not in the main diagonal and that, in this situation ($k \neq 1$ and $k \neq p+1$), all the other elements of the matrix γ are zeros).

On the other hand, if $k = 1$ then $\gamma_{\rho(i+1),1} = \gamma_{\rho(i),1}$ because $S_{\rho(i)} = S_{\rho(i+1)}$ (to see this, note once more that in this situation the elements $\gamma_{\rho(i+1),1}$, $\gamma_{\rho(i),1}$ are not in the main diagonal). In a similar way, we can see that if $k = p+1$ then $\gamma_{\rho(i+1),p+1} = \gamma_{\rho(i),p+1}$.

(2) Suppose now that $k = \rho(i)$. For simplicity we subdivide this proof in the following three cases:

(**2.1**) If $\rho(i) \neq 1$ and $\rho(i) \neq p+1$ then we have:

$$X_{i,\rho(i)} = -\varepsilon(S_{\rho(i)})\ ,$$
$$Y_{i,\rho(i)} = -\gamma_{\rho(i),\rho(i)}\ ;$$

note that $\gamma_{\rho_(i+1),\rho(i)} = 0$ because this element is not in the main diagonal and by hypothesis is not in the first column nor in the $(p+1)$-column and all the other elements of γ are zeros. But, by definition of γ:

$$\gamma_{\rho(i)\rho(i)} = \varepsilon(S_{\rho(i)})$$

and so $X_{i,\rho(i)} = Y_{i,\rho(i)}$.

(**2.2**) If $\rho(i) = 1$ then $X_{i,1} = X_{\rho^{-1}(1),1} = 1$. On the other hand, we have:

$$Y_{i,1} = \gamma_{\rho(i+1),1} - \gamma_{11} = \gamma_{\rho(i+1),1} = 1\ .$$

Note that according to the hypothesis $\rho(i) = 1$ and so $S_i = A$. Then $S_{i+1}=M$ and thus, by definition of γ, $\gamma_{\rho(i+1),1} = 1$.

(**2.3**) If $\rho(i) = p+1$ then:

$$X_{i,p+1} = X_{\rho^{-1}(p+1),p+1} = -1$$

$$Y_{i,p+1} = Y_{\rho^{-1}(p+1),p+1} = \gamma_{\rho(i+1),p+1} - \gamma_{\rho(i),p+1} = -1 \ .$$

(Note that in this case $S_i = B$ and so $S_{i+1} = R$).

(3) The proof of the case $k = \rho(i+1)$ is very similar to the previous one and so the proof of Proposition 1 is completed. ∎

In what follows we denote by θ the product of γ by the rotation ω introduced before. Thus the square-matrix θ of $(p+q)(p+q)$ elements is given by:

$$\begin{aligned}
\theta_{i,2} &= \delta(S_i) \quad \text{ for } \ i = 1,2,...,p+q \ , \\
\theta_{i,p+2} &= \nu(S_i) \quad \text{ for } \ i = 1,2,...,p+q \ , \\
\theta_{p,1} &= \varepsilon(S_p) \ , \\
\theta_{i,i+1} &= \varepsilon(S_i) \quad \text{ for } \ i = 2,3,...,p-1,p+2,...,p+q-1 \ , \\
\theta_{p+q,p+1} &= \varepsilon(S_{p+q}) \ ,
\end{aligned}$$

and all the others elements of the matrix θ are zeros where S_i is the i-element of the $p+q$-tuple $(A, P_1, \ldots, P_{p-1}, B, Q_1, \ldots, Q_{q-1})$ and $\varepsilon(L) = \varepsilon(R) = 1$, $\varepsilon(M) = -1$, $\delta(R) = \delta(M) = \delta(B) = 2$, $\delta(A) = 1$, $\delta(L) = 0$, $\nu(L) = \nu(M) = \nu(A) = 0$, $\nu(B) = -1$ and $\nu(R) = -2$.

Then the characteristic polynomial of the matrix θ satisfies the next proposition.

Proposition 2. *To each pair of sequences $(P_1 \ldots P_{p-1}A, Q_1 \ldots Q_{q-1}B)$ we have:*

$$P_\theta(t) = \det(I - \theta t) =$$

$$= \left[1 - \delta(P_1)\,t - \sum_{i=2}^{p} \delta(P_i) \Big(\prod_{j=1}^{i-1} \varepsilon(P_j)\Big)\, t^i\right].\left[1 - \nu(Q_1)\,t - \sum_{i=2}^{q} \nu(Q_i) \Big(\prod_{j=1}^{i-1} \varepsilon(Q_j)\Big)\, t^i\right]$$

$$- \left[\nu(P_1)\,t + \sum_{i=2}^{p} \nu(P_i) \Big(\prod_{j=1}^{i-1} \varepsilon(P_j)\Big)\, t^i\right].\left[\delta(Q_1)\,t + \sum_{i=2}^{q} \delta(Q_i) \Big(\prod_{j=1}^{i-1} \varepsilon(Q_j)\Big)\, t^i\right] .$$

Proof: Let $\bar{\theta} = I - \theta\, t =$

$$\begin{bmatrix} 1 & -\delta(A)t & 0 & \ldots & 0 & 0 & -\nu(A)t & 0 & \ldots & 0 \\ 0 & 1-\delta(P_1)t & -\varepsilon(P_1)t & \ldots & 0 & 0 & -\nu(P_1)t & 0 & \ldots & 0 \\ 0 & -\delta(P_2)t & 1 & \ldots & 0 & 0 & -\nu(P_2)t & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ -\varepsilon(P_{p-1})t & -\delta(P_{p-1})t & 0 & \ldots & 1 & 0 & -\nu(P_{p-1})t & 0 & \ldots & 0 \\ 0 & -\delta(B)t & 0 & \ldots & 0 & 1 & -\nu(B)t & 0 & \ldots & 0 \\ 0 & -\delta(Q_1)t & 0 & \ldots & 0 & 0 & 1-\nu(Q_1)t & -\varepsilon(Q_1)t & \ldots & 0 \\ 0 & -\delta(Q_2)t & 0 & \ldots & 0 & 0 & -\nu(Q_2)t & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & -\delta(Q_{q-1})t & 0 & \ldots & 0 & \varepsilon(Q_{q-1}) & -\nu(Q_{q-1})t & 0 & \ldots & 1 \end{bmatrix},$$

where I is the identity matrix. Now, if we expand the determinant of $\bar{\theta}$ in terms of the p-row elements, according to Laplace theorem we get four terms associated to the elements $\bar{\theta}_{p,1}$, $\bar{\theta}_{p,2}$, $\bar{\theta}_{p,p}$ and $\bar{\theta}_{p,p+2}$ of the matrix $\bar{\theta}$ (note that these are the non-zero elements of the p-row of the matrix $\bar{\theta}$).

The expansion associated to the $\bar{\theta}_{p,2}$ element leads to:

$$\begin{aligned} b(t) &= \Big[-(-1)^{p+2}\,\delta(P_{p-1})\,t\,(-1)^{p-2}\,\varepsilon(P_1)\cdots\varepsilon(P_{p-2})\,t^{p-2}\Big]\,(1-\nu_Q) \\ &= -\varepsilon(P_1)\cdots\varepsilon(P_{p-2})\,\delta(P_{p-1})\,t^{p-1}(1-\nu_Q) \end{aligned}$$

where

$$(1-\nu_Q) = 1-\nu(Q_1)\,t - \sum_{i=2}^{q}\nu(Q_i)\,\Big(\prod_{j=1}^{i-1}\varepsilon(Q_j)\Big)\,t^i$$

is the determinant of the matrix:

$$\begin{bmatrix} 1 & -\nu(Q_q)t & 0 & \ldots & 0 & 0 \\ 0 & 1-\nu(Q_1)t & -\varepsilon(Q_1)t & \ldots & 0 & 0 \\ 0 & -\nu(Q_2)t & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & -\nu(Q_{q-2})t & 0 & \ldots & 1 & -\varepsilon(Q_{q-2})t \\ -\varepsilon(Q_{q-1})t & -\nu(Q_{q-1})t & 0 & \ldots & 0 & 1 \end{bmatrix}$$

which can be obtained from the expansion in terms of the last row.

The expansion associated to the $\bar{\theta}_{p,1}$ element leads to a determinant that can be obtained from the non-zero elements of the first row ($\delta(P_p)$ and $\nu(P_p)$) giving:

$$\begin{aligned} c(t) &= \Big[-(-1)^{p+1}\,\varepsilon(P_{p-1})\,t\,(-\delta(P_p)\,t)\,(-1)^{p-2}\,\varepsilon(P_1)\cdots\varepsilon(P_{p-2})\,t^{p-2}\Big]\,(1-\nu_Q) \\ &\quad + (-1)^{p+1}\,\varepsilon(P_{p-1})\,t\,\nu(P_p)\,t\,(-1)^{p+2}\,\varepsilon(P_1)\cdots\varepsilon(P_{p-2})\,t^{p-2}\,\delta_Q \\ &= -\varepsilon(P_1)\cdots\varepsilon(P_{p-1})\,\delta(P_p)\,t^p\,(1-\nu_Q) - \varepsilon(P_1)\cdots\varepsilon(P_{p-1})\,\nu(P_p)t^p\,\delta_Q \end{aligned}$$

where

$$\delta_Q = \delta(Q_1)\, t + \sum_{i=2}^{q} \delta(Q_i) \Big(\prod_{j=1}^{i-1} \varepsilon(Q_j)\Big)\, t^i$$

is the determinant of the matrix:

$$\begin{bmatrix} -\delta(Q_q)t & 1 & 0 & \dots & 0 \\ -\delta(Q_1)t & 0 & -\varepsilon(Q_1)t & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -\delta(Q_{q-2})t & 0 & 0 & \dots & -\varepsilon(Q_{q-2})t \\ -\delta(Q_{q-1})t & -\varepsilon(Q_{q-1})t & 0 & \dots & 1 \end{bmatrix}$$

which can be easily obtained from the expansion in terms of the first column.

On the other hand, the expansion associated to the $\bar\theta_{p,p+2}$ element leads to:

$$\begin{aligned} e(t) &= \Big[-(-1)^{2p+2}\, \nu(P_{p-1})\, t\, (-1)^{p-2}\, (-1)^{p-2}\, \varepsilon(P_1) \cdots \varepsilon(P_{p-2})\, t^{p-2}\Big]\, \delta_Q \\ &= -\varepsilon(P_1) \cdots \varepsilon(P_{p-2})\, \nu(P_{p-1})\, t^{p-1}\, \delta_Q \ . \end{aligned}$$

Finally, the expansion associated to the $\bar\theta_{p,p}$ element leads to a determinant that is obtained from the first column giving a determinant $\bar\gamma$ of $(p+q-2)(p+q-2)$ elements. Note that the first $(p-2)(p-2)$ elements of this determinant are given by:

$$\begin{vmatrix} 1-\delta(P_1)t & -\varepsilon(P_1)t & \dots & 0 \\ -\delta(P_2)t & 1 & \dots & 0 \\ -\delta(P_3)t & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ -\delta(P_{p-3})t & 0 & \dots & -\varepsilon(P_{p-3})t \\ -\delta(P_{p-2})t & 0 & \dots & 1 \end{vmatrix} .$$

Now, after interchanging the last row with all other rows and the last column with all other columns, we get a new determinant which is equal to the initial one (because we only made elementary operations). Note once more, that this determinant is formally identical to the initial determinant of the matrix $\bar\theta$ and that the first $(p-2)(p-2)$ elements of this new determinant are:

$$\begin{vmatrix} 1 & -\delta(P_{p-2})t & \dots & 0 \\ 0 & 1-\delta(P_1)t & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ -\varepsilon(P_{p-3})t & -\delta(P_{p-3})t & \dots & 1 \end{vmatrix} .$$

So by a recursive method we get:

$$\det \bar{\gamma} =$$

$$= \left[1 - \delta(P_1)t - \sum_{i=2}^{p-2} \delta(P_i) \left(\prod_{j=1}^{i-1} \varepsilon(P_j)\right) t^i\right] \left[1 - \nu(Q_1)t - \sum_{i=2}^{q} \nu(Q_i) \left(\prod_{j=1}^{i-1} \varepsilon(Q_j)\right) t^i\right]$$

$$- \left[\nu(P_1)t + \sum_{i=2}^{p-2} \nu(P_i) \left(\prod_{j=1}^{i-1} \varepsilon(P_j)\right) t^i\right] \left[\delta(Q_1)t + \sum_{i=2}^{q} \delta(Q_i) \left(\prod_{j=1}^{i-1} \varepsilon(Q_j)\right) t^i\right] .$$

Adding the terms $b(t)$, $c(t)$, $e(t)$ to the previous one we get the desired result. ∎

This result is very important because it is used to prove the next proposition which establishes how the kneading-determinant and the characteristic polynomial of the matrix θ are related and also because it allows for a easy computation of the topological entropy associated to a map of the interval with two critical points (see [6]). In fact, we have:

Proposition 3. *To each pair of sequences* $(P_1 \dots P_{p-1}A, Q_1 \dots Q_{q-1}B)$ *we have:*

$$P_\theta(t) = (1-t)\,(1-t^p)\,(1-t^q)\,D(t)$$

where $D(t)$ *is the kneading-determinant.*

Proof: According to the previous proposition and denoting by $\tau_k = \prod_{i=1}^{k-1} \varepsilon(S_i)$, $k > 1$, $\tau_1 = 1$ when $k = 1$, we have:

$$\det(I - \theta t) =$$

$$= \left[1 - \delta(P_1)t - \sum_{i=2}^{p} \delta(P_i) \left(\prod_{j=1}^{i-1} \varepsilon(P_j)\right) t^i\right] . \left[1 - \nu(Q_1)\,t - \sum_{i=2}^{q} \nu(Q_i) \left(\prod_{j=1}^{i-1} \varepsilon(Q_j)\right) t^i\right]$$

$$- \left[\nu(P_1)t + \sum_{i=2}^{p} \nu(P_i) \left(\prod_{j=1}^{i-1} \varepsilon(P_j)\right) t^i\right] . \left[\delta(Q_1)t + \sum_{i=2}^{q} \delta(Q_i) \left(\prod_{j=1}^{i-1} \varepsilon(Q_j)\right) t^i\right] .$$

Then we conclude that:

$$\det(I-\theta t) = \Big(1 - 2\sum_{\substack{1\le i\le p-1\\ P_i=M}} \tau_i t^i - 2\sum_{\substack{1\le i\le p-1\\ P_i=R}} \tau_i\, t^i - \tau_p\, t^p\Big)$$
$$\times\Big(1 + 2\sum_{\substack{1\le i\le q-1\\ Q_i=R}} \hat\tau_i\, t^i + \hat\tau_q\, t^q + 2\sum_{\substack{1\le i\le q-1\\ P_i=R}} \tau_i\, t^i\Big)$$
$$\times\Big(2\sum_{\substack{1\le i\le q-1\\ Q_i=M}} \hat\tau_i\, t^i + 2\sum_{\substack{1\le i\le q-1\\ Q_i=R}} \hat\tau_i\, t^i + 2\,\hat\tau_q\, t^q\Big)$$

where $\tau_i = \prod_{j=1}^{i-1}\varepsilon(P_j)$ and $\hat\tau_i = \prod_{j=1}^{i-1}\varepsilon(Q_j)$. In what follows we denote M_P by:

$$M_P = \sum_{\substack{1\le i\le p-1\\ P_i=M}} \tau_i\, t^i$$

(in a similar way, $M_Q = \sum_{\substack{1\le i\le q-1\\ Q_i=M}} \hat\tau_i\, t^i$, $R_P = \ldots$, $R_Q = \ldots$) to get an abbreviated formula for $\det(I-\theta t)$. In fact, with these notations, we have:

$$\begin{aligned}\det(I-\theta t) &= (1-2M_P-2R_P-\tau_p\, t^p)\,(1+2R_Q+\hat\tau_q\, t^q)\\ &\quad + 2R_P(2M_Q+2R_Q+2\hat\tau_q\, t^q)\\ &= 1+2R_Q+\hat\tau_q\, t^q-2M_P-4M_PR_Q-2M_P\,\hat\tau_q\, t^q\\ &\quad -2R_P+2R_P\,\hat\tau_q\, t^q-\tau_p\, t^p-2R_Q\,\tau_p\, t^p-\tau_p\,\hat\tau_q\, t^{p+q}+4M_QR_P\ .\end{aligned}$$

In order to prove our proposition we suppose that $P_1\ldots P_{p-1}$ has odd parity and that $Q_1\ldots Q_{q-1}$ has even parity (the others three cases works analogously). Then we have $\tau_p=-1$, $\hat\tau_q=1$ and so:

$$\begin{aligned}\det(I-\theta t) = 1&+2R_Q+t^q-2M_P-4M_PR_Q-2M_P\, t^q-2R_P\\ &+2R_P\, t^q+t^p+2R_Q\, t^p+t^{p+q}+4M_QR_P\ .\end{aligned}$$

On the other hand, following the Milnor–Thurston techniques we have:

$$\begin{aligned}\nu_1(t) &= M-L-2\tau_1P_1\, t-2\tau_2P_2\, t^2-\cdots-2\tau_{p-1}P_{p-1}\, t^{p-1}-2\tau_pP_p\, t^p\\ &\quad -2\tau_1P_1\, t^{p+1}-2\tau_2P_2\, t^{p+2}-\cdots-2\tau_{p-1}P_{p-1}\, t^{2p-1}-2\tau_p\, P_p\, t^{2p}-\ldots\\ &= M-L-\frac{2\tau_1P_1\, t+2\tau_2P_2\, t^2+\cdots+2\tau_p\, P_p\, t^p}{1-t^p}\end{aligned}$$

and

$$\nu_2(t) = R-M+\frac{2\hat\tau_1Q_1\, t+2\hat\tau_2Q_2\, t^2+\cdots+2\hat\tau_qQ_q\, t^q}{1-t^q}\ .$$

Thus, with the previous assumptions $\tau_p = -1$, $\hat{\tau}_q = 1$ we have (see examples of section 1):

$$\nu_1(t) = \frac{M - M\,t^p - L + L\,t^p - 2\tau_1 P_1\,t - \cdots - 2\tau_{p-1}P_{p-1}\,t^{p-1} + 2M\,t^p}{1 - t^p}$$

and,

$$\nu_2(t) = \frac{R - R\,t^q - M + M\,t^q + 2\hat{\tau}_1 Q_1\,t + \cdots + 2\hat{\tau}_{q-1}Q_{q-1}\,t^{q-1} + 2R\,t^q}{1 - t^q}$$

hence,

$$D(t)\,(1-t) = \det \begin{bmatrix} \dfrac{1 - 2M_p + t^p}{1 - t^p} & \dfrac{-2R_p}{1 - t^p} \\ \dfrac{-1 + 2M_Q + t^q}{1 - t^q} & \dfrac{1 + 2R_Q + t^q}{1 - t^q} \end{bmatrix}$$

then, we have:

$$d(t) = \frac{(1 - 2M_P + t^p)\,(1 + 2R_Q + t^q) + 2R_P(-1 + 2M_Q + t^q)}{1 - t}$$

and so:

$$\begin{aligned}(1-t)\,d(t) &= 1 + 2R_Q + t^q - 2M_P - 4M_PR_Q - 2M_P\,t^q + t^p + 2R_Q\,t^p + t^{p+q} \\ &\quad + 4R_PM_Q - 2R_P + 2R_P\,t^q \\ &= \det(I - \theta t)\ . \ \blacksquare\end{aligned}$$

Example. If we return once more to the kneading sequence $(RRLMA, LLRB)$ we get:

$$\theta = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 1 & -2 & 0 & 0 \end{bmatrix}$$

and by proposition 2 we have:

$$P_\theta(t) = 1 - 2t - 2t^2 + 2t^3 - t^4 + 3t^5 + 2t^6 - 4t^7 + t^9$$

following proposition 3 we also have:

$$\begin{aligned} P_\theta(t) &= (1-t)\,(1-t^5)\,(1-t^4)\,D(t) \\ &= (1-t)\,(1-t^5)\,(1-t^4)\,\frac{1-2t-2t^2+2t^3-t^4+3t^5+2t^6-4t^7+t^9}{(1-t)\,(1-t^5)\,(1-t^4)}\ . \end{aligned}$$

In what follows let's denote by ψ the non-negative matrix defined by $\psi = \alpha\beta$. This matrix defines a subshift of finite type that is "equivalent" to the matrix θ, according to the next commutative diagram:

$$\begin{array}{ccc} C_0 & \xrightarrow{\ \eta\ } & B_0 \\ {\scriptstyle\theta}\big\downarrow & & {\scriptstyle\psi}\big\downarrow \\ C_0 & \xrightarrow{\ \eta\ } & B_0 \end{array}$$

Proposition 4. *The characteristic polynomial of the matrix θ satisfies the following result:*

$$P_\theta(t) = (1-t)\det(I-\psi\,t)\ .$$

Proof: This result follows from the commutativity of the previous diagram, from a generalization of this result to an exact sequence and from an homological algebra theorem [7]. ∎

Example. We return to our kneading sequence ($RRLMA$, $LLRB$) and we get:

$$\psi = \alpha\,\beta = \begin{bmatrix} 0&0&1&1&0&0&0&0\\ 0&0&0&0&1&1&1&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&1&1&1&1&1\\ 1&1&1&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&1&1&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix}$$

and $\det(I-\psi t) = 1-t-3t^2-t^3-2t^4+t^5+3t^6-t^7-t^8$.

To conclude we point out the simplicity of the expression in proposition 2 which according to proposition 3 allows for a simple computation of symbolic invariants of all the bimodal maps of the interval, this being important in applications. Note that θ is directly obtained from the kneading sequence while the "traditional" Markov matrix (ψ, here) is obtained with a great deal more difficulty.

This novel approach has the obvious consequence of providing a simplified alternative method to the $0-1$ matrices.

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REFERENCES

[1] Chavoya-Aceves, O., Angulo-Broussard, F. and Piña, E. – Symbolic dynamics of the cubic maps, *Physica,* 14D (1985), 374–386.

[2] Bowen, R. and Franks, J. – Homology for zero-dimensional nonwandering sets, *Ann. of Math.,* 106 (1977), 73–92.

[3] Coste, J. and Peyraud, N. – A new type of period-doubling bifurcation in one-dimensional transformations with two extrema, *Physica,* 5D (1982), 415–420.

[4] Hockett, K. and Holmes, P. – Bifurcation to badly ordered orbits in one-parameter families of circle maps or angels fallen from the devil's staircase, *Proc. of the American Math. Soc.,* 102(4) (1988), 1031–1051.

[5] Lampreia, J.P., Rica da Silva, A. and Sousa Ramos, J. – *Homology for Iterated Maps of the Interval*, Actas das XII Jornadas Luso Espanholas – Braga, 1987.

[6] Lampreia, J.P. and Sousa Ramos, J. – *Computing the Topological Entropy of Bimodal Maps*, European Conference on Iteration Theory (ECIT87), World Scientific Publishing Co. Pte. Ltd., 1989, 431–437.

[7] Lang, L. – *Algebra*, Addison–Wesley, 1965.

[8] Mackay, R.S. and Tresser, C. – Some flesh on the skeleton: the bifurcation structure of bimodal maps, *Physica,* 27D (1987), 412–422.

[9] Mackay, R.S. and Tresser, C. – Boundary of topological chaos for bimodal maps of the interval, *J. London Math. Soc.,* 37(2) (1988), 164–181.

[10] May, R. – *Chaotic Behavior of Deterministic Systems*, Les Houches Summer School, 1981.

[11] Milnor, J. and Thurston, W. – *On Iterated Maps of the Interval*, in "Dynamical System" (J.C. Alexander, ed.), Procededings Univ. Maryland 1986–1987, Lect. Notes in Math. n.1342, Springer-Verlag, 1988.

[12] Perez, R. and Glass, L. – Bistability, period-doubling bifurcations and chaos in a periodically forced oscillator, *Phys. Lett.,* A90 (1982), 441–443.

[13] Rogers, T.D. and Whitley, D.C. – Chaos in the cubic mapping, *Math. Modelling,* 4 (1983), 9–25.

[14] Testa, J. and Held, G.A. – Study of a one-dimensional map with multiple basins, *Phys. Rev.,* A28 (1983), 3085–3089.

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