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EXACT DISTRIBUTED CONTROLLABILITY FOR THE SEMILINEAR WAVE EQUATION *

Wei-Jiu Liu

Abstract: In this paper we generalize the theorems of exact controllability for the linear wave equation with a distributed control to the semilinear case, showing that, given T large enough, for every initial state in a sufficiently small neighbourhood of the origin in a certain function space, there exists a distributed control, supported on a part of a domain, driving the system to rest. Also, if the control is allowed to support on the entire domain, then we prove that the system is globally exactly controllable at any time T.

1 – Introduction

The main purpose of this paper is to generalize the theorems of exact controllability for the linear wave equation to the following semilinear case

$$
\begin{cases}
y'' - \Delta y + f(y) = h & \text{in } Q, \\
y(x,0) = y^0, \quad y'(x,0) = y^1 & \text{in } \Omega, \\
y = 0 & \text{on } \Sigma\,.
\end{cases}
\tag{1.1}
$$

In (1.1), Ω is a bounded domain (nonempty, open, and connected) in $\mathbb{R}^n$ with suitably smooth boundary $\Gamma = \partial\Omega$ (say C^2); $Q = \Omega\times(0,T)$ and $\Sigma = \Gamma\times(0,T)$ for $T > 0$; the prime $'$ denotes the time derivative, $h = h(x,t)$ represents a distributed control and $f(y)$ is a given function.

The exact controllability can be defined as follows.

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Definition 1.1. System (1.1) is said to be globally exactly controllable in a suitable Hilbert space $\mathcal{H}$ if, for every initial state $(y^0, y^1) \in \mathcal{H}$ and every terminal state $(z^0, z^1) \in \mathcal{H}$, there exists a control h such that the solution of (1.1) satisfies

$$y(x,T;h) = z^0, \quad y'(x,T;h) = z^1 \quad \text{in } \ \Omega\ . \tag{1.2}$$ □

Let $\mathcal{C}$ be the set of all initial states (y^0, y^1) in a suitable Hilbert space $\mathcal{H}$, each of which can be steered to rest by a control h, that is, the solution of (1.1) satisfies

$$y(x,T;h) = 0, \quad y'(x,T;h) = 0 \quad \text{in } \ \Omega\ . \tag{1.3}$$

The set $\mathcal{C}$ is called *the set of null-controllability*.

Definition 1.2. System (1.1) is said to be locally exactly null-controllable if the set $\mathcal{C}$ of null-controllability contains an open neighborhood of 0 in the suitable Hilbert space $\mathcal{H}$. □

Definition 1.2 follows the definition of local controllability for control processes in $\mathbb{R}^n$ (see [11, p. 364]).

For the problem of local controllability for nonlinear distributed systems, the earliest definitive results appears to be the paper [17] of Markus. Based on the implicit function theorem, Markus [17] studied the local controllability problem for nonlinear finite dimensional ordinary differential equations. Subsequently, the implicit function type method was applied to nonlinear wave equations by Fattorini [5], Chewning [3] and Russell [19] and nonlinear plate equations [16].

More recently, Lagnese [9] developed a method of contraction mapping principle type to prove local controllability for nonlinear partial differential equations governing the evolution of the von Karman plate. On the other hand, using Schauder's fixed point theorem, Zuazua [21, 22, 23] studied the problem of exact boundary or distributed controllability for the semilinear wave equation

$$\begin{cases} y'' - \Delta y + f(y) = 0 & \text{in } \ Q, \\ y(x,0) = y^0, \quad y'(x,0) = y^1 & \text{in } \ \Omega\ . \end{cases} \tag{1.4}$$

Under the globally Lipschitz hypothesis on the nonlinearity, Zuazua proved that the semilinear wave equation is globally exactly controllable in $H_0^s(\Omega) \times H^{s-1}(\Omega)$ with Dirichlet boundary controls $\phi \in H_0^s(0,T;L^2(\Gamma))$ for $0 < s < 1$. The limit case $s = 0$ was left as an open question due to the lack of compactness which is required by Schauder fixied point theorem. Later, this open question was affirmatively answered by Lasiecka and Triggiani [10], who considered the exact controllability for semilinear abstract systems by using a direct approach based on the explicit construction of the controllability map, and then applied their abstract results to boundary control problems for the semilinear wave equation. Also, Zuazua proved that if the nonlinearity f is super-linear, that is, if $f(y) \in W^{1,\infty}_{\rm loc}(\mathbb{R})$ with $f(0) \equiv 0$, and there exist constants $k > 0$ and $p > 1$ such that

$$|f'(y)| \le k\,|y|^{p-1}\,, \qquad y \in \mathbb{R}\,, \tag{1.5}$$

with

$$p \le 2 \;\text{ if }\; n = 1 \quad\text{ and }\quad p < 1 + \frac{2}{n} \;\text{ if }\; n \ge 2\,, \tag{1.6}$$

then the semilinear wave equation is locally exactly null-controllable with boundary controls and globally exactly controllable with internal controls and some additional assumptions.

We note that the case where $p = (n+2)/n$ $(n \ge 2)$ is excluded in (1.6). This case is critical as it results in the lack of compactness. Therefore, we here consider such a critical exponential case. On one hand, we prove that (1.1) is globally exactly controllable with controls supported on the whole domain. On the other hand, we prove that (1.1) is locally exactly null-controllable with controls supported on only a part of the domain.

Our results apply for more general open subset ω of Ω. Roughly speaking, provided the linear wave equation is exactly controllable with controls supported in ω, our methods allow to show that the semilinear wave equation is also locally controllable. We refer to [2] for sharp geometric conditions on ω guaranteeing the exact controllability of the wave equation.

Although we study only the null-controllability property, we can show that any sufficiently small initial data may be driven to any sufficiently small final state (not necessarily the zero one) by using the same arguments.

Our main results are presented in Section 2 and proved in Section 3. The tools used in the proofs are the Hilbert uniqueness method, the multiplier method and the Banach contraction fixed point theorem.

2 – Main Results

Throughout this paper, let Ω be a bounded domain (nonempty, open, and connected) in $\mathbb{R}^n$ with suitably smooth boundary $\Gamma = \partial\Omega$ (the precise smoothness will be specified later) and let ν be the unit normal of Γ directed towards the exterior of Ω. Let $T > 0$ and set $Q = \Omega\times(0,T)$ and $\Sigma = \Gamma\times(0,T)$.

In the sequel, $H^s(\Omega)$ denotes the usual Sobolev space and $\|\cdot\|_s$ denotes its norm for any $s \in \mathbb{R}$ (see [1, 13]). For $s \geq 0$, $H_0^s(\Omega)$ denotes the completion of $C_0^\infty(\Omega)$ in $H^s(\Omega)$, where $C_0^\infty(\Omega)$ denotes the space of all infinitely differentiable functions on Ω which have compact support in Ω. Let X be a Banach space. We denote by $C^k([0,T];X)$ the space of all k times continuously differentiable functions defined on $[0,T]$ with values in X, and write $C([0,T];X)$ for $C^0([0,T];X)$.

We further introduce some standard notation (see, e.g., [12]). Let $x^0 \in \mathbb{R}^n$ and set

$$m(x) = x - x^0 = (x_k - x_k^0) , \tag{2.1}$$

$$\Gamma(x^0) = \Big\{ x \in \Gamma :\ m(x) \cdot \nu(x) = m_k(x) \cdot \nu_k(x) > 0 \Big\} , \tag{2.2}$$

$$\Gamma_*(x^0) = \Gamma - \Gamma(x^0) = \Big\{ x \in \Gamma :\ m(x) \cdot \nu(x) \leq 0 \Big\} , \tag{2.3}$$

$$\Sigma(x^0) = \Gamma(x^0) \times (0,T) , \tag{2.4}$$

$$\Sigma_*(x^0) = \Gamma_*(x^0) \times (0,T) . \tag{2.5}$$

For a subset $G \subset \Omega$, we denote

$$R(x^0, G) = \max_{x \in \bar{G}} |m(x)| = \max_{x \in \bar{G}} \left| \sum_{k=1}^{n} (x_k - x_k^0)^2 \right|^{1/2} . \tag{2.6}$$

We make the following assumption on $f = f(y)$:

(H) Assume $f \in W^{1,\infty}_{\text{loc}}(\mathbb{R})$ and there exist constants $k > 0$ and $p \geq 1$ such that

$$|f'(y)| \leq k\,|y|^{p-1} , \qquad \forall\, y \in \mathbb{R} . \tag{2.7}$$

Theorem 2.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary Γ of class C^2 and let O be a neighborhood of $\overline{\Gamma(x^0)}$ and $\omega = O \cap \Omega$. Let $T > 2\,R(x^0, \Omega/\omega)$. Assume (H) holds and $f(0) = 0$. Suppose that p satisfies*

$$1 < p \leq 2 \ \text{ if } n = 1 \quad \text{ and } \quad 1 < p \leq 1 + \frac{2}{n} \ \text{ if } n \geq 2 . \tag{2.8}$$

Then, system (1.1) is locally exactly null-controllable in $L^2(\Omega) \times H^{-1}(\Omega)$ with controls $h \in C([0,T]; H^{-1}(\Omega))$ supported in ω.

If the controls are allowed to support on the entire domain Ω, we have the following global exact controllability theorems.

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary Γ and let $T>0$. Assume (H) holds and the exponent p satisfies*

$$1 \le p < \infty \quad \text{if } n \le 2 \qquad \text{and} \qquad 1 \le p \le \frac{n}{n-2} \quad \text{if } n \ge 3 \; . \tag{2.9}$$

Then, system (1.1) is globally exactly controllable in $H_0^1(\Omega)\times L^2(\Omega)$. That is, for every initial state (y^0, y^1) and every terminal state (z^0, z^1) in $H_0^1(\Omega)\times L^2(\Omega)$, there exists a control $h \in C([0,T]; L^2(\Omega))$ such that the solution $y = y(x,t;h)$ of (1.1) satisfies (1.2).

If we wish to enlarge the control space $H_0^1(\Omega)\times L^2(\Omega)$, for example, to $L^2(\Omega)\times H^{-1}(\Omega)$, we have to impose a stronger condition on p. In this case the nonlinearity f is required to map $L^2(\Omega)$ into $H^{-1}(\Omega)$ so that the problem (1.1) is well posed. By the Sobolev imbedding theorem (see [1, p. 97]), we have $H_0^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$ and then $L^{2n/(n+2)}(\Omega) \hookrightarrow H^{-1}(\Omega)$. Therefore, we require that $y^p \in L^{2n/(n+2)}(\Omega)$ for $y \in L^2(\Omega)$. In other words, $p \le (n+2)/n$.

Theorem 2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a Lipschitz boundary Γ and let $T>0$. Assume (H) holds and the exponent p satisfies*

$$1 \le p \le 2 \quad \text{if } n = 1 \qquad \text{and} \qquad 1 \le p \le 1 + \frac{2}{n} \quad \text{if } n \ge 2 \; . \tag{2.10}$$

Then, system (1.1) is globally exactly controllable in $L^2(\Omega)\times H^{-1}(\Omega)$ with controls $h \in C([0,T]; H^{-1}(\Omega))$.

3 – Proofs

In this section we prove our main results by using the method of contraction mapping principle type developed by Lagnese [9].

For completeness, we quote from [14, 15] an observability inequality for the wave equation

$$\begin{cases} u'' - \Delta u = 0 & \text{in } Q, \\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma \; . \end{cases} \tag{3.1}$$

We define the energy $E(u,t)$ of (3.1) by

$$E(u,t) \;=\; \frac{1}{2}\Big[\|\nabla u(t)\|_0^2 + \|u'(t)\|_0^2\Big] \;, \tag{3.2}$$

where

$$\|\nabla u(t)\|_0^2 \;=\; \int_\Omega |\nabla u(t)|^2 \, dx \; . \tag{3.3}$$

The following observability inequality was proved in [15] as a special case.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a boundary Γ of class C^2. Let O be a neighborhood of $\overline{\Gamma(x^0)}$ and $\omega = O \cap \Omega$. Assume that $T > 2\,R(x^0, \Omega/\omega)$. Then there exist a nonnegative function $\varphi \in C^\infty(\Omega)$ with $\varphi(x) = 0$ in $\Omega - \omega$ and a positive constant c such that for all solutions of (3.1)*

$$E(u,0) \leq c \int_0^T \int_\omega \varphi\Big(|u|^2 + |\nabla u|^2\Big)\, dx\, dt\ . \blacksquare \tag{3.4}$$

We are now in the position to prove Theorem 2.1.

Proof of Theorem 2.1: Given initial conditions $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, we consider

$$\begin{cases} u'' - \Delta u = 0 & \text{in } Q, \\ u(0) = u^0, \quad u'(0) = u^1 & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma\ . \end{cases} \tag{3.5}$$

It is well known that (3.5) has a unique solution u with

$$u \in C\Big([0,T]; H_0^1(\Omega)\Big) \cap C^1\Big([0,T]; L^2(\Omega)\Big)\ . \tag{3.6}$$

Using the solution u of (3.5), we then consider the problem

$$\begin{cases} y'' - \Delta y + f(y) = -\varphi\, u + \operatorname{div}(\varphi\, \nabla u) & \text{in } Q, \\ y(T) = 0, \quad y'(T) = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma\ , \end{cases} \tag{3.7}$$

where φ is the function function given in Lemma 3.1. The solution y of (3.7) can be written as

$$y = w + z\ ,$$

where w and z are repectively the solutions of

$$\begin{cases} w'' - \Delta w = -\varphi\, u + \operatorname{div}(\varphi\, \nabla u) & \text{in } Q, \\ w(T) = 0, \quad w'(T) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Sigma\ , \end{cases} \tag{3.8}$$

$$\begin{cases} z'' - \Delta z + f(w+z) = 0 & \text{in } Q, \\ z(T) = 0, \quad z'(T) = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \Sigma\ . \end{cases} \tag{3.9}$$

Since $-\varphi\, u + \operatorname{div}(\varphi\, \nabla u) \in H^{-1}(\Omega)$, problem (3.8) has a weak solution with

$$w \in C\Big([0,T]; L^2(\Omega)\Big) \cap C^1\Big([0,T]; H^{-1}(\Omega)\Big) \;. \tag{3.10}$$

Moreover, there exists a constant c such that

$$\begin{aligned} \|w\|_{L^\infty(0,T;L^2(\Omega))} + \|w'\|_{L^\infty(0,T;L^2H^{-1}(\Omega))} &\le c\, \|\nabla u\|_{L^1(0,T;L^2(\Omega))} \\ &\le c\, \Big[\|u^0\|_1 + \|u^1\|_0\Big] \;. \end{aligned} \tag{3.11}$$

On the other hand, by Lemma 3.3 below, there exists some positive constant r such that, for every

$$(u^0,u^1) \in \overline{B}_r(0) = \Big\{(u^0,u^1) \in H_0^1(\Omega){\times}L^2(\Omega)\colon\ \|u^0\|_1 + \|u^1\|_0 \le r\Big\} \;, \tag{3.12}$$

problem (3.9) has a unique solution z with

$$z \in C\Big([0,T]; H_0^s(\Omega)\Big) \cap C\Big([0,T]; H^{s-1}(\Omega)\Big) \tag{3.13}$$

for some $0 \le s < 1$. Moreover, there exists some constant $c > 0$, independent of (u^0,u^1), such that for all $t \in [0,T]$

$$\|z(t)\|_s + \|z'(t)\|_{s-1} \le c\, \Big[\|u^0\|_1 + \|u^1\|_0\Big]^p \;. \tag{3.14}$$

We now define the nonlinear operator W by

$$\begin{aligned} W(u^0,u^1) &= \Big(y'(0), -y(0)\Big) \\ &= \Big(w'(0), -w(0)\Big) + \Big(z'(0), -z(0)\Big) \\ &= \Lambda(u^0,u^1) + K(u^0,u^1) \;, \end{aligned} \tag{3.15}$$

where

$$\Lambda(u^0,u^1) = \Big(w'(0), -w(0)\Big)\,, \qquad K(u^0,u^1) = \Big(z'(0), -z(0)\Big) \;. \tag{3.16}$$

By Lemma 3.1, the "controllability operator" Λ associated with the linear wave equation is an isomorphism from $H_0^1(\Omega){\times}L^2(\Omega)$ onto $H^{-1}(\Omega){\times}L^2(\Omega)$.

We look at the problem

$$W(u^0,u^1) = (y^1, -y^0) \;, \tag{3.17}$$

for $(y^0,y^1) \in L^2(\Omega){\times}H^{-1}(\Omega)$. Once we have shown that there exists a neighbourhood ϑ of $(0,0)$ in $L^2(\Omega){\times}H^{-1}(\Omega)$ such that for any $(y^0,y^1) \in \vartheta$ problem (3.17) has a solution, the problem of controllability is solved with control $h = -\varphi\, u + \operatorname{div}(\varphi\, \nabla u) \in C([0,T]; H^{-1}(\Omega))$ supported in ω.

We apply Banach contraction fixed point theorem to solve (3.17). Since the operator Λ is an isomorphism from $H_0^1(\Omega)\times L^2(\Omega)$ onto $H^{-1}(\Omega)\times L^2(\Omega)$, problem (3.17) is equivalent to the following fixed point problem

$$\begin{aligned}(u^0,u^1) &= -\Lambda^{-1}K(u^0,u^1)+\Lambda^{-1}(y^1,-y^0)\\ &= G(u^0,u^1)\ .\end{aligned}\tag{3.18}$$

For $\xi_1=(u_1^0,u_1^1)$, $\xi_2=(u_2^0,u_2^1)\in\overline{B}_r(0)$, by Lemma 3.4 below, we have

$$\begin{aligned}&\|G(\xi_1)-G(\xi_2)\|_{H_0^1(\Omega)\times L^2(\Omega)} =\\ &\quad= \|\Lambda^{-1}K(\xi_2)-\Lambda^{-1}K(\xi_1)\|_{H_0^1(\Omega)\times L^2(\Omega)}\\ &\quad\le c\,\|K(\xi_2)-K(\xi_1)\|_{H^{-1}(\Omega)\times L^2(\Omega)}\\ &\quad\le c\,\|K(\xi_2)-K(\xi_1)\|_{H^{s-1}(\Omega)\times H_0^s(\Omega)}\\ &\quad\le c\,(r+r^p)^{(p-1)/2}\exp\Big[c\Big(1+(r+r^p)^{p-1}\Big)\Big]\,\|\xi_1-\xi_2\|_{H_0^1(\Omega)\times L^2(\Omega)}\ ,\end{aligned}\tag{3.19}$$

where the constant c is independent of r. Therefore, there exists $r_0>0$ such that if $r\le r_0$, then G is a strict contraction on $\overline{B}_r(0)$.

On the other hand, we prove that there exists $\tau\in(0,r_0]$ such that G maps $\overline{B}_\tau(0)$ into $\overline{B}_\tau(0)$. In fact, by (3.14) we deduce, for some constant $c>0$,

$$\begin{aligned}\|G(u^0,u^1)\|_{H_0^1(\Omega)\times L^2(\Omega)} &\le \|-\Lambda^{-1}K(u^0,u^1)\|_{H_0^1(\Omega)\times L^2(\Omega)}\\ &\qquad+\ \|\Lambda^{-1}(y^1,-y^0)\|_{H_0^1(\Omega)\times L^2(\Omega)}\\ &\le c\,\tau^p+\|\Lambda^{-1}(y^1,-y^0)\|_{H_0^1(\Omega)\times L^2(\Omega)}\ ,\end{aligned}\tag{3.20}$$

for any $(u^0,u^1)\in\overline{B}_\tau(0)$. Thus, it is enough to choose $\tau\in(0,r_0]$ such that

$$c\,\tau^p+\|\Lambda^{-1}(y^1,-y^0)\|_{H_0^1(\Omega)\times L^2(\Omega)}\le\tau\ .$$

This is possible if we take

$$\|\Lambda^{-1}(y^1,-y^0)\|_{H_0^1(\Omega)\times L^2(\Omega)}\le\min\left\{\frac{1}{(c\,p)^{\frac{1}{p-1}}}\left(1-\frac{1}{p}\right),\ |r_0-c\,r_0^p|\right\}.\tag{3.21}$$

By Banach contraction fixed point theorem, G has a fixed point for $(y^0,y^1)\in L^2(\Omega)\times H^{-1}(\Omega)$ satisfying (3.21). Consequently, equation (3.17) has a solution. Thus, the proof of Theorem 2.1 is complete provided we can prove the following lemmas.

Let us introduce the operator $A=-\Delta\colon\ D(A)\subset L^2(\Omega)\to L^2(\Omega)$ with domain $D(A)=H^2(\Omega)\cap H^1_0(\Omega)$. It is well known that A is a strictly positive self-adjoint operator on $L^2(\Omega)$.

Since problems (3.8) and (3.9) are time-reversible, we may consider the following problems instead of (3.8) and (3.9):

$$\begin{cases} w''-\Delta w=-\varphi\, u+\operatorname{div}(\varphi\,\nabla u) & \text{in } Q,\\ w(0)=0,\quad w'(0)=0 & \text{in } \Omega,\\ w=0 & \text{on } \Sigma\,, \end{cases} \tag{3.22}$$

$$\begin{cases} z''-\Delta z+f(w+z)=0 & \text{in } Q,\\ z(0)=0,\quad z'(0)=0 & \text{in } \Omega,\\ z=0 & \text{on } \Sigma\,. \end{cases} \tag{3.23}$$

Lemma 3.2. *Suppose assumption (H) in Section 2 holds and p satisfies (2.10). Set*

$$s\;=\;1-\frac{n\,(p-1)}{2}\;\le\;1\;. \tag{3.24}$$

Then $f(y)\colon\ L^2(\Omega)\to H^{s-1}(\Omega)$ is locally Lipschitz continuous in y, that is, for every constant $c\ge0$ there is a constant $l(c)$ such that

$$\|f(y_1)-f(y_2)\|_{s-1}\;\le\;l(c)\,\|y_1-y_2\|_0\;, \tag{3.25}$$

for all $y_1,y_2\in L^2(\Omega)$ with $\|y_1\|_0\le c$, $\|y_2\|_0\le c$.

Proof: If $p=1$, it is clear that $f\colon\ L^2(\Omega)\to L^2(\Omega)$ is globally Lipschitz. Thus, we now assume that $p>1$. For any $y_1,y_2\in L^2(\Omega)$, it follows from Hölder's inequality and assumption (H) that

$$\begin{aligned} \|f(y_1)-f(y_2)\|_{0,\frac{2}{p}}\;&\le\;k\,\Big\||y_1|^{p-1}+|y_2|^{p-1}\Big\|_{0,\frac{2}{p-1}}\;\|y_1-y_2\|_0\\ &\le\;k\Big(\|y_1\|_0^{p-1}+\|y_2\|_0^{p-1}\Big)\;\|y_1-y_2\|_0\;. \end{aligned} \tag{3.26}$$

On the other hand, by Sobolev's embedding theorem (see [1, p. 218]), we have

$$H_0^{1-s}(\Omega)\hookrightarrow L^q(\Omega)\qquad\text{with}\;\; q=\frac{2\,n}{n-2(1-s)}\;. \tag{3.27}$$

Therefore we have

$$L^r(\Omega)\hookrightarrow H^{s-1}(\Omega)\;, \tag{3.28}$$

where $r>0$ and $\frac{1}{r}+\frac{1}{q}=1$. By (3.24), we obtain that $r=2/p$. Hence, $L^{\frac{2}{p}}(\Omega)$ is continuously embeded into $H^{s-1}(\Omega)$. Thus (3.25) follows from (3.26). ∎

Lemma 3.3. *Suppose that (H) holds and $f(0)=0$. Assume that p satisfies (2.10). Then there exists a positive constant r such that for every*

$$(u^0,u^1)\in \overline{B}_r(0)=\Big\{(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega):\ \|u^0\|_1+\|u^1\|_0\le r\Big\}\ , \tag{3.29}$$

problem (3.23) has a unique weak solution z with

$$z\in C\Big([0,T];H_0^s(\Omega)\Big)\cap C^1\Big([0,T];H^{s-1}(\Omega)\Big)\ , \tag{3.30}$$

where s is given by (3.24). Moreover, there exists a constant $c>0$, independent of (u^0,u^1), such that for all $t\in[0,T]$

$$\|z(t)\|_s+\|z'(t)\|_{s-1}\ \le\ c\Big[\|u^0\|_1+\|u^1\|_0\Big]^p\ . \tag{3.31}$$

Proof: Since $w\in C([0,T];L^2(\Omega))$, it follows from Lemma 3.2 that $f(w+\cdot)$ maps $L^2(\Omega)$ into $H^{s-1}(\Omega)$. It then follows from the standard theory of semigroups (see [18]) that for each $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$ there exists some $t_{\max}$ depending on (u^0,u^1) such that problem (3.23) has a unique solution with

$$z\in C\Big([0,t_{\max});H_0^s(\Omega)\Big)\cap C\Big([0,t_{\max});H^{s-1}(\Omega)\Big)\ . \tag{3.32}$$

Moreover, by Theorem 1.4 of [18, p. 185], we deduce the following alternative holds: either $t_{\max}>T$ and (3.30) holds, or $t_{\max}\le T$ and

$$\lim_{t\to t_{\max}}\|z(t)\|_s+\|z'(t)\|_{s-1}\ =\ +\infty\ . \tag{3.33}$$

We are going to prove that $t_{\max}>T$ for (u^0,u^1) small enough. As a consequence of this, (3.30) and (3.31) will hold immediately.

Set

$$E(z,t)\ =\ \frac{1}{2}\int_\Omega\Big[|A^{s/2}\,z(t)|^2+|A^{(s-1)/2}\,z'(t)|^2\Big]\,dx\ . \tag{3.34}$$

Multiplying (3.23) by $A^{s-1}z'$ and integrating over $Q_t=\Omega\times(0,t)$, it follows that (the following c's denoting various constants)

$$\begin{aligned}
E(z,t)\ &=\ -\int_{Q_t} f\Big(w(t)+z(t)\Big)\,A^{s-1}z'(t)\ dx\,dt\\
&\le\ c\int_0^t\Big\|f\Big(w(t)+z(t)\Big)\Big\|_{s-1}\,\|A^{(s-1)/2}\,z'(t)\|_0\ dt\\
&\le\ c\int_0^t\Big\|f\Big(w(t)+z(t)\Big)\Big\|_{0,2/p}\,\|A^{(s-1)/2}\,z'(t)\|_0\ dt\ \le\\
&\qquad\text{(use (2.7) and } f(0)=0)
\end{aligned}$$

$$\le c \int_0^t \|w(t)+z(t)\|_0^p \, \|A^{(s-1)/2} z'(t)\|_0 \, dt$$

$$\le c\,\|w\|^{2p}_{L^\infty(0,T;L^2(\Omega))} + c\int_0^t \Big(E(z,t) + \|z(t)\|_0^p \, \|A^{(s-1)/2} z'(t)\|_0\Big)\, dt$$

(use (3.11))

$$\le c\,\|(u^0,u^1)\|^{2p}_{H_0^1(\Omega)\times L^2(\Omega)} + c\int_0^t \Big[E(z,t) + E^{(p+1)/2}(z,t)\Big]\, dt$$

$$= d + c\int_0^t \Big[E(z,t) + E^{(p+1)/2}(z,t)\Big]\, dt\ , \tag{3.35}$$

where

$$d = c\,\|(u^0,u^1)\|^{2p}_{H_0^1(\Omega)\times L^2(\Omega)}\ . \tag{3.36}$$

On the other hand, the solution of the initial value problem

$$\begin{cases} \psi' = c(\psi + \psi^{(p+1)/2}), \\ \psi(0) = d\ , \end{cases} \tag{3.37}$$

is

$$\psi = \frac{d\, e^{ct}}{\Big[1 + d^{(p-1)/2} - d^{(p-1)/2}\exp\Big(\tfrac12\, c(p-1)\,t\Big)\Big]^{2/(p-1)}}\ . \tag{3.38}$$

It therefore follows from Corollary 6.5 of [6, p. 35] that

$$E(z,t) \le \frac{d\, e^{ct}}{\Big[1 + d^{(p-1)/2} - d^{(p-1)/2}\exp\Big(\tfrac12\, c(p-1)t\Big)\Big]^{2/(p-1)}}$$

$$\le 2\, d\, e^{ct}\ , \tag{3.39}$$

for $0 \le t \le T$ if

$$\Big[1 + d^{(p-1)/2} - d^{(p-1)/2}\exp\Big(\frac12\, c(p-1)\,T\Big)\Big]^{2/(p-1)} \ge \frac12\ , \tag{3.40}$$

that is,

$$d = c\,\|(u^0,u^1)\|^{2p}_{H_0^1(\Omega)\times L^2(\Omega)} < \frac{\Big(1 - 2^{(1-p)/2}\Big)^{2/(p-1)}}{\Big[\exp\Big(c(p-1)\,T/2\Big) - 1\Big]^{2/(p-1)}}\ . \tag{3.41}$$

Thus we have proved (3.30) and (3.31). ∎

In the following, the constants c's denote various constants depending on T, μ, Ω, the constants k, p in (2.7). In addition, using problems (3.22) and (3.23), the operator K defined in (3.16) is now given by

$$K(u^0,u^1) \;=\; \Big(z'(T), -z(T)\Big)\;. \tag{3.42}$$

Lemma 3.4. *Suppose that (H) holds and $f(0)=0$. Assume that p satisfies (2.10). Then we have*

$$\begin{aligned} &\|K(\xi_1)-K(\xi_2)\|_{H^{s-1}(\Omega)\times H^s_0(\Omega)} \;\le \\ &\le\; c\,(r+r^p)^{(p-1)/2}\,\exp\Big[c\Big(1+(r+r^p)^{p-1}\Big)\Big]\;\|\xi_1-\xi_2\|_{H^1_0(\Omega)\times L^2(\Omega)} \end{aligned} \tag{3.43}$$

for any $\xi_1=(u^0_1,u^1_1)$, $\xi_2=(u^0_2,u^1_2)\in\overline{B}_r(0)\subset H^1_0(\Omega)\times L^2(\Omega)$, where r is the constant obtained in Lemma 3.3 and s is as in (3.24).

Proof: Given $\xi_1=(u^0_1,u^1_1)$, $\xi_2=(u^0_2,u^1_2)\in\overline{B}_r(0)$, by Lemma 3.3, (3.23) has unique solutions z_1, z_2, respectively. Let w_1, w_2 be the solutions of (3.22) corresponding to ξ_1, ξ_2, respectively. By (3.11) and (3.31), we have

$$\|w_i(t)\|_0\le c\,r,\quad \|z_i(t)\|_0\le c\,r^p\,,\qquad i=1,2,\quad \forall\,t\in[0,T]\;. \tag{3.44}$$

From (3.23) it follows that

$$\begin{cases} (z_1-z_2)''-\Delta(z_1-z_2)+f(w_1+z_1)-f(w_2+z_2) \;=\; 0 & \text{in } Q,\\ z_1(0)-z_2(0)=0,\quad z'_1(0)-z'_2(0)=0 & \text{in } \Omega,\\ z_1-z_2=0 & \text{on } \Sigma\;. \end{cases} \tag{3.45}$$

As in the proof of (3.35), multiplying (3.45) by $A^{s-1}(z_1-z_2)'$ and integrating over $Q_t=\Omega\times(0,t)$, it follows that

$$\begin{aligned} &E(z_1-z_2,\,t)\;\le\\ &\le\; c\int_0^t\Big(\|w_1(t)+z_1(t)\|_0^{p-1}+\|w_2(t)+z_2(t)\|_0^{p-1}\Big)\\ &\qquad\times\Big(\|w_2(t)-w_1(t)\|_0+\|z_2(t)-z_1(t)\|_0\Big)\,\|A^{(s-1)/2}(z_1-z_2)'(t)\|_0\;\;dt\\ &\qquad\text{(use (3.44))}\\ &\le\; c\int_0^t(r+r^p)^{p-1}\\ &\qquad\times\Big(\|w_2(t)-w_1(t)\|_0+\|z_2(t)-z_1(t)\|_0\Big)\,\|A^{(s-1)/2}(z_1-z_2)'(t)\|_0\;\;dt\;\le \end{aligned}$$

$$\leq\; c\,(r+r^p)^{p-1}\,\|w_2-w_1\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$+\; c\Big[1+(r+r^p)^{p-1}\Big]\int_0^t E(z_1-z_2,\,t)\;dt\;, \tag{3.46}$$

where the positive constant c is independent of r. It therefore follows from Gronwall's inequality (see [6, p. 36]) that

$$E(z_1-z_2,t)\;\leq\; c\,(r+r^p)^{p-1}\;\exp\Big[c\,\Big(1+(r{+}r^p)^{p-1}\Big)\,t\Big]\;\|w_2-w_1\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$\leq\; c\,(r+r^p)^{p-1}\;\exp\Big[c\Big(1+(r{+}r^p)^{p-1}\Big)\,t\Big]\;\|\xi_2-\xi_1\|^2_{H^1_0(\Omega)\times L^2(\Omega)}\;. \tag{3.47}$$

This implies (3.43). ∎

Until now, we have finished the proof of Theorem 2.1. ∎

We are now in the position to prove Theorem 2.2.

Proof Proof of Theorem 2.2: It is known from [12, Theorem 2.1, p. 405] that, for every initial state (y^0,y^1) and terminal state (z^0,z^1) in $H^1_0(\Omega)\times L^2(\Omega)$, there exists a control

$$v\,\in\,C\Big([0,T];L^2(\Omega)\Big)\;, \tag{3.48}$$

such that

$$\begin{cases} y''-\Delta y=v & \text{in } Q,\\ y(x,0)=y^0,\;\; y'(x,0)=y^1 & \text{in } \Omega,\\ y(x,T)=z^0,\;\; y'(x,T)=z^1 & \text{in } \Omega,\\ y=0 & \text{on } \Sigma\;. \end{cases} \tag{3.49}$$

By setting

$$h\;=\;v+f(y)\;, \tag{3.50}$$

then we have

$$\begin{cases} y''-\Delta y+f(y)=h & \text{in } Q,\\ y(x,0)=y^0,\;\; y'(x,0)=y^1 & \text{in } \Omega,\\ y(x,T)=z^0,\;\; y'(x,T)=z^1 & \text{in } \Omega,\\ y=0 & \text{on } \Sigma\;. \end{cases} \tag{3.51}$$

It remains to prove that

$$h\,\in\,C\Big([0,T];L^2(\Omega)\Big)\;. \tag{3.52}$$

Since $v\in C([0,T];L^2(\Omega))$ and $y\in C([0,T];H^1_0(\Omega))$, it suffices to prove

$$f(y)\,\in\,C\Big([0,T];L^2(\Omega)\Big)\;. \tag{3.53}$$

This follows from the following lemma. ∎

In what follows, we denote by $\|\cdot\|_{0,r}$ the norm in $L^r(\Omega)$ and we recall that $\|\cdot\|_s$ denotes the norm of $H^s(\Omega)$ for any $s\in\mathbb{R}$.

Lemma 3.5. *If assumption (H) in Section 2 holds and p satisfies (2.9), then $f(y)\colon H_0^1(\Omega)\to L^2(\Omega)$ is locally Lipschitz continuous, that is, for every constant $c\ge 0$, there is a constant $l(c)$ such that*

$$\|f(y_1)-f(y_2)\|_0 \;\le\; l(c)\,\|y_1-y_2\|_1\;, \tag{3.54}$$

for all $y_1,y_2\in H_0^1(\Omega)$ with $\|y_1\|_1\le c$, $\|y_2\|_1\le c$.

Proof: If $p=1$, it is easy to see that f is actually globally Lipschitz. Thus, we now assume that $p>1$. If $n\ge 3$, by (2.9), we have

$$(p-1)\,(n-2)\;\le\;2\;. \tag{3.55}$$

Set

$$q_1=\frac{n}{n-(n-2)\,(p-1)}\,,\qquad q_2=\frac{n}{(n-2)\,(p-1)}\;. \tag{3.56}$$

Then we have

$$1<q_1\le\frac{n}{n-2}\,,\qquad 1\le(p-1)\,q_2\le\frac{n}{n-2}\,,\qquad \frac{1}{q_1}+\frac{1}{q_2}=1\;. \tag{3.57}$$

If $n\le 2$, we take

$$q_1=p\,,\qquad q_2=\frac{p}{p-1}\;. \tag{3.58}$$

Then we have

$$1<q_1<\infty\,,\qquad 1\le(p-1)\,q_2\,,\qquad \frac{1}{q_1}+\frac{1}{q_2}=1\;. \tag{3.59}$$

It therefore follows from the differential mean value theorem, (2.7) and Hölder's inequality that

$$\begin{aligned}\|f(y_1)-f(y_2)\|_0 \;&\le\; c\,\Big\|(y_1-y_2)\,\Big(|y_1|^{p-1}+|y_2|^{p-1}\Big)\Big\|_0\\ &\le\; c\,\|y_1-y_2\|_{0,2q_1}\,\Big\||y_1|^{p-1}+|y_2|^{p-1}\Big\|_{0,2q_2}\\ &\le\; c\,\|y_1-y_2\|_{0,2q_1}\Big(\|y_1\|^{p-1}_{0,2q_2(p-1)}+\|y_2\|^{p-1}_{0,2q_2(p-1)}\Big)\;.\end{aligned} \tag{3.60}$$

On the other hand, by the Sobolev imbedding theorem (see [1, p. 97]), we have the following continuous imbeddings:

$$H^1(\Omega)\subset L^r(\Omega)\,,\qquad 1\le r\le\frac{2\,n}{n-2},\quad n\ge 3\;, \tag{3.61}$$

$$H^1(\Omega)\subset L^r(\Omega)\,,\qquad 1\le r<\infty,\quad n=2\;, \tag{3.62}$$

$$H^1(\Omega)\subset C(\bar\Omega)\,,\qquad n=1\;. \tag{3.63}$$

Therefore, (3.54) follows from (3.60) and the above imbeddings. ∎

Finally, we prove Theorem 2.3.

Proof of Theorem 2.3: The proof is similar to that of Theorem 2.2 by using Theorem 2.2 of [12, p. 408] about the linear wave equation. In this case, by Theorem 2.2 of [12, p. 408] we have

$$v \in C\Big([0,T]; H^{-1}(\Omega)\Big) , \tag{3.64}$$

$$y \in C\Big([0,T]; L^2(\Omega)\Big) . \tag{3.65}$$

Hence, by Lemma 3.2, we have

$$f(y) \in C\Big([0,T]; H^{s-1}(\Omega)\Big) \subset C\Big([0,T]; H^{-1}(\Omega)\Big) . \tag{3.66}$$

Thus, we deduce that

$$h \in C\Big([0,T]; H^{-1}(\Omega)\Big) . \blacksquare \tag{3.67}$$

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Wei-Jiu Liu,
Department of Mathematics and Statistics, Dalhousie University,
Halifax, Nova Scotia, B3H 3J5 – CANADA
E-mail: `weiliu@mathstat.dal.ca`