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# ON QUASIMONOTONE INCREASING SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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**Abstract:** We prove an uniqueness theorem for the initial value problem $x'(t) = f(t, x(t))$, $x(t_0) = x_0$, in case that $f$ is quasimonotone increasing with respect to an arbitrary cone, and is satisfying a one-sided Lipschitz condition with respect to a single linear functional. An inequality concerning the difference of solutions is obtained.

## 1

Let $\mathbb{R}^n$ be ordered by a cone $K$ with $\operatorname{Int} K \neq \emptyset$. A cone $K$ is a closed convex subset of $E$ with $\lambda K \subseteq K$, $\lambda \geq 0$, and $K \cap (-K) = \{0\}$. We define $x \leq y \Leftrightarrow y - x \in K$, and we use the notation $x \ll y$ for $y - x \in \operatorname{Int} K$ and $K^*$ for the dual cone, i.e., the set of all $\varphi \in (\mathbb{R}^n)^*$ with $\varphi(x) \geq 0$, $x \geq 0$. We have $\operatorname{Int} K^* \neq \emptyset$. For $x, y \in E$, $x \leq y$ we define the order interval $[x, y] = \{z \in E \colon x \leq z \leq y\}$.

Now we fix $p \gg 0$. Then $\mathbb{R}^n$ can be normed by a norm $\|\cdot\|_p$, such that $\{x \in \mathbb{R}^n \colon \|x\|_p \leq 1\} = [-p, p]$ (see e.g. [4]). For $x, y \in \mathbb{R}^n$ we have $0 \leq x \leq y \Rightarrow \|x\|_p \leq \|y\|_p$, $\|x\|_p \leq c \Leftrightarrow -cp \leq x \leq cp$, and $-y \leq x \leq y \Rightarrow \|x\|_p \leq \|y\|_p$.

Now let $f \colon [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, and let $(t_0, x_0) \in [0, \infty) \times \mathbb{R}^n$. We consider the initial value problem

$$x'(t) = f(t, x(t)), \qquad x(t_0) = x_0 \; . \tag{1}$$

The function $f$ is called quasimonotone increasing (in $x$) (in the sense of Volkmann [7]) if

$$(t,x), (t,y) \in [0,\infty) \times \mathbb{R}^n, \;\; x \leq y, \;\; \varphi \in K^*, \\ \varphi(x) = \varphi(y) \implies \varphi(f(t,x)) \leq \varphi(f(t,y)) \; .$$

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According to a theorem of Volkmann (see [7]) we have the following assertion if $f$ is quasimonotone increasing:

Let $[a,b] \subset [0,\infty)$, and let $u, v \in C^1([a,b],\mathbb{R}^n)$. Then

$$u'(t) - f(t,u(t)) \ll v'(t) - f(t,v(t)), \qquad t \in [a,b] ,$$
$$u(a) \ll v(a) \implies u(t) \ll v(t), \qquad t \in [a,b] .$$

Next, if $\psi \in \operatorname{Int} K^*$, there are constants $\alpha, \beta > 0$ with

$$\alpha \|x\|_p \le \psi(x) \le \beta \|x\|_p , \qquad x \ge 0 .$$

We consider the following condition (P):

There exists $\psi \in \operatorname{Int} K^*$ and $L \in C([0,\infty),\mathbb{R})$ with

$$\psi\Big(f(t,y) - f(t,x)\Big) \le L(t)\,\psi(y-x) ,$$
$$(t,x),(t,y) \in [0,\infty) \times \mathbb{R}^n , \qquad x \le y .$$

We will prove the following theorem:

**Theorem 1.** *Let $f\colon [0,\infty)\times\mathbb{R}^n \to \mathbb{R}^n$ be continuous, quasimonotone increasing and satisfy (P). Then we have:*

**1**) *Problem (1) is uniquely solvable on $[t_0,\infty)$.*

**2**) *Let $x(t;t_0,x_0)$, $t \ge t_0$ denote the solution of problem (1). The following inequality holds:*

$$\Big\| x(t;t_0,x_0) - x(t;t_0,y_0) \Big\|_p \le \frac{\beta}{\alpha} \exp\Big( \int_{t_0}^{t} L(s)\,ds \Big) \, \|x_0 - y_0\|_p ,$$
$$(t_0,x_0),(t_0,y_0) \in [0,\infty) \times \mathbb{R}^n , \qquad t \ge t_0 .$$

**Remarks.**

**1**. The amazing on Theorem 1 is that a one-sided estimate with respect to a single linear functional leads to the norm inequality in Theorem 1.

**2**. Theorem 1 leads to stability results for the differential equation in problem (1). For example if $f(t,0) = 0$, $t \ge 0$, and $\limsup\limits_{t\to\infty} \int_0^t L(s)\,ds < \infty$, then the origin is stable in the sense of Liapunov.

**3**. Theorem 1 also leads to results on periodic solutions of the differential equation in problem (1). If for example $f$ is periodic in $t$ with period $T$ then there exists a unique $T$-periodic solution on $[0,\infty)$ if $\log(\frac{\beta}{\alpha})+\int_0^T L(s)\,ds<0$.

**4**. Condition (P) does not imply uniqueness to the left. Consider for example $f\colon \mathbb{R}^2\to\mathbb{R}^2$, $f(x,y)=(-\sqrt[3]{x},\sqrt[3]{x})$, with $K=[0,\infty)^2$ and $\psi(x,y)=x+y$.

**5**. Conditions related to (P) can imply existence and uniqueness of the solution of initial value problems in ordered Banach spaces (see [2]).

**6**. For related uniqueness conditions in case that $K$ is the natural cone we refer to [6]. □

**Proof:**

**1**) Let $x\colon [t_0,\omega)\to\mathbb{R}^n$ be a solution of problem (1), with $[t_0,\omega)$ a right maximal interval of existence. We will show that $\omega=\infty$. We define $g\colon [0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ as $g(t,x)=f(t,x)-f(t,0)$. The function $g$ is quasimonotone increasing. Let $\lambda\colon [0,\infty)\to[1,\infty)$ be defined as $\lambda(t)=\|f(t,0)\|_p+1$. We have

$$-f(t,0)+\lambda(t)\,p\gg 0,\quad -f(t,0)-\lambda(t)\,p\ll 0\,,\qquad t\in[0,\infty)\ .$$

Now choose $\mu>0$ with $-\mu p\ll x_0\ll\mu p$, and let $u\colon [t_0,\omega_1)\to\mathbb{R}^n$, respectively $v\colon [t_0,\omega_2)\to\mathbb{R}^n$ be solutions (both defined on a right maximal interval of existence) of the initial value problems

$$u'(t)=g(t,u(t))-\lambda(t)\,p\,,\qquad u(t_0)=-\mu\,p\ ,$$

respectively

$$v'(t)=g(t,v(t))+\lambda(t)\,p\,,\qquad v(t_0)=\mu\,p\ .$$

We have $u'(t)-g(t,u(t))\ll 0$, $t\in[t_0,\omega_1)$, $u(t_0)\ll 0$, which implies $u(t)\ll 0$, $t\in[t_0,\omega_1)$ (since $g(t,0)=0$, $t\geq 0$). Analoguous $v(t)\gg 0$, $t\in[t_0,\omega_2)$. Moreover we have

$$\psi(u'(t))=\psi\Big(f(t,u(t))-f(t,0)\Big)-\lambda(t)\,\psi(p)\ \geq\ L(t)\,\psi(u(t))-\lambda(t)\,\psi(p)\ ,$$
$$\psi(u(t_0))=-\mu\,\psi(p)\ .$$

Applying common results on differential inequalities (see e.g. [8]) we get that there is a function

$$h\in C\Big([t_0,\infty),(0,\infty)\Big)$$

with

$$\psi(u(t))\geq -h(t)\,,\qquad t\in[t_0,\omega_1)\ .$$

Hence
$$\|u(t)\|_p \leq \frac{\psi(-u(t))}{\alpha} \leq \frac{h(t)}{\alpha}\,, \qquad t \in [t_0, \omega_1)\ .$$
This implies by standard reasoning that $\omega_1 = \infty$. Analoguous we get $\omega_2 = \infty$. We have for $t \in [t_0, \omega)$ that

$$u'(t) - f(t, u(t)) = -f(t, 0) - \lambda(t)\, p \ll 0 = x'(t) - f(t, x(t))\ ,$$

and
$$x'(t) - f(t, x(t)) = 0 \ll -f(t, 0) + \lambda(t)\, p = v'(t) - f(t, v(t))\ .$$
Since $u(t_0) \ll x(t_0) \ll v(t_0)$ we have

$$u(t) \ll x(t) \ll v(t)\,, \qquad t \in [t_0, \omega)\ .$$

This implies $\omega = \infty$.

Next we show that problem (1) is uniquely solvable. Let $x_i\colon\ [t_0, \infty) \to \mathbb{R}^n$, $i = 1, 2$ be solutions of problem (1). We redefine $u$ and $v$. Let $\mu > 0$ and let $u\colon\ [t_0, \infty) \to \mathbb{R}^n$, respectively $v\colon\ [t_0, \infty) \to \mathbb{R}^n$ be a solution of the initial value problems
$$u'(t) = f(t, u(t)) - \mu p\,, \qquad u(t_0) = x_0 - \mu p\ ,$$
respectively
$$v'(t) = f(t, v(t)) + \mu p\,, \qquad v(t_0) = x_0 + \mu p\ .$$
We have $u(t) \ll x_i(t) \ll v(t)$, $t \in [t_0, \infty)$, $i = 1, 2$. Hence

$$u(t) - v(t) \ll x_1(t) - x_2(t) \ll v(t) - u(t)\,, \qquad t \in [t_0, \infty)\ .$$

Therefore
$$\|x_1(t) - x_2(t)\|_p \leq \|v(t) - u(t)\|_p\,, \qquad t \in [t_0, \infty)\ .$$
Since

$$\psi\Big(v'(t) - u'(t)\Big)\ \leq\ L(t)\, \psi\Big(v(t) - u(t)\Big) + 2\, \mu\, \psi(p)\,, \qquad t \in [t_0, \infty)\ ,$$

and
$$\psi\Big(v(t_0) - u(t_0)\Big) = 2\, \mu\, \psi(p)\ ,$$
we have that $\psi(v(t) - u(t))$ (and therefore $\|v(t) - u(t)\|_p$) tends to 0 as $\mu$ tends to $0+$. Hence $x_1 = x_2$.

**2**) Let $x_0, y_0 \in \mathbb{R}^n$, $\mu > 0$, and let

$$u_0 = \frac{x_0 + y_0 - \|x_0 - y_0\|_p\, p}{2} - \mu\, p\,, \qquad v_0 = \frac{x_0 + y_0 + \|x_0 - y_0\|_p\, p}{2} + \mu\, p\ .$$

We have

$$\|v_0 - u_0\|_p = \|x_0 - y_0\|_p + 2\mu\ ,$$

and

$$u_0 \ll x_0 \ll v_0\,, \qquad u_0 \ll y_0 \ll v_0\ .$$

We again redefine $u$ and $v$. Let $u\colon\ [t_0, \infty) \to \mathbb{R}^n$, respectively $v\colon\ [t_0, \infty) \to \mathbb{R}^n$ be the solution of the initial value problem

$$u'(t) = f(t, u(t)) - \mu\, p\,, \qquad u(t_0) = u_0\ ,$$

respectively

$$v'(t) = f(t, v(t)) + \mu\, p\,, \qquad v(t_0) = v_0\ .$$

We have for $t \in [t_0, \infty)$ that

$$u(t) \ll x(t; t_0, x_0) \ll v(t)\,, \qquad u(t) \ll x(t; t_0, y_0) \ll v(t)\ ,$$

and therefore

$$u(t) - v(t) \ll x(t; t_0, x_0) - x(t; t_0, y_0) \ll v(t) - u(t)\ ,$$

which implies

$$\Big\| x(t; t_0, x_0) - x(t; t_0, y_0) \Big\|_p \le \Big\| v(t) - u(t) \Big\|_p\ .$$

Now

$$\psi\Big(v'(t) - u'(t)\Big)\ \le\ L(t)\,\psi\Big(v(t) - u(t)\Big) + 2\,\mu\,\psi(p)\,, \qquad t \in [t_0, \infty)\ .$$

We get

$$\psi\Big(v(t) - u(t)\Big)\ \le\ \exp\Big(\int_{t_0}^{t} L(s)\,ds\Big)\,\psi(v_0 - u_0)\ + \int_{t_0}^{t} \exp\Big(\int_{s}^{t} L(\tau)\,d\tau\Big)\,2\,\mu\,\psi(p)\,ds\ ,$$

for $t \in [t_0, \infty)$. Therefore

$$\Big\| x(t; t_0, x_0) - x(t; t_0, y_0) \Big\|_p \le \Big\| v(t) - u(t) \Big\|_p \le \frac{\psi\Big(v(t) - u(t)\Big)}{\alpha} \le$$

$$\le \frac{\beta}{\alpha}\ \exp\Big(\int_{t_0}^{t} L(s)\,ds\Big)\,\Big(\|x_0 - y_0\|_p + 2\,\mu\Big) + \frac{2\,\mu\,\psi(p)}{\alpha}\int_{t_0}^{t} \exp\Big(\int_{s}^{t} L(\tau)\,d\tau\Big)\,ds\ ,$$

for $t \in [t_0, \infty)$.

As $\mu$ tends to $0+$ we get for $t \in [t_0, \infty)$ that

$$\Big\|x(t;t_0,x_0) - x(t;t_0,y_0)\Big\|_p \le \frac{\beta}{\alpha}\exp\Big(\int_{t_0}^{t} L(s)\,ds\Big)\,\|x_0 - y_0\|_p\,. \blacksquare$$

**2**

Now we will illustrate Theorem 1 by some examples:

**1**. Let $\mathbb{R}^3$ be ordered by the natural cone $K = \{(x_1,x_2,x_3)\colon x_i \ge 0,\ i=1,2,3\}$. Now consider the quasimonotone increasing function $f\colon\ [0,\infty)\times\mathbb{R}^3 \to \mathbb{R}^3$,

$$f(t,x) = \begin{pmatrix} -2(1+t)\,(\sqrt[3]{x_1} + x_1) + \sqrt[3]{x_2} + \sqrt[3]{x_3} \\ \sqrt[3]{x_1} - 2\,(1+t)\,(\sqrt[3]{x_2} + x_2) + \sqrt[3]{x_3} \\ \sqrt[3]{x_1} + \sqrt[3]{x_2} - 2\,(1+t)\,(\sqrt[3]{x_3} + x_3) \end{pmatrix}.$$

Choosing $\psi(x) = x_1 + x_2 + x_3$ we find

$$\psi\Big(f(t,y) - f(t,x)\Big) \le -2\,(1+t)\,\psi(y-x)\,, \qquad t \in [0,\infty), \quad x \le y\,.$$

Choosing $p = (1,1,1)$ we have $\|x\|_p = \max\{|x_1|,|x_2|,|x_3|\}$, $\alpha = 1$, and $\beta = 3$. According to Theorem 1 we have

$$\Big\|x(t;t_0,x_0) - x(t;t_0,y_0)\Big\|_p \le 3\exp\Big(-(t-t_0)\,(2+t-t_0)\Big)\,\|x_0 - y_0\|_p\,,$$

$t \ge t_0$, $x_0, y_0 \in \mathbb{R}^3$.

**2**. Let $\mathbb{R}^2$ be ordered by the natural cone $K = \{(x_1,x_2)\colon x_i \ge 0,\ i = 1,2\}$. Now let $g \in C([0,\infty),(0,\infty))$ and consider

$$A(t) = \begin{pmatrix} -1 & \dfrac{g(t)}{2} \\ \dfrac{1}{2g(t)} & -1 \end{pmatrix}.$$

The function $f:\ [0,\infty)\times\mathbb{R}^2 \to \mathbb{R}^2$, $f(t,x) = A(t)\,x$ is quasimonotone increasing. Although the eigenvalues of $A(t)$ are (independent of t) $-\frac{1}{2}$ and $-\frac{3}{2}$, there are functions $g$ such that the origin is unstable. Setting

$g(t) = \exp(5\sin^2(t))$ it is numerically evident that $x_i(t; 0, (1,1)) \to \infty$, $i = 1, 2$ as $t \to \infty$ (comp. the example of Markus and Yamabe [5]). Setting $\psi(x) = x_1 + x_2$ we have

$$\psi(A(t)x) \le \Bigl(\max\Bigl\{\frac{g(t)}{2}, \frac{1}{2g(t)}\Bigr\} - 1\Bigr)\,\psi(x)\,, \qquad x \ge 0\ .$$

According to Theorem 1 $x(t; t_0, x_0) \to 0$ as $t \to \infty$ for every $x_0 \in \mathbb{R}^2$ if for example there exists $q < 1$ with

$$\max\Bigl\{\frac{g(t)}{2}, \frac{1}{2g(t)}\Bigr\} \le q\,, \qquad t \ge 0\ .$$

**3**. Again let $\mathbb{R}^2$ be ordered by the natural cone $K = \{(x_1, x_2) : x_i \ge 0,\ i = 1, 2\}$. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuous with $g_1(x_1, .)$ increasing for every $x_1 \in \mathbb{R}$ and $g_2(\cdot, x_2)$ increasing for every $x_2 \in \mathbb{R}$. Then $f\colon\ \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x) = \begin{pmatrix} \max\{x_1, 0\}\, g_1(x_1, x_2) \\ \max\{x_2, 0\}\, g_2(x_1, x_2) \end{pmatrix},$$

is quasimonotone increasing. Equations of the type $x' = f(x)$ are describing the interaction of two cooperating species (see e.g. [1]). If for example $\psi(x) = x_1 + x_2$, $p = (1, 1)$, and there is a constant $L \in \mathbb{R}$ with

$$\psi\Bigl(f(y) - f(x)\Bigr) \le L\,\psi(y - x)\,, \qquad x \le y\ ,$$

then we have according to Theorem 1 that

$$\Bigl\|x(t; t_0, x_0) - x(t; t_0, y_0)\Bigr\|_p \;\le\; 2\,\exp\Bigl(L(t - t_0)\Bigr)\, \|x_0 - y_0\|_p\ ,$$

for $t \ge t_0$, $\ x_0, y_0 \in \mathbb{R}^2$.

**4**. Let $\mathbb{R}^3$ be ordered by the cone $K = \{(x_1, x_2, x_3) :\ x_3 \ge \sqrt{x_1^2 + x_2^2}\}$. It holds that $K = K^*$, and now we choose $\psi(x) = x_3$ ($\psi \in \operatorname{Int} K^*$), and $p = (0, 0, 1) \in \operatorname{Int} K$. We have $\|x\|_p = |x_3| + \sqrt{x_1^2 + x_2^2}$, $\alpha = \frac{1}{2}$, and $\beta = 1$. For the cone $K$ in this example it is much more complicated to decide wether a given function is quasimonotone increasing or not, than for the natural cone. Applying the Mean Value Theorem the following assertion is easy to prove (for general cones):

Let $f: [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let $f(t,\cdot) \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $t \geq 0$. Then $f$ is quasimonotone increasing if and only if $f_x(t,x)$ is a quasimonotone increasing matrix for every $(t,x) \in [0,\infty) \times \mathbb{R}^n$. (Here a matrix is called quasimonotone increasing if the linear mapping induced by this matrix is quasimonotone increasing.)

For matrices we have the following sufficient condition to be quasimonotone increasing in our example:

The matrix

$$A = \begin{pmatrix} a_1 & b & c \\ -b & a_2 & d \\ c & d & a_3 \end{pmatrix}$$

is quasimonotone increasing if $a_3 \geq \max\{a_1, a_2\}$, compare [3]. It is worth to be mentioned, that there are quasimonotone matrices $A$ of this type such that $A + \lambda I$ is never monotone ($\lambda \in \mathbb{R}$). (Example of Lemmert and Volkmann: Set $c = 1$ and all other entries 0).

Now consider the quasimonotone function $f\colon [0,\infty) \times \mathbb{R}^3 \to \mathbb{R}^3$,

$$f(t,x) = \begin{pmatrix} -3\,x_1 - x_2 - x_3 + \cos(x_1 - x_2 + 6\,t) \\ x_1 - 3\,x_2 + \cos(x_1 - x_2 + 6\,t) \\ -x_1 - 2\,x_3 \end{pmatrix} .$$

We have

$$\psi\Big(f(t,y) - f(t,x)\Big) \leq -\psi(y-x)\,, \qquad t \in [0,\infty), \quad x \leq y \;.$$

Hence according to Theorem 1

$$\Big\|x(t;t_0,x_0) - x(t;t_0,y_0)\Big\|_p \;\leq\; 2\,\exp\Big(-(t-t_0)\Big)\,\|x_0 - y_0\|_p \;,$$

for $t \geq t_0$, $x_0, y_0 \in \mathbb{R}^3$. Since $2\exp(-\frac{\pi}{3}) < 1$ we have that the differential equation in problem (1) has a unique $\frac{\pi}{3}$-periodic solution on $[0,\infty)$, which is assymptotically stable.

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