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TOPOLOGICAL PROPERTIES OF SOLUTION SETS FOR SWEEPING PROCESSES WITH DELAY

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Abstract: Let $r > 0$ be a finite delay and $\mathcal{C}_0 = \mathcal{C}([-r,0],H)$ the Banach space of continuous vector-valued functions defined on $[-r,0]$ taking values in a real separable Hilbert space H. This paper is concerned with topological properties of solution sets for the functional differential inclusion of sweeping process type:

$$\frac{du}{dt} \in -N_{K(t)}(u(t)) + F(t,u_t) ,$$

where K is a γ-Lipschitzean multifunction from $[0,T]$ to the set of nonempty compact convex subsets of H, $N_{K(t)}(u(t))$ is the normal cone to $K(t)$ at $u(t)$ and F: $[0,T]\times\mathcal{C}_0 \to H$ is an upper semicontinuous convex weakly compact valued multifunction. As an application, we obtain periodic solutions to such functional differential inclusions, when K is T-periodic, i.e. when $K(0) = K(T)$ with $T \geq r$.

Introduction

The existence of solutions for functional differential equations (FDE) governed by nonlinear operators in Banach spaces has been studied extensively (see, for example, [17], [24], [25], [26], [29], [31]). The basic source of reference for general FDE is [23]. Functional differential inclusions (FDI) have been studied e.g. in [18], [20], [21], [22]. Topological properties of the solution sets of differential inclusions have been considered by many authors, for example, [4], [14], [21] and the references therein. However, not much study has been done for topological properties of solution sets for the functional differential inclusions governed by sweeping process [27] (another type of FDI is considered in [21] and [22]). The

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purpose of this paper is to prove existence of solutions and to study uniqueness (section 2) and also to characterize topological properties of solution sets for this class of FDI (section 3). As an application, we present in section 4 a new result of existence of periodic solutions to such FDI that is a continuation of our recent work on periodic solutions for perturbations of sweeping process associated to a *periodic closed convex moving set* (see [9], [10]). This sheds a new light on the study of FDI governed by subdifferential operators or accretive operators since we deal with the normal cone to a closed convex *moving* set in a Hilbert space H.

1 – Notations and preliminary results

We will use the following notations:

– H is a real separable Hilbert space, $\langle x, y\rangle$ the scalar product in H.

– $c(H)$ (resp. $cc(H)$) (resp. $ck(H)$) (resp. $cwk(H)$) the set of all nonempty closed (resp. convex closed) (resp. convex compact) (resp. convex weakly compact) subsets of H.

– $\psi_A(\cdot)$ the indicator function of a subset A of H (it takes value 0 on A, $+\infty$ elsewhere).

– $\delta^*(\cdot, A)$ is the support function of a subset A of H.

– $N_A(y)$ is the normal cone to $A \in cc(H)$ at $y \in A$. One has

$$n \in N_A(y) \iff y \in A \text{ and } \langle n, y\rangle = \delta^*(n, A) \iff y \in A,\ n \in \partial\psi_A(y) ,$$

where $\partial\psi_A$ is the subdifferential in the sense of convex analysis of ψ_A.

– If A and B are nonempty subsets of H, the excess of A over B is

$$e(A, B) = \sup\Big\{d(a, B)\colon\ a \in A\Big\} ,$$

where $d(a, B) := \inf\{d(a, b) : b \in B\}$ and their Hausdorff distance is $h(A, B) = \max(e(A, B), e(B, A))$. For every nonempty subset A of H we denote by $|A| := h(A, \{0\})$.

– A multifunction K from a topological space X to a topological space Y is said to be upper semicontinuous (usc) at x_0 if, for any open subset U of Y, $\{x \in \Omega\colon\ K(x) \subset U\}$ is a neighborhood of x_0 or is an empty set.

– A multifunction $K : X \to cwk(H)$ is upper semicontinuous for the weak topology $\sigma(H, H)$ iff, for every $e \in H$, the scalar function $\delta^*(e, K(\cdot))$ is upper semicontinuous on X (shortly K is scalarly upper semicontinuous). (See [13], Theorem II-20).

– If $(\Omega, \mathcal{A})$ is a measurable space and if $\Gamma\colon\ (\Omega, \mathcal{A}) \to cwk(H)$ is a scalarly measurable multifunction, that is, for every $e \in H$, the scalar function $\delta^*(e, \Gamma(\cdot))$ is $\mathcal{A}$-measurable, then Γ admits an $\mathcal{A}$-measurable selection. (See [13]).

– A multifunction $K\colon\ [0,T] \to c(H)$ is γ-Lipschitzean ($\gamma > 0$) if for any s, t in [0,T], we have $h(K(t), K(s)) \le \gamma|t-s|$.

– If X is a topological space, $\mathcal{B}(X)$ is the Borel tribe of X.

We will deal with a finite delay $r > 0$. If $u\colon\ [-r,T] \to H$ with $T > 0$, then for every $t \in [0,T]$, we introduce the function

$$u_t(\tau) = u(t+\tau)\,, \qquad \tau \in [-r,0]\ .$$

Clearly, if $u \in \mathcal{C}_T := \mathcal{C}([-r,T],H)$ then $u_t \in \mathcal{C}_0 := \mathcal{C}([-r,0],H)$ and the mapping $u \mapsto u_t$ is continuous from $\mathcal{C}_T$ onto $\mathcal{C}_0$ in the sense of uniform convergence.

We refer to Barbu [1] and Brézis [5] for the concepts and results on nonlinear evolution equations. See Castaing–Valadier [13] for the theory of measurable multifunctions, Castaing–Duc Ha–Valadier [6], Castaing–Marques ([7], [8]), Duc Ha–Marques [15] for the differential inclusions governed by the sweeping process. We give a useful result.

Lemma 1.1. *Let $K:\ [0,T] \to cc(H)$ be a γ-Lipschitzean multifunction and $h \in L^\infty_H([0,T], dt)$ with $\|h(t)\| \le m$, dt-a.e. Then any absolutely continuous solution u of the differential inclusion*

$$\begin{cases} -u'(t) \in N_{K(t)}(u(t)) + h(t) & dt\text{-a.e. } t \in [0,T] \\ u(t) \in K(t)\,, \quad \forall\, t \in [0,T] \\ u(0) = a \in K(0) \end{cases}$$

is λ-Lipschitzean with $\lambda = \gamma + 2m$.

Proof: Let us define, for $t \in [0,T]$

$$v(t) := u(t) - \int_0^t h(s)\,ds\,, \qquad \hat{K}(t) = K(t) - \int_0^t h(s)\,ds\ .$$

Then v is a solution of the sweeping process

$$\begin{cases} -v'(t) \in N_{\hat{K}(t)}(v(t)) & dt\text{-a.e. } t \in [0,T] \\ v(t) \in \hat{K}(t), \quad \forall\, t \in [0,T] \\ v(0) = a \in \hat{K}(0) = K(0) \ . \end{cases}$$

Since $\hat{K}$ is easily shown to be $(\gamma+m)$-Lipschitzean, it follows from [27] that the solution v is necessarily $(\gamma+m)$-Lipschitzean. The conclusion is obvious. ∎

2 – Existence and uniqueness

We consider first the problem of existence of solutions to the following functional differential inclusion of sweeping process type:

$$u'(t) \ \in \ -N_{K(t)}(u(t)) + F(t,u_t) \quad \text{ a.e. } \ t \in [0,T] \ .$$

Our existence theorem is stated under the following assumptions:

(**H1**) $K\colon [0,T] \to ck(H)$ is γ-Lipschitzean.

(**H2**) $F\colon [0,T]\times\mathcal{C}_0 \to cwk(H)$ is scalarly upper semicontinuous on $\mathcal{C}_0$ for each $t\in[0,T]$, scalarly $\mathcal{L}([0,T])\otimes\mathcal{B}(\mathcal{C}_0)$-measurable on $[0,T]\times\mathcal{C}_0$, where $\mathcal{L}([0,T])$ is the σ-algebra of Lebesgue measurable sets of $[0,T]$ and $\mathcal{B}(\mathcal{C}_0)$ is the Borel tribe of $\mathcal{C}_0$ and $|F(t,u)| \le m$ for all $(t,u)\in[0,T]\times\mathcal{C}_0$ for some positive constant $m>0$.

Now we are able to state the main result on the existence of solutions of the above mentioned FDI.

Theorem 2.1. *If $K\colon [0,T]\to ck(H)$ satisfies (H1) and $F\colon [0,T]\times\mathcal{C}_0 \to cwk(H)$ satisfies (H2), then, for any $\varphi\in\mathcal{C}_0$ with $\varphi(0)\in K(0)$, there exists a continuous mapping $u\colon [-r,T]\to H$ such that u is Lipschitzean on $[0,T]$ and*

$$\text{(2.1.1)} \qquad \begin{cases} u'(t) \ \in \ -N_{K(t)}(u(t)) + F(t,u_t) & \text{a.e. } \ t\in[0,T] \\ u(t)\in K(t)\,, \quad \forall\, t\in[0,T] \\ u_0 = \varphi \quad \textit{in} \ \ [-r,0] \ . \end{cases}$$

Proof: 1. We first assume that F *is scalarly upper semicontinuous on* $[0,T]\times\mathcal{C}_0$ and we proceed by approximation: a sequence of continuous mappings (x_n) in $\mathcal{C}_T := \mathcal{C}([-r,T],H)$ will be defined such that a subsequence of it converges uniformly in $[-r,T]$ to a solution of (2.1.1). The sequence is defined via discretization. We put

$$x_n(t) = \varphi(t)\,, \qquad t\in[-r,0]\ . \tag{2.1.2}$$

We partition $[0,T]$ by points

$$t_i^n = i\,\frac{T}{n} \qquad (i=0,...,n)$$

and define x_n by linear interpolation, where $x_n(t_i^n) = x_i^n$ are obtained by induction starting with

$$x_0^n = x_n(0) := \varphi(0)\in K(0)\ . \tag{2.1.3}$$

If the x_j^n with $0\le j\le i$ are known, then $x_n(t)$ is known for $t\le t_i^n$ and $(x_n)_{t_i^n}(\tau) = x_n(t_i^n+\tau)$ is well defined in $[-r,0]$; we shall also write

$$T(t_i^n)\,x_n = (x_n)_{t_i^n}$$

for notational clarity. As a special element in $F(t,\varphi)$, we can pick the minimal norm element $f_0(t,\varphi)$, so that by (H2)

$$\|f_0(t,\varphi)\| = \min\Big\{\|y\|\colon\ y\in F(t,\varphi)\Big\}\le m\ . \tag{2.1.4}$$

An implicit discrete version of (2.1.1) gives,

$$\frac{1}{h_n}(x_{i+1}^n - x_i^n)\ \in\ -N_{K(t_{i+1}^n)}(x_{i+1}^n) + f_0(t_i^n, T(t_i^n)x_n)\ , \tag{2.1.5}$$

where $h_n = \frac{T}{n}$. Equivalently, because we deal with a cone:

$$x_{i+1}^n - x_i^n - h_n\,f_0(t_i^n, T(t_i^n)x_n)\ \in\ -N_{K(t_{i+1}^n)}(x_{i+1}^n)\ ,$$

or, by a standard property of projections onto closed convex sets:

$$x_{i+1}^n := \mathrm{proj}\Big(x_i^n + h_n f_0(t_i^n, T(t_i^n)x_n), K(t_{i+1}^n)\Big)\ . \tag{2.1.6}$$

Thus, by construction

$$x_i^n\in K(t_i^n)\,, \qquad 0\le i\le n\ . \tag{2.1.7}$$

Since x_n is defined by linear interpolation, we have

$$\|x_n'(t)\| \le \sup_i \frac{\|x_{i+1}^n - x_i^n\|}{t_{i+1}^n - t_i^n} = \frac{1}{h_n} \sup_i \|x_{i+1}^n - x_i^n\| ,$$

for a.e. $t \in [0,T]$. Since projections are non-expansive, then

$$\Big\|x_{i+1}^n - \mathrm{proj}(x_i^n, K(t_{i+1}^n))\Big\| \le \Big\|h_n f_0(t_i^n, T(t_i^n)x_n)\Big\| \le h_n\, m ;$$

while by (2.1.7)

$$\Big\|\mathrm{proj}(x_i^n, K(t_{i+1}^n)) - x_i^n\Big\| \le h(K(t_{i+1}^n), K(t_i^n)) \le \gamma |t_{i+1}^n - t_i^n| = \gamma\, h_n .$$

Thus, $\|x_{i+1}^n - x_i^n\| \le (m+\gamma) h_n$ and

$$\|x_n'(t)\| \le m+\gamma\,, \quad \forall\, n, \ \text{ a.e. } t \in [0,T] . \tag{2.1.8}$$

Let $t \in [0,T]$. For each $n, t \in [t_i^n, t_{i+1}^n[$ for some i (for $t = T$ consider this interval to be $\{T\}$). By (2.1.7), (2.1.8) we have the estimate

$$\begin{aligned} d(x_n(t), K(t)) &\le \|x_n(t) - x_n(t_i^n)\| + h(K(t_i^n), K(t)) \\ &\le (2\gamma + m)\, |t - t_i^n| \\ &\le \frac{T}{n} (2\gamma + m) . \end{aligned} \tag{2.1.9}$$

Since $K(t)$ is compact, (2.1.9) implies that the sequence $(x_n(t))_n$ is relatively compact in H. Thus by (2.1.2), (2.1.8), (2.1.9) and Arzelà–Ascoli's theorem, we conclude that there is a subsequence of (x_n), still denoted (x_n) for simplicity, which converges uniformly on $[-r,T]$ to a continuous function x which clearly satisfies $x_0 = \varphi$. By letting $n \to \infty$ in (2.1.9), we obtain

$$x(t) \in K(t)\,, \quad \forall\, t \in [0,T] . \tag{2.1.10}$$

It remains to prove that (2.1.1) holds a.e. in $[0,T]$. For $t \in [0,T[$ and $n \ge 1$, let us define

$$\sigma_n(t) = t_i^n, \quad \theta_n(t) = t_{i+1}^n \quad \text{ if } \ t \in [t_i^n, t_{i+1}^n[.$$

Then, by (2.1.5) we have

$$x_n'(t) \ \in\ -N_{K(\theta_n(t))}(x_n(\theta_n(t))) + f_0(\sigma_n(t), T(\sigma_n(t))x_n) , \tag{2.1.11}$$

a.e. in $[0,T]$. Since $|\theta_n(t) - t| \le \frac{T}{n}$ and $|\sigma_n(t) - t| \le \frac{T}{n}$, then

$$\theta_n(t) \to t, \quad \sigma_n(t) \to t \quad \text{ uniformly on } \ [0,T[. \tag{2.1.12}$$

It follows that

$$h(K(\theta_n(t)), K(t)) \le \gamma|\theta_n(t) - t| \to 0, \quad \text{as } n \to \infty \; ; \tag{2.1.13}$$

and, by (2.1.8) and (2.1.12)

$$\begin{aligned} \lim_n \|x_n(\theta_n(t)) - x(t)\| &= \lim_n \|x_n(\theta_n(t)) - x_n(t)\| \\ &\le \lim_n (m+\gamma)\,|\theta_n(t) - t| = 0 \; . \end{aligned} \tag{2.1.14}$$

The study of the perturbation term is a little bit more involved. Let us denote the modulus of continuity of a function ψ defined on an interval I of $\mathbb{R}$ by

$$\omega(\psi, I, \varepsilon) := \sup\Big\{ \|\psi(t) - \psi(s)\| \colon\; s,t \in I,\; |s-t| \le \varepsilon \Big\} \; .$$

Then

$$\begin{aligned} \|T(\sigma_n(t))x_n - T(t)x_n\| &= \sup\Big\{ \Big\| x_n(\tau + \sigma_n(t)) - x_n(\tau + t) \Big\| \colon\; \tau \in [-r, 0] \Big\} \\ &\le \omega\Big(x_n, [-r,T], \frac{T}{n}\Big) \\ &\le \omega\Big(\varphi, [-r,0], \frac{T}{n}\Big) + \omega\Big(x_n, [0,T], \frac{T}{n}\Big) \\ &\le \omega\Big(\varphi, [-r,0], \frac{T}{n}\Big) + (m+\gamma)\,\frac{T}{n} \; . \end{aligned}$$

Thus, by continuity of φ:

$$\|T(\sigma_n(t))x_n - T(t)x_n\| \to 0, \quad \text{as } n \to \infty \; ;$$

and since the uniform convergence of x_n to x on $[-r,T]$ implies $T(t)x_n \to T(t)x$ uniformly on $[-r,0]$, we deduce that

$$T(\sigma_n(t))x_n \to T(t)x = x_t \quad \text{in } \mathcal{C}_0 := \mathcal{C}([-r,0], H) \; . \tag{2.1.15}$$

Let us denote $f_n(t) := f_0(\sigma_n(t), T(\sigma_n(t))x_n)$; hence (f_n) is a bounded sequence in $L^\infty_H([0,T], dt)$. Since by (2.1.8) (x_n') is also bounded in $L^\infty_H([0,T], dt)$, by extracting subsequences we may suppose that $f_n \to f$ and $x_n' \to x'$ weakly-$*$ in $L^\infty_H([0,T], dt)$. Therefore, from

$$f_n(t) \;\in\; F\Big(\sigma_n(t), T(\sigma_n(t))x_n\Big) \; ,$$

(2.1.12) and (2.1.15), we can classically (see [13], Theorem V-14) conclude that

$$f(t) \in F(t, x_t) \quad \text{a.e. } t \in [0,T] \; , \tag{2.1.16}$$

because by hypothesis F is scalarly upper semicontinuous with convex weakly compact values. It is now clear that (2.1.1) will follow from (2.1.16) and

$$x'(t) \in -N_{K(t)}(x(t)) + f(t) \quad \text{a.e. } t \in [0,T] \,, \tag{2.1.17}$$

which in turn is a consequence of taking limits in (2.1.11), by standard procedure. We rewrite (2.1.11) as

$$\begin{aligned} \int_0^T \delta^*\Big(-x_n'(t) + f_n(t),\, K(\theta_n(t))\Big)\, dt \le \\ \le \int_0^T \Big\langle -x_n'(t) + f_n(t),\, x_n(\theta_n(t))\Big\rangle\, dt \;. \end{aligned} \tag{2.1.18}$$

Now set $\psi(t,x) := \delta^*(x, K(t))$ for all $(t,x) \in [0,T] \times H$ and using (2.1.12)–(2.1.14) and a well known lower semicontinuity result (see e.g. [30]) for convex integral functionals, we obtain

$$\liminf_n \int_0^T \psi\Big(\theta_n(t), -x_n'(t) + f_n(t)\Big)\, dt \ge \int_0^T \psi\Big(t, -x'(t) + f(t)\Big)\, dt \;.$$

Since the second integral in (2.1.18) obviously converges to $\int_0^T \langle -x'(t)+f(t), x(t)\rangle\, dt$ we have

$$\int_0^T \delta^*\Big(-x'(t) + f(t), K(t)\Big)\, dt \le \int_0^T \Big\langle -x'(t) + f(t),\, x(t)\Big\rangle\, dt \;. \tag{2.1.19}$$

So the desired inclusion follows from (2.1.10), (2.1.16) and (2.1.19).

2. Now we go to the *general case, namely F satisfies (H2).*

We will proceed again by approximation which allows to use the global upper semicontinuity in the first step via a convergence result. Let (r_n) be a sequence of strictly positive numbers such that $\lim_{n\to\infty} r_n = 0$. For each $n \ge 1$, put

$$G_n(t,u) := \frac{1}{r_n} \int_{I_{t,r_n}} F(s,u)\, ds$$

for all $(t,u) \in [0,T] \times \mathcal{C}_0$, where $I_{t,r_n} := [t, t + r_n] \cap [0,T]$. By (H2) it easy to check that each multifunction G_n is globally scalarly upper semicontinuous on $[0,T] \times \mathcal{C}_0$ with $|G_n(t,u)| \le m$ for all $n \ge 1$ and for all $(t,u) \in [0,T] \times \mathcal{C}_0$, so that we can apply the result of the first step. For each $n \ge 1$, there exists a continuous mapping $x_n \in \mathcal{C}_T := \mathcal{C}([-r,T], H)$ satisfying the FDI

$$\begin{cases} x_n'(t) \in -N_{K(t)}(x_n(t)) + G_n(t, (x_n)_t) & \text{a.e. } t \in [0,T] \\ x_n(t) \in K(t)\,, \quad \forall\, t \in [0,T] \\ (x_n)_0 = \varphi \quad \text{in } [-r, 0] \;. \end{cases} \tag{2.1.20}$$

It is not difficult to check that the sequence $(x_n)_n$ is relatively compact in $\mathcal{C}_T$. By (2.1.20), for each $n \geq 1$, there is a measurable mapping $h_n \colon [0,T] \to H$ such that $h_n(t) \in G_n(t,(x_n)_t)$ a.e. and that

$$\begin{cases} x_n'(t) \in -N_{K(t)}(x_n(t)) + h_n(t) \quad \text{a.e. } t \in [0,T] \\ x_n(t) \in K(t)\,, \quad \forall\, t \in [0,T] \\ (x_n)_0 = \varphi \quad \text{in } [-r,0]\,. \end{cases} \tag{2.1.21}$$

By extracting a subsequence we may suppose that $x_n \to u \in \mathcal{C}_T$ uniformly, hence

$$u(t) \in K(t), \quad \forall\, t \in [0,T] \quad \text{and} \quad u_0 = \varphi \ \text{ in } [-r,0]\,. \tag{2.1.22}$$

On the other hand, it is obvious that the sequences (x_n') and (h_n) are relatively $\sigma(L^1, L^\infty)$ compact in $L^1_H([0,T],dt)$. By extracting subsequences, we may suppose that $x_n' \to u'$ and $h_n \to h$ weakly in $L^1_H([0,T],dt)$. Since $(x_n)_t \to u_t$ in $\mathcal{C}_0$ for each $t \in [0,T]$ and $h_n \to h$ for $\sigma(L^1, L^\infty)$ with $h_n(t) \in G_n(t,(x_n)_t)$ a.e., by ([12], Lemma 6.5) we conclude that

$$h(t) \in F(t,u_t) \ \text{ a.e.}\,. \tag{2.1.23}$$

By rewriting the first inclusion in (2.1.20) as

$$\int_0^T \delta^*\Big(-x_n'(t) + h_n(t), K(t)\Big)\,dt \leq \int_0^T \Big\langle -x_n'(t) + h_n(t), x_n(t)\Big\rangle\,dt \tag{2.1.24}$$

and using a well known lower semicontinuity result for convex integral functionals (see e.g. [13], Theorem VII-7), we obtain

$$\liminf_n \int_0^T \delta^*\Big(-x_n'(t) + h_n(t), K(t)\Big)\,dt \geq \int_0^T \delta^*\Big(-u'(t) + h(t), K(t)\Big)\,dt\,. \tag{2.1.25}$$

Since the second integral in (2.1.24) tends to $\int_0^T \langle -u'(t) + h(t), u(t)\rangle\,dt$, we conclude that

$$\int_0^T \delta^*\Big(-u'(t) + h(t), K(t)\Big)\,dt \leq \int_0^T \Big\langle -u'(t) + h(t), u(t)\Big\rangle\,dt\,. \tag{2.1.26}$$

Then (2.1.22), (2.1.23) and (2.1.26) yield

$$u'(t) \in -N_{K(t)}(u(t)) + h(t) \subset -N_{K(t)}(u(t)) + F(t,u_t) \quad \text{a.e. } t \in [0,T]$$

thus completing the proof. ∎

It is possible to prove the result in step 2 by invoking a recent extension of the multivalued Scorza Dragoni's theorem [11], which allows to reduce to the globally upper semicontinuous assumption in the first step. We do not emphasize this fact and details are left to the reader.

Now we present a useful corollary of Theorem 2.1 concerning uniqueness of solution for (2.1.1) when the perturbation is single-valued.

Proposition 2.2. *Assume that K satisfies (H1). Let $g\colon [0,T]\times\mathcal{C}_0 \to H$ be a bounded mapping satisfying:*

a) *There exists $c>0$ such that $\|g(t,u)-g(t,v)\| \le c\|u-v\|_0$, for all u,v in $\mathcal{C}_0$, where $\|\cdot\|_0$ is the sup-norm in $\mathcal{C}_0$.*

b) *For each $u\in\mathcal{C}_0$, $g(\cdot,u)$ is Lebesgue measurable on $[0,T]$.*

Let $\varphi\in\mathcal{C}_0$ with $\varphi(0)\in K(0)$. Then there exists a unique continuous mapping $u\colon [-r,T]\to H$ such that

$$
\text{(2.2.1)}\qquad \begin{cases} u'(t) \in -N_{K(t)}(u(t)) + g(t,u_t) & \text{a.e. } t\in[0,T] \\ u(t)\in K(t)\,, \quad \forall\, t\in[0,T] \\ u_0=\varphi \quad \textit{in} \ \ [-r,0]\ . \end{cases}
$$

Proof: Existence follows from Theorem 2.1.

Uniqueness. Let us assume that x and y are two solutions of (2.2.1). Then from

$$-(x'(t)-g(t,x_t)) \in N_{K(t)}(x(t))\,, \qquad -(y'(t)-g(t,y_t)) \in N_{K(t)}(y(t))$$

we obtain that, for a.e. $t\in[0,T]$

$$\Big\langle [x'(t)-g(t,x_t)] - [y'(t)-g(t,y_t)],\, x(t)-y(t)\Big\rangle \le 0$$

by monotonicity. Integrating on $[0,s]$

$$\frac{1}{2}\,\|x(s)-y(s)\|^2 - \frac{1}{2}\,\|x(0)-y(0)\|^2 \le \int_0^s \Big\langle g(t,x_t)-g(t,y_t),\, x(t)-y(t)\Big\rangle\, dt\ .$$

Since $x(0)=y(0)=\varphi(0)$ and g is c-Lipschitzean, then

$$\frac{1}{2}\,\|x(s)-y(s)\|^2 \le c\int_0^s \|x_\tau - y_\tau\|_0\, \|x(\tau)-y(\tau)\|\, d\tau\ .$$

So
$$\frac{1}{2}\,\|x(s)-y(s)\|^2 \le c\int_0^t \|x_\tau - y_\tau\|_0\,\|x(\tau)-y(\tau)\|\,d\tau\,, \qquad 0\le s\le t\ .$$
Since $x=\varphi=y$ in $[-r,0]$, we have, in the norm $\|\cdot\|_t$ of $\mathcal{C}_t=\mathcal{C}([-r,t],H)$,
$$\frac{1}{2}\,\|x-y\|_t^2 \le c\int_0^t \|x_\tau-y_\tau\|_0\,\|x_\tau-y_\tau\|_0\,d\tau \le c\int_0^t \|x-y\|_\tau^2\,d\tau\ .$$
Because $t\mapsto\|x-y\|_t$ is continuous, by applying Gronwall's lemma we conclude that
$$\|x-y\|_t=0\,, \qquad \forall\,t\in[0,T]\ .$$
Hence $x=y$ in $[-r,T]$. ∎

3 – Solution sets

Let $\mathcal{K}$ be a nonempty compact subset of $\mathcal{C}_0$ such that $\varphi(0)\in K(0)$ for all $\varphi\in\mathcal{K}$. For each $\varphi\in\mathcal{K}$ we denote by $\mathcal{S}_F(\varphi)$ the set of all continuous mappings $u\colon [-r,T]\to H$ such that

$$\text{(2.1.1)}\qquad \begin{cases} u'(t)\ \in\ -N_{K(t)}(u(t))+F(t,u_t) & \text{a.e. } \ t\in[0,T]\\ u(t)\in K(t)\,, \quad \forall\,t\in[0,T] & \\ u_0=\varphi \quad \text{in } \ [-r,0]\ . & \end{cases}$$

Remark 3.1. By using Lemma 1.1 it is easily seen that each $u\in\mathcal{S}_F(\varphi)$ is necessarily $(\gamma{+}2m)$-Lipschitzean on $[0,T]$. Indeed by (2.1.1) there is a measurable mapping $h\colon [0,T]\to H$ such that $h(t)\in F(t,u_t)$ a.e. and that

$$\begin{cases} u'(t)\ \in\ -N_{K(t)}(u(t))+h(t) & \text{a.e. } \ t\in[0,T]\\ u(t)\in K(t)\,, \quad \forall\,t\in[0,T] & \\ u_0=\varphi \quad \text{in } \ [-r,0]\ . & \end{cases}$$

Since $\|h(t)\|\le m$ by (H2), then Lemma 1.1 implies that u is $(\gamma{+}2m)$-Lipschitzean on $[0,T]$.

Proposition 3.2. *Assume that (H1) and (H2) are satisfied. Then*
$$\mathcal{S}_F(\mathcal{K}):=\bigcup_{\varphi\in\mathcal{K}}\mathcal{S}_F(\varphi)$$

is relatively compact in $\mathcal{C}_T := \mathcal{C}([-r,T],H)$.

Proof: By Remark 3.1 each $u \in \mathcal{S}_F(\varphi)$ is $(\gamma+2m)$-Lipschitzean on $[0,T]$. Moreover, for every $t \in [0,T]$, $u(t) \in K(t)$, which is a compact subset of H by hypothesis. It follows from Arzelà–Ascoli's theorem that the restrictions of $\mathcal{S}_F(\mathcal{K})$ to $[0,T]$ form a relatively compact subset of $\mathcal{C}([0,T],H)$. Since the restriction of $\mathcal{S}_F(\mathcal{K})$ to $[-r,0]$ coincides with $\mathcal{K}$, the result follows. ∎

Proposition 3.3. *Assume that (H1) and (H2) are satisfied. Then the multifunction* $\mathcal{S}_F$ *has closed graph in* $\mathcal{K} \times \mathcal{C}_T$.

Proof: Let us consider sequences (φ_n) and (x_n) with $\varphi_n \in \mathcal{K}$, $\varphi_n \to \varphi \in \mathcal{K}$ uniformly, $x_n \in \mathcal{S}_F(\varphi_n)$ and $x_n \to x \in \mathcal{C}_T$ uniformly. By Remark 3.1, x is λ-Lipschitzean on $[0,T]$ where $\lambda = \gamma+2m$. Obviously $x_n(t) \in K(t)$ implies that $x(t) \in K(t)$ for all $t \in [0,T]$. Similarly $(x_n)_0 = \varphi_n$ implies that $x_0 = \varphi$. It remains to prove that x satisfies

$$x'(t) \in -N_{K(t)}(x(t)) + F(t,x_t) \quad \text{a.e. } t \in [0,T] \,. \tag{3.3.1}$$

For every n, there exists a measurable selection h_n such that

$$h_n(t) \in F(t,(x_n)_t) \quad \text{a.e. } t \in [0,T] \tag{3.3.2}$$

and that

$$-x_n'(t) + h_n(t) \in N_{K(t)}(x_n(t)) \quad \text{a.e. } t \in [0,T] \,. \tag{3.3.3}$$

Since $\|x_n'(t)\| \le \lambda$ and $\|h_n(t)\| \le m$ a.e., we may assume w.l.o.g. that $x_n \to x'$ and $h_n \to h$ weakly in $L^1_H([0,T],dt)$. Since $F(t,\cdot)$ is scalarly upper semicontinuous with convex weakly compact values, it follows from (3.3.2) and a classical closure theorem (see [13], Theorem VI-4) that

$$h(t) \in F(t,x_t) \ \text{ a.e.} \tag{3.3.4}$$

Next, we notice that (3.3.3) is equivalent to

$$\int_0^T \delta^*\Big(-x_n'(t)+h_n(t),\, K(t)\Big)\,dt + \int_0^T \Big\langle x_n'(t)-h_n(t), x_n(t)\Big\rangle\,dt \le 0 \,. \tag{3.3.5}$$

It is obvious that

$$\lim_{n\to\infty} \int_0^T \Big\langle x_n'(t)-h_n(t), x_n(t)\Big\rangle\,dt = \int_0^T \Big\langle x'(t)-h(t), x(t)\Big\rangle\,dt \,. \tag{3.3.6}$$

Set $f(t,x) := \delta^*(x, K(t))$ for all $(t,x) \in [0,T] \times H$ and

$$I_f(u) = \int_0^T f(t,u(t))\,dt\,, \qquad \forall\, u \in L^1_H([0,T],dt)\ .$$

Then it is easily checked that f is a convex normal integrand satisfying the conditions of Theorem VII-7 in [13] so that

$$\liminf_{n\to\infty} I_f(-x'_n + h_n) \geq I_f(-x' + h)\ , \tag{3.3.7}$$

because $-x'_n + h_n \to -x' + h$ weakly in $L^1_H([0,T],dt)$. So (3.3.5) and (3.3.6) yield

$$\int_0^T \Big[\delta^*\Big(-x'(t) + h(t), K(t)\Big) + \Big\langle x'(t) - h(t), x(t)\Big\rangle\Big]\,dt \leq 0$$

which is equivalent to

$$-x'(t) + h(t) \ \in\ N_{K(t)}(x(t)) \qquad \text{a.e. } \ t \in [0,T]$$

because $x(t) \in K(t)$. ∎

If H is $\mathbb{R}^d$ equipped with its euclidean norm and if F is Lipschitzean, then the multifunction $\mathcal{S}_F$ defined above enjoys a remarkable property. Namely we have the following:

Proposition 3.4. *Let $F\colon\ [0,T] \times \mathcal{C}_0 \to ck(\mathbb{R}^d)$ be a multifunction satisfying:*

(**1**) *There exists $c > 0$ such that*

$$h(F(t,u), F(t,v)) \leq c\|u - v\|_0\,, \qquad \forall\,(t,u,v) \in [0,T] \times \mathcal{C}_0 \times \mathcal{C}_0\ ,$$

where $\|\cdot\|_0$ is the sup-norm in $\mathcal{C}_0$.

(**2**) *For every $u \in \mathcal{C}_0$ the multifunction $F(\cdot,u)$ is measurable.*

(**3**) *There is $m > 0$ such that $|F(t,u)| \leq m$, $\forall\,(t,u) \in [0,T] \times \mathcal{C}_0$.*

Then there exists a mapping $s\colon\ [0,T] \times \mathcal{C}_0 \to \mathbb{R}^d$ which enjoys the following properties:

a) $\forall\,(t,u) \in [0,T] \times \mathcal{C}_0$, $s(t,u) \in F(t,u)$.

b) $\|s(t,u) - s(t,v)\| \leq c\sqrt{d}\,\|u - v\|_0$, $\forall\,(t,u,v) \in [0,T] \times \mathcal{C}_0 \times \mathcal{C}_0$.

c) *For every $u \in \mathcal{C}_0$, $s(\cdot,u)$ is measurable.*

d) *The single valued mapping $\varphi \mapsto \mathcal{S}_{\{s\}}(\varphi)$ from $\mathcal{K}$ into $\mathcal{C}_T$ is a continuous selection of $\mathcal{S}_F$.*

Proof: For each $(t,u) \in [0,T] \times \mathcal{C}_0$, we associate the Steiner point $s(t,u)$ of the convex compact set $F(t,u)$ (see [28], for example). By classical properties of Steiner point, it is easy to check that the mapping $s\colon\ (t,u) \mapsto s(t,u)$ from $[0,T] \times \mathcal{C}_0$ into $\mathbb{R}^d$ satisfies (a), (b) and (c). Whereas (d) follows directly from Proposition 2.2 so that $\mathcal{S}_{\{s\}}$ is a continuous selection of $\mathcal{S}_F$. ∎

In order to study topological properties of solution sets to the FDI (2.1.1) we introduce the following definition.

Let (X,d) be a Polish space. A multifunction $F\colon\ [0,T] \times X \to cwk(H)$ is *Lipschitzean approximable* if there exists a sequence $F_n\colon\ [0,T] \times X \to cwk(H)$ $(n \geq 1)$ satisfying the following properties:

a) Each F_n is λ_n-Lipschitzean, that is

$$\forall\, t \in [0,T], \quad \forall\, (x,y) \in X \times X\,, \qquad h\Big(F_n(t,x), F_n(t,y)\Big) \leq \lambda_n\, d(x,y)\ .$$

b) For every fixed $x \in X$, $F(\cdot,x)$ is measurable on $[0,T]$.

c) $\forall\, t \in [0,T]$, $\forall\, x \in X$, $\lim_{n\to\infty} h(F_n(t,x), F(t,x)) = 0$.

d) $\forall\, n \geq 1$, $\forall\, t \in [0,T]$, $\forall\, x \in X$, $F_{n+1}(t,x) \subset F_n(t,x)$.

e) $\forall\, t \in [0,T]$, $\forall\, x \in X$, $F(t,x) = \bigcap_{n\geq 1} F_n(t,x)$.

Remarks 1. Each F_n has measurable graph, that is the graph of F_n belongs to $\mathcal{L}([0,T]) \otimes \mathcal{B}(X) \otimes B(H)$ where $\mathcal{L}([0,T])$ is the σ-algebra of Lebesgue measurable sets in $[0,T]$, $\mathcal{B}(X)$ and $B(H)$ are the Borel tribe of X and H respectively. Hence, by (e) F also has measurable graph. Since for every fixed $t \in [0,T]$, each $F_n(t,\cdot)$ has closed graph in $X \times H$, $F(t,\cdot)$ also has closed graph in $X \times H$.

2. Conversely, every upper semicontinuous multifunction F defined on a metric space M with convex weakly compact values in a reflexive Banach space satisfying some growth condition can be approximated by means of a decreasing sequence of multifunctions which are Lipschitzean with respect to the Hausdorff distance (see, for example, Gavioli [19]). Recently Benassi and Gavioli ([2], [3]) state some analogous Lipschitzean approximation results for upper semicontinuous multifunctions defined on a metric space M taking compact connected values in $\mathbb{R}^d$.

3. If X is a compact metric space, $F\colon\ [0,T] \times X \to ck(\mathbb{R}^d)$ is a bounded multifunction such that the graph of F is a borelian subset of $[0,T] \times X \times \mathbb{R}^d$ and for every fixed $t \in [0,T]$, the multifunction $F(t,\cdot)$ is upper semicontinuous, then

F is Lipschitzean-approximable by the F_n with (F_n) *equibounded* and satisfying (a), (b), (c), (d), (e) in the preceding definition (see, for example, El Arni [16]).

Using the Lipschitzean approximation we have the following:

Proposition 3.5. *Let $F\colon [0,T]\times\mathcal{C}_0 \to cwk(H)$ be such that for every t in $[0,T]$, $F(t,\cdot)$ is upper semicontinuous on $\mathcal{C}_0$. Assume that F is Lipschitzean approximable by an equibounded sequence $(F_n)_{n\geq 1}$. Then*

$$\forall\,\varphi\in\mathcal{K}\,,\qquad \mathcal{S}_F(\varphi) = \bigcap_{n\geq 1} \mathcal{S}_{F_n}(\varphi)\ . \tag{3.5.1}$$

Proof: Clearly $F(\cdot,u)$ is measurable on $[0,T]$ for every fixed $u\in\mathcal{C}_0$ and we have

$$\forall\, n\geq 1,\ \ \forall\varphi\in\mathcal{K}\,,\qquad \mathcal{S}_F(\varphi)\subset\mathcal{S}_{F_n}(\varphi)\ . \tag{3.5.2}$$

Now let $x\colon [-r,T]\to H$ be a Lipschitzean mapping such that $x_0=\varphi$ and

$$x'(t)\ \in\ -N_{K(t)}(x(t)) + F_n(t,x_t)\ \text{ a.e.}$$

for every $n\geq 1$. Then, there exists $h_n\in L^\infty_H([0,T],dt)$ such that $h_n(t)\in F_n(t,x_t)$ a.e. and

$$x'(t)-h_n(t)\ \in\ -N_{K(t)}(x(t))\ \text{ a.e.} \tag{3.5.3}$$

By extracting a subsequence with $h_n\to h$, $\sigma(L^\infty,L^1)$, and by using the convexity of the right hand side of (3.5.3), we conclude classically that

$$-x'(t)+h(t)\in N_{K(t)}(x(t))\ \text{ a.e.} \tag{3.5.4}$$

Since $\varepsilon_n(t):=h(F_n(t,x_t),F(t,x_t))\to 0$ and

$$h_n(t)\ \in\ F(t,x_t)+\varepsilon_n(t)\,\overline{B}_H\ \text{ a.e.}$$

where $\overline{B}_H$ is the closed unit ball in H, by passing to convex combinations of $(h_n(t))$, denoted by $\widetilde{h}_n(t)$, we further have $\widetilde{h}_n(t)\to h(t)$ a.e. in H and

$$\widetilde{h}_n(t)\ \in\ \sum_{m\geq n}\alpha_m(t)\Big[F(t,x_t)+\varepsilon_m(t)\,\overline{B}_H\Big]\ \text{ a.e. ,}$$

where $\sum_{m\geq n}\alpha_m(t)=1$, $\alpha_m(t)\geq 0$. Since F takes convex values, we have

$$\widetilde{h}_n(t)\ \in\ F(t,x_t)+(\sup_{m\geq n}\varepsilon_m(t))\,\overline{B}_H$$

so that in the limit

$$h(t) \in F(t, x_t) \quad \text{a.e.} \tag{3.5.5}$$

since $F(t, x_t)$ is closed. By (3.5.4) and (3.5.5), $x \in \mathcal{S}_F$. ∎

The following result provides the convergence of the sequence $\mathcal{S}_{F_n}(\varphi)$ towards $\mathcal{S}_F(\varphi)$ for the Hausdorff distance Δ associated to the sup-norm $\|\cdot\|_T$ in $\mathcal{C}_T$.

Proposition 3.6. *Let $F\colon [0,T]\times\mathcal{C}_0 \to cwk(H)$ such that for every t in $[0,T]$, $F(t,\cdot)$ is continuous on $\mathcal{C}_0$ with respect to the Hausdorff distance h on $cwk(H)$. If F is Lipschitzean approximable by an equibounded sequence $(F_n)_{n\geq 1}$, then, for all $\varphi \in \mathcal{K}$:*

$$\lim_{n\to\infty} \Delta(\mathcal{S}_{F_n}(\varphi), \mathcal{S}_F(\varphi)) = 0\ . \tag{3.6.1}$$

Proof: Since $\mathcal{S}_F(\varphi) \subset \mathcal{S}_{F_n}(\varphi)$, we only need to show that

$$\lim_{n\to\infty} \sup_{u\in\mathcal{S}_{F_n}(\varphi)} d(u, \mathcal{S}_F(\varphi)) = 0\ . \tag{3.6.2}$$

Since $\mathcal{S}_{F_n}(\varphi)$ is compact and the function $d(\cdot, \mathcal{S}_F(\varphi))$ is continuous, there is $u_n \in \mathcal{S}_{F_n}(\varphi)$ such that

$$d(u_n, \mathcal{S}_F(\varphi)) = \sup_{u\in\mathcal{S}_{F_n}(\varphi)} d(u, \mathcal{S}_F(\varphi))\ . \tag{3.6.3}$$

Recall that $u_n : [-r,T] \to H$ is a Lipschitzean mapping on $[0,T]$ such that $u_{n,0} = \varphi$ and that

$$\begin{cases} u_n'(t) \ \in\ -N_{K(t)}(u_n(t)) + F_n(t,(u_n)_t) & \text{a.e. } t \in [0,T] \\ u_n(t) \in K(t)\,, \quad \forall\, t \in [0,T]\ . \end{cases} \tag{3.6.4}$$

Then, there exists $h_n \in L^\infty_H([0,T], dt)$ such that

$$h_n(t) \ \in\ F_n(t,(u_n)_t) \quad \text{a.e. } t \in [0,T] \tag{3.6.5}$$

and that

$$-u_n'(t) + h_n(t) \ \in\ N_{K(t)}(u_n(t)) \quad \text{a.e. } t \in [0,T]\ . \tag{3.6.6}$$

Since $(u_n)_n$ is relatively compact in $\mathcal{C}_T$ and $(u_n')_n$ is relatively $\sigma(L^1, L^\infty)$ compact, by extracting subsequences, we may ensure that $u_n \to u \in \mathcal{C}_T$ for the uniform convergence with $u_0 = \varphi$ and $u_n' \to u' \in L^1_H([0,T], dt)$ for $\sigma(L^1, L^\infty)$.

Now we claim that $u \in \mathcal{S}_F(\varphi)$.

Fact 1:

$$-u'(t) + h(t) \;\in\; N_{K(t)}(u(t)) \quad \text{a.e. } \; t \in [0,T] \; . \tag{3.6.7}$$

Note that by (3.6.4) and (3.6.6) we have

$$\begin{cases} u_n(t) \in K(t)\,, \qquad \forall\, t \in [0,T] \\ \displaystyle\int_0^T \Big[\delta^*\Big(-u_n'(t) + h_n(t), K(t)\Big) + \Big\langle u_n'(t) - h_n(t), u_n(t)\Big\rangle\Big]\, dt \le 0 \; . \end{cases} \tag{3.6.8}$$

Since $u_n \to u \in \mathcal{C}_T$, by repeating the arguments of the proof of Proposition 3.3 we obtain

$$\begin{cases} u(t) \in K(t)\,, \qquad \forall\, t \in [0,T] \\ \displaystyle\int_0^T \Big[\delta^*\Big(-u'(t) + h(t), K(t)\Big) + \Big\langle u'(t) - h(t), u(t)\Big\rangle\Big]\, dt \le 0 \; , \end{cases} \tag{3.6.9}$$

by which Fact 1 follows.

Fact 2:

$$h(t) \in F(t, u_t) \quad \text{a.e. } \; t \in [0,T] \; . \tag{3.6.10}$$

Let $(e_k)_{k\ge1}$ be a sequence in H which separates the points of H. Observe that for every $k \ge 1$ and every $(t,u) \in [0,T] \times \mathcal{C}_0$, we have

$$\downarrow \lim_{n\to\infty} \delta^*(e_k, F_n(t,u)) = \delta^*(e_k, F(t,u)) \tag{3.6.11}$$

and since $F(t,\cdot)$ is continuous, by (3.6.11) for every fixed $t \in [0,T]$, and every fixed $k \ge 1$, we have

$$\downarrow \lim_{n\to\infty} \delta^*(e_k, F_n(t,u)) = \delta^*(e_k, F(t,u)) \tag{3.6.12}$$

uniformly on $\{(u_n)_t :\; n \ge 1\} \cup \{u_t\}$ by virtue of Dini's theorem. Now (3.6.5) obviously is equivalent to

$$\langle e_k, h_n(t)\rangle \le \delta^*(e_k, F_n(t,(u_n)_t)) \quad \text{a.e. } \; t \in [0,T] \tag{3.6.13}$$

for all $k \ge 1$ and all $n \ge 1$. By (3.6.12), it is not difficult to see that for every $k \ge 1$ and every $t \in [0,T]$

$$\lim_{n\to\infty} \delta^*(e_k, F_n(t,(u_n)_t)) = \delta^*(e_k, F(t,u_t)) \tag{3.6.14}$$

because $(u_n)_t \to u_t$ in $\mathcal{C}_0$ and $F(t,\cdot)$ is continuous. Using (3.6.13) and integrating, we have

$$\int_A \langle e_k, h_n(t)\rangle\, dt \le \int_A \delta^*(e_k, F_n(t,(u_n)_t))\, dt \tag{3.6.15}$$

for all $k \ge 1$ and for all Lebesgue measurable sets $A \subset [0,T]$. Since $h_n \to h$ weakly in $L^1_H([0,T],dt)$, (3.6.13), (3.6.14), (3.6.15) and Lebesgue dominated convergence yield

$$\int_A \langle e_k, h(t)\rangle\, dt \le \int_A \delta^*(e_k, F(t,u_t))\, dt\ . \tag{3.6.16}$$

Since (3.6.16) holds for all Lebesgue measurable sets $A \subset [0,T]$ and for all $k \ge 1$, we get (3.6.10). Hence we conclude that $u \in \mathcal{S}_F(\varphi)$ and consequently $\mathcal{S}_{F_n}(\varphi)$ converges towards $\mathcal{S}_F(\varphi)$ for the Hausdorff distance associated to the sup-norm in $\mathcal{C}_T$. ■

Let us focus our attention to the special case when H is $\mathbb{R}^d$ and F is bounded and Carathéodory-Lipschitzean, i.e. $F(\cdot,u)$ is measurable on $[0,T]$ for every $u \in \mathcal{C}_0$ and

$$\forall\, t \in [0,T],\quad \forall\,(x,y) \in \mathcal{C}_0 \times \mathcal{C}_0\,, \qquad h(F(t,x),F(t,y)) \le c\|x-y\|_0\ ,$$

with $c > 0$ and $\|\cdot\|_0$ denotes the sup-norm in the Banach space $\mathcal{C}_0 := \mathcal{C}([-r,0],\mathbb{R}^d)$ and $K(t) = K$, $\forall\, t \in [0,T]$, where K is a fixed non empty convex compact subset of $\mathbb{R}^d$. We establish a topological property for solution sets of FDI (2.1.1) which follows easily from Propositions 2.2, 3.3 and 3.4.

Proposition 3.7. *For every fixed $\varphi_0 \in \mathcal{K}$, $\mathcal{S}_F(\varphi_0)$ is contractible, that is, there exists a continuous map $H\colon\ [0,1] \times \mathcal{S}_F(\varphi_0) \to \mathcal{S}_F(\varphi_0)$ such that*

a) *$H(1,u) = u$ for every $u \in \mathcal{S}_F(\varphi_0)$;*

b) *There exists $\widetilde{u} \in \mathcal{S}_F(\varphi_0)$ such that $H(0,u) = \widetilde{u}$ for every $u \in \mathcal{S}_F(\varphi_0)$.*

Proof: Let g be a Carathéodory–Lipschitzean selection of F which is ensured by the properties of Steiner points of convex compact sets in $\mathbb{R}^d$. For each $s \in [0,T]$ and each $\varphi \in \mathcal{K}$ we denote by $z(\cdot; s,\varphi)\colon\ [s,T] \to \mathbb{R}^d$ the unique solution of the FDI (see Proposition 2.2)

$$\begin{cases} z'(t) \ \in\ -N_K(z(t)) + g(t,z_t) \qquad \text{a.e. } \ t \in [s,T] \\ z(t) \in K\,, \qquad \forall\, t \in [s,T] \\ z_s = \varphi\ . \end{cases} \tag{3.7.1}$$

For each $(\tau, u) \in [0,1] \times \mathcal{S}_F(\varphi_0)$, we set

$$H(\tau,u)(t) = \begin{cases} u(t) & \text{if } t \in [0, \tau T] \\ z(t; \tau T, u_{\tau T}) & \text{if } t \in]\tau T, T] \ . \end{cases}$$

Hence we get a continuous mapping $H\colon\ [0,1] \times \mathcal{S}_F(\varphi_0) \to \mathcal{S}_F(\varphi_0)$ because of the continuous dependence of the solutions of (3.7.1) upon the data (see Proposition 3.3 and 3.4) with $H(1,u) = u$ and $H(0,u) = z(\cdot; 0, \varphi_0) \in \mathcal{S}_F(\varphi_0)$. ∎

Comments. If $H = \mathbb{R}^d$, if the perturbation $F:\ [0,T] \times \mathcal{C}_0 \to ck(\mathbb{R}^d)$ is a bounded Carathéodory multifunction and if the multifunction K is constant, then the previous results show that the *solution sets multifunction*

$$\mathcal{S}_F\colon\ \mathcal{K} \to \mathcal{C}_T\colon\ [\varphi \mapsto \mathcal{S}_F(\varphi)]$$

is the intersection of a decreasing sequence $(\mathcal{S}_{F_n})_{n\geq 1}$ of contractible compact valued upper semicontinuous multifunctions, namely

$$(*) \qquad \forall\, \varphi \in \mathcal{K}\,, \quad \mathcal{S}_F(\varphi) = \bigcap_{n\geq 1} \mathcal{S}_{F_n}(\varphi)$$

and

$$(**) \qquad \forall\, \varphi \in \mathcal{K}\,, \quad \lim_{n\to\infty} \Delta(\mathcal{S}_{F_n}(\varphi), \mathcal{S}_F(\varphi)) = 0 \ .$$

Unfortunately we are unable to prove property $(**)$ when F is only measurable in $t \in [0,T]$ and upper semicontinuous in $u \in \mathcal{C}_0$. This is an open problem.

4 – Periodic solutions

To end this paper we would like to mention an interesting application to the existence of periodic solutions of the FDI (2.1.1) when $K(0) = K(T)$ and $T \geq r$. This result is a continuation of our recent work on the existence of absolutely continuous and BV *periodic* solutions for differential inclusions governed by the sweeping process (see [9], [10]).

Proposition 4.1. *Let $T \geq r$. Let $K\colon\ [0,T] \to ck(\mathbb{R}^d)$ be a γ-Lipschitzean multifunction with $K(0) = K(T)$. Let $F\colon\ [0,T] \times \mathcal{C}([-r,0], \mathbb{R}^d) \to ck(\mathbb{R}^d)$. Assume that F is Lipschitzean-approximable by an equibounded sequence $(F_n)_{n\geq 1}$ of measurable Lipschitzean convex compact valued multifunctions ($|F_n(t,u)| \leq$*

$m < \infty$, $\forall\, n \geq 1$, $\forall\, (t,u) \in [0,T] \times \mathcal{C}([-r,0],\mathbb{R}^d))$. *Then there exists a* $(\gamma+2m)$-*Lipschitzean function* $u\colon\ [-r,T] \to \mathbb{R}^d$ *such that*

$$\begin{cases} u'(t) \ \in\ -N_{K(t)}(u(t)) + F(t,u_t) \quad \text{a.e. } \ t \in [0,T] \\ u(t) \in K(t)\,, \quad \forall\, t \in [0,T] \\ u_0 = u_T\ . \end{cases}$$

Proof: Since $T \geq r$, then by taking

$$\widetilde{K}(\tau) = K(T+\tau)\,, \qquad \tau \in [-r,0]\ , \tag{4.1.1}$$

we define an extension of K to the whole interval $[-r,T]$. Moreover, $\widetilde{K}$ is γ-Lipschitzean (the assumption $K(0) = K(T)$ implies its continuity at zero). We introduce a subset of the Banach space $\mathcal{C}_0 := \mathcal{C}([-r,0],\mathbb{R}^d)$:

$$\mathcal{K}_\lambda := \Big\{\varphi \in \mathcal{C}_0\colon\ \varphi(\tau) \in \widetilde{K}(\tau),\ \|\varphi(\tau) - \varphi(\overline{\tau})\| \leq \lambda|\tau - \overline{\tau}|,\ \forall\, \tau, \overline{\tau} \in [-r,0]\Big\},$$

where $\lambda = \gamma + 2m$; $\mathcal{K}_\lambda$ is convex compact and nonempty; in fact, any solution of the sweeping process by $\widetilde{K}$ is an element of $\mathcal{K}_\lambda$.

The outline of the proof is the following. For notational convenience, we set $F_\infty = F$.

For each $n \in \mathbb{N}^* \cup \{\infty\}$ and each $\varphi \in \mathcal{K}_\lambda$, we denote by $\mathcal{S}_{F_n}(\varphi)$ the set of all absolutely continuous mappings $u\colon\ [-r,T] \to \mathbb{R}^d$ such that

$$\begin{cases} u'(t) \ \in\ -N_{K(t)}(u(t)) + F_n(t,u_t) \quad \text{a.e. } \ t \in [0,T] \\ u(t) \in K(t)\,, \quad \forall\, t \in [0,T] \\ u_0 = \varphi \quad \text{in } \ [-r,0]\ . \end{cases} \tag{4.1.2$_n$}$$

By Lemma 2.1, Theorem 2.2 and the definition of $\mathcal{K}_\lambda$, any solution $u \in \mathcal{S}_{F_n}(\varphi)$ is λ-Lipschitzean. Now we introduce the set

$$\mathcal{S}_{F_n,T}(\varphi) := \Big\{u_T \in \mathcal{C}_0\colon\ u \in \mathcal{S}_{F_n}(\varphi)\Big\} \tag{4.1.3$_n$}$$

and we consider the multifunction

$$\mathcal{S}_{F_n,T}\colon\ \varphi \mapsto \mathcal{S}_{F_n}(\varphi)\ . \tag{4.1.4$_n$}$$

Since every $u \in \mathcal{S}_{F_n}(\varphi)$ satisfies $u(t) \in K(t)$, $\forall\, t \in [0,T]$, by (4.1.1) we have

$$\forall\, \tau \in [-r,0]\,, \quad u_T(\tau) := u(T+\tau) \;\in\; K(T+\tau) = \widetilde{K}(\tau)\ .$$

Hence $\mathcal{S}_{F_n,T}(\varphi) \subset \mathcal{K}_\lambda$ for all $\varphi \in \mathcal{K}_\lambda$.

By Proposition 3.3, the multifunctions $\mathcal{S}_{F_n}\colon \mathcal{K}_\lambda \to \mathcal{K}_\lambda$ ($n \in \mathbb{N}^* \cup \{\infty\}$) have compact graph and satisfy (see Proposition 3.5)

$$\forall\, \varphi \in \mathcal{K}_\lambda\,, \quad \mathcal{S}_{F_\infty}(\varphi) = \bigcap_{n\geq 1} \mathcal{S}_{F_n}(\varphi)\ .$$

Moreover, by Proposition 3.4 each $\mathcal{S}_{F_n}$ ($n \in \mathbb{N}^*$) admits a continuous selection. Hence it is not difficult to see that the multifunctions $\mathcal{S}_{F_n,T}$ enjoy the same properties. Namely,

a) $\mathcal{S}_{F_n,T}\colon \mathcal{K}_\lambda \to \mathcal{K}_\lambda$ ($n \in \mathbb{N}^* \cup \{\infty\}$) has compact graph.

b) $\forall\, \varphi \in \mathcal{K}_\lambda$, $\mathcal{S}_{F_\infty,T}(\varphi) = \bigcap\limits_{n\geq 1} \mathcal{S}_{F_n,T}(\varphi)$.

c) $\mathcal{S}_{F_n,T}$ ($n \in \mathbb{N}^*$) admits a continuous selection.

Hence (a), (b), (c) imply that the multifunction $\mathcal{S}_{F_\infty,T}$ admits a fixed point $\varphi \in \mathcal{S}_{F_\infty,T}(\varphi)$ (see [22], Theorem A.III.1). Whence there is $u \in \mathcal{S}_{F_\infty}(\varphi)$ such that $\varphi = u_T$. So we conclude that u solves

$$\begin{cases} u'(t) \;\in\; -N_{K(t)}(u(t)) + F(t,u_t) \quad \text{a.e. } \; t \in [0,T] \\ u(t) \in K(t)\,, \quad \forall\, t \in [0,T] \\ u_0 = u_T\ . \ \blacksquare \end{cases}$$

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