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A VIABILITY RESULT FOR A FIRST-ORDER DIFFERENTIAL INCLUSION

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Abstract: This paper deals with the existence of solutions of a first-order viability problem of the type

$$\dot{x} \in f(t,x) + F(x)\ , \qquad x(t) \in K$$

where K a closed subset of $\mathbb{R}^n$, F is upper semicontinuous with compact values contained in the subdifferential $\partial V(x)$ of a convex proper lower semicontinuous function V and f is a Carathéodory single valued map.

1 – Introduction

Bressan, Cellina and Colombo [1] proved the existence of solutions of the problem $\dot{x} \in F(x)$, $x(0) = x_0 \in K$, where F is an upper semicontinuous multifunction contained in the subdifferential of a convex proper lower semicontinuous function in the finite dimensional space. This result has been generalized by Ancona and Colombo [2] by proving the existence of solutions of the perturbed problem $\dot{x} \in F(x) + f(t,x)$, $x(0) = x_0$, with f satisfying the Carathéodory conditions. The proof is based on approximate solutions; to overcome the weak convergence of derivatives of such solutions, the authors use the following basic relation:

$$\frac{d}{dt}\,(V(x(t)) = \left\|\dot{x}(t)\right\|^2\ .$$

The aim of the present paper is to prove a viability result of the following problem:

$$\begin{cases} \dot{x} \in f(t,x) + F(x) \quad \text{a.e. } t \in [0,T]\,, \\ x(0) = x_0 \in K\,, \\ x(t) \in K \quad \forall\, t \in [0,T]\ . \end{cases} \tag{1.1}$$

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where F is an upper semicontinuous with compact valued multifunction such that $F(x) \subset \partial V(x)$, for some convex proper lower semicontinuous function V and f is a Carathéodory function.

This paper is a generalization of the work of Rossi [5]. Our argument is different from the one appearing in Rossi's paper.

2 – The result

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with scalar product $\langle .,.\rangle$ and norm $\|.\|$. Let K be a closed subset of $\mathbb{R}^n$. Let F be a multifunction from $\mathbb{R}^n$ into the set of all nonempty compact subsets of $\mathbb{R}^n$. Let f be a function from $\mathbb{R}\times\mathbb{R}^n$ into $\mathbb{R}^n$. Assume that F and f satisfy the following conditions:

A$_1$) F is upper semicontinuous, i.e. for all $x \in \mathbb{R}^n$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|x - x'\| \le \delta$ then $F(x') \subseteq F(x) + \varepsilon B$, where B is the unit ball of $\mathbb{R}^n$.

A$_2$) There exists a convex proper and lower semicontinuous function $V\colon \mathbb{R}^n \to \mathbb{R}$ such that $F(x) \subset \partial V(x)$, where ∂V is the subdifferential of the function V.

A$_3$) $f\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, i.e. for every $x \in \mathbb{R}^n$, $t \to f(t,x)$ is measurable and for all $t \in \mathbb{R}$, $x \to f(t,x)$ is continuous.

A$_4$) There exists $m \in L^2(\mathbb{R})$ such that

$$\|f(t,x)\| \le m(t) \qquad \forall\,(t,x) \in \mathbb{R}\times\mathbb{R}^n \ .$$

A$_5$) (*Tangential condition*) $\forall\,(t,x) \in \mathbb{R}\times K$, $\exists\, v \in F(x)$ such that

$$\liminf_{h\to 0^+} \frac{1}{h}\, d_K\left(x + hv + \int_t^{t+h} f(s,x)\,ds\right) = 0 \ .$$

Let $x_0 \in K$, let f and F satisfying assumptions $A_1, ..., A_5$, then we shall prove the following result:

Theorem 1. *There exist $T > 0$ and $x\colon [0,T] \to \mathbb{R}^n$ such that*

$$\begin{cases} \dot{x}(t) \in f(t,x(t)) + F(x(t)) \quad \text{a.e. on } [0,T]\,, \\ x(0) = x_0 \in K\,, \\ x(t) \in K \quad \forall\, t \in [0,T]\ . \end{cases}$$

3 – Proof of the main result

Lemma 2. *Let V be a convex proper lower semicontinuous function such that for all $x \in \mathbb{R}^n$, $F(x) \subset \partial V(x)$. Then there exist $r = r_x > 0$ and $M = M_x > 0$ such that $\|F(x)\| = \sup_{z \in F(x)} \|z\| \le M$ and V is lipschitz continuous with constant M on $B(x, r)$.* ∎

For the proof, see [1].

Let r be the real given by Lemma 2 associated to x_0. Choose $T > 0$ such that

$$\int_0^T \Big(m(s) + M + 1\Big)\, ds \;<\; \frac{r}{2} \;.$$

In all the sequel, denote by K_0 the compact subset $K \cap \overline{B}(x_0, r)$.

Lemma 3. *Assume that F and f satisfy A_1,...,A_5. Then for all $\varepsilon > 0$, there exists $\eta > 0$ ($\eta < \varepsilon$) with the following properties:*

for all $(t, x) \in [0, T] \times K_0$, there exist $u \in F(x) + \dfrac{\varepsilon}{T} B$ and $h_{t,x} \in [\eta, \varepsilon]$ such that

$$x + h_{t,x} u + \int_t^{t+h_{t,x}} f(s, x)\, ds \;\in\; K \;.$$

Proof: Let $(t, x) \in [0, T] \times K_0$, let $\varepsilon > 0$. Since F is upper semicontinuous, then there exists $\delta_x > 0$ such that

$$F(y) \;\subset\; F(x) + \varepsilon B \;, \qquad \forall\, y \in B(x, \delta_x) \;.$$

Let $(s, y) \in [0, T] \times K$. By the tangential condition there exist $h_{s,y} \in\,]0, \varepsilon]$ and $v \in F(y)$ such that

$$d_K\left(y + h_{s,y} v + \int_s^{s+h_{s,y}} f(\tau, y)\, d\tau\right) \;<\; h_{s,y}\, \frac{\varepsilon}{4T} \;.$$

Consider the subset

$$N(s, y) \;=\; \left\{(t, z) \in \mathbb{R} \times \mathbb{R}^n \,/\; d_K\left(z + h_{s,y} v + \int_t^{t+h_{s,y}} f(\tau, z)\, d\tau\right) < h_{s,y}\, \frac{\varepsilon}{4T}\right\} \;.$$

Since

$$\|f(s, z)\| \le m(s) \quad \text{ a.e. on } \; [0, T], \;\; \forall z \in \mathbb{R}^n$$

then, the dominated convergence theorem applied to the sequence of functions $\left(\chi_{[t,t+h_{s,y}]}\, f(.,.)\right)_t$ shows that the function

$$(l,z) \;\to\; z + h_{s,y}v + \int_l^{l+h_{s,y}} f(\tau,z)\,d\tau$$

is continuous. So that, the function

$$(l,z) \;\to\; d_K\left(z + h_{s,y}v + \int_l^{l+h_{s,y}} f(\tau,z)\,d\tau\right)$$

is continuous and consequently the subset $N(s,y)$ is open.

Moreover, since (s,y) belongs to $N(s,y)$, there exists a ball $B((s,y),\eta_{\tau,y})$ of radius $\eta_{(\tau,y)} < \delta_x$ contained in $N(s,y)$. Therefore, the compact subset $[0,T]\times K_0$ can be covered by q such balls $B((s_i,y_i),\eta_{s_i,y_i})$. For simplicity, we set $h_{s_i,y_i} = h_i$, $i = 1,...,q$. Put $\eta = \min\limits_{i=1,...,q} h_i > 0$.

Let $(t,x)\in[0,T]\times K_0$ be fixed. Since $(t,x)\in B((s_i,y_i),\eta_{s_i,y_i})$ which is contained in $N(s_i,y_i)$, then there exist $x_i\in K$ and $u_i\in F(y_i)$ such that

$$\left\|u_i - \frac{1}{h_i}\left(x_i - x - \int_t^{t+h_i} f(s,x)\,ds\right)\right\| \;\le$$

$$\le\; \frac{1}{h_i}\, d_K\left(x + h_i u_i + \int_t^{t+h_i} f(\tau,z)\,d\tau\right) + \frac{\varepsilon}{4T} \;\le\; \frac{\varepsilon}{2T}\;.$$

Set

$$u \;=\; \frac{1}{h_i}\left(x_i - x - \int_t^{t+h_i} f(s,x)\,ds\right)$$

hence

$$x + h_i u + \int_t^{t+h_i} f(s,x)\,ds \;\in\; K$$

and

$$\|u_i - u\| \;\le\; \frac{\varepsilon}{2T}\;.$$

Since

$$\|x - y_i\| < \eta_{(\tau,y)} < \delta_x$$

then

$$F(y_i) \;\subset\; F(x) + \frac{\varepsilon}{2T}\,B$$

so that

$$u \;\in\; F(x) + \frac{\varepsilon}{T}\,B\;.$$

Hence the Lemma 3 is proved. ∎

Now, our purpose is to define on $[0,T]$ a family of approximate solutions and show that a subsequence converges to a solution of the problem (1.1).

4 – Construction of approximate solutions

Let $x_0 \in K_0$ and $\varepsilon < T$. By Lemma 3, there exist $\eta > 0$, $h_0 \in [\eta, \varepsilon]$ and $u_0 \in F(x_0) + \frac{\varepsilon}{T} B$ such that

$$x_1 \;=\; x_0 + h_0 u_0 + \int_0^{h_0} f(s, x_0)\, ds \;\in\; K$$

then if $h_0 \leq T$ we have

$$\|x_1 - x_0\| \;=\; \left\| h_0 u_0 + \int_0^{h_0} f(s, x_0)\, ds \right\| \;\leq\; \left\| \int_0^T \bigl(M + 1 + m(s)\bigr)\, ds \right\| \;\leq\; \frac{r}{2}$$

and thus $x_1 \in K_0$. Hence for (h_0, x_1) there exist $h_1 \in [\eta, \varepsilon]$ and $u_1 \in F(x_1) + \frac{\varepsilon}{T} B$ such that

$$x_2 \;=\; x_1 + h_1 u_1 + \int_{h_0}^{h_0 + h_1} f(s, x_1)\, ds \;\in\; K$$

we have

$$\|x_2 - x_0\| \;=\; \left\| h_0 u_0 + \int_0^{h_0} f(s, x_0)\, ds \;+\; h_1 u_1 \;+\; \int_{h_0}^{h_0 + h_1} f(s, x_1)\, ds \right\|$$

then if $h_0 + h_1 < T$ we have

$$\|x_2 - x_0\| \;\leq\; \left\| \int_0^T \bigl(M + 1 + m(s)\bigr)\, ds \right\| \;\leq\; \frac{r}{2}$$

thus $x_2 \in K_0$.

Set $h_{-1} = 0$, by induction, since h_i belongs to $[\eta, \varepsilon]$, then there exists an integer s such that $\sum_{i=0}^{s-1} h_i < T \leq \sum_{i=0}^{s} h_i$. Hence we construct the sequences $(h_p)_p \subset [\eta, \varepsilon]$, $(x_p)_p \subset K_0$, and $(u_p)_p$ such that for every $p = 0, ..., s-1$, we have

$$\begin{cases} x_{p+1} \;=\; x_p + h_p u_p + \displaystyle\int_{h_{p-1}}^{h_{p-1} + h_p} f(s, x_p)\, ds \;\in\; K \\ u_p \;\in\; F(x_p) + \dfrac{\varepsilon}{T} B\ . \end{cases}$$

By induction, for all $p \geq 2$ we have

$$x_p \;=\; x_0 + \sum_{i=0}^{i=p-1} h_i u_i \;+ \sum_{i=1}^{i=p-1} \int_{\sum_{j=0}^{i-1} h_j}^{\sum_{j=0}^{i} h_j} f(\tau, x_i)\, d\tau$$

$$u_p \;\in\; F(x_p) + \frac{\varepsilon}{T} B$$

and the estimates

$$\begin{aligned}\|x_p - x_0\| &= \left\| \sum_{i=0}^{i=p-1} h_i u_i + \sum_{i=0}^{i=p-1} \int_{\sum_{j=0}^{i-1} h_j}^{\sum_{j=0}^{i} h_j} f(\tau, x_i)\, d\tau \right\| \\ &\le (M+1) \sum_{i=1}^{i=p-1} h_i + \int_0^T m(\tau)\, d\tau \ .\end{aligned}$$

Since $\sum_{i=0}^{i=p-1} h_i \le T$, then we obtain $\|x_p - x_0\| \le \frac{r}{2}$.

For any nonzero integer k and for every integer $q = 0, ..., s$, denote by h_q^k a real associated to $\varepsilon = \frac{1}{k}$ and $x = x_q$ given by Lemma 3, consider the sequence $(\tau_k^q)_k$

$$\begin{cases} \tau_k^0 = 0\,, \quad \tau_k^s = T \\ \tau_k^q = h_0^k + \cdots + h_{q-1}^k \end{cases}$$

and define on $[0, T]$ the sequence of functions $(x_k(.))_k$ by

$$\begin{aligned} x_k(t) &= x_{q-1} + (t - \tau_k^{q-1}) u_{q-1} + \int_{\tau_k^{q-1}}^t f(s, x_{q-1})\, ds \qquad \forall\, t \in [\tau_k^{q-1}, \tau_k^q] \\ x_k(0) &= x_0 \end{aligned}$$

then for all $t \in [\tau_k^{q-1}, \tau_k^q]$

$$\dot{x}_k(t) = u_{q-1} + f(t, x_{q-1}) \ .$$

5 – Convergence of approximate solutions

Observe that the sequence $(x_k(.))_k$ satisfies the following relations

1) $\|\dot{x}_k(t)\| \le \|u_{q-1}\| + \|f(t, x_{q-1}\ )\| \le M + 1 + m(t)$,

2)
$$\begin{aligned} \|x_k(t)\| &= \left\| x_k(\tau_k^{q-1}) + \int_{\tau_k^{q-1}}^t \dot{x}_k(\tau)\, d\tau \right\| \\ &\le \|x_{q-1}\| + \left\| \int_0^T \big(M + 1 + m(t)\big)\, d\tau \right\| \\ &\le \|x_0\| + \frac{r}{2} + \frac{r}{2} \le \|x_0\| + r \ . \end{aligned}$$

Hence

$$\int_0^T \|\dot{x}_k(t)\|^2\,dt \;\le\; \int_0^T \big(M+1+m(t)\big)^2\,dt$$

the sequence $(\dot{x}_k(.))_k$ is bounded in $L^2([0,T],\mathbb{R}^n)$ and therefore $(x_k(.))_k$ is equi-uniformly continuous. Hence there exists a subsequence, still denoted by $(x_k(.))_k$ and an absolutely continuous function $x(.)\colon [0,T]\to\mathbb{R}^n$ such that $x_k(.)$ converges to $x(.)$ uniformly and $\dot{x}_k(.)$ converges weakly in $L^2([0,T],\mathbb{R}^n)$ to $\dot{x}(.)$.

The family of approximate solutions $x_k(.)$ has the following property:

Proposition 4. *For every $t\in[0,T]$ there exists $q\in\{1,...,s\}$ such that*

$$\lim_{k\to\infty} d_{grF}\Big(x_k(t),\,\dot{x}_k(t)-f\big(t,x_k(\tau_k^{q-1})\big)\Big) \;=\; 0\ .$$

Proof: Let $t\in[0,T]$. By construction of τ_k^q there exists q such that $t\in[\tau_k^{q-1},\tau_k^q[$ and $(\tau_k^q)_k$ converges to t.

Since

$$\dot{x}_k(t)-f\big(t,x_k(\tau_k^{q-1})\big)=u_{q-1}\;\in\; F\big(x_k(\tau_k^{q-1})\big)+\frac{1}{kT}$$

then

$$\lim_{k\to\infty} d_{gr(F)}\Big(x_k(t),\,\dot{x}_k(t)-f\big(t,x_k(\tau_k^{q-1})\big)\Big) \;\le\; \lim_{k\to\infty}\Big(\big\|x_k(t)-x_k(\tau_k^{q-1})\big\|+\frac{1}{kT}\Big)$$

hence

$$\lim_{k\to\infty} d_{gr(F)}\Big(x_k(t),\,\dot{x}_k(t)-f\big(t,x_k(\tau_k^{q-1})\big)\Big) \;=\; 0\ .$$

This completes the proof. ∎

Since the sequences $x_k(.)\to x(.)$ uniformly, $\dot{x}_k(.)\to\dot{x}(.)$ weakly in $L^2([0,T],\mathbb{R}^n)$, $\big(f(.,x_k(\tau_k^q))\big)_k$ converges to $f(.,x(.))$ in $L^2([0,T],\mathbb{R}^n)$ and F is upper semi-continuous, then by theorem 1.4.1 in [3], $x(.)$ is a solution of the following convexified problem:

$$\begin{cases} \dot{x}(t)\;\in\; f(t,x(t))+\operatorname{co}F(x(t))\\ x(0)=x_0\ . \end{cases}$$

Consequently, for all $t\in[0,T]$ we have that

(5.1) $$\dot{x}(t)-f(t,x(t))\;\in\;\partial V(x(t))\ .$$

Proposition 5. *The application $x(.)$ is a solution of the problem (1.1).*

Proof: To begin with, we prove that $(\|\dot{x_k}\|_2)_k$ converges to $\|\dot{x}\|_2$. Since the map $x(.)$ and $V(x(.))$ are absolutely continuous, we obtain from (5.1) by applying Lemma 3.3 in [4] that

$$\frac{d}{dt}V(x(t)) = \Big\langle \dot{x}(t),\, \dot{x}(t) - f(t, x(t))\Big\rangle \quad \text{a.e. on } [0,T]$$

therefore

$$V(x(T)) - V(x_0) = \int_0^T \|\dot{x}(s)\|^2\, ds - \int_0^T \Big\langle \dot{x}(s),\, f(s, x(s))\Big\rangle\, ds\ . \tag{5.2}$$

On the other hand, since for all $q = 1, ..., s$

$$\dot{x}_k(t) - f\big(t, x_k(\tau_k^{q-1})\big) = \dot{x}_k(t) - f(t, x_{q-1}) \in \partial V\big(x_k(\tau_k^{q-1})\big) + \frac{1}{kT}B\ .$$

there exists $b_q \in B$ such that

$$\dot{x}_k(t) - f(t, x_{q-1}) + \frac{1}{kT}b_q \in \partial V\big(x_k(\tau_k^{q-1})\big)\ .$$

Moreover the subdifferential properties of a convex function imply that for every $z \in \partial V(x_k(\tau_k^{q-1}))$

$$V(x_k\big(\tau_k^q)\big) - V\big(x_k(\tau_k^{q-1})\big) \geq \Big\langle x_k(\tau_k^q) - x_k(\tau_k^{q-1}),\, z\Big\rangle \tag{5.3}$$

particularly, for

$$z = \dot{x}_k(t) - f(t, x_{q-1}) + \frac{1}{kT}b_q$$

we have

$$V\big(x_k(\tau_k^q)\big) - V\big(x_k(\tau_{k^{q-1}})\big) \geq \left\langle \int_{\tau_{k^{q-1}}}^{\tau_k^q} \dot{x}_k(s)\, ds,\ \dot{x}_k(t) - f(t, x_{q-1}) + \frac{1}{kT}b_q \right\rangle$$

thus

$$\begin{aligned} V\big(x_k(\tau_k^q)\big) - V\big(x_k(\tau_k^{q-1})\big) \geq & \int_{\tau_k^{q-1}}^{\tau_k^q} \Big\langle \dot{x}_k(s), \dot{x}_k(s)\Big\rangle\, ds + \int_{\tau_k^{q-1}}^{\tau_k^q} \Big\langle \dot{x}_k(s), \frac{1}{kT}b_q\Big\rangle\, ds \\ & - \int_{\tau_k^{q-1}}^{\tau_k^q} \Big\langle \dot{x}_k(s), f\big(s, x_k(\tau_k^{q-1})\big)\Big\rangle\, ds \end{aligned}$$

hence, it is clear that

$$\begin{aligned} V(x_k(T)) - V(x_0) \geq & \int_0^T \|\dot{x}_k(s)\|^2\, ds - \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \Big\langle \dot{x}_k(s), f\big(s, x_k(\tau_k^{q-1})\big)\Big\rangle\, ds \\ & + \sum_{q=1}^{s} \frac{1}{kT} \int_{\tau_k^{q-1}}^{\tau_k^q} \Big\langle \dot{x}_k(s), b_q\Big\rangle\, ds\ . \end{aligned} \tag{5.4}$$

Claim. *The sequence*

$$\left(\sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \Big\langle \dot{x}_k(s), f\big(s, x_k(\tau_k^{q-1})\big) \Big\rangle \, ds \right)_k$$

converges to

$$\int_0^T \Big\langle \dot{x}(s), f(s, x(s)) \Big\rangle \, ds \; .$$

Proof: We have

$$\left\| \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \Big\langle \dot{x}_k(s), f\big(s, x_k(\tau_k^{q-1})\big) \Big\rangle \, ds \; - \int_0^T \Big\langle \dot{x}(s), f(s, x(s)) \Big\rangle \, ds \right\| \; =$$

$$= \left\| \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \Big(\Big\langle \dot{x}_k(s), f(s, x_k(\tau_k^{q-1})) \Big\rangle - \Big\langle \dot{x}(s), f(s, x(s)) \Big\rangle \Big) \, ds \right\|$$

$$\leq \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \left\| \Big\langle \dot{x}_k(s), f(s, x_k(\tau_k^{q-1})) \Big\rangle - \Big\langle \dot{x}(s), f(s, x(s)) \Big\rangle \right\| ds$$

$$\begin{aligned} \leq & \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \left\| \Big\langle \dot{x}_k(s), f(s, x_k(\tau_k^{q-1})) \Big\rangle - \Big\langle \dot{x}_k(s), f(s, x_k(s)) \Big\rangle \right\| ds \\ & + \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \left\| \Big\langle \dot{x}_k(s), f(s, x_k(s)) \Big\rangle - \Big\langle \dot{x}_k(s), f(s, x(s)) \Big\rangle \right\| ds \\ & + \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \left\| \Big\langle \dot{x}_k(s), f(s, x(s)) \Big\rangle - \Big\langle \dot{x}(s), f(s, x(s)) \Big\rangle \right\| ds \end{aligned}$$

$$\begin{aligned} = & \sum_{q=1}^{s} \int_{\tau_k^{q-1}}^{\tau_k^q} \left\| \Big\langle \dot{x}_k(s), f(s, x_k(\tau_k^{q-1})) \Big\rangle - \Big\langle \dot{x}_k(s), f(s, x_k(s)) \Big\rangle \right\| ds \\ & + \int_0^T \left\| \Big\langle \dot{x}_k(s), f(s, x_k(s)) \Big\rangle - \Big\langle \dot{x}_k(s), f(s, x(s)) \Big\rangle \right\| ds \\ & + \int_0^T \left\| \Big\langle \dot{x}_k(s), f(s, x(s)) \Big\rangle - \Big\langle \dot{x}(s), f(s, x(s)) \Big\rangle \right\| ds \; . \end{aligned}$$

Since f is a Carathéodory function, $x_k(.) \to x(.)$ uniformly, $\|\dot{x}_k(s)\| \leq M + 1 + m(s)$, $m(.) \in L^2([0,T], \mathbb{R}^n)$ and $\dot{x}_k(.) \to \dot{x}(.)$ weakly in $L^2([0,T], \mathbb{R}^n)$ then the last term converges to 0. This completes the proof of the claim. ∎

Since

$$\lim_{k\to\infty} \sum_{q=1}^{s} \frac{1}{k} \int_{\tau_k^{q-1}}^{\tau_k^q} \Big\langle \dot{x}_k(s), b_q \Big\rangle\, ds \;=\; 0$$

then by passing to the limit for $k \to \infty$ in (5.4) and using the continuity of the function V on the ball $B(x_0, r)$, we obtain the following inequality

$$V\big(x(T) - V(x_0)\big) \;\geq\; \lim_{k\to\infty} \sup \int_0^T \|\dot{x}_k(s)\|^2\, ds \;-\; \int_0^T \Big\langle \dot{x}(s), f(s, x(s)\Big\rangle\, ds \;.$$

Moreover, by the equality (5.2) we have

$$\|\dot{x}\|_2^2 \;\geq\; \lim_{k\to\infty} \sup \|\dot{x}_k\|_2^2$$

and by the weak lower semicontinuity of the norm, it follows that

$$\|\dot{x}\|_2^2 \;\leq\; \lim_{k\to\infty} \inf \|\dot{x}_k\|_2^2 \;.$$

Finally, since $(\dot{x}_k)_k$ converges to $\dot{x}(.)$ strongly in $L^2([0,T], \mathbb{R}^n)$, then there exists a subsequence denoted by $\dot{x}_k(.)$ which converges pointwisely to $\dot{x}(.)$. Therefore, we conclude, in view of Proposition 4, that

$$d_{grF}\Big(x(t), \dot{x}(t) - f(t, x(t))\Big) \;=\; 0 \qquad \text{a.e. on} \quad [0,T] \;.$$

Since the graph of F is closed we have

$$\dot{x}(t) \;\in\; f(t, x(t)) + F(x(t)) \qquad \text{a.e. on} \quad [0,T] \;.$$

Finally, let $t \in [0,T]$. Recall that there exists $(\tau_k^q)_k$ such that $\lim_{k\to\infty} \tau_k^q = t$ for all $t \in [0,T]$. Since

$$\lim_{k\to\infty} \|x(t) - x_k(\tau_k^q)\| \;=\; 0$$

$x_k(\tau_k^q) \in K$, K is closed, by passing to the limit we obtain $x(t) \in K$.

This completes the proof. ∎

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