PORTUGALIAE MATHEMATICA
Vol. 56 Fasc. 3 – 1999

SKEW SEMI-INVARIANT SUBMANIFOLDS OF A LOCALLY PRODUCT MANIFOLD

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Abstract: In this paper, we defined and studied a new class of submanifolds of a locally Riemannian product manifold, i.e., skew semi-invariant submanifolds. We give two sufficient conditions for submanifolds to be skew semi-invariant submanifolds. Moreover, we discussed the sectional curvature of skew semi-invariant submanifolds and obtained many interesting results.

1 – Introduction

In the early years of the sixties, S. Tachibana [1] introduced and studied a class if important manifolds, i.e., locally product manifolds. After that, some authors discussed this class of manifolds, they obtained many very interesting results (cf. [2], [3], [4] and [5]). In [6], A. Bejancu defined and studied semi-invariant submanifolds of a locally product manifold. In this paper, we defined and discussed a new class of submanifolds of a locally product manifold, i.e., skew semi-invariant submanifolds, which contain semi-invariant submanifolds as a special case.

There are two parts in this paper, in section one we give the definition of skew semi-invariant submanifolds and some preliminaries which we will use later. In section two we discuss the parallelism of the canonical structures P and Q and the sectional curvature of skew semi-invariant submanifolds.

Received: December 2, 1997.
1991 Mathematics Subject Classification: 53C40, 53C15.
Keywords and Phrases: Sectional curvature, Locally product manifold, Skew semi-invariant submanifold.

2 – Definitions and preliminaries

In this paper, we suppose that all manifolds and maps are C^∞-differentiable.

Let $(\bar{M}, g, F)$ be an almost product Riemannian manifold, where g is a Riemannian metric and F is a non-trivial tensor field of type $(1,1)$, F is called an almost product structure. Moreover g and F satisfying the following conditions

$$F^2 = I \quad (F \neq \pm I)\,, \qquad g(FX, FY) = g(X, Y)\,, \tag{1}$$

where $X, Y \in T\bar{M}$ and I is the identity transformation.

We denote by $\bar{\nabla}$ the Levi–Civita connection on $\bar{M}$ with respect to g, if $\bar{\nabla}_X F = 0$, $X \in T\bar{M}$, we call $\bar{M}$ a locally product Riemannian manifold.

Let M be a Riemannian manifold isometrically immersed in $\bar{M}$ and denote by the same symbol g the Riemannian metric induced on M, for $p \in M$ and tangent vector $X_p \in T_pM$, we write

$$FX_p = PX_p + QX_p \tag{2}$$

where $PX_p \in T_pM$ is tangent to M and $QX_p \in T_p^\perp M$ is normal to M.

For any two vectors $X_p, Y_p \in T_pM$, we have $g(FX_p, Y_p) = g(PX_p, Y_p)$, which implies that $g(PX_p, Y_p) = g(X_p, PY_p)$. So P and P^2 are all symmetric operators on the tangent space T_pM. If $\alpha(p)$ is the eigenvalue of P^2 at $p \in M$, since P^2 is a composition of an isometry and a projection, hence $\alpha(p) \in [0,1]$.

For each $p \in M$, we set $D_p^\alpha = \operatorname{Ker}(P^2 - \alpha(p)\, I)$, where I is the identity transformation on T_pM, and $\alpha(p)$ is an eigenvalue of P^2 at $p \in M$, obviously, we have $D_p^0 = \operatorname{Ker} P$, $D_p^1 = \operatorname{Ker} Q$, D_p^1 is the maximal F invariant subspace of T_pM and D_p^0 is the maximal F anti-invariant subspace of T_pM. If $\alpha_1(p), ..., \alpha_k(p)$ are all eigenvalues of P^2 at p, then T_pM can be decomposed as the direct sum of the mutually orthogonal eigenspaces, that is,

$$T_pM = D_p^{\alpha_1} \oplus \cdots \oplus D_p^{\alpha_k}\ .$$

Now we give the following definition.

Definition. A submanifold M of a locally product manifold $\bar{M}$ is called a skew semi-invariant submanifold if there exists an integer k and constant functions α_i, $1 \le i \le k$, defined on M with values in $(0,1)$ such that

(**i**) Each α_i, $1 \le i \le k$, is a distinct eigenvalue of P^2 with $T_pM = D_p^0 \oplus D_p^1 \oplus D_p^{\alpha_1} \oplus \cdots \oplus D_p^{\alpha_k}$, for $p \in M$.

(**ii**) The dimensions of D_p^0, D_p^1 and $D_p^{\alpha_i}$, $1 \le i \le k$, are independent of $p \in M$.

Remark. Condition (ii) in the above definition implies that D_p^0, D_p^1 and $D_p^{\alpha_i}$, $1 \le i \le k$, defined P invariant, mutually orthogonal distributions which we denote by D^0, D^1 and D^{α_i}, $1 \le i \le k$, respectively. Moreover the tangent bundle of M has the following decomposition

$$TM = D^0 \oplus D^1 \oplus D^{\alpha_1} \oplus \cdots \oplus D^{\alpha_k} \ .$$

Particularly if $k = 0$ then M is a semi-invariant submanifold [6]. If $k = 0$, and $D_p^0(D_p^1)$ is trivial, then M is an invariant (anti-invariant) submanifold of $\bar{M}$ [4].

Denote the induced connection in M by ∇, we have the formulas of Gauss and Weingarten

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \ , \tag{3}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \ , \tag{4}$$

for all vector fields $X, Y \in TM$ and $N \in T^{\perp}M$. Here h denotes the second fundamental form and $T^{\perp}M$ denotes the normal bundle of M in $\bar{M}$. Moreover we have

$$g\Big(h(X,Y), N\Big) = g(A_N X, Y) \ . \tag{5}$$

For $N \in T^{\perp}M$, we set

$$FN = tN + fN \tag{6}$$

where $tN \in TM$, $fN \in T^{\perp}M$.

From $F(\bar{\nabla}_X Y) = \bar{\nabla}_X FY$, (3), (4) and (6) we have

$$\begin{aligned} P(\nabla_X Y) + Q(\nabla_X Y) + t\,h(X,Y) + f\,h(X,Y) &= \\ = \nabla_X PY + h(X,PY) - A_{QY} X &+ \nabla_X^{\perp} QY \ , \end{aligned} \tag{7}$$

for $X, Y \in TM$. Comparing tangential and normal components in (7) we obtain

$$P\,\nabla_X Y = \nabla_X PY - t\,h(X,Y) - A_{QY} X \ , \tag{8}$$

$$Q\,\nabla_X Y = h(X,PY) + \nabla_X^{\perp} QY - f\,h(X,Y) \ , \tag{9}$$

for $X, Y \in TM$. From (8) and (9) we can get

$$P[X,Y] = \nabla_X PY - \nabla_Y PX + A_{QX} Y - A_{QY} X \ , \tag{10}$$

$$Q[X,Y] = h(X,PY) - h(PX,Y) + \nabla_Y^{\perp} QX - \nabla_X^{\perp} QX \ . \tag{11}$$

We have the following lemma immediately from (10) and (11)

Lemma 1.1. *Let M be a skew semi-invariant submanifold of a locally product manifold $\bar{M}$, then*

- (**i**) *The distribution D^0 is integrable if and only if $A_{FX}Y = A_{FY}X$ for all $X, Y \in D^0$.*
- (**ii**) *The distribution D^1 is integrable if and only if $h(X, FY) = h(FX, Y)$ for all $X, Y \in D^1$.*

We define the covariant derivatives of P and Q in a manner as follows

$$(\nabla_X P)\, Y = \nabla_X PY - P\,\nabla_X Y \ , \tag{12}$$

$$(\nabla_X Q)\, Y = \nabla^{\perp}_X QY - Q\,\nabla_X Y \ , \tag{13}$$

for all $X, Y \in TM$. Using (8) and (9) we have

$$(\nabla_X P)\, Y = t\, h(X, Y) + A_{QY} X \ , \tag{14}$$

$$(\nabla_X Q)\, Y = f\, h(X, Y) - h(X, PY) \ . \tag{15}$$

Let D^1 and D^2 be two distributions defined on a manifold M. We say that D^1 is parallel with respect to D^2 if for all $X \in D^2$ and $Y \in D^1$, we have $\nabla_X Y \in D^1$. D^1 is called parallel if for $X \in TM$ and $Y \in D^1$, we have $\nabla_X Y \in D^1$, it is easy to verify that D^1 is parallel if and only if the orthogonal complementary distribution of D^1 is also parallel.

Let M be a submanifold of $\bar{M}$. A distribution D on M is said to be totally geodesic if for all $X, Y \in D$ we have $h(X, Y) = 0$. In this case we say also that M is D totally geodesic. For two distributions D^1 and D^2 defined on M, we say that M is D^1-D^2 mixed totally geodesic if for all $X \in D^1$ and $Y \in D^2$ we have $h(X, Y) = 0$.

Proposition 1.1. *Let M be a skew semi-invariant submanifold of a locally product manifold $\bar{M}$, for any distribution D^α, if $A_N PX = PA_N X$, for all $X \in D^\alpha$ and $N \in T^\perp M$, then M is D^α-D^βmixed totally geodesic, where $\alpha \neq \beta$.*

Proof: From the assumption we have $P^2 A_N X - \alpha\, A_N X = 0$, which implies that $A_N X \in D^\alpha$. So for all $Y \in D^\beta$, $N \in T^\perp M$, $\alpha \neq \beta$, we have $0 = g(A_N X, Y) = g(h(X, Y), N)$, that is $h(X, Y) = 0$, hence M is D^α-D^β mixed totally geodesic. ∎

From (2) and (6) we can obtain

$$f\,QX_p = -Q\,PX_p \,, \tag{16}$$

$$Q\,t\,N = N - f^2 N \,, \tag{17}$$

for all $X_p \in T_pM$, $N \in T_p^{\perp}M$. Furthermore, for $X_p \in D_p^{\alpha_i}$, $1 \le i \le k$, we have

$$f^2\,QX_p = \alpha_i\,QX_p \,. \tag{18}$$

Also if $X_p \in D_p^0$ then it is clear that $f^2\,QX_p = 0$. Thus if X_p is an eigenvector of P^2 corresponding to the eigenvalue $\alpha(p) \neq 1$, then QX_p is an eigenvector of f^2 with the same eigenvalue $\alpha(p)$. (17) implies that $\alpha(p)$ is an eigenvalue of f^2 if and only if $\gamma(p) = 1-\alpha(p)$ is an eigenvalue of Qt. Since Qt and f^2 are symmetric operators on the normal bundle $T^{\perp}M$, their eigenspaces are orthogonal. The dimension of the eigenspace of Qt corresponding to the eigenvalue $1-\alpha(p)$ is equal the dimension of D_p^{α} if $\alpha(p) \neq 1$. Consequently, we have

Lemma 1.2. *Let M be a submanifold of a locally product Riemannian manifold $\bar{M}$. M is a skew semi-invariant submanifold if and only if the eigenvalues of Qt are constant and the eigenspaces of Qt have constant dimension.*

3 – Skew semi-invariant submanifold

Theorem 2.1. *Let M be a submanifold of a locally product manifold $\bar{M}$, if $\nabla P = 0$, then M is a skew semi-invariant submanifold. Furthermore each of the P invariant distributions D^0, D^1 and D^{α_i}, $1 \le i \le k$, is parallel.*

Proof: Fix $p \in M$, for any $Y_p \in D_p^{\alpha_i}$ and any vector field $X \in TM$, let Y be the parallel translation of Y_p along the integral curve of X. Since $(\nabla_X P)\,Y = 0$, we have by (8)

$$\nabla_X(P^2 - \alpha(p)\,Y) = P^2\,\nabla_X Y - \alpha(p)\,\nabla_X Y = 0$$

since $P^2Y - \alpha(p)\,Y = 0$ at p, it is identical 0 on M. Thus the eigenvalues of P^2 are constant. Moreover, parallel translation of T_pM along any curve is an isometry which preserves each D^{α}. Thus the dimension of each D^{α} is constant and M is a skew semi-invariant submanifold.

Now if Y is any vector field in D^{α}, we have $P^2Y = \alpha\,Y$ (α constant), i.e., $P^2\,\nabla_X Y = \alpha\,\nabla_X Y$ which implies that D^{α} is parallel. ∎

Next we turn our attention to the vanishing of ∇Q. For $X, Y \in TM$, if $(\nabla_X Q)\, Y = 0$ then (15) yields

$$f\, h(X,Y) = h(X, PY)\ . \tag{19}$$

In particular, if $Y \in D^\alpha$ then (19) implies

$$f^2\, h(X,Y) = \alpha\, h(X,Y) \tag{20}$$

consequently we have

Proposition 2.1. *Let M be a skew semi-invariant submanifold of a locally product manifold $\bar{M}$, if $\nabla Q \equiv 0$, then M is D^α-D^β mixed totally geodesic for all $\alpha \neq \beta$. Moreover, if $X \in D^\alpha$ then either $h(X,X) = 0$ or $h(X,X)$ is an eigenvector of f^2 with eigenvalue α.*

The next lemma is easy to prove so we omit the proof.

Lemma 2.1. *Let M be a submanifold of a locally product manifold $\bar{M}$, then $\nabla Q = 0$ if and only if $\nabla_X tN = t\, \nabla^\perp_X N$ for all $X \in TM$ and $N \in T^\perp M$.*

Theorem 2.2. *Let M be a submanifold of a locally product manifold $\bar{M}$, if $\nabla Q = 0$, then M is skew semi-invariant submanifold.*

Proof: If $TM = D^1$ then we are done. Otherwise, we may find a point $p \in M$ and a vector $X_p \in D^\alpha_p$, $\alpha \neq 1$. Set $N_p = QX_p$, then N_p is an eigenvector of Qt with eigenvalue $\gamma(p) = 1 - \alpha(p)$. Now, let $Y \in TM$ and N be the translation of N_p in the normal bundle $T^\perp M$ along an integral curve of Y, we have

$$\nabla^\perp_Y (Qt\, N - \gamma(p)\, N) = \nabla^\perp_Y Qt\, N - \gamma(p)\, \nabla^\perp_Y N = Q(\nabla_Y tN) - \gamma(p)\, \nabla^\perp_Y N\ .$$

By Lemma 2.1, this becomes $\nabla^\perp_Y (Qt\, N - \gamma(p)\, N) = Qt\, \nabla^\perp_Y N - \gamma(p)\, \nabla^\perp_Y N = 0$. Since $Qt\, N - \gamma(p)\, N = 0$ at p, $Qt\, N - \gamma(p)\, N \equiv 0$ on M. It follows from Lemma 1.2 that M is a skew semi-invariant submanifold. ∎

For a submanifold M of a locally product manifold $\bar{M}$, let $\bar{R}$ (resp. R) denote the curvature tensor of $\bar{M}$ (resp. M), then the equation of Gauss is given by

$$\begin{aligned} g\Big(R(X,Y)\, Z,\, W\Big) &= g\Big(\bar{R}(X,Y)\, Z,\, W\Big) + g\Big(h(X,W),\, h(Y,Z)\Big) \\ &\quad - g\Big(h(X,Z),\, h(Y,W)\Big) \end{aligned} \tag{21}$$

for $X, Y, Z, W \in TM$.

The sectional curvature of a plane section of $\bar{M}$ determined by two orthogonal unit vectors $X, Y \in T\bar{M}$ is given by

$$K_{\bar{M}}(X \wedge Y) = g\Big(\bar{R}(X,Y)\,Y,\, X\Big)\ . \tag{22}$$

The sectional curvature of a plane section of M determined by two orthogonal unit vectors $X, Y \in TM$ is given by

$$K_M(X \wedge Y) = g\Big(R(X,Y)\,Y,\, X\Big)\ . \tag{23}$$

For $X, Y \in TM$, from (21), (22) and (23) we can obtain

$$K_M(X \wedge Y) = K_{\bar{M}}(X \wedge Y) + g\Big(h(X,X),\, h(Y,Y)\Big) - |h(X,Y)|^2\ . \tag{24}$$

Proposition 2.2. *Let M be a skew semi-invariant submanifold of a locally product manifold $\bar{M}$, if $\nabla Q = 0$, then for any unit vectors $X \in D^\alpha$ and $Y \in D^\beta$, $\alpha \neq \beta$, we have $K_M(X \wedge Y) = K_{\bar{M}}(X \wedge Y)$.*

Proof: It can be followed easily from Proposition 2.1. ∎

Lemma 2.2. *Let M be a skew semi-invariant submanifold of a locally product manifold $\bar{M}$, then the followings are equivalent*

(**i**) $(\nabla_X Q)\,Y - (\nabla_Y Q)\,X = 0$ *for all* $X, Y \in D^\alpha$.

(**ii**) $h(P,X,Y) = h(X, PY)$ *for all* $X, Y \in D^\alpha$.

(**iii**) $Q[X,Y] = \nabla^\perp_X QY - \nabla^\perp_Y QX$ *for all* $X, Y \in D^\alpha$.

(**iv**) $A_N PY - PA_N Y$ *is perpendicular to* D^α *for all* $Y \in D^\alpha$ *and* $N \in T^\perp N$.

The proof is very trivial, we omit it here.

We call P α commutative if any of the equivalent conditions in the above Lemma holds.

For each P invariant D^α, let $n(\alpha) = \dim D^\alpha$. For each D^α we may choose a local orthonormal basis $E^1, ..., E^{n(\alpha)}$. Define the D^α mean curvature vector by $H^\alpha = \sum_{i=1}^{n(\alpha)} h(E^i, E^i)$, then the mean curvature vector is given by $H = \frac{1}{n}\,(H^0 + H^1 + H^{\alpha_1} + \cdots + H^{\alpha_k})$, $n = \dim M$.

A skew semi-invariant submanifold M of a locally product manifold $\bar{M}$ is called D^α minimal if $H^\alpha = 0$ and minimal if $H = 0$.

For any unit vector $X \in D^\alpha$, $\alpha \neq 0$, defined the α sectional curvature of $\bar{M}$ and M by

$$\bar{H}_\alpha(X) = K_{\bar{M}}(X \wedge Y)\,, \qquad H_\alpha(X) = K_M(X \wedge Y)$$

respectively, where $Y = \frac{PX}{\sqrt{\alpha}}$. From (24) we have

$$H_\alpha(X) = \bar{H}_\alpha(X) - \frac{1}{\alpha}\, g\Big(h(X,X),\, h(PX,PX)\Big) - \frac{1}{\alpha}\, |h(X,PX)|^2 \ . \tag{25}$$

Then we have the following proposition

Proposition 2.3. *Let M be a skew semi-invariant submanifold of a locally product manifold $\bar{M}$, if P is α commutative, $\alpha \neq 0$, then*

$$H_\alpha(X) = \bar{H}_\alpha(X) + |h(X,X)|^2 - \frac{1}{\alpha}\, |h(X,PX)|^2 \ .$$

Let $\{E^1, ..., E^{n(\alpha)}\}$ and $\{F^1, ..., F^{n(\beta)}\}$ be the local orthonormal bases for D^α and D^β, respectively. We define α-β sectional curvatures of $\bar{M}$ and M by

$$\bar{\rho}_{\alpha\beta} = \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} K_{\bar{M}}(E^i \wedge F^j)\,, \qquad \rho_{\alpha\beta} = \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} K_M(E^i \wedge F^j)\ ,$$

respectively.

From (24) we see that for $\alpha \neq \beta$ we have

$$\rho_{\alpha\beta} = \bar{\rho}_{\alpha\beta} + g(H^\alpha, H^\beta) - \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} |h(E^i \wedge F^j)|^2 \ , \tag{26}$$

for $\alpha = \beta$ we have

$$\rho_{\alpha\alpha} = \bar{\rho}_{\alpha\alpha} - \sum_{i=1}^{n(\alpha)} \sum_{j=1}^{n(\beta)} |h(E^i \wedge F^j)|^2 \ . \tag{27}$$

Using (26) and (27) we have the following proposition

Proposition 2.4. *Let M be a skew semi-invariant submanifold of a locally product manifold $\bar{M}$.*

- **(i)** *If H^α is perpendicular to H^β, $\alpha \neq \beta$, then $\rho_{\alpha\beta} \leq \bar{\rho}_{\alpha\beta}$, and the equality holds if and only if M is D^α-D^β mixed totally geodesic.*
- **(ii)** *If M is D^α minimal, then $\rho_{\alpha\alpha} \leq \bar{\rho}_{\alpha\alpha}$, and the equality holds if and only if M is D^α totally geodesic.*

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