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COMPLEX POWERS OF THE WAVE OPERATOR

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1 – Introduction

In the late 1940s, Marcel Riesz in his paper "L'intégrale de Riemann–Liouville et le Problème de Cauchy" ([7]) constructed a holomorphic family of kernels, K_s, $s \in \mathbb{C}$, associated to the wave operator, $\Box$, on the product of a Riemannian manifold and $\mathbb{R}$ which can be regarded as its complex powers.

The family has the properties

$$K_1 = \Box ,$$
$$K_r \circ K_s = K_{r+s} , \qquad r, s \in \mathbb{C} ,$$
$$K_0 = \mathrm{Id} ,$$

and K_{-1} equals the forward fundamental solution of the wave operator. Our purpose, in this paper, is to give a new direct and short construction and to use this construction to obtain a good understanding of the micro-local properties of these kernels. We proceed by using a modification of the method of descent to construct the kernels.

The complex powers of elliptic operators and their micro-local properties are well understood — Seeley showed that they form a holomorphic family of pseudo-differential operators in [8]. However for the wave operator, it is clear that such a result can not be true as its fundamental solutions have singularities which are not on the diagonal reflecting the fact it is not hypo-elliptic and so even the -1^{th} power is not pseudo-differential.

In fact, the forward fundamental solution lies in a class of distributions which is now well understood. It is a paired Lagrangian distribution or a pseudo-differential operator with singular symbol. That is its wavefront set lies in a

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union of two Lagrangian submanifolds, the conormal bundle to the diagonal, Δ, and the conormal bundle to the light cone, C, and it is a Lagrangian distribution away from the intersection of these two submanifolds and its behaviour at the intersection is controlled by the behaviour elsewhere (see for example [3] or [5].) This is therefore a natural class in which to look for the complex powers. We show that

$$K_s \in I_{phg}^{2s,s-\frac{1}{2}}(N^*(\Delta), N^*(C))$$

and compute the principal symbol on each Lagrangian submanifold. We thus achieve a good micro-local understanding of the complex powers. For the reader not familiar with paired Lagrangian distributions, this implies that if P is a zeroth order pseudo-differential operator which is smoothing near $N^*(\Delta) \cap N^*(C)$ then

$$PK_s \in I^{2s}(N^*(\Delta)) + I^{s-\frac{1}{2}}(N^*(C))$$

and in addition nothing special happens at the intersection.

We set about proving this by using the theory of Fourier integral operators to reduce to the constant coefficient case. We understand the constant coefficient case by showing an equivalence with an alternative definition in terms of the inverse Fourier transform of a regularisation of $(\tau^2 - \xi^2)^s$. This allows us to determine the micro-local properties directly and to compute the principal symbols.

2 – Complex powers of the wave operator

In this section, we define the complex powers of the wave operator on the cartesian product of $\mathbb{R}$ and a Riemannian manifold M; we then proceed to show that they form a holomorphic family satisfying the group law.

We use the holomorphic family of distributions, χ_+^α, on $\mathbb{R}$ these are a holomorphic extension of the distributions

$$\chi_+^a = H(x)\, x^a/\Gamma(a+1)\,, \qquad \Re a > -1\ ,$$

where H is the Heaviside function and Γ is the Gamma function. These have the property that

$$\chi_+^{-k} = \delta_0^{(k-1)}\,, \qquad k = 0, 1, 2, 3, \ldots$$

A full account of these distributions can be found in section 3.2 of [2], Vol. 1.

The forward fundamental solution of the wave operator on $M \times \mathbb{R} \times \mathbb{R}$ can be written as $K(x, x', t, r)$, since the wave operator is constant coefficient in t and r.

Let π_r denote the projection

$$(x, x', t, r) \mapsto (x, x', t) \ .$$

Definition 2.1. For $\Re s \ll 0$, $L_s = 2e^{i\pi(s+1)}(\pi_r)_* (\chi_+^{2(-s-1)}(r)\, K(x, x', t, r))$.

The problem here is that we are taking a product not permitted by the calculus of wavefront sets but this is permissible provided one of the factors is sufficiently smooth, which is why we take $\Re s \ll 0$. The push forward is well defined because for (x, x', t) in a compact set, it is compactly supported in r. We will use analytic continuation to extend to all $s \in \mathbb{C}$. The family is holomorphic where defined because it is the product of a holomorphic function and a linear function of a holomorphic family of distributions. The fact that K is supported in the set

$$\Big\{(x, x', t, r)\colon\ t \geq 0,\ d(x, x')^2 + r^2 \leq t^2\Big\}$$

immediately implies that L_s is supported in the set

$$\Big\{(x, x', t)\colon\ t \geq 0,\ d(x, x')^2 \leq t^2\Big\}$$

which is the forward light cone.

To carry out the analytic continuation we need to show that

$$(D_t^2 - \Delta)^k\, L_{s-k}$$

for any s and large k is independent of k. We commence by showing that

$$(D_t^2 - \Delta)^k\, L_{-k} = \mathrm{Id} \ .$$

To do this we use an alternative representation for s a negative integer,

$$L_s = e^{i\pi(s+1)}\, \frac{(\pi_r)_* \Big(r^{2(-1-s)} K(x, x', t, r)\Big)}{\Gamma\Big(-2(s+1) + 1\Big)} \ , \tag{2.1}$$

which follows from the fact that K is even in r. This definition will make sense for *any* negative integer s as in this case $r^{2(-1-s)}$ is smooth; we denote these additional kernels L'_s. Now, let $\phi \in C_0^\infty(X \times X \times \mathbb{R})$ and then in the case of $s = -1$, we have

$$\Big\langle (D_t^2 - \Delta) L'_{-1},\, \phi \Big\rangle = \Big\langle K,\, (D_t^2 - \Delta)^t\, \pi_r^* \,\phi \Big\rangle \tag{2.2}$$

$$= \Big\langle K,\, (D_t^2 - \Delta - D_r^2)^t\, \pi_r^* \,\phi \Big\rangle \tag{2.3}$$

$$= \Big\langle \delta(t, r)\, \delta_{x'}(x),\, \pi_r^* \,\phi \Big\rangle \tag{2.4}$$

$$= \Big\langle \delta(t, r)\, \delta_{x'}(x), \phi \Big\rangle \ . \tag{2.5}$$

This means that $(D_t^2-\Delta)L'_{-1} = \mathrm{Id}$ when the kernels are interpreted as operators. And for $-s-1$ an integer greater than zero we have

$$\left\langle (D_t^2-\Delta)L'_s, \phi \right\rangle = e^{i\pi(s+1)} \frac{\left\langle K,\, (D_t^2-\Delta)^t\, r^{2(-s-1)}\, \pi_r^*\,\phi \right\rangle}{\Gamma\Big(2(-s-1)+1\Big)} \tag{2.6}$$

$$= e^{i\pi(s+1)} \frac{\left\langle K,\, (D_t^2-\Delta-D_r^2)^t\, r^{2(-s-1)}\, \pi_r^*\,\phi \right\rangle}{\Gamma\Big(2(-s-1)+1\Big)} \tag{2.7}$$

$$+\, e^{i\pi(s+1)} \frac{\left\langle K,\, D_r^2 r^{2(-s-1)}\, \pi_r^*\,\phi \right\rangle}{\Gamma\Big(2(-s-1)+1\Big)} \tag{2.8}$$

$$= -e^{i\pi(s+1)} \frac{\left\langle K,\, \left(\frac{\partial}{\partial r}\right)^2 r^{2(-s-1)}\, \pi_r^*\,\phi \right\rangle}{\Gamma\Big(2(-s-1)+1\Big)} \tag{2.9}$$

$$= \langle L'_{s+1}, \phi\rangle\ . \tag{2.10}$$

The first term in (2.8) vanishes because $r^{2(-s-1)}\delta(t,r) = 0$.

So using the equality of L_{-k} and L'_k, for k a large positive integer, we have that

$$(D_t^2-\Delta)^k L_{-k} = \delta_{x'}(x)\,\delta(t)\ .$$

Now, using a similar argument, we can show that

$$(D_t^2-\Delta)^k L_s = L_{s+k}$$

where both are defined. Putting these two facts together and regarding the L_s as operators we have that

$$L_{-k}\circ(D_t^2-\Delta)^k\circ L_s = L_{-k}\circ L_{k+s}$$

and thus

$$L_p\circ L_s = L_{s+p}$$

for p a large negative integer.

It remains to extend the composition law to all $p\in\mathbb{C}$ with $\Re p \ll 0$. We do this (see [7], p.197) by using

Theorem 2.1. *If f is a holomorphic function on a half p1ane $\Re z < \alpha$, dominated by a non-vanishing holomorphic function and vanishes on the integers then f is identically zero.*

Now, $K \in (C^0)'$ so if we let $\phi \in C_0^\infty(X \times X \times \mathbb{R})$ and choose $\psi(r) \in C_0^\infty(\mathbb{R})$ so that $\psi \equiv 1$ on a sufficiently large set then as K is supported inside the light cone, we have

$$|\langle L_s, \phi\rangle| \leq \frac{2}{|\Gamma(-2s-1)|} \|K\|_0 \left\| \chi_+^{-2s-2}(r)\, \psi(r)\, \phi(x,x',t)\right\|_{C^0} \tag{2.11}$$

$$\leq \frac{C'}{|\Gamma(-2s-1)|} |C^{2s}|\, |e^{\pi i(s+1)}| \ . \tag{2.12}$$

Thus, we have that

$$|\langle L_{s+p}, \phi\rangle| \leq \frac{C'}{\left|\Gamma(-2(s+p)-1)\right|} |C^{2(s+p)}|\, |e^{\pi i(s+p+1)}| \ . \tag{2.13}$$

We need to establish a similar bound for $\langle L_s \circ L_p, \phi\rangle$. Picking $\psi(r) \in C_0^\infty(\mathbb{R})$ identically 1 on a sufficiently large set we obtain

$$\begin{aligned}\langle L_s \circ L_p,\, \phi\rangle = 2e^{i\pi(s+p)} &\int \cdots \int \chi_+^{2(-s-1)}\,(r-r')\,\chi_+^{2(-p-1)}(r') \\ &\cdot K(x,x',t-t',r-r')\,K(x',x'',t',r')\,dr'\,dt'\,dx' \cdot \\ &\cdot \psi(r)\,\phi(x,x',t)\,dt\,dx\,dx''\end{aligned} \tag{2.14}$$

$$= 2e^{i\pi(s+p)} \int\!\!\int \chi_+^{2(-s-1)}(r-r')\,\chi_+^{2(-p-1)}(r')\,b(r',r-r')\,\psi(r)\,dr\,dr' \ , \tag{2.15}$$

where

$$b(r,s) = \int \cdots \int K(x,x',t-t',s)\,K(x',x'',t',r)\,\phi(x,x'',t)\,dt\,dx\,dx'\,dt'\,dx'' \ . \tag{2.16}$$

The advantage of this representation is that $b(r,s)$ is continuous which allows us to remove it from the integral. Now, $b(r,s)$ is defined by a push-forward of a paired Lagrangian distribution but points near the flow out are wiped out by the push-forward and we only retain points that are conormal to $r=0$ or to $s=0$. Now, since on the diagonal K is a conormal distribution of order -2 we obtain

$$b(r,s) = \int\!\!\int e^{i(r\gamma+s\tau)}\, c(r,s,\gamma,\tau)\,d\gamma\,d\tau + C^\infty \ , \tag{2.17}$$

where c is a product type symbol in (γ,τ) of order $(-2,-2)$. This means that b is paired Lagrangian distribution associated to $N^*(s=0)$, $N^*(t=0)$ and $N^*(s=t=0)$ with symbolic order $-2,-2,-4$. This shows that $b \in H^{\frac{7}{4}}(\mathbb{R}^2)$ and so by the Sobolev embedding theorem b is continuous.

Hence, we have

$$\text{(2.18)}\quad |\langle L_s \circ L_p,\, \phi\rangle| \le \sup(|b|) \int |\psi(r)| \int \chi_+^{2(-s-1)}(r-r')\, \chi_+^{2(-p-1)}(r')\, dr'\, dr$$

$$\text{(2.19)}\quad = \sup(|b|) \int |\psi(r)|\, \chi_+^{2(-s-p-1)}(r)\, dr$$

$$\text{(2.20)}\quad \le \frac{\sup(|b|)}{\Big|\Gamma(2(-s-p-1)+1)\Big|}\, |C^{2(-s-p-1)}|\, |e^{i\pi(s+p)}| \;.$$

Thus, if we regard $\langle L_p \circ L_s - L_{p+s},\, \phi\rangle$ as a function of p, it satisfies the hypotheses of Theorem 2.1 and so is identically zero, which proves the composition law.

We are now in a position to define L_s, for all complex s, by analytic extension.

Definition 2.2. $L_s = (D_t^2 - \Delta)^k L_{s-k}$ where $\Re(s-k) \ll 0$.

We must check that this is independent of the choice of k. But if $k > l$ we have

$$\text{(2.21)}\quad (D_t^2 - \Delta)^k L_{s-k} = (D_t^2 - \Delta)^l\, (D_t^2 - \Delta)^{k-l} L_{s-k} = (D_t^2 - \Delta)^l\, L_{s-l} \;,$$

so it is well defined.

Using analyticity, we can extend the previously proven relations to the whole complex plane. Thus, we have

Theorem 2.2. *L_s is an entire holomorphic family of kernels supported in the forward light cone such that*

$$\text{(2.22)}\quad L_p \circ L_s = L_{p+s} \quad \textit{where} \;\; s, p \in \mathbb{C} \;,$$

$$\text{(2.23)}\quad L_0 = \mathrm{Id} \;,$$

$$\text{(2.24)}\quad L_1 = D_t^2 - \Delta \;,$$

$$\text{(2.25)}\quad L_{-1} = \textit{the forward fundamental solution} \;.$$

3 – Paired Lagrangian distributions

Paired Lagrangian distributions were first introduced by Melrose and Uhlmann in [6] to give a symbolic method of constructing parametrices to real principal operators. However the calculus introduced there is too narrow for our purposes as it has strong constraints on the allowable singularities — it only contains L_j, for $j = -1, 0, 1, 2, \ldots$. We therefore work within the calculus developed in [3] of which a full account will appear in [5]. We recall the necessary facts from [3], [5] in this section.

Definition 3.1. If X is a smooth manifold then the pair of embedded, conic, Lagrangian submanifolds $(\Lambda_0, \Lambda_1) \subset T^*(X) - \{0\}$ form a one-sided pair, if they intersect cleanly and Λ_1 is a submanifold with boundary equal to the intersection.

Proposition 3.1. *If f is a homogeneous symplectomorphism from an open conic neighbourhood U in $T^*(X)$ to a neighbourhood V in $T^*(Y)$ and (Λ_0, Λ_1) is a one-sided intersecting pair then if F is a k^{th} order Fourier integral operator associated to f then*

$$F\colon\ I^{m,p}_{phg}\Big(\Lambda_0 \cap U,\, \Lambda_1 \cap U\Big) \to I^{m+k,p+k}_{phg}\Big(f(\Lambda_0 \cap U),\, f(\Lambda_1 \cap U)\Big)\ .$$

Proposition 3.2. *The distribution equal to the inverse Fourier transform of the distribution $(\xi_1 - i0)^s\, a(\xi)$ on $\mathbb{R}^n$, where a is a classical symbol of order m, is an element of*

$$I^{m+s+\frac{n}{4},m+\frac{n}{4}-\frac{1}{2}}_{phg}\Big(N^*(x=0),\ N^*(x''=0,\ x_1 \geq 0)\Big)\ ,$$

where $x = (x_1, x'')$.

The most important thing we will need is a version of Egorov's theorem for operators with Schwartz kernels equal to paired Lagrangian distributions.

Theorem 3.1. *If (Λ_0, Λ_1) is a one-sided intersecting pair in*

$$(T^*(X) - \{0\}) \times (T^*(X) - \{0\})\ ,$$

f is a homogeneous symplectomorphism from $T^*(Y) - \{0\}$ to $T^*(X) - \{0\}$ and F is an elliptic Fourier operator associated to f with parametrix G then for $P \in I^{m,p}_{phg}(\Lambda_0, \Lambda_1)$ we have that

$$FPG \in I^{m,p}_{phg}\Big((f \times f)^{-1}\Lambda_0,\ (f \times f)^{-1}\Lambda_1\Big)$$

and the symbols are the pull-backs by $f \times f$.

4 – Complex powers of the constant coefficient wave operator

We shall deduce the micro-local properties of our kernels by reducing to the constant coefficient wave operator, Δ_F, on $\mathbb{R}^n$. In this section, we therefore study Δ_F on $\mathbb{R}^n$. Applying the constant coefficient wave operator is equivalent to multiplying the Fourier transform by $\tau^2 - \xi^2$, so naively, we would like to define the s^{th} complex power to be multiplication of the Fourier transform by $(\tau^2 - \xi^2)^s$ but the problem is that the zeroes of $\tau^2 - \xi^2$ introduce singularities. We can surmount this problem by regarding $(\tau^2 - \xi^2)^s$ as the boundary value of a holomorphic function defined in an open cone. (For a discussion of distributions as holomorphic boundary values see [2].)

For the complex powers of the constant coefficient wave operator, we therefore take the representation

$$K_s = \left(\frac{1}{2\pi}\right)^{n+1} \lim_{\epsilon\to 0^+} \int e^{i(x.\xi+t\tau)} \Big((\tau - i\,\epsilon)^2 - \xi^2\Big)^s \,d\xi\,d\tau \ . \tag{4.1}$$

Provided we can show this makes sense, it is clear that

$$K_1 = D_t^2 - D_x^2 \ ,$$

$$K_0 = \text{Id} \ ,$$

$$K_{r+s} = K_r \circ K_s \ ,$$

when we regard K_s as a convolution operator.

Proposition 4.1. *The boundary value distribution* $((\tau - i0)^2 - \xi^2)^s$ *is well defined and tempered.*

Proof: The imaginary part of $(\tau - i\epsilon)^2 - \xi^2$ zero if and only $\epsilon\tau = 0$ and so then τ will be zero and then $(\tau - i\epsilon)^2 - \xi^2 < 0$. So, if we cut along the positive real axis $(\tau^2 - \xi^2)^s$ is well-defined on $\Im\tau < 0$. We take the argument of -1 to be π.

It is easily checked that $|((\tau - i\epsilon)^2 - \xi^2)|^{-1} \le C\epsilon^{-2}$ locally and so the boundary value distribution exists. The distribution $((\tau - i0)^2 - \xi^2)^s$ is tempered as it is homogeneous of order $2s$. ∎

We now establish the properties of K_s as paired Lagrangian distributions. In this case, the Lagrangians are the conormal bundle to the origin, Λ_0, and the closure of the conormal bundle of the light cone, Λ_1^e, i.e. $\overline{N^*(|x| = t,\ t > 0)}$.

Theorem 4.1. $K_s \in I_{phg}^{2s+\frac{n+1}{4}, s+\frac{n+1}{4}-\frac{1}{2}}(\Lambda_0, \Lambda_1^e)$. *On* Λ_0 *away from* Λ_1^e, *the symbol of* K_s *is* $((\tau - i0)^2 - \xi^2)^s$ *and on* Λ_1^e *away from* Λ_0 *the symbol has the asymptotic expansion*

$$\sum_{j=0}^{\infty} (\pm 1)^j \, e^{i\frac{\pi}{2}(s+j)} \, \frac{s(s-1)\cdots(s-j+1)}{j!} \, (2|\xi|)^{s-j} \, t^{-s-j-1} / \Gamma(-s-j)$$

with respect to the phase functions $x.\xi \pm t|\xi|$.

Note our orders here are slightly different as we are working with the constant coefficient kernel. If we take the full kernel, we get

$$K_s(x-y, \, t-t') \; \in \; I_{phg}^{2s, s-\frac{1}{2}}(N^*(\Delta), N^*(C)) \; .$$

Proof: We have defined K_s to be the inverse Fourier transform of $((\tau - i0)^2 - \xi^2)^s$, so away from $\tau^2 - \xi^2 = 0$, it is clear that we have an element of $I_{phg}^{2s+\frac{n+1}{4}}(\Lambda_0)$. The characteristic variety has two components $\tau = |\xi|$ and $\tau = -|\xi|$ and thus the flow out will also. We discuss $\tau = |\xi|$, as $\tau = -|\xi|$ will be the same with a few sign changes. We cut off close to $\tau = |\xi|$, letting $\psi(\xi, \tau) = \phi(1 - |\xi|^{-1} |\tau|)$, where ϕ is a cut-off function, we compute

$$K_s' = \left(\frac{1}{2\pi}\right)^{n+1} \int e^{i(x.\xi + t\tau)} \, \psi(\xi, \tau) \left((\tau - i0)^2 - \xi^2\right)^s d\xi \, d\tau \tag{4.2}$$

$$= \left(\frac{1}{2\pi}\right)^{n+1} \int e^{i(x.\xi + t\tau)} \, \psi(\xi, \tau) \, (\tau - i0 - |\xi|)^s \, (\tau + |\xi|)^s \, d\xi \, d\tau \tag{4.3}$$

$$= \left(\frac{1}{2\pi}\right)^{n+1} \int e^{i(x.\xi + t\tau + t|\xi|)} \, (\tau - i0)^s \, \phi(|\xi|^{-1} \tau) \, (\tau + 2|\xi|)^s \, d\xi \, d\tau \; . \tag{4.4}$$

We want to write K_s' as a Fourier integral operator applied to a distribution associated with the model; we can rewrite the last integral as

$$\int e^{i(\langle x-y, \xi\rangle + \langle t-r, \tau\rangle + t|\xi|)} \, a(x, \xi, t, \tau) \int e^{i(\langle y, \eta\rangle + r\gamma)} (\gamma - i0)^s \, \phi(|\eta|^{-1} \gamma) \, (\gamma + 2|\eta|)^s \, d\eta \, d\gamma \, dy \, dr \, d\tau \, d\xi + C^\infty \; , \tag{4.5}$$

where a is a symbol which is identically one in a conic neighbourhood of $\tau = 0$ and zero outside a conic neighbourhood. From Proposition 3.2 and Proposition 3.1, it now follows that $K_s \in I_{phg}^{2s+\frac{n+1}{4}, s+\frac{n+1}{4}-\frac{1}{2}}(\Lambda_0, \Lambda_1^e)$. To find the symbol on the flow out, we just take the Fourier transform of the Taylor expansion of $(\tau + 2|\xi|)^s$ term by term. ∎

For the general case we will need a more invariant formulation of the symbols' values. The forward light cone is generated by the flow out of the characteristic variety at the origin, $\{x = 0,\, t = 0\}$, by the Hamiltonians of $p_\pm = \tau \pm |\xi|$. These therefore yield natural coordinates, (ξ, t), on Λ_1^e. We can also think of these coordinates as being given by the geodesic flow at time t with initial point

$$x = 0\,, \quad t = 0\,, \quad \tau = \pm|\xi|\ .$$

The symbol of a Lagrangian distribution is a section of the Maslov bundle tensored with the half-density bundle over the Lagrangian submanifold so to specify it we need to choose a trivialization of the Maslov bundle. Now over any Lagrangian submanifold which is a conormal bundle there is a canonical trivialization of the Maslov bundle — see [1] and when two Lagrangian submanifolds intersect cleanly there is a canonical isomorphism between the two Maslov bundles over the intersection — see [6] or [5]. This means that there is a canonical trivialization of the Maslov bundle of Λ_1^e over $\Lambda_0 \cap \Lambda_1^e$ this trivialization can then be extended to all of Λ_1^e by transport by the Hamiltonians of $\tau = \pm|\xi|$. This construction of a trivialization will also work in the curved case — if we use the geodesic flow on the manifold. We shall refer to this trivialization as the natural trivialization induced by the origin and the geodesic flow.

Corollary 4.1. *The principal symbol on Λ_1^e away from the intersection with Λ_0 is*

$$\frac{t^{-s-1}}{\Gamma(-s)}\,(2|\xi|)^s\, e^{i\frac{\pi}{2}s}\,|dt|^{\frac{1}{2}}\,|d\xi|^{\frac{1}{2}}$$

in the coordinates induced by the geodesic flow from $(0, 0, \xi, \pm|\xi|)$ with respect to the natural trivialization of the Maslov bundle induced by the origin and the geodesic flow.

Proof: We must check that the trivialization of the Maslov bundle given by the phase $\phi = x.\xi \pm t|\xi|$ agrees with the natural trivialization on the flow out. Let $\lambda_1 = T_p(\Lambda_1^e)$, λ_2 be the tangent space to the fibre and let $\psi(x, t)$ be a generating function parametrizing a Lagrangian, $\Gamma = (x, t, \psi'_{x,t})$, transversal to the fibre and to Λ. Letting $\mu = T_p(\Gamma)$ we have that the transition function is given by

$$e^{i\frac{\pi}{4}(\sigma(\lambda_1,\lambda_2;\mu)+S(\phi,\psi))}$$

where σ is the cross ratio as defined in [1] and $S(\phi, \psi)$ is the signature of the matrix

$$\begin{pmatrix} \phi''_{\xi\xi} & \phi''_{\xi(x,t)} \\ \phi''_{(x,t)\xi} & \phi''_{(x,t)(x,t)} - \psi''_{(x,t)(x,t)} \end{pmatrix}\ .$$

It will be enough to do all this at some fixed point. Take $\phi = x.\xi + |\xi|t$. We choose a point, $p = (0, 0, \xi, |\xi|)$, at the intersection. We then have that λ_1 is the span of $\{\xi_j \frac{\partial}{\partial\tau} + |\xi| \frac{\partial}{\partial\xi_j}, \sum_l \frac{\partial}{\partial x_l} - \frac{\xi_j}{|\xi|}\frac{\partial}{\partial\tau}\}$ and λ_2 is the span of $\{\frac{\partial}{\partial\xi_j}, \frac{\partial}{\partial\tau}\}$. Thus we need ψ giving μ transversal to λ_1. (Transversality to the fibre is automatic.) Let $\psi(x,t) = x.\xi + |\xi|t + \frac{t^2}{2}$ we have

$$\Gamma = \{(x, t, \xi, |\xi| + t)\}$$

and μ is the span of $\{\frac{\partial}{\partial x_j}, \frac{\partial}{\partial t} + \frac{\partial}{\partial\tau}\}$. Transversality is clear and our matrix is

$$\begin{pmatrix} 0 & \mathrm{Id} & \frac{\xi}{|\xi|} \\ \mathrm{Id} & 0 & 0 \\ \frac{\xi^t}{|\xi|} & 0 & -1 \end{pmatrix}. \tag{4.6}$$

Executing a change of variables, this will have the same signature as

$$\begin{pmatrix} \mathrm{Id} & 0 & \frac{1}{2}\frac{\xi}{|\xi|} \\ 0 & -\mathrm{Id} & -\frac{1}{2}\frac{\xi}{|\xi|} \\ \frac{1}{2}\frac{\xi^t}{|\xi|} & -\frac{1}{2}\frac{\xi^t}{|\xi|} & -1 \end{pmatrix}. \tag{4.7}$$

This has signature -1 as signature is a homotopy type invariant in the space of invertible matrices and the off diagonal elements can be shrunk to zero.

We are left to compute $\sigma(\lambda_1, \lambda_2; \mu)$ this is defined to be equal to $\sigma(\lambda_1^\rho, \lambda_2^\rho; \mu^\rho)$ where the superscript ρ denotes that the subspaces have been intersected by $(\lambda_1 \cap \lambda_2)^\perp$ and reduced by $\lambda_1 \cap \lambda_2$ then λ_1^ρ, λ_2^ρ, μ^ρ are all transversal one-dimensional Lagrangian subspaces $\frac{(\lambda_1\cap\lambda_2)^\perp}{\lambda_1\cap\lambda_2}$. Picking symplectic linear coordinates, (r, η), such that

$$\lambda_1 = \{r = 0\}\ ,$$

$$\lambda_2 = \{\eta = 0\}\ ,$$

$$\mu = \{\xi = Ax\}\ ,$$

$\sigma(\lambda_1^\rho, \lambda_2^\rho; \mu^\rho)$ is the signature of A. Of course, A is a 1×1 matrix here, so we just need to know its sign. It is easily calculated to be 1.

The argument is similar for $\phi = x.\xi - t|\xi|$. ∎

5 – The symbols in the general case

Our final task is to reduce the variable coefficient case to the constant coefficient case. We do this by reducing to the constant coefficient case using Fourier integral operators. Our proof mimics that of Duistermaat and Hörmander to construct a parametrix for an operator of real principal type. We commence by checking that our two representations in the constant coefficient case are equal.

Lemma 5.1. *If $M = \mathbb{R}^n$ then $L_s(x, x', t) = K_s(x - x', t)$.*

Proof: As we have two holomorphic families, it is enough to show equality for $\Re s \ll 0$ and $\Im s > 0$. Letting $K(x, t, r)$ be the forward fundamental solution of the wave equation on $\mathbb{R}^n_x \times \mathbb{R}_t \times \mathbb{R}_r$, we compute

$$L_s = e^{i\pi(s+1)}\, 2(\pi_r)_* \,(\chi_+^{2(-1-s)}(r)\, K) \tag{5.1}$$

$$= e^{i\pi(s+1)}\, \frac{2}{(2\pi)^{n+2}} \int e^{i(\langle x,\xi\rangle + t\tau)} \left[\lim_{\delta\to 0^+} \int e^{ir\eta}\, \chi_+^{2(-1-s)}(r) \cdot \frac{1}{(\tau - i\delta)^2 - \xi^2 - \eta^2}\, dr\, d\eta\right] d\xi\, d\tau \tag{5.2}$$

$$= i e^{i\pi(2s+1)}\, \frac{2}{(2\pi)^{n+2}} \int e^{i(\langle x,\xi\rangle + t\tau)} \left[\lim_{\delta\to 0^+} \int \frac{(-\eta - i0)^{1+2s}}{a_\delta^2 - \eta^2}\, d\eta\right] d\xi\, d\tau\ , \tag{5.3}$$

where $(\tau - i\delta)^2 - \xi^2 = a_\delta^2$ and we can take $\Im a_\delta < 0$.

We want to evaluate the inner integral, this is equal to a contour integral along the real axis with a small semi-circular detour below the axis at the origin, using Cauchy's theorem. Taking $\Im s > 0$ and considering a large semi-circular contour below the axis, we conclude from Cauchy's Residue theorem that

$$\int \frac{(-\eta - i0)^{1+2s}}{a_\delta^2 - \eta^2}\, d\eta = -2\pi i\, \mathrm{Res}\left(\frac{(-\eta - i0)^{1+2s}}{a_\delta^2 - \eta^2},\, a_\delta\right) \tag{5.4}$$

$$= -i\pi\, e^{-i\pi(2s+1)} (a_\delta^2)^s \tag{5.5}$$

and hence,

$$L_s = \frac{1}{(2\pi)^{n+1}} \int e^{i(\langle x,\xi\rangle + t\tau)} \Big((\tau - i0)^2 - \xi^2\Big)^s\, d\xi\, d\tau\ . \ \blacksquare \tag{5.6}$$

Now, the principal symbol of the variable coefficient wave operator is $\tau^2 - \sum g^{ij}(x)\, \xi_i\, \xi_j$ where g is the Riemann metric on M whereas that of the constant coefficient wave operator, $D_t^2 - \Delta_F$, is $\tau^2 - \xi^2$.

Lemma 5.2. *Let $q \in T^*(M \times \mathbb{R}) - 0$ be a point in the characteristic variety of the wave operator then there exists a homogeneous symplectomorphism, f, from a conic neighbourhood U in $T^*(\mathbb{R}^{n+1}) - 0$ to a conic neighbourhood $V \subset T^*(M \times \mathbb{R}) - 0$ containing the bicharacteristic through q such that $f^*(\tau^2 - \sum g^{ij}(x)\,\xi_i\,\xi_j) = \tau^2 - \xi^2$.*

Proof: We prove that there exist homogeneous symplectic coordinates, (y, η), in a conic neighbourhood of the bicharacteristic such that in these coordinates

$$\tau^2 - \sum g^{ij}(x)\,\xi_i\,\xi_j = \eta_1\,\eta_2\ .$$

This will mean that we can in particular do so for the constant coefficient wave operator and so the result follows.

We can write

$$(5.7)\quad \tau^2 - \sum g^{ij}(x)\,\xi_i\,\xi_j = \left(\tau - \left(\sum g^{ij}(x)\,\xi_i\,\xi_j\right)^{\frac{1}{2}}\right)\left(\tau + \left(\sum g^{ij}(x)\,\xi_i\,\xi_j\right)^{\frac{1}{2}}\right)\ .$$

These factors are homogeneous of degree one, Poisson commute and have non-vanishing differentials near the characteristic variety and we can therefore extend them to a homogeneous, symplectic coordinate system, (y, η), in a small conic neighbourhood W (see [2], Chap. 21). To get the coordinates along a bicharacteristic, we simply extend via the bicharacteristic flows. ∎

To reduce to the model we now work with quantizations of f that is operators which are Fourier integral operators with respect to Γ'_f — the twisted graph of f. We denote the twisted wavefront set of Fourier integral operators by WF′

$$\mathrm{WF}'(K) = \left\{(x, t, \xi, x', t', \xi'\tau') \colon\ (x, t, \xi, \tau, x', t', -\xi', -\tau') \in \mathrm{WF}(K)\right\}\ .$$

Proposition 5.1. *Given $q \in T^*(M \times \mathbb{R}) - 0$, there exist classical Fourier integral operators $A \in I^0(M \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R},\ \Gamma'_f)$ and $B \in I^0(\mathbb{R}^n \times \mathbb{R} \times M \times \mathbb{R},\ \Gamma'_{f^{-1}})$ such that $q \notin \mathrm{WF}'(AB - I)$ and $(q, f(q)) \notin \mathrm{WF}'((D_t^2 - \Delta)A - A(D_t^2 - \Delta_F))$. If q is in the characteristic variety of the wave operator then the result holds for the entire bicharacteristic through q.*

Proof: If q is characteristic, let f be the symplectomorphism from Lemma 5.2. Otherwise considering the square root of the modulus of $\tau^2 - \sum g^{ij}(x)\,\xi_i\,\xi_j$ as a symplectic coordinate we see that such coordinates exist in a conic neighbourhood, V, of q. To do the two cases at once, let W be the set $\{q\}$, if q is non-characteristic and the set of points in the bicharacteristic through q, if q is characteristic.

Let A_1 be an element of $I^0(M\times\mathbb{R}\times\mathbb{R}^n\times\mathbb{R},\,\Gamma'_f)$ which is elliptic on $(W,f(W))$ and has a classical symbol and let $B_1\in I^0(\mathbb{R}^n\times\mathbb{R}\times M\times\mathbb{R},\,\Gamma'_{f^{-1}})$ satisfy

$$W\cap \mathrm{WF}'(A_1B_1-I)=\emptyset$$

then from the calculus of FIOs, we have that the principal symbol of

$$Q=B_1(D_t^2-\Delta)\,A_1$$

is $\tau^2-\xi^2$ and that

$$(W,f(W))\cap \mathrm{WF}'\Big((D_t^2-\Delta)\,A-A(D_t^2-\Delta_F)\Big)=\emptyset\ .$$

This reduces us to proving that if Q is a first order classical pseudo-differential operator then there exists zeroth-order, classical, pseudo-differential operator A_2 which is micro-elliptic on W, such that

$$(D_t^2-\Delta_F+Q)\,A_2-A_2(D_t^2-\Delta_F)\ \in\ \Psi^{-\infty} \tag{5.8}$$

and putting $A=A_1A_2$ our result follows.

To solve (5.8), we rewrite it in the form

$$[D_t^2-\Delta_F,A_2]+QA_2\ \in\ \Psi^{-\infty}\ . \tag{5.9}$$

If we denote the principal symbol of Q by q_1 and of A_2 by a_0 then the vanishing of the principal symbol requires,

$$\Big(2\tau\,\frac{\partial}{\partial t}-2\xi.\frac{\partial}{\partial x}\Big)a_0+q_1\,a_0=0\ .$$

Dividing this through by $|\tau,\xi|$ gives us

$$\Big(2\,\frac{\tau}{|\tau,\xi|}\,\frac{\partial}{\partial t}-2\,\frac{\xi}{|\tau,\xi|}.\frac{\partial}{\partial x}\Big)a_0+\frac{q_1}{|\tau,\xi|}\,a_0=0\ .$$

The coefficients of this equation are homogeneous of degree zero and so if we specify non-zero, degree 0 homogeneous initial data on a non-characteristic, conic hypersurface, we can solve to obtain a non-zero degree 0 homogeneous function. So, letting A_0 have principal symbol a_0, we have solved our equation up to zeroth order. We now use an iterative process to obtain lower order terms. We now choose $A_{-1},A_{-2},...$ such that A_j is classical and has homogeneous principal symbol of degree j and such that

$$[D_t^2-\Delta_F,\,A_0+A_{-1}+...+A_{-N}]+q(A_0+A_{-1}+...+A_{-N})\ \in\ \Psi^{-N-1}. \tag{5.10}$$

The condition on A_N is therefore

$$[D_t^2 - \Delta_F, A_{-N}] + q\, A_{-N} = R_{-N} \ \in \ \Psi^{-N} \tag{5.11}$$

or, in terms of principal symbols, we have

$$\Big(2\tau \frac{\partial}{\partial t} - 2\xi . \frac{\partial}{\partial x}\Big) a_{-N} + q_1\, a_{-N} = r_{-N} \ . \tag{5.12}$$

If we divide through as before and pick initial data of homogeneity $-N-1$ this will have a solution of homogeneity $-N-1$.

And so having constructed $\{A_j\}$ we pick $A_2 \sim \sum A_j$ and we are done. ∎

Theorem 5.1. *If A and B are as in Proposition 5.1 then*

$$(q, q) \notin \mathrm{WF}'(AK_{F,s}B - K_s)$$

and if q is characteristic, the bicharacteristic through (q, q) in the first variable does not meet $\mathrm{WF}'(AK_{F,s}B - K_s)$.

Proof: We will first of all, assume that $\Re s \ll 0$ and then use the micro-locality of the wave operator to deduce the general case.

In this proof, all our operators are constant coefficient in r and so we will regard our kernels as functions of (x, x', t, t', r) and we let

$$\widetilde{\mathrm{WF}}(L) = \Big\{(x, t, \xi, \tau, x, t, \xi, \tau, r, \eta) \colon\ (x, t, \xi, \tau, x, t, -\xi, -\tau, r, \eta) \in \mathrm{WF}(L)\Big\} \ .$$

We denote by $\bar{A}$ and $\bar{B}$, the operators with kernels $A(x, t, x', t')\, \delta(r)$ and $B(x, t, x', t')\, \delta(r)$.

We can write

$$\begin{aligned} AK_{F,s}B - K_s &= A\Big((\pi_r)_* \, (\chi_+^s(r) K_F)\Big)B - (\pi_r)_* \, (\chi_+^s(r) K) \\ &= (\pi_r)_* \Big(\chi_+^s(r)\, (\bar{A} K_F \bar{B} - K)\Big) \ . \end{aligned}$$

So as wavefront which is not conormal to the fibres is killed by a push-forward we have

$$(x_0, t_0, \xi_0, \tau_0, x_1, t_1, \xi_1, \tau_1) \in \widetilde{\mathrm{WF}}(AK_{F,s}B - K_s)$$

implies

$$\exists\, r \ \ (x_0, t_0, \xi_0, \tau_0, x_1, t_1, \xi_1, \tau_1, r, 0) \in \widetilde{\mathrm{WF}}\Big(\chi_+^s(r)\, (\bar{A} K_F \bar{B} - K)\Big) \ .$$

Now the wavefront set of $\chi_+^s(r)$ is the conormal bundle to $r = 0$ so this means that either

$$\exists\, r \quad (x_0, t_0, \xi_0, \tau_0, x_1, t_1, \xi_1, \tau_1, r, 0) \in \widetilde{\mathrm{WF}}(\bar{A}K_F\bar{B} - K)$$

or

$$\exists\, \eta \quad (x_0, t_0, \xi_0, \tau_0, x_1, t_1, \xi_1, \tau_1, 0, \eta) \in \widetilde{\mathrm{WF}}(\bar{A}K_F\bar{B} - K)\ .$$

Thus, in order to show $(q, q') \notin \widetilde{\mathrm{WF}}(AK_{F,s}B - K_s)$, it is certainly sufficient to show that there does not exist (r, η) such that

$$(q, q', r, \eta) \in \widetilde{\mathrm{WF}}(\bar{A}K_F\bar{B} - K)\ . \tag{5.13}$$

We prove this by using the microlocal uniqueness of the forward fundamental solution of the wave equation, which we reprove using propagation of singularities. For simplicity, we write $\Box = D_t^2 - \Delta$ and $\Box_F = D_t^2 - \Delta_F$. We compute

$$\begin{aligned}(\Box - D_r^2)\,(\bar{A}K_F\bar{B} - K) &= \Box\,\bar{A}K_F\bar{B} - \bar{A}\,D_r^2\bar{B} - \mathrm{Id}\\ &= \bar{A}(\Box_F - D_r^2)K_F\bar{B} + (\Box\,\bar{A} - \bar{A}\,\Box_F)K_F\bar{B} - \mathrm{Id}\\ &= (\bar{A}\,\bar{B} - \mathrm{Id}) + (\Box\,\bar{A} - \bar{A}\,\Box_F)K_F\bar{B}\ .\end{aligned}$$

Now, our construction of A and B ensures that $W \cap \mathrm{WF}'(AB - I) = \emptyset$ and so we have that $(q', q, r, \eta) \notin \widetilde{\mathrm{WF}}(\bar{A}\bar{B} - I)$ for all q' in W. The same holds for $(\Box\,\bar{A} - \bar{A}\,\Box_F)K_F\bar{B}$ this is true because $(\Box\,A - A\,\Box_F)$ is smoothing at (q', p) for any p (remember A is associated to a canonical graph) and so the lifted version will be smooth at (q', p, r, η). Hence,

$$(q', q, r, \eta) \notin \widetilde{\mathrm{WF}}\Big((\Box\,\bar{A} - \bar{A}\,\Box_F)K_F\bar{B}\Big)$$

by the composition law for wavefront sets.

Now, any wavefront set of $\bar{A}K_F\bar{B} - K$ with non-zero (ξ, τ, ξ', τ') component must be supported in the forward flow out as this is true of K and K_F and any additional singularities introduced by $\bar{A}$ and $\bar{B}$ will have zero (ξ, τ, ξ', τ') component. Thus if $(q', q) \in W$, we have

$$(q', q, r, \eta) \notin \widetilde{\mathrm{WF}}(\bar{A}K_F\bar{B})$$

since otherwise the entire backward bicharacteristic through (q', q, r, η) would be in $\mathrm{WF}'(\bar{A}K_F\bar{B} - K)$.

So, this establishes for $\Re s \ll 0$ that

$$(q', q) \notin \mathrm{WF}'(AK_FB - K)\ ,$$

which implies the result, for $\Re s \ll 0$.

Using the micro-locality of $\Box$, we have

$$(q', q) \notin \mathrm{WF}'\Big(\Box(AK_{F,s}B - K_s)\Big)$$

and

$$\Box(AK_{F,s}B - K_s) = A\,\Box_F\,K_{F,s}B - K_{s+1} + (\Box\,A - A\,\Box_F)K_{F,s}B\ .$$

The third term is smoothing at (q', q) and so this implies that

$$(q', q) \notin \mathrm{WF}'(AK_{F,s+1}B - K_{s+1})$$

and the general case now follows by induction. ∎

Letting Λ_0 denote the conormal bundle to the diagonal and Λ_1 the flow-out of the characteristic variety's intersection with the diagonal in positive time, i.e. the conormal bundle to the forward light cone, we are now ready to prove the main result

Theorem 5.2. *The kernels $L_s(x, x', t - t')$ are elements of $I^{2s,s-1/2}_{phg}(\Lambda_0, \Lambda_1)$. The principal symbol of L_s on Λ_0 off Λ_1 is*

$$\Big((\tau - i0)^2 - \sum g^{ij}(x)\,\xi_i\,\xi_j\Big)^s$$

and on Λ_1 off Λ_0, the principal symbol is

$$\frac{e^{i\frac{\pi}{2}s}}{\Gamma(-s)}\Big(4\sum g^{ij}(\xi)\Big)^{s/2} t^{-s-1}\,|d\xi|^{\frac{1}{2}}\,|dx|^{\frac{1}{2}}\,|dt|^{\frac{1}{2}}$$

at the point induced by the geodesic flow, in the first variable, at time t from the point $(x, \xi, x, \xi, 0, \pm|\sum g^{ij}(x)\,\xi_i\,\xi_j|^{\frac{1}{2}})$ on the diagonal with respect to the natural trivialization of the Maslov bundle given by Λ_0 and the geodesic flow.

Note that the symbol on Λ_0 is a function rather than a section of the Maslov bundle tensored with the half-density bundle because the conormal structure yields a natural trivialisation of the Maslov bundle and the half-density bundle has the natural trivialization $|d\xi|^{\frac{1}{2}}\,|dx|^{\frac{1}{2}}$ here as it is the square root of the symplectic density on $T^*(M)$ which is naturally isomorphic to $N^*(\Delta)$ via the projection in either coordinate. This reflects the fact that the principal symbol of a pseudo-differential operator is a function.

Proof: We already know this for K_s in the constant coefficient case and using the equality of K_s and L_s in that case, this is an immediate consequence of Egorov's theorem (Proposition 3.1). ∎

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