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Exponential functionals of Brownian motion, II: Some related diffusion processes*

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Abstract: This is the second part of our survey on exponential functionals of Brownian motion. We focus on the applications of the results about the distributions of the exponential functionals, which have been discussed in the first part. Pricing formula for call options for the Asian options, explicit expressions for the heat kernels on hyperbolic spaces, diffusion processes in random environments and extensions of Lévy's and Pitman's theorems are discussed.

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1. Introduction

Let $B = \{B_t, t \geqq 0\}$ be a one-dimensional Brownian motion starting from 0 and defined on a probability space $(\Omega, \mathcal{F}, P)$. Denoting by $B^{(\mu)} = \{B_t^{(\mu)} = B_t + \mu t\}$ the corresponding Brownian motion with constant drift $\mu \in \mathbf{R}$, we consider the exponential functional $A^{(\mu)} = \{A_t^{(\mu)}\}$ defined by

$$A_t^{(\mu)} = \int_0^t \exp(2B_s^{(\mu)})ds, \quad t \geqq 0. \tag{1.1}$$

In Part I [49] of our survey, we have discussed about the probability law of $A_t^{(\mu)}$ for fixed t and about several related topics.

*This is an original survey paper.

Among the results, we have shown some explicit (integral) representations for the density of $A_t^{(\mu)}$. In particular, we have proven the following formula originally obtained in Yor [63]:

$$P(A_t^{(\mu)} \in du, B_t^{(\mu)} \in dx) = e^{\mu x - \mu^2 t/2} \exp\left(-\frac{1+e^{2x}}{2u}\right)\theta(e^x/u, t)\frac{dudx}{u}, \quad (1.2)$$

where, for $r > 0$ and $t > 0$,

$$\theta(r,t) = \frac{r}{(2\pi^3 t)^{1/2}} e^{\pi^2/2t} \int_0^\infty e^{-\xi^2/2t} e^{-r\cosh(\xi)} \sinh(\xi) \sin\left(\frac{\pi\xi}{t}\right) d\xi. \quad (1.3)$$

The function $\theta(r,t)$ appears in the representation for the (unnormalized) density of the so-called Hartman-Watson distribution and satisfies

$$\int_0^\infty e^{-\alpha^2 t/2}\theta(r,t)dt = I_\alpha(r), \quad \alpha > 0, \quad (1.4)$$

where I_α is the usual modified Bessel function. For details, see Part I and the references cited therein.

Another important fact, which has been used in several domains and also discussed in Part I, is the following identity in law due to Dufresne [18]. Let $\mu > 0$. Then one has

$$A_\infty^{(-\mu)} \equiv \int_0^\infty \exp(2B_s^{(-\mu)})ds \overset{\text{(law)}}{=} \frac{1}{2\gamma_\mu}, \quad (1.5)$$

where γ_μ is a gamma random variable with parameter μ, that is,

$$P(\gamma_\mu \in dx) = \frac{1}{\Gamma(\mu)} x^{\mu-1} e^{-x} dx, \quad x \geqq 0.$$

The purpose of this second part of our surveys is to present some results obtained by applying the formulae and identities mentioned in Part I to Brownian motion and some related stochastic processes.

In Section 2 we discuss about the pricing formula for the average option, so called, Asian option in the Black-Scholes model.

In Section 3 we present some formulae for the heat kernels of the semigroups generated by the Laplacians on hyperbolic spaces. By reasoning in probabilistic terms, we obtain not only the classical formulae but also new expressions.

In Section 4 we apply the results on exponential functionals to a question pertaining to a class of diffusion processes in random environments.

In Section 5 we show Dufresne's recursion relation for the probability density of $A_t^{(\mu)}$ with respect to μ which, as we have seen in Part I, plays an important role in several studies on exponential functionals.

Dufresne's relation is important in studying extensions or analogues of Lévy's and Pitman's theorems about, respectively, $\{M_t^{(\mu)} - B_t^{(\mu)}\}$ and $\{2M_t^{(\mu)} - B_t^{(\mu)}\}$, where $M_t^{(\mu)} = \max_{0 \leqq s \leqq t} B_s^{(\mu)}$, by means of exponential functionals. These topics are finally discussed in Section 6 of this survey, where we consider the stochastic process defined by

$$M_t^{(\mu),\lambda} = \frac{1}{\lambda} \log\biggl(\int_0^t \exp(\lambda B_s^{(\mu)}) ds \biggr), \quad t > 0.$$

By the Laplace method, we easily see that, as $\lambda \to \infty$, $M_t^{(\mu),\lambda}$ converges to $M_t^{(\mu)}$, and we prove that $\{M_t^{(\mu),\lambda} - B_t^{(\mu)}\}$ and $\{2M_t^{(\mu),\lambda} - B_t^{(\mu)}\}$ are diffusion processes for any $\lambda \in \mathbf{R}$. Hence, the classical Lévy and Pitman theorems may be seen as limiting results of those as $\lambda \to \infty$.

2. Asian options

In this section we consider the Asian or average call option in the framework of the Black-Scholes model and present some identities for the pricing formula.

By the Black-Scholes model, we mean a market model which consists of a riskless bond $b = \{b_t\}$ with a constant interest rate and a risky asset $S = \{S_t\}$ with a constant appreciation rate and volatility. That is, letting $r > 0$, $\mu \in \mathbf{R}$ and $\sigma > 0$ be constants, we let b and S be given by the stochastic differential equation

$$\frac{db_t}{b_t} = r dt, \quad \frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

where $B = \{B_t\}$ is a one-dimensional Brownian motion with $B_0 = 0$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

For simplicity we normalize them by setting $b_0 = 1$. Then we have

$$b_t = \exp(rt) \quad \text{and} \quad S_t = S_0 \exp(\sigma B_t + (\mu - \sigma^2/2)t).$$

Following the standard procedure, we consider the discounted stock price $\widetilde{S} = \{\widetilde{S}_t\}$ given by

$$\widetilde{S}_t = e^{-rt} S_t = S_0 \exp(\sigma B_t + (\mu - r - \sigma^2/2)t).$$

Then, by Girsanov's theorem, there exists a unique probability measure Q which is absolutely continuous with respect to P and under which $\widetilde{S}$ is a martingale. Q is called the martingale measure for $\widetilde{S}$ and we have

$$\frac{dQ}{dP}\bigg|_{\mathcal{F}_T} = \exp\biggl(-\frac{\mu - r}{\sigma} B_T - \frac{(\mu - r)^2}{2\sigma^2} T \biggr),$$

where $\mathcal{F}_T = \sigma\{B_s, s \leqq T\}$. $\widetilde{B} = \{\widetilde{B}_t = B_t + \sigma^{-1}(\mu - r)t\}$ is a Brownian motion under Q.

Let us consider the European and the Asian call options with fixed strike price $k > 0$ and maturity T. The payoffs are given by

$$(S_T - k)_+ \quad \text{and} \quad (\mathcal{A}(T) - k)_+,$$

respectively, where $x_+ = \max\{x, 0\}$ and

$$\mathcal{A}(t) = \frac{1}{t}\int_0^t S_u du, \quad 0 < t \leqq T.$$

By the Black-Scholes formula or by the non-arbitrage argument, we can show that the theoretical price $C_E(k,T)$ and $C_A(k,T)$ of these call options at time $t = 0$ are given by

$$C_E(k,T) = e^{-rT} E^Q[(S_T - k)_+]$$

and

$$C_A(k,T) = e^{-rT} E^Q[(\mathcal{A}(T) - k)_+],$$

where E^Q denotes the expectation with respect to the martingale measure Q.

Proposition 2.1. *If $r \geqq 0$, one has $C_A(k,T) \leqq C_E(k,T)$ for every $k > 0$ and $T > 0$.*

Proof. We have

$$C_A(k,T) = e^{-rT} E^Q\left[\left(\frac{1}{T}\int_0^T S_t dt - k\right)_+\right].$$

Using Jensen's inequality, we get

$$C_A(k,T) \leqq e^{-rT}\frac{1}{T}\int_0^T E^Q[(S_t - k)_+]dt.$$

Since $r \geqq 0$, $\{S_t = S_0 \exp(\sigma\widetilde{B}_t + (r - \sigma^2/2)t)\}$ is a submartingale under Q. Therefore, using Jensen's inequality again, we see that $\{(S_t - k)_+\}$ is also a submartingale. Hence we obtain

$$\begin{aligned} C_A(k,T) &\leqq e^{-rT}\frac{1}{T}\int_0^T E^Q[(S_T - k)_+]dt \\ &= e^{-rT} E^Q[(S_T - k)_+] = C_E(k,T). \end{aligned}$$

□

For more discussions on $C_A(k,T)$, see Geman-Yor [24], Rogers-Shi [55] and the references cited therein.

By using explicit expressions for the density of $A_t^{(\mu)}$ discussed in Part I, we obtain several integral representations for $C_A(k,T)$. However, they are complicated. Hence we omit this approach and consider instead the Laplace transform of $C_A(k,T)$ in T.

In the following we set $\sigma = 2$ and consider

$$A_t^{(\mu)} = \int_0^t \exp(2B_s^{(\mu)})ds$$

under the original probability measure P to follow the same convention as in Part I and in other parts of the present article.

Let T_λ be an exponential random variable with parameter $\lambda > 0$ independent of B. Yor [62] (see also Part I) has shown the identity in law

$$A^{(\mu)}_{T_\lambda} \overset{\text{(law)}}{=} \frac{Z_{1,a}}{2\gamma_b},$$

where $a = (\nu+\mu)/2, b = (\nu-\mu)/2, \nu = \sqrt{2\lambda+\mu^2}$, $Z_{1,a}$ is a beta variable with parameters $(1,a)$, γ_b is a gamma variable with parameter b and $Z_{1,a}$ and γ_b are independent.

From this identity we deduce the following result.

Theorem 2.2. *For all* $\mu \in \mathbf{R}$, $\lambda > \max\{2(1+\mu), 0\}$ *and* $k > 0$, *we have*

$$\begin{aligned}\lambda \int_0^\infty & e^{-\lambda t} E[(A^{(\mu)}_t - k)_+] dt \\ &= \frac{1}{(\lambda - 2(1+\mu))\Gamma(b-1)} \int_0^{1/2k} e^{-t} t^{b-2} (1-2kt)^{a+1} dt.\end{aligned}$$

The same formula has been proven in [24] with the help of some properties of Bessel processes.

We next present another proof of Theorem 2.2, following Donati-Martin, Ghomrasni and Yor [16], who have used, as an auxiliary tool, the stochastic process $Y^{(\mu)}(x) = \{Y^{(\mu)}_t(x)\}$ given by

$$Y^{(\mu)}_t(x) = \exp(2B^{(\mu)}_t)\biggl(x + \int_0^t \exp(-2B^{(\mu)}_s) ds\biggr).$$

$Y^{(\mu)}(x)$ is a diffusion process with generator

$$\mathcal{L}^{(\mu)} = 2x^2 \frac{d^2}{dx^2} + (2(1+\mu)x+1)\frac{d}{dx} = 2x^{1-\mu} e^{1/2x} \frac{d}{dx}\biggl(x^{1+\mu} e^{-1/2x} \frac{d}{dx}\biggr).$$

In fact, in [16], the authors have taken advantage of the identity in law

$$\int_0^t \exp(2B^{(\mu)}_s) ds \overset{\text{(law)}}{=} Y^{(\mu)}_t(0) \equiv \exp(2B^{(\mu)}_t) \int_0^t \exp(-2B^{(\mu)}_s) ds$$

for every fixed $t > 0$ and have computed the Laplace transform of $E[(Y^{(\mu)}_t(0) - k)_+]$ in t by using the general theory of the Sturm-Liouville operators.

We present an explicit form of the Green function for $\mathcal{L}^{(\mu)}$. For this purpose we recall the confluent hypergeometric functions $\Phi(\alpha,\gamma;z)$ and $\Psi(\alpha,\gamma;z)$ of the first and second kinds defined by

$$\Phi(\alpha,\gamma;z) = \sum_{k=0}^\infty \frac{(\alpha)_k}{(\gamma)_k} \frac{z^k}{k!} \tag{2.1}$$

and

$$\Psi(\alpha,\gamma;z)=\frac{\Gamma(1-\gamma)}{\Gamma(1+\alpha-\gamma)}\Phi(\alpha,\gamma;z)+\frac{\Gamma(\gamma-1)}{\Gamma(\alpha)}z^{1-\gamma}\Phi(1+\alpha-\gamma,2-\gamma;z),$$

where $(\alpha)_0=1$ and

$$(\alpha)_k=\frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}=\alpha(\alpha+1)\cdots(\alpha+k-1),\quad k=1,2,...$$

For details about the confluent hypergeometric functions, we refer to Lebedev [38]. $\Phi(\alpha,\gamma;z)$ and $\Psi(\alpha,\gamma;z)$ are linearly independent solutions for the linear differential equation

$$zu''+(\gamma-z)u'-\alpha u=0.$$

We set $\nu=\sqrt{2\lambda+\mu^2}$ and define the functions u_1 and u_2 on $(0,\infty)$ by

$$u_1(x)=x^{-(\mu+\nu)/2}\Psi\left(\frac{\mu+\nu}{2},1+\nu;\frac{1}{2x}\right)\tag{2.2}$$

and

$$u_2(x)=x^{-(\mu+\nu)/2}\Phi\left(\frac{\mu+\nu}{2},1+\nu;\frac{1}{2x}\right),\tag{2.3}$$

respectively. Then, by straightforward computations, we can check

$$\mathcal{L}^{(\mu)}u_i=\lambda u_i,\quad i=1,2.$$

Moreover, $u_2(x)$ is monotone decreasing in $x\in(0,\infty)$. On the other hand, recalling the integral representation for $\Psi(\alpha,\gamma;z)$:

$$\Psi(\alpha,\gamma;z)=\frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-zt}t^{\alpha-1}(1+t)^{\gamma-\alpha-1}dt,\qquad \alpha>0,z>0,$$

(cf. [38], p.268), we can easily show that $u_1(x)$ is monotone increasing. In fact, we have

$$u_1(x)=\frac{1}{\Gamma((\nu+\mu)/2)}\int_0^\infty e^{-\xi/2}\xi^{(\mu+\nu)/2-1}(1+x\xi)^{(\nu-\mu)/2}d\xi,\quad x\geqq 0.$$

About the Wronskian, it is known ([38], p.265) that

$$\Phi(\alpha,\gamma;z)\Psi'(\alpha,\gamma;z)-\Phi'(\alpha,\gamma;z)\Psi(\alpha,\gamma;z)=-\frac{\Gamma(\gamma)}{\Gamma(\alpha)}z^{-\gamma}e^z,$$

which yields

$$\frac{1}{x^{-(1+\mu)}e^{1/2x}}(u_1'(x)u_2(x)-u_1(x)u_2'(x))=\frac{2^\nu\Gamma(1+\nu)}{\Gamma((\mu+\nu)/2)}.$$

Here the function $x^{-(1+\mu)}e^{1/2x}$ is the derivative of the scale function for $Y^{(\mu)}(x)$.

Checking the boundary conditions, we obtain the following.

Proposition 2.3. *Let $u_1(x)$ and $u_2(x)$ be the functions defined by* (2.2) *and* (2.3). *Then the Green function $G^{(\mu)}(x,y;\lambda)$ for $\mathcal{L}^{(\mu)}$ with respect to the Lebesgue measure is given by*

$$G^{(\mu)}(x,y;\lambda) = \frac{\Gamma((\mu+\nu)/2)}{2^{1+\nu}\Gamma(1+\nu)} y^{\mu-1} e^{-1/2y} u_1(x) u_2(y), \quad 0 \leqq x \leqq y.$$

In order to proceed to a proof of Theorem 2.2, we recall the following identity presented in [16]:

$$\begin{aligned}\int_a^\infty (\xi - a)\xi^{-1-(\nu-\mu)/2} e^{-1/\xi} \Phi\Big(\frac{\nu+\mu}{2}, 1+\nu; \frac{1}{\xi}\Big) d\xi \\ = \frac{\Gamma((\nu-\mu)/2-1)}{\Gamma((\nu-\mu)/2+1)} a^{1-(\nu-\mu)/2} e^{-1/a} \Phi\Big(\frac{\nu+\mu}{2}+2, 1+\nu; \frac{1}{a}\Big),\end{aligned}$$

which may be proven by Kummer's transformation

$$\Phi(\alpha,\gamma;x) = e^x \Phi(\gamma-\alpha,\gamma;-x)$$

and the series expansion (2.1) of Φ. It is a special case of a general formula given on page 279, Problem 21, Lebedev [38]. See also [23].

Then, noting that $u_1(0) = 2^{(\mu+\nu)/2}$, we obtain

$$\begin{aligned}\lambda \int_0^\infty e^{-\lambda t} E[(A_t^{(\mu)} - k)_+] dt &= \lambda \int_k^\infty (y-k) G^{(\mu)}(0,y;\lambda) dy \\ &= \frac{\lambda \Gamma((\mu+\nu)/2)\Gamma((\nu-\mu)/2-1)}{2^{1+(\nu-\mu)/2}\Gamma(1+\nu)\Gamma((\nu-\mu)/2+1)} \\ &\quad \times k^{1-(\nu-\mu)/2} e^{-1/2k} \Phi\Big(\frac{\mu+\nu}{2}+2, 1+\nu; \frac{1}{2k}\Big).\end{aligned}$$

Moreover, we recall the integral representation of Φ:

$$\Phi(\alpha,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 e^{zu} u^{\alpha-1}(1-u)^{\gamma-\alpha-1} du.$$

Then, after some elementary computations, we arrive at

$$\begin{aligned}\lambda \int_0^\infty e^{-\lambda t} E[(A_t^{(\mu)} - k)_+] dt \\ = \frac{\lambda \Gamma((\mu+\nu)/2)}{4\Gamma((\nu-\mu)/2+1)\Gamma((\nu+\mu)/2+2)} \int_0^{1/2k} e^{-u} b^{b-2} (1-2ku)^{a+1} du.\end{aligned}$$

Finally, by using the identity $z\Gamma(z) = \Gamma(z+1)$, we obtain

$$\frac{\lambda \Gamma((\mu+\nu)/2)}{4\Gamma((\nu-\mu)/2+1)\Gamma((\nu+\mu)/2+2)} = \frac{1}{(\lambda - 2(\mu+1))\Gamma((\nu-\mu)/2-1)}$$

and Theorem 2.2.

3. Heat kernels on hyperbolic spaces

Let $\mathbf{H}^n$ be the upper half space in $\mathbf{R}^n$ given by

$$\{z = (x, y) = (x^1, ..., x^{n-1}, y); x \in \mathbf{R}^{n-1}, y > 0\},$$

endowed with the Poincaré metric $ds^2 = y^{-2}(dx^2 + dy^2)$. The Riemannian volume element is given by $dv = y^{-n}dxdy$ and the distance $d(z, z')$ between $z, z' \in \mathbf{H}^n$ is given by the formula

$$\cosh(d(z, z')) = \frac{|x - x'|^2 + y^2 + (y')^2}{2yy'}, \tag{3.1}$$

where $|x - x'|$ is the Euclidean distance between $x, x' \in \mathbf{R}^{n-1}$.

The Laplace-Beltrami operator Δ_n is written as

$$\Delta_n = y^2 \sum_{i=1}^{n-1} \left(\frac{\partial}{\partial x^i} \right)^2 + y^2 \left(\frac{\partial}{\partial y} \right)^2 - (n-2)y \frac{\partial}{\partial y}. \tag{3.2}$$

We denote by $p_n(t, z, z')$ the heat kernel with respect to the volume element dv of the semigroup generated by $\Delta_n/2$. Since $p_n(t, z, z')$ is a function of $r = d(z, z')$ for a fixed $t > 0$, we occasionally write $p_n(t, r)$ for $p_n(t, z, z')$.

Then, for $n = 2$ and 3, the following formulae are well known:

$$p_2(t, r) = \frac{\sqrt{2}e^{-t/8}}{(2\pi t)^{3/2}} \int_r^\infty \frac{be^{-b^2/2t}}{(\cosh(b) - \cosh(r))^{1/2}} db, \tag{3.3}$$

$$p_3(t, r) = \frac{1}{(2\pi t)^{3/2}} \frac{r}{\sinh(r)} \exp\left(-\frac{t}{2} - \frac{r^2}{2t} \right). \tag{3.4}$$

Moreover, the following recursion formula due to Millson is also well known and we also have explicit expressions of $p_n(t, r)$ for every $n \geqq 4$:

$$p_{n+2}(t, r) = -\frac{e^{-nt/2}}{2\pi \sinh(r)} \frac{\partial}{\partial r} p_n(t, r). \tag{3.5}$$

For details about the real hyperbolic space $\mathbf{H}^n$ and the classical formulae for the heat kernels, we refer the reader to Davies [14].

Gruet [28] has considered the Brownian motion on $\mathbf{H}^n$, which is a diffusion process generated by $\Delta_n/2$, and has derived a new integral representation for $p_n(t, r)$ by using the explicit expression (1.2) for the joint density of $(A_t^{(\mu)}, B_t^{(\mu)})$. While the classical expressions for $p_n(t, r)$ have different forms for odd and even dimensions, Gruet's formula (3.6) below holds for every n.

Theorem 3.1. *For every $n \geqq 2, t > 0, z, z' \in \mathbf{H}^n$, it holds that*

$$\begin{aligned} p_n(t, z, z') = &\frac{e^{-(n-1)^2 t/8}}{\pi(2\pi)^{n/2} t^{1/2}} \Gamma\left(\frac{n+1}{2} \right) \\ &\times \int_0^\infty \frac{e^{(\pi^2 - b^2)/2t} \sinh(b) \sin(\pi b/t)}{(\cosh(b) + \cosh(r))^{(n+1)/2}} db, \end{aligned} \tag{3.6}$$

where $r = d(z, z')$.

Before giving a proof of (3.6), we mention its relationship to the classical formulae. First of all we note that Millson's formula (3.5) is easily obtained from (3.6) if we differentiate both hand sides of (3.6) with respect to r.

When $n = 3$, the integrand on the right hand side of (3.6) may be extended to a meromorphic function in b on $\mathbf{C}$. Hence we can apply residue calculus and obtain (3.4).

In the case $n = 2$, which is the most interesting and important, we compute the Laplace transform in t of the right hand sides of (3.3) and (3.6). Then, using the Hankel-Lipschitz formula for the modified Bessel functions (see Watson [57], p.386), we can check the coincidence of the Laplace transforms or of the expressions for the Green function. For details, see [28], [40], [41].

We give a proof of (3.6) and see how the exponential functional $A^{(\mu)}$ comes into the story.

Proof of Theorem 3.1. Let $(W, \mathcal{B}, P)$ be the n-dimensional standard Wiener space with the canonical filtration $\{\mathcal{B}_s\}_{s \geqq 0}$: W is the space of all $\mathbf{R}^n$-valued continuous paths $w_\cdot = (w_\cdot^1, ..., w_\cdot^{n-1}, w_\cdot^n)$ starting from 0 with the topology of uniform convergence on compact intervals, $\mathcal{B}$ is the topological σ-field, $\mathcal{B}_s$ is the sub σ-field of $\mathcal{B}$ generated by $\{w_u, 0 \leqq u \leqq s\}$ and P is the n-dimensional Wiener measure.

The Brownian motion on $\mathbf{H}^n$ may be obtained as the unique solution of the stochastic differential equation

$$\begin{cases} dX^i(s) = Y(s)dw_s^i, & i = 1, ..., n-1, \\ dY(s) = Y(s)dw_s^n - \frac{n-2}{2} Y(s)ds. & \end{cases}$$

We denote by $Z_z = \{Z_z(t,w) = (X_z(t,w), Y_z(t,w)), t \geqq 0\}$ the unique strong solution satisfying $Z_z(0) = z = (x, y)$. Then we have

$$\begin{cases} X_z^i(t,w) = x^i + \int_0^t y \exp(B_s^{(\mu)}) dw_s^i, & i = 1, ..., n-1, \\ Y_z(t,w) = y \exp(B_t^{(\mu)}), & \end{cases} \tag{3.7}$$

where $B_s^{(\mu)} = w_s^n + \mu s$ and $\mu = -(n-1)/2$.

Let $F_t^{(\mu)}$ be the $\mathbf{R}^{n-1}$-valued random variable defined by

$$F_t^{(\mu)} = \left(\int_0^t \exp(B_s^{(\mu)}) dw_s^1, ..., \int_0^t \exp(B_s^{(\mu)}) dw_s^{n-1} \right).$$

Then the conditional distribution of $F_t^{(\mu)}$ given $\{Y_z(s), 0 \leqq s \leqq t\}$ or $\{w_s^n, 0 \leqq s \leqq t\}$ is the $(n-1)$-dimensional Gaussian distribution with mean 0 and covariance matrix $A_t^{(\mu)} \mathrm{I}_{n-1}$, I_{n-1} being the $(n-1)$-dimensional identity matrix.

Note that the heat kernel $p_n(t, z, z')$ may be written as

$$p_n(t, z, z') = \int_W \widetilde{\delta}_{z'}(Z_z(t,w)) dP(w) = (y')^n \int_W \delta_{z'}(Z_z(t,w)) dP(w),$$

where $\widetilde{\delta}_{z'}$ and $\delta_{z'}$ are the Dirac delta functions concentrated at z' with respect to the volume element dv and the Lebesgue measure $dz = dxdy$, respectively, and $\delta_{z'}(Z_z(t,w))$ is the composition of the distribution $\delta_{z'}$ and the smooth Wiener functional $Z_z(t,w)$ in the sense of Malliavin calculus (see [32]). Therefore, we obtain

$$
\begin{aligned}
&p_n(t,z,z') \\
&= (y')^n \int_W \delta_{(x',y')}(x + yF_t^{(\mu)}, y\exp(B_t^{(\mu)}))dP \\
&= \left(\frac{y'}{y}\right)^n \int_W \delta_{((x'-x)/y, y'/y)}(F_t^{(\mu)}, \exp(B_t^{(\mu)}))dP \\
&= \left(\frac{y'}{y}\right)^n \int_W \frac{1}{(2\pi A_t^{(\mu)})^{(n-1)/2}} \exp\left(-\frac{|x'-x|^2}{2y^2 A_t^{(\mu)}}\right)\delta_{y'/y}(\exp(B_t^{(\mu)}))dP \\
&= \left(\frac{y'}{y}\right)^{n-1} \int_W \frac{1}{(2\pi A_t^{(\mu)})^{(n-1)/2}} \exp\left(-\frac{|x'-x|^2}{2y^2 A_t^{(\mu)}}\right)\delta_{\log(y'/y)}(B_t^{(\mu)})dP,
\end{aligned}
\tag{3.8}
$$

where we have used the same notation for the Dirac delta functions on $\mathbf{R}^n$ and $\mathbf{R}$.

Now we apply formula (1.2) for the last expression. Then we obtain

$$
\begin{aligned}
p_n(t,z,z') = \left(\frac{y'}{y}\right)^{n-1} &\int_0^\infty \frac{1}{(2\pi u)^{(n-1)/2}} \exp\left(-\frac{|x'-x|^2}{2y^2 u}\right) \\
&\times \left(\frac{y'}{y}\right)^{-(n-1)/2} e^{-(n-1)^2 t/8} \frac{1}{u} \exp\left(-\frac{1+(y'/y)^2}{2u}\right)\theta(y'/yu, t)du.
\end{aligned}
$$

Moreover, changing variables by $v = y'/yu$ and using (3.1), we obtain

$$
p_n(t,z,z') = \frac{e^{-(n-1)^2 t/8}}{(2\pi)^{(n-1)/2}} \int_0^\infty v^{(n-3)/2} \exp(-v\cosh(r))\theta(v,t)dv \tag{3.9}
$$

for $r = d(z,z')$.

Finally we use the integral representation (1.3) for $\theta(v,t)$. Then, changing the order of the integrations by Fubini's theorem, we obtain (3.6) after some elementary computations. □

Remark 3.1. Recalling formula (1.4), we can easily obtain an explicit expression of the Green function for Δ_n from formula (3.9).

Remark 3.2. From the last expression of (3.8), we obtain

$$
\begin{aligned}
p_n(t,z,z') = \left(\frac{y'}{y}\right)^{n-1} &\int_{\mathbf{R}^{n-1}} e^{-\sqrt{-1}\langle x'-x, \lambda\rangle} d\lambda \\
&\times \int_W \exp\left(-\frac{1}{2}|\lambda|^2 y^2 A_t^{(\mu)}\right)\delta_{\log(y'/y)}(B_t^{(\mu)})dP.
\end{aligned}
$$

Hence, we see that the Laplacian Δ_n on $\mathbf{H}^n$ and the Schrödinger operator on $\mathbf{R}$ with the Liouville potential $-\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}|\lambda|^2 e^{2x}$ are unitary equivalent, which may be directly verified by Fourier analysis and has been already pointed out in Comtet [11], Debiard-Gaveau [15], Grosche [26] and so on. See also [31].

In the rest of this section, we restrict ourselves to the case $n = 2$ and consider two questions related to the results and formulae presented above. For other related topics, see, e.g., [2] and [29].

Let us consider the following Schrödinger operator H_k, $k \in \mathbf{R}$, on $\mathbf{H}^2$ with a magnetic field:

$$H_k = \frac{1}{2}y^2\left(\sqrt{-1}\frac{\partial}{\partial x} + \frac{k}{y}\right)^2 - \frac{1}{2}y^2\left(\frac{\partial}{\partial y}\right)^2.$$

The differential 1-form $\alpha = ky^{-1}dx$ is called the vector potential and its exterior derivative $d\alpha = ky^{-2}dx \wedge dy$ represents the corresponding magnetic field. Since $d\alpha$ is equal to constant k times the volume element dv, we call H_k a Schrödinger operator with a constant magnetic field. It is essentially the same as the Maass Laplacian which plays an important role in several domains of mathematics, e.g., number theory, representation theory and so on. For details, see [22], [31] and the references cited therein.

In [31], the authors have started their arguments from the Brownian motion on $\mathbf{H}^2$ given in the above proof of Theorem 3.1 and have discussed about explicit and probabilistic expressions for the heat kernel $q_k(t, z, z')$ of the semigroup generated by H_k. They have also applied the results to a study of the Selberg trace formula on compact quotient spaces, i.e., compact Riemannian surfaces, and have shown close relationship between the spectrum and the action integrals for the corresponding classical paths. It should be mentioned that some physicists have shown similar results in the context of Feynman path integrals prior to [31]. See, e.g., [12], [25], [27].

Explicit formulae for several quantities related to the operator H_k, e.g., the Green functions, the heat kernels, have been obtained by Fay [22] by harmonic analysis. On the other hand, starting from computations by Feynman path integrals, Comtet [11] and Grosche [26] have obtained explicit forms of the Green functions.

From the point of view of probability theory along the line of [31], another explicit representation for the heat kernel $q_k(t, z, z')$ has been shown in [1] by using an extension of formula (1.2) and Gruet's formula (3.6). We introduce the result in [1] together with some arguments taken from [31].

To show an explicit representation for $q_k(t, z, z')$, we recall from Proposition 2.2 in [31] (see also the references therein) that $q_k(t, z, z')$ may be written in the form

$$q_k(t, z, z') = \left(\frac{z' - \bar{z}}{z - \bar{z'}}\right)^k g_k(t, d(z, z')) \tag{3.10}$$

for some positive function $g_k(t, r)$. This is a consequence of the group action of $\mathrm{SL}(2; \mathbf{R})$ on $\mathbf{H}^2$. Here a point $z = (x, y) \in \mathbf{H}^2$ is identified with $z = x + \sqrt{-1}y \in \mathbf{C}$, $d(z, z')$ is the hyperbolic distance given by (3.1), and, for $\omega = |\omega| \exp(\sqrt{-1}\theta) \in \mathbf{C}$ with $-\pi < \theta \leqq \pi$, $\omega^k = |\omega|^k \exp(\sqrt{-1}k\theta)$. Therefore, if $x = x'$, we have $q_k(t, z, z') = g_k(t, d(z, z'))$.

Theorem 3.2. *The function* $g_k(t,r)$ *on the right hand side of* (3.10) *is given by*

$$g_k(t,r) = \frac{\sqrt{2}e^{-t/8-k^2t/2}}{(2\pi t)^{3/2}} \int_r^\infty \frac{\cosh(2k\varphi(b,r))be^{-b^2/2t}}{(\cosh(b)-\cosh(r))^{1/2}}db, \tag{3.11}$$

where

$$\varphi(b,r) = \mathrm{Argcosh}\Big(\frac{\cosh(b/2)}{\cosh(r/2)}\Big), \quad 0 \leqq r \leqq b.$$

Remark 3.3. When $k=0$, $g_0(t,r)$ coincides with the classical formula (3.3) for the heat kernel $p_2(t,r)$ on $\mathbf{H}^2$.

Proof. We show (3.11) when $|k|<1/2$. Formula (3.11) for a general value of k follows from this result on the special case by analytic continuation. We use the same notations as those in the proof of Theorem 3.1.

Let $I_t(\alpha)$ denote the stochastic line integral (cf. [32]) of the differential 1-form $\alpha = ky^{-1}dx$ along the path $\{Z_z(s), 0 \leqq s \leqq t\}$ of the Brownian motion Z_z on $\mathbf{H}^2$:

$$I_t(\alpha) = \int_0^t \alpha(Z_z(s)) \circ dZ_z(s).$$

In fact, it is easy to show

$$I_t(\alpha) = \int_0^t \frac{k}{Y_z(s)} \circ dX_z(s) = kw_t^1.$$

By using the Itô formula, we have

$$\begin{aligned} q_k(t,z,z') &= \int_W \exp(-\sqrt{-1}I_t(\alpha,w))\widetilde{\delta}_{z'}(Z_z(t,w))dP(w) \\ &= \int_W \exp(-\sqrt{-1}kw_t^1)\widetilde{\delta}_{(x',y')}(X_z(t,w),Y_z(t,w))dP(w). \end{aligned}$$

As in the proof of Theorem 3.1, we consider the conditional distribution of $(w_t^1, \int_0^t \exp(B_s^{(-1/2)})dw_s^1)$ given $\{B_s^{(-1/2)} = w_s^2 - s/2, 0 \leqq s \leqq t\}$. Then it is easy to see that this conditional distribution is a two-dimensional Gaussian distribution with mean 0 and covariance matrix

$$\begin{pmatrix} t & a_t^{(-1/2)} \\ a_t^{(-1/2)} & A_t^{(-1/2)} \end{pmatrix},$$

where

$$a_t^{(\mu)} = \int_0^t \exp(B_s^{(\mu)})ds \quad \text{and} \quad A_t^{(\mu)} = \int_0^t \exp(2B_s^{(\mu)})ds.$$

Taking the conditional expectation and using the Cameron-Martin theorem, we obtain

$$\begin{aligned} q_k(t,z,z') = e^{-t/8-k^2t/2}\Big(\frac{y'}{y}\Big)^{3/2} \int_W \frac{1}{\sqrt{2\pi A_t(w^2)}} \\ \times \exp\Big(-\frac{1}{2A_t(w^2)}\Big(\frac{x'-x}{y} + \sqrt{-1}ka_t(w^2)\Big)^2\Big)\delta_{y'/y}(\exp(w_t^2))dP, \end{aligned}$$

where
$$a_t(w^2) = \int_0^t \exp(w_s^2)ds \quad \text{and} \quad A_t(w^2) = \int_0^t \exp(2w_s^2)ds.$$

In the same way as is mentioned in Remark 3.2, we may write

$$\begin{aligned} q_k(t,z,z') &= e^{-t/8-k^2t/2}\frac{\sqrt{yy'}}{2\pi}\int_{\mathbf{R}} e^{-\sqrt{-1}(x'-x)\lambda}d\lambda \\ &\times \int_W \exp\Big(-\frac{1}{2}\lambda^2 y^2 A_t(w^2) + \lambda k y a_t(w^2)\Big)\delta_{\log(y'/y)}(w_t^2)dP \qquad (3.12) \\ &= e^{-t/8-k^2t/2}\frac{\sqrt{yy'}}{2\pi}\int_{\mathbf{R}} e^{-\sqrt{-1}(x'-x)\lambda} q_{\lambda,k}(t,\log y,\log y')d\lambda, \end{aligned}$$

where $q_{\lambda,k}(t,\xi,\eta)$ denotes the heat kernel of the semigroup generated by the Schrödinger operator $H_{\lambda,k}$ on $\mathbf{R}$ with the Morse potential given by

$$H_{\lambda,k} = -\frac{1}{2}\frac{d^2}{d\xi^2} + V_{\lambda,k}, \qquad V_{\lambda,k}(\xi) = \frac{1}{2}\lambda^2 e^{2\xi} - \lambda k e^{\xi}.$$

In [1] and Part I, we have shown an explicit representation for $q_{\lambda,k}(t,\xi,\eta)$: for $\lambda > 0$,

$$q_{\lambda,k}(t,\xi,\eta) = \int_0^\infty e^{2ku}\frac{1}{2\sinh(u)}\exp(-\lambda(e^\xi + e^\eta)\coth(u))\theta(\phi,t/4)du, \quad (3.13)$$

where the function $\theta(r,t)$ is given by (1.3) and $\phi = 2\lambda e^{(\xi+\eta)/2}/\sinh(u)$.

For $\lambda < 0$, we have $q_{\lambda,k}(t,\xi,\eta) = q_{-\lambda,-k}(t,\xi,\eta)$.

We now recall the remark following the statement of Theorem 3.2 and consider the case $x' = x$. Then, combining (3.12) and (3.13), we obtain

$$\begin{aligned} &q_k(t,z,z') = g_k(t,r) \\ &= e^{-t/8-k^2t/2}\frac{\sqrt{yy'}}{2\pi}\int_0^\infty (q_{\lambda,k}(t,\log y,\log y') + q_{-\lambda,k}(t,\log y,\log y'))d\lambda \\ &= e^{-t/8-k-2t/2}\frac{\sqrt{yy'}}{2\pi}\int_0^\infty d\lambda \int_0^\infty \frac{\cosh(2ku)}{\sinh(u)}\exp(-\lambda(y+y')\coth(u)) \\ &\qquad\qquad \times \theta\Big(\frac{2\lambda\sqrt{yy'}}{\sinh(u)}, \frac{t}{4}\Big)du. \end{aligned}$$

Note that the integral is convergent if $|k| < 1/2$.

Then, using the integral representation (1.3) for $\theta(r,t)$ and carrying out the integration in λ first, we obtain

$$\begin{aligned} g_k(t,r) &= \frac{e^{-t/8-k^2t/2}}{\pi(2\pi)^{3/2}t^{1/2}}\int_0^\infty \cosh(2ku)F_t(u)du, \\ F_t(u) &= \int_0^\infty \frac{e^{2(\pi^2-\xi^2)/t}\sinh(\xi)\sin(4\pi\xi/t)}{(\cosh(r/2)\cosh(u)+\cosh(\xi))^2}d\xi. \end{aligned}$$

By Gruet's formula (3.6), we have

$$p_3(t/4, r) = \frac{2e^{-t/8}}{\pi(2\pi)^{3/2}t} \int_0^\infty \frac{e^{2(\pi^2 - b^2)/t} \sinh(b) \sin(4\pi b/t)}{(\cosh(b) + \cosh(r))^2} db$$
$$= \left(\frac{2}{\pi t}\right)^{3/2} \frac{r}{\sinh(r)} \exp\left(-\frac{t}{8} - \frac{r^2}{t}\right).$$

It is now easy to show (3.11) from these formulae. □

Similar arguments to those in the proofs of Theorems 3.1 and 3.2 are available to study the Laplace-Beltrami operators on the complex and quaternion hyperbolic spaces. Also on these symmetric spaces of rank one, we have explicit expressions of Brownian motions as Wiener functionals and we can show explicit representations for the heat kernels and for the Green functions. For details, see [40].

Next we consider the diffusion process on $\mathbf{H}^2$ associated to the infinitesimal generator

$$\mathcal{L}_{\nu,\mu} = \frac{1}{2}y^2\left(\frac{\partial}{\partial x}\right)^2 + \frac{1}{2}y^2\left(\frac{\partial}{\partial y}\right)^2 - \nu y\frac{\partial}{\partial x} - \left(\mu - \frac{1}{2}\right)y\frac{\partial}{\partial y},$$

where $\nu \geqq 0$ and $\mu > 0$. The operator $\mathcal{L}_{\nu,\mu}$ is invariant under the special transforms on $\mathbf{H}^2$ of the form $z \mapsto az+b$, $a > 0$ and $b \in \mathbf{R}$, while the operator H_k and, in particular, the Laplacian Δ_2 are invariant under the action of $\mathrm{SL}(2;\mathbf{R})$.

The diffusion process starting from $z = (x, y)$ associated to $\mathcal{L}_{\nu,\mu}$ may be realized as the unique strong solution $\{Z_t^{(\nu,\mu)} = (X_t^{(\nu,\mu)}, Y_t^{(\nu,\mu)})\}$ of the stochastic differential equation

$$\begin{cases} dX_t = Y_t dw_t^1 - \nu Y_t dt, & X_0 = x, \\ dY_t = Y_t dw_t^2 - \left(\mu - \dfrac{1}{2}\right)Y_t dt, & Y_0 = y, \end{cases}$$

defined on a two-dimensional Wiener space. As in the case of Brownian motion on $\mathbf{H}^2$, $Z^{(\nu,\mu)}$ is also represented as a Wiener functional by

$$\begin{cases} X_t^{(\nu,\mu)} = x + y\displaystyle\int_0^t \exp(B_s^{(-\mu)}) dW_s^{(-\nu)}, \\ Y_t^{(\nu,\mu)} = y\exp(B_t^{(-\mu)}), \end{cases}$$

where $W_s^{(-\nu)} = w_s^1 - \nu s$ and $B_s^{(-\mu)} = w_s^2 - \mu s$.

Since μ is assumed to be positive, $Y_t^{(\nu,\mu)}$ converges to 0 as t tends to ∞. Following [4], we show that the distribution of $X_t^{(\nu,\mu)}$ converges weakly as $t \to \infty$ and that we can specify the limiting distribution. It is enough to consider the special case $x = 0$ and $y = 1$.

Theorem 3.3. *When $x = 0$ and $y = 1$, the distribution of $X_t^{(\nu,\mu)}$ on $\mathbf{R}$ converges weakly to the distribution with density*

$$f(\xi) = C_{\nu,\mu} \frac{\exp(-2\nu \mathrm{Arctan}(\xi))}{(1+\xi^2)^{\mu+1/2}}, \quad \xi \in \mathbf{R}.$$

For details on the normalizing constant $C_{\nu,\mu}$, see [4]. Note that, if $\nu = 0$ and $\mu = 1/2$, that is, if we consider the Brownian motion on $\mathbf{H}^2$, the limiting distribution is the Cauchy distribution as in the case of the hitting distribution on lines of standard Brownian motion on $\mathbf{R}^2$. In general, the limiting distribution belongs to the type IV family of Pearson distributions (cf. [34]).

It should be mentioned that the functional $\int_0^\infty \exp(B_s^{(-\mu)}) dW_s^{(-\nu)}$ has been much studied in the context of risk theory. See Paulsen [50] and the references cited therein about this. In [50] the density is derived when $\nu > 1$. See also [2] and [3] about some results in special cases. For further related discussions, see [47] and [64].

We present a probabilistic proof taken from [4], where we also find an analytic proof.

Proof. The limiting distribution coincides with that for the stochastic process $\{\bar{X}_t^{(\nu,\mu)}\}$ given by

$$\bar{X}_t^{(\nu,\mu)} = x \exp(B_t^{(-\mu)}) + \int_0^t \exp(B_s^{(-\mu)}) dW_s^{(-\nu)}.$$

We also consider the diffusion process $\{\widetilde{X}_t^{(\nu,\mu)}\}$ given by

$$\widetilde{X}_t^{(\nu,\mu)} = \exp(B_t^{(-\mu)}) \left(x + \int_0^t \exp(-B_s^{(-\mu)}) dW_s^{(-\nu)} \right)$$

with infinitesimal generator

$$\widetilde{\mathcal{L}}^{(\nu,\mu)} = \frac{1+x^2}{2} \frac{d^2}{dx^2} - \left(\nu + \left(\mu - \frac{1}{2} \right) x \right) \frac{d}{dx}.$$

By the invariance of the law of Brownian motion under time reversal from a fixed time, $\bar{X}_t^{(\nu,\mu)}$ and $\widetilde{X}_t^{(\nu,\mu)}$ are identical in law for any fixed $t > 0$. Therefore, to prove the theorem, we only have to check $\widetilde{\mathcal{L}}^{(\nu,\mu)*} f = 0$ for the adjoint operator $\widetilde{\mathcal{L}}^{(\nu,\mu)*}$ to $\widetilde{\mathcal{L}}^{(\nu,\mu)}$. □

Remark 3.4. Set

$$A_t^{(-\mu)} = \int_0^t \exp(2B_s^{(-\mu)}) ds \quad \text{and} \quad a_t^{(-\mu)} = \int_0^t \exp(B_s^{(-\mu)}) ds.$$

The joint distribution of $(A_t^{(-\mu)}, a_t^{(-\mu)}, B_t^{(-\mu)})$ or the Laplace transform of the conditional distribution of $A_t^{(-\mu)}$ given $(a_t^{(-\mu)}, B_t^{(-\mu)})$ has been studied in [1] (see also Part I).

We have, using some obvious notation,

$$X_\infty^{(\nu,\mu)} \overset{\text{(law)}}{=} \gamma_{A_\infty^{(-\mu)}} - \nu a_\infty^{(-\mu)}$$

for a Brownian motion $\{\gamma_t\}$ independent of B. Hence we obtain

$$f(\xi) = E\left[\frac{1}{\sqrt{2\pi A_\infty^{(-\mu)}}} \exp\left(-\frac{(\xi+\nu a_\infty^{(-\mu)})^2}{2A_\infty^{(-\mu)}}\right)\right].$$

However, we have not succeeded in obtaining Theorem 3.3 from this expression.

Remark 3.5. The limiting distribution with density $f(\xi)$ belongs to the domain of attraction of a stable distribution, whose characteristic function ϕ is of the form

$$\phi(t) = \exp\left(\sqrt{-1}zt + c|t|^\alpha\left(1+\sqrt{-1}\gamma\mathrm{sgn}(t)\tan\left(\frac{\alpha\pi}{2}\right)\right)\right), \quad 0<\alpha \leqq 2, \alpha \neq 1,$$

or

$$\phi(t) = \exp\left(\sqrt{-1}zt + c|t|\left(1\sqrt{-1}\gamma\mathrm{sgn}(t)\log(|t|)\frac{2}{\pi}\right)\right),$$

where $c>0, -1<\gamma<1$ and $z \in \mathbf{R}$.

It is also the case if we consider the hitting distribution on $\{\mathrm{Im}(z)=a\}$, that is, the distribution of $X_{\tau_a}^{(\nu,\mu)}$ when $Y_0^{(\nu,\mu)} = y > a > 0$, where τ_a is the first hitting time of a by $\{Y_t^{(\nu,\mu)}\}$. For details, see the original paper [4].

4. Maximum of a diffusion process in random environment

The purpose of this section is to survey the work by Kawazu-Tanaka [35] on the maximum of a diffusion process in a drifted random environment. In [35], several equalities and inequalities for the exponential functionals of Brownian motion are used.

Let $W = \{W(y), y \in \mathbf{R}\}$ be a Brownian environment defined on a probability space $(\Omega_1, \mathcal{F}_1, P_1)$: $\{W(y), y \geqq 0\}$ and $\{W(-y), y \geqq 0\}$ are independent one-dimensional Brownian motions with $W(0)=0$. For $c \in \mathbf{R}$, we set $W^{(c)}(y) = W(y)+cy$.

For each $\omega_1 \in \Omega_1$, we consider a diffusion process $X(W(\omega_1)) = X(W) = \{X(t,W), t \geqq 0\}$, $W(\omega_1) = W(\cdot,\omega_1)$, with $X(0,W)=0$, whose infinitesimal generator is given by

$$\frac{1}{2}\exp(W^{(c)}(x))\frac{d}{dx}\left(\exp(-W^{(c)}(x))\frac{d}{dx}\right).$$

We denote by P_{ω_1} the probability law of the diffusion process $\{X(t,W(\omega_1))\}$ and consider the full probability law $\mathcal{P} = \int P_{\omega_1} P_1(d\omega_1)$ of $\{X(t,\cdot)\}$.

A scale function $S^{(c)}(x) = S^{(c)}_W(x)$ for $X(W)$ is given by

$$S^{(c)}(x) = \int_0^x \exp(W^{(c)}(y))dy \text{ for } x \geqq 0, \quad = -\int_x^0 \exp(W^{(c)}(y))dy \text{ for } x \leqq 0.$$

By the general theory of the one-dimensional diffusion processes, $\{S^{(c)}(X_t(W))\}$ may be represented as a random time change of another Brownian motion and, based on this representation, several interesting results have been obtained. For these results, see, e.g., Brox [8], Hu-Shi-Yor [30], Kawazu-Tanaka [36].

In the rest of this section, we assume $c > 0$. Then we have $S^{(c)}(\infty) = \infty$ and $S^{(c)}(-\infty) > -\infty$, and therefore, $\max_{t\geqq 0} X(t) < \infty$, $\mathcal{P}$-a.s.

The question we discuss in the present section is how the tail probability $\mathcal{P}(\max_{t\geqq 0} X(t) > x)$ decays as $x \to \infty$. We have

$$\mathcal{P}\left(\max_{t\geqq 0} X(t) > x\right) = E^{P_1}\left[\frac{-S^{(c)}(-\infty)}{S^{(c)}(x) - S^{(c)}(-\infty)}\right], \quad x > 0, \tag{4.1}$$

and several results on the exponential functional given by (1.1) are quite useful in this study. The random variables $S^{(c)}(x)$ and $S^{(c)}(-\infty)$ are independent. We also note (cf. (1.5)) that $-S^{(c)}(-\infty)$ is distributed as $2\gamma_{2c}^{-1}$, where γ_{2c} is a gamma random variable with parameter $2c$.

Theorem 4.1. (i) *If $c > 1$, then one has*

$$\mathcal{P}\left(\max_{t\geqq 0} X(t) > x\right) = \frac{2(c-1)}{2c-1} \exp\left(-\left(c - \frac{1}{2}\right)x\right)(1 + o(1)), \quad x \to \infty.$$

(ii) *If $c = 1$, then*

$$\mathcal{P}\left(\max_{t\geqq 0} X(t) > x\right) = \sqrt{\frac{2}{\pi}} x^{-1/2} e^{-x/2}(1 + o(1)), \quad x \to \infty.$$

(iii) *If $0 < c < 1$, then*

$$\mathcal{P}\left(\max_{t\geqq 0} X(t) > x\right) = C x^{-3/2} e^{-c^2 x/2}(1 + o(1)), \quad x \to \infty,$$

where the constant C is given by

$$C = \frac{2^{5/2-2c}}{\Gamma(2c)} \int_0^\infty \cdots \int_0^\infty \frac{za^{2c-1}e^{-a/2}}{a+z} y^{2c} e^{-\lambda(y,u)z} u \sinh(u)\, dadydzdu,$$
$$\lambda(y,u) = \frac{1+y^2}{2} + y\cosh(u).$$

Before proceeding to a proof for each assertion, we rewrite the right hand side of (4.1) into different forms. We set $A^{(c)} = -S^{(c)}(-\infty)$ and

$$f^{(c)}(a,x) = E^{P_1}[(a + S^{(c)}(x))^{-1}], \qquad a > 0, x > 0.$$

Then we have

$$\mathcal{P}\left(\max_{t\geqq 0} X(t) > x\right) = E[A^{(c)} f^{(c)}(A^{(c)}, x)] \tag{4.2}$$

$$= \frac{1}{\Gamma(2c)} \int_0^\infty \frac{2}{\xi} f^{(c)}\left(\frac{2}{\xi}, x\right) \xi^{2c-1} e^{-\xi} d\xi. \tag{4.3}$$

Moreover, considering the time reversal of W, we easily obtain

$$f^{(c)}(a,x) = e^{-(c-1/2)x} E^{P_1}\left[\left(a \exp(W^{(-c)}(x)) + \int_0^x \exp(W^{(-c)}(y))dy\right)^{-1} \times \exp(W(x) - x/2)\right].$$

By the Cameron-Martin theorem, we also obtain

$$f^{(c)}(a,x) = e^{-(c-1/2)x} E^{P_1}\left[\left(a \exp(W^{(1-c)}(x)) + \int_0^x \exp(W^{(1-c)}(y))dy\right)^{-1}\right]. \tag{4.4}$$

By considering the time reversal again, we may write

$$f^{(c)}(a,x) = e^{-(c-1/2)x} E^{P_1}\left[\left(a + \int_0^x \exp(W^{(c-1)}(y))dy\right)^{-1} \exp(W^{(c-1)}(x))\right].$$

Proof of (i). From (4.2) and (4.4), we have

$$e^{(c-1/2)x}\mathcal{P}\left(\max_{t\geqq 0} X(t) > x\right) = E^{P_1}\left[A^{(c)}\left(A^{(c)} \exp(W^{(1-c)}(x)) + \int_0^x \exp(W^{(1-c)}(y))dy\right)^{-1}\right]$$

and, by the independence of $A^{(c)}$ and $\{W^{(1-c)}(y), y \geqq 0\}$, we obtain

$$\lim_{x\to\infty} e^{(c-1/2)x}\mathcal{P}\left(\max_{t\geqq 0} X(t) > x\right) = E\left[\frac{2}{\gamma_{2c}}\right] E\left[\frac{\gamma_{2(c-1)}}{2}\right],$$

where γ_μ is a gamma random variable with parameter $\mu > 0$. Easy evaluation of the right hand side yields the assertion. □

Before proceeding to a proof of (ii), we prepare two lemmas.

Lemma 4.1. *Setting*

$$\psi(x) = E^{P_1}\left[\left(\int_0^x \exp(W(y))dy\right)^{-1} \exp(W(x))\right],$$

we have

$$\psi(x) = E^{P_1}\left[\left(\int_0^x \exp(W(y))dy\right)^{-1}\right]$$

and

$$\lim_{x\to\infty} \sqrt{2\pi x}\,\psi(x) = 1. \tag{4.5}$$

Proof. The first assertion is easily shown by time reversal.

We can show the second assertion from the identity

$$E^{P_1}\Big[\Big(\int_0^x \exp(2W(y))dy\Big)^{-1}\Big|W(x)=u\Big]=\frac{ue^{-u}}{x\sinh(u)},\quad x>0, u\in\mathbf{R},$$

which has been shown in Part I, Proposition 5.9. However, we give another direct proof.

Set

$$\varphi(x)=E^{P_1}\Big[\log\Big(\int_0^x \exp(W(y))dy\Big)\Big].$$

Then we have $\varphi'(x)=\psi(x)$ and, if we show

$$\lim_{x\to\infty}\frac{\varphi(x)}{\sqrt{x}}=\sqrt{\frac{2}{\pi}},$$

we obtain (4.5) by L'Hospital's theorem.

By the scaling property of Brownian motion, we have

$$\frac{\varphi(x)}{\sqrt{x}}=E^{P_1}\Big[\frac{1}{\sqrt{x}}\log\Big(\int_0^1 \exp(\sqrt{x}W(y))dy\Big)\Big]+\frac{1}{\sqrt{x}}\log(x).$$

and, by the Laplace principle, we also have

$$\lim_{x\to\infty}\frac{1}{\sqrt{x}}\log\Big(\int_0^1 \exp(\sqrt{x}W(y))dy\Big)=\max_{0\leqq y\leqq 1}W(y).$$

Hence, applying the dominated convergence theorem, we obtain

$$\lim_{x\to\infty}\frac{\varphi(x)}{\sqrt{x}}=E^{P_1}\Big[\max_{0\leqq y\leqq 1}W(y)\Big]=\sqrt{\frac{2}{\pi}}.$$

□

Lemma 4.2. *For all $x>0$, one has*

$$E^{P_1}\Big[\Big(\int_0^x \exp(W(y))dy\Big)^{-2}\exp(W(x))\Big]\leqq\Big\{\psi\Big(\frac{x}{2}\Big)\Big\}^{1/2}.$$

Proof. We write

$$\begin{aligned}
&E^{P_1}\Big[\Big(\int_0^x \exp(W(y))dy\Big)^{-2}\exp(W(x))\Big]\\
&\leqq E^{P_1}\Big[\Big(\int_0^{x/2} \exp(W(y))dy\Big)^{-1}\Big(\int_{x/2}^x \exp(W(y))dy\Big)^{-1}\exp(W(x))\Big]\\
&=E^{P_1}\Big[\Big(\int_0^{x/2} \exp(W(y))dy\Big)^{-1}\\
&\qquad\times\Big(\int_{x/2}^x \exp(W(y)-W(x/2))dy\Big)^{-1}\exp(W(x)-W(x/2))\Big].
\end{aligned}$$

Hence, using the independence of increments of Brownian motion, we obtain

$$\begin{aligned}
&E^{P_1}\left[\left(\int_0^x \exp(W(y))dy\right)^{-2}\exp(W(x))\right] \\
&\leqq E^{P_1}\left[\left(\int_0^{x/2} \exp(W(y))dy\right)^{-1}\right]E^{P_1}\left[\left(\int_0^{x/2} \exp(W(y))dy\right)^{-1}\exp(W(x/2))\right] \\
&= \{\psi(x/2)\}^2.
\end{aligned}$$

□

Proof of (ii). Set $A^{(1)} = -S^{(1)}(-\infty)$. Then, since $A^{(1)}$ and $B_x^{(1)}$ are independent, we have

$$\begin{aligned}
E^{P_1}\left[\frac{A^{(1)}}{S^{(1)}(x)}\right] &= E^{P_1}[A^{(1)}]E^{P_1}\left[\frac{1}{S^{(1)}(x)}\right] \\
&= E\left[\frac{2}{\gamma_2}\right]E^{P_1}\left[\left(\int_0^x \exp(W(y))dy\right)^{-1}\exp(W(x)-x/2)\right],
\end{aligned}$$

where γ_2 is a gamma variable with parameter 2 and we have used the Cameron-Martin theorem for the second equality. Therefore we obtain

$$\lim_{x\to\infty} x^{1/2}e^{x/2}E^{P_1}\left[\frac{A^{(1)}}{S^{(1)}(x)}\right] = 2\lim_{x\to\infty}\sqrt{x}\psi(x) = \sqrt{\frac{2}{\pi}} \tag{4.6}$$

from (4.5).

We next prove

$$E^{P_1}\left[\frac{A^{(1)}}{S^{(1)}(x)}\right] - E^{P_1}\left[\frac{A^{(1)}}{A^{(1)}+S^{(1)}(x)}\right] \leqq Cx^{-3/4}e^{-x/2} \tag{4.7}$$

for some absolute constant C. Combining this with (4.6) above, we obtain the assertion.

For this purpose we note the elementary inequality

$$0 \leqq \frac{a}{b} - \frac{a}{a+b} \leqq \frac{1}{2}\left(\frac{a}{b}\right)^{3/2}, \quad a,b>0.$$

Then we obtain

$$E^{P_1}\left[\frac{A^{(1)}}{S^{(1)}(x)}\right] - E^{P_1}\left[\frac{A^{(1)}}{A^{(1)}+S^{(1)}(x)}\right] \leqq \frac{1}{2}E^{P_1}[(A^{(1)})^{3/2}]E[(S^{(1)}(x))^{-3/2}].$$

For the first term on the right hand side, we have

$$E^{P_1}[(A^{(1)})^{3/2}] = 2^{3/2}\int_0^\infty x^{-3/2}xe^{-x}dx < \infty.$$

For the second term, we use the Cauchy-Schwarz inequality to show

$$
\begin{aligned}
&E^{P_1}[(S^{(1)}(x))^{-3/2}] = E^{P_1}\left[\left(\int_0^x \exp(W(y))dy\right)^{-3/2} \exp(W(x)-x/2)\right] \\
&\leqq e^{-x/2}\left\{E^{P_1}\left[\left(\int_0^x \exp(W(y))dy\right)^{-1} \exp(W(x))\right]\right\}^{1/2} \\
&\qquad \times \left\{E^{P_1}\left[\left(\int_0^x \exp(W(y))dy\right)^{-2} \exp(W(x))\right]\right\}^{1/2}.
\end{aligned}
$$

Then, using Lemmas 4.1 and 4.2, we obtain (4.7) and the result of (ii). □

Proof of (iii). We prove this case by using formula (1.2). To do this in a direct way, we note that

$$
S_x^{(c)} = \int_0^x \exp(W^{(c)}(y))dy \overset{(\text{law})}{=} 4\int_0^{x/4} \exp(2W^{(2c)}(y))dy,
$$

and that the latter is $4A_{x/4}^{(2c)}$, where $A_t^{(\mu)}$ is defined by (1.1).

Then, by using (4.1) and (1.2), we obtain

$$
\begin{aligned}
&\mathcal{P}\left(\max_{t\geqq 0} X(t) > x\right) = E^{P_1}[A^{(c)} f^{(c)}(A^{(c)}, x)] \\
&= \int_0^\infty \frac{2}{a}\frac{1}{\Gamma(2c)} a^{2c-1}e^{-a}da \cdot e^{-c^2x/2} \int_0^\infty du \int_{\mathbf{R}} d\xi \int_0^\infty db \\
&\quad \times e^{2c\xi}u^{-1}\exp\left(-\frac{1+e^{2\xi}}{2u}\right)\left(\frac{2}{a}+4u\right)^{-1} \\
&\quad \times \sqrt{\frac{2}{\pi^3 x}}\frac{e^\xi}{u} e^{(2\pi^2-2b^2)/x} \exp\left(-\frac{e^\xi \cosh(b)}{u}\right)\sinh(b)\sin\left(\frac{4\pi\xi}{x}\right).
\end{aligned}
$$

From this identity we see that the order of decay is $x^{-3/2}e^{-c^2x/2}$ and, by using the dominated convergence theorem and changing the variables in the integration, we obtain the assertion. For details, see the original paper [35]. □

5. Exponential functionals with different drifts

The purpose of this section is to show a relationship between the laws of the exponential functionals of Brownian motions with different drifts.

In this and the next sections, we consider several stochastic processes or transforms on path space related to the exponential functional $\{A_t^{(\mu)}\}$. In particular, the following transform Z plays an important role. For a continuous function $\phi : [0,\infty) \to \mathbf{R}$, we associate $A(\phi) = \{A_t(\phi)\}$ and $Z(\phi) = \{Z_t(\phi)\}$ defined by

$$
A_t(\phi) = \int_0^t \exp(2\phi(s))ds \quad \text{and} \quad Z_t(\phi) = \exp(-\phi(t))A_t(\phi). \tag{5.1}
$$

Let $\nu < \mu$ and consider two exponential functionals $A^{(\nu)} = A(B^{(\nu)})$ and $A^{(\mu)} = A(B^{(\mu)})$:

$$A_t^{(\nu)} = \int_0^t \exp(2B_s^{(\nu)})ds \quad \text{and} \quad A_t^{(\mu)} = \int_0^t \exp(2B_s^{(\mu)})ds,$$

where $B_s^{(\nu)} = B_s + \nu s$ and $B = \{B_s\}$ is a one-dimensional Brownian motion with $B_0 = 0$ as in the previous sections.

We first consider the case where $\nu = -\mu$ and $\mu > 0$ and, using the result in this special case, we will give the general result in Theorem 5.4 below.

Theorem 5.1. *Let $\mu > 0$ and let γ_μ be a gamma random variable with density $\Gamma(\mu)^{-1}x^{\mu-1}e^{-x}$, $x > 0$, independent of B. Then one has the identity in law*

$$\left\{\frac{1}{A_t^{(-\mu)}}, t > 0\right\} \stackrel{\text{(law)}}{=} \left\{\frac{1}{A_t^{(\mu)}} + 2\gamma_\mu, t > 0\right\}. \tag{5.2}$$

The identity in law for a fixed $t > 0$ has been obtained by Dufresne [19] and the extension (5.2) at the process level has been given in [47].

Proof. We sketch a proof based on the theory of initial enlargements of filtrations (cf. Yor [59]). Another proof based on some properties of Bessel processes has been given in [47].

Let $\mathcal{B}_t^{(-\mu)} = \sigma\{B_s^{(-\mu)}, s \leqq t\}$ and $\widehat{\mathcal{B}}_t^{(-\mu)} = \mathcal{B}_t^{(-\mu)} \vee \sigma\{A_\infty^{(-\mu)}\}$. Then we can show that there exists a $\{\widehat{\mathcal{B}}_t^{(-\mu)}\}$-Brownian motion $\{\widehat{B}_t\}$ independent of $A_\infty^{(-\mu)}$ such that

$$B_t^{(-\mu)} = \widehat{B}_t + \mu t - \int_0^t \frac{\exp(2B_s^{(-\mu)})}{A_\infty^{(-\mu)} - A_s^{(-\mu)}}ds. \tag{5.3}$$

We regard this identity as an equation for $\{B_s^{(-\mu)}\}$ with given initial data $\{\widehat{B}_t\}$ and $A_\infty^{(-\mu)}$. Then, solving (5.3), we obtain

$$B_t^{(-\mu)} = \widehat{B}_t^{(\mu)} - \log\left(1 + \frac{\widehat{A}_t^{(\mu)}}{A_\infty^{(-\mu)}}\right),$$

where $\widehat{B}_t^{(\mu)} = \widehat{B}_t + \mu t$ and $\{\widehat{A}_t^{(\mu)}\} = \{A_t(\widehat{B}^{(\mu)})\}$.

As a consequence, it follows that

$$A_t^{(-\mu)} = \int_0^t \left(1 + \frac{\widehat{A}_t^{(\mu)}}{A_\infty^{(-\mu)}}\right)^{-2} \exp(2\widehat{B}_s^{(\mu)})ds = \frac{\widehat{A}_t^{(\mu)}}{1 + \widehat{A}_t^{(\mu)}/A_\infty^{(-\mu)}},$$

hence

$$\frac{1}{A_t^{(-\mu)}} = \frac{1}{\widehat{A}_t^{(\mu)}} + \frac{1}{A_\infty^{(-\mu)}}.$$

Since $(A_\infty^{(-\mu)})^{-1} \stackrel{\text{(law)}}{=} 2\gamma_\mu$ by (1.5) and $\{\widehat{B}_t^{(\mu)}\}$ is independent of $A_\infty^{(-\mu)}$, we obtain the theorem. ∎

Next we consider the stochastic processes $Z^{(-\mu)} = Z(B^{(-\mu)})$ and $\widehat{Z}^{(\mu)} = Z(\widehat{B}^{(\mu)})$:

$$Z_t^{(-\mu)} = \exp(-B_t^{(-\mu)})A_t^{(-\mu)} \quad \text{and} \quad \widehat{Z}_t^{(\mu)} = \exp(-\widehat{B}_t^{(\mu)})\widehat{A}_t^{(\mu)}.$$

Then we have

$$\frac{d}{dt}\left(\frac{1}{A_t^{(-\mu)}}\right) = \frac{1}{(Z_t^{(-\mu)})^2}. \tag{5.4}$$

Hence, from the identity (5.2), we obtain $Z(B^{(-\mu)}) \overset{\text{(law)}}{=} Z(B^{(\mu)})$ for any $\mu > 0$. Moreover, by (5.3), we have the pathwise identity $Z_t^{(-\mu)} = \widehat{Z}_t^{(\mu)}$, $t \geqq 0$.

The study of the stochastic process $Z(B^{(\mu)})$ is in fact the main object of the next section. We will show that, for any $\mu \in \mathbf{R}$, $Z(B^{(\mu)}) = \{Z_t^{(\mu)}\}$ is a diffusion process with respect to its natural filtration $\{\mathcal{Z}_t^{(\mu)}\}$, $\mathcal{Z}_t^{(\mu)} = \sigma\{Z_s^{(\mu)}, s \leqq t\}$, and that this result gives rise to an extension of Pitman's theorem ([51],[54]).

A key fact in the proof of the above mentioned result is the following Proposition 5.2, which also plays an important role in the rest of this section.

Before mentioning the proposition, we note another important fact. By (5.4), we easily obtain the following: for every $t > 0$,

$$\mathcal{B}_t^{(\mu)} = \sigma\{A_s^{(\mu)}, s \leqq t\} = \mathcal{Z}_t^{(\mu)} \vee \sigma\{A_t^{(\mu)}\}. \tag{5.5}$$

In particular, $\mathcal{Z}_t^{(\mu)}$ is strictly smaller than the original filtration $\mathcal{B}_t^{(\mu)}$ of the Brownian motion B, as is also shown very clearly in the next proposition.

Proposition 5.2. *Let $\mu \in \mathbf{R}$. Then, for any $t > 0$, the conditional distribution of $\exp(B_t^{(\mu)})$ given $\mathcal{Z}_t^{(\mu)}$ is a generalized inverse Gaussian distribution and is given by*

$$P(\exp(B_t^{(\mu)}) \in dx | \mathcal{Z}_t^{(\mu)}) = \frac{1}{2K_\mu(1/z)} x^{\mu-1} \exp\left(-\frac{1}{2z}\left(x + \frac{1}{x}\right)\right)dx, \;\; x > 0, \tag{5.6}$$

where $z = Z_t^{(\mu)}$ and K_μ is the modified Bessel (Macdonald) function.

Corollary 5.3. *Let $\mu = 0$ and write B_t, $\mathcal{Z}_t$ for $B_t^{(0)}$, $\mathcal{Z}^{(0)}$, respectively. Then,*
(i) *the conditional distributions of* $\exp(B_t)$ *and* $\exp(-B_t)$ *given $\mathcal{Z}_t$ are identical*;
(ii) *letting γ_δ be a gamma random variable independent of $\{B_t\}$, one has*

$$E[f(e^{B_t})e^{\delta B_t} | \mathcal{Z}_t] = E[f(e^{B_t} + 2z\gamma_\delta)e^{-\delta B_t} | \mathcal{Z}_t]. \tag{5.7}$$

Proof of the corollary. The first assertion is easily obtained.

For the second assertion, we consider random variables $I_z^{(\pm\delta)}$ whose densities are given by the right hand side of (5.6), replacing $\pm\delta$ for μ. Then, assuming that $I_z^{(-\delta)}$ and γ_z are independent, we have $I_z^{(\delta)} \overset{\text{(law)}}{=} I_z^{(-\delta)} + 2z\gamma_\delta$. In fact, more general identities in law for generalized inverse Gaussian and gamma random variables are well known. See Seshadri [56], [46] and the references therein.

Hence, from the identity (5.6) considered for $\mu = 0$, we obtain

$$\begin{aligned} E[f(e^{B_t})e^{\delta B_t}|\mathcal{Z}_t] &= \frac{K_\delta(1/z)}{K_0(1/z)} E[f(I_z^{(\delta)})] \\ &= \frac{K_\delta(1/z)}{K_0(1/z)} E[f(I_z^{(-\delta)} + 2z\gamma_\delta)] \\ &= E[f(e^{B_t} + 2z\gamma_\delta)e^{-\delta B_t}|\mathcal{Z}_t]. \end{aligned}$$

□

We postpone a proof of Proposition 5.2 to the next section and, admitting this proposition as proven, we show a general relationship between the probability laws of the exponential functionals of Brownian motions with different drifts.

Theorem 5.4. *Let $\nu < \mu$ and set $\delta = (\mu - \nu)/2$ and $m = (\mu + \nu)/2$. Then, for every $t \geqq 0$ and for every non-negative functional F on $C([0, t] \to \mathbf{R})$, one has*

$$\begin{aligned} &e^{\nu^2 t/2} E\left[F\left(\frac{1}{A_s^{(\nu)}}, s \leqq t\right)\left(\frac{1}{A_t^{(\nu)}}\right)^m\right] \\ &\qquad = e^{\mu^2 t/2} E\left[F\left(\frac{1}{A_s^{(\mu)}} + 2\gamma_\delta, s \leqq t\right)\left(\frac{1}{A_t^{(\mu)}}\right)^m\right], \end{aligned} \tag{5.8}$$

where γ_δ is a gamma random variable with parameter δ independent of $\{B_s^{(\mu)}\}$.

Proof. We start from (5.7). Then, for any non-negative function ψ on $\mathbf{R}_+$, we have

$$E\left[\psi\left(\frac{e^{B_t}}{z}\right)e^{\delta B_t}\middle|\mathcal{Z}_t\right] = E\left[\psi\left(\frac{e^{B_t}}{z} + 2\gamma_\delta\right)e^{-\delta B_t}\middle|\mathcal{Z}_t\right].$$

We also deduce from the first assertion of the corollary

$$E\left[\psi\left(\frac{e^{-B_t}}{z}\right)e^{-\delta B_t}\middle|\mathcal{Z}_t\right] = E\left[\psi\left(\frac{e^{-B_t}}{z} + 2\gamma_\delta\right)e^{\delta B_t}\middle|\mathcal{Z}_t\right]$$

or, since $Z_t = \exp(-B_t)A_t$,

$$E\left[\psi\left(\frac{1}{A_t}\right)\left(\frac{z}{A_t}\right)^{\delta}\middle|\mathcal{Z}_t\right] = E\left[\psi\left(\frac{1}{A_t} + 2\gamma_\delta\right)\left(\frac{z}{A_t}\right)^{-\delta}\middle|\mathcal{Z}_t\right].$$

Multiplying by $(Z_t)^{-\nu}$ on both hand sides, we rewrite the last identity into

$$E\left[\psi\left(\frac{1}{A_t}\right)\left(\frac{1}{A_t}\right)^m e^{\nu B_t}\middle|\mathcal{Z}_t\right] = E\left[\psi\left(\frac{1}{A_t} + 2\gamma_\delta\right)\left(\frac{1}{A_t}\right)^m e^{\mu B_t}\middle|\mathcal{Z}_t\right].$$

Then we obtain, for any non-negative functional $\widetilde{F}$,

$$E\left[\widetilde{F}(Z_s, s \leqq t)\psi\left(\frac{1}{A_t}\right)\frac{\exp(\nu B_t)}{(A_t)^m}\right] = E\left[\widetilde{F}(Z_s, s \leqq t)\psi\left(\frac{1}{A_t} + 2\gamma_\delta\right)\frac{\exp(\mu B_t)}{(A_t)^m}\right].$$

By the Cameron-Martin theorem, we now obtain

$$\begin{aligned} & e^{\nu^2 t/2} E\Big[\widetilde{F}(Z_s^{(\nu)}, s \leqq t)\psi\Big(\frac{1}{A_t^{(\nu)}}\Big)\Big(\frac{1}{A_t^{(\nu)}}\Big)^m\Big] \\ & \qquad = e^{\mu^2 t/2} E\Big[\widetilde{F}(Z_s^{(\mu)}, s \leqq t)\psi\Big(\frac{1}{A_t^{(\mu)}} + 2\gamma_\delta\Big)\Big(\frac{1}{A_t^{(\mu)}}\Big)^m\Big]. \end{aligned}$$

This identity is equivalent to (5.8) because of (5.4) or (5.5). □

Let $f^{(\mu)}(a,t)$ be the density of $(2A_t^{(\mu)})^{-1}$. Then the following "recursion" formula, originally due to Dufresne [20], is deduced from (5.8).

Theorem 5.5. *Let $\nu < \mu$. Then, for any $t > 0$, one has, with $\delta = (\mu - \nu)/2$,*

$$e^{\nu^2 t/2} f^{(\nu)}(a,t) = e^{\mu^2 t/2} \frac{a^{-m} e^{-a}}{\Gamma(\delta)} \int_0^a (a-b)^{\delta-1} b^m e^b f^{(\mu)}(b,t) db, \quad a > 0.$$

6. Some exponential analogues of Lévy's and Pitman's theorems

In this section we consider the two stochastic processes $\xi^{(\mu)} = \{\xi_t^{(\mu)}\}$ and $Z^{(\mu)} = \{Z_t^{(\mu)}\}$ defined by

$$\xi_t^{(\mu)} = \exp(-2B_t^{(\mu)}) A_t^{(\mu)} = \exp(-2B_t^{(\mu)}) \int_0^t \exp(2B_s^{(\mu)}) ds$$

and

$$Z_t^{(\mu)} = \exp(-B_t^{(\mu)}) A_t^{(\mu)},$$

where $B_t^{(\mu)} = B_t + \mu t$ and $B = \{B_t\}$ is a one-dimensional Brownian motion starting from 0.

Our purpose here is to show that both $\xi^{(\mu)}$ and $Z^{(\mu)}$ are diffusion processes, that is, they give representations for some diffusion processes starting from 0, with respect to their natural filtrations and that this result gives rise to analogues or extensions of the celebrated Lévy and Pitman theorems.

We start by recalling these classical theorems. Set

$$M_t^{(\mu)} = \max_{0 \leqq s \leqq t} B_s^{(\mu)}.$$

Then the Lévy and Pitman theorems may be stated in the following general form with any $\mu \in \mathbf{R}$.

Theorem 6.1. (i) *Let $X^{(\mu)} = \{X_t^{(\mu)}\}$ be the bang-bang process with $X_0^{(\mu)} = 0$ and with parameter μ, that is, the diffusion process with infinitesimal generator $\frac{1}{2}\frac{d^2}{dx^2} - \mu \mathrm{sgn}(x)\frac{d}{dx}$ and let $\{\ell_t^{(\mu)}\}$ be the local time of $X^{(\mu)}$ at 0. Then, the following identity in law holds :*

$$\{(M_t^{(\mu)} - B_t^{(\mu)}, M_t^{(\mu)}), t \geqq 0\} \overset{(\mathrm{law})}{=} \{(|X_t^{(\mu)}|, \ell_t^{(\mu)}), t \geqq 0\}.$$

(ii) $\sigma\{M_s^{(\mu)} - B_s^{(\mu)}, s \leqq t\} = \mathcal{B}_t^{(\mu)} \equiv \sigma\{B_s^{(\mu)}, s \leqq t\}.$

Theorem 6.2. (i) *Let* $\{\rho_t^{(\mu)}\}$ *be the diffusion process starting from* 0 *with infinitesimal generator*

$$\frac{1}{2}\frac{d^2}{dx^2}+\mu\coth(\mu x)\frac{d}{dx}$$

and set $j_t^{(\mu)}=\inf_{s\geqq t}\rho_s^{(\mu)}$. *Then, the following identity in law holds :*

$$\{(2M_t^{(\mu)}-B_t^{(\mu)},M_t^{(\mu)}),t\geqq 0\}\overset{\text{(law)}}{=}\{(\rho_t^{(\mu)},j_t^{(\mu)}),t\geqq 0\}.$$

(ii) $\sigma\{2M_s^{(\mu)}-B_s^{(\mu)},s\leqq t\}\subsetneqq\mathcal{B}_t^{(\mu)}$.
(iii) *As a consequence of* (i), *the diffusion processes* $\{2M_t^{(-\mu)}-B_t^{(-\mu)}\}$ *and* $\{2M_t^{(\mu)}-B_t^{(\mu)}\}$ *have the same probability law.*

When $\mu=0$, $\{|X_t^{(0)}|\}$ and $\{\rho_t^{(0)}\}$ are respectively a reflecting Brownian motion and a three-dimensional Bessel process, and the theorems give their representations in terms of the maximum process $\{M_t^{(0)}\}$ of a Brownian motion B. For the proofs and related references, see [44], [46], [53] *et al.* It should be noted that the stochastic processes $\{kM_t^{(\mu)}-B_t^{(\mu)}\}$, $k\in\mathbf{R}$, are not Markovian except for these two interesting cases $k=1,2$ and the trivial case $k=0$. For a rigorous and detailed proof, see [42].

Now, for $\lambda>0$, we set

$$M_t^{(\mu),\lambda}=\frac{1}{2\lambda}\log\biggl(\int_0^t\exp(2\lambda B_s^{(\mu)})ds\biggr).$$

Then, the Laplace principle implies

$$\lim_{\lambda\to\infty}M_t^{(\mu),\lambda}=M_t^{(\mu)},$$

and, by the scaling property of Brownian motion, we deduce

$$\{M_t^{(\mu),\lambda},t>0\}\overset{\text{(law)}}{=}\biggl\{\frac{1}{\lambda}M_{\lambda^2 t}^{(\mu/\lambda),1}-\frac{1}{2\lambda}\log\lambda^2,t>0\biggr\}.$$

Moreover, we have

$$\log(\xi_t^{(\mu)})=\log\biggl(\int_0^t\exp(2B_s^{(\mu)})ds\biggr)-2B_t^{(\mu)}\equiv 2(M_t^{(\mu),1}-B_t^{(\mu)})$$

and

$$\log(Z_t^{(\mu)})=\log\biggl(\int_0^t\exp(2B_s^{(\mu)})ds\biggr)-B_t^{(\mu)}\equiv 2M_t^{(\mu),1}-B_t^{(\mu)}.$$

Hence, if we show that $\xi^{(\mu)}$ and $Z^{(\mu)}$ are diffusion processes for every $\mu\in\mathbf{R}$, then we see that $\{M_t^{(\mu),\lambda}-B_t^{(\mu)}\}$ and $\{2M_t^{(\mu),\lambda}-B_t^{(\mu)}\}$ are also diffusion processes for every $\lambda > 0$ and we can recover the Lévy and Pitman theorems as the limiting cases by letting $\lambda \to \infty$.

Furthermore, there is an exponential analogue of the second assertion of Theorem 6.2 since we have shown in the previous section (see (5.5)) that $\mathcal{Z}_t^{(\mu)} \equiv \sigma\{Z_s^{(\mu)}, s \leqq t\} \subsetneqq \mathcal{B}_t^{(\mu)}$ holds for every $t > 0$.

We can easily show that $\xi^{(\mu)}$ is a diffusion process. In fact, from the Itô formula, we deduce

$$d\xi_t^{(\mu)} = -2\xi_t^{(\mu)} dB_t + ((2-2\mu)\xi_t^{(\mu)} + 1)dt,$$

which implies the following.

Theorem 6.3. *Let $\mu \in \mathbf{R}$.*
(i) *$\xi^{(\mu)}$ is a diffusion process with respect to the natural filtration $\{\mathcal{B}_t^{(\mu)}\}$ of $B^{(\mu)}$ and its infinitesimal generator is given by*

$$2x^2 \frac{d^2}{dx^2} + ((2-2\mu)x + 1)\frac{d}{dx}.$$

(ii) *For every fixed $t > 0$, one has*

$$\xi_t^{(\mu)} \overset{\text{(law)}}{=} \int_0^t \exp(-2B_s^{(\mu)})ds \overset{\text{(law)}}{=} \int_0^t \exp(2B_s^{(-\mu)})ds.$$

On the other hand, it is not as easy to show that $Z^{(\mu)}$ is a diffusion process. By the Itô formula, we have

$$Z_t^{(\mu)} = -\int_0^t Z_s^{(\mu)} dB_s + \int_0^t \left(\frac{1}{2} - \mu\right) Z_s^{(\mu)} ds + \int_0^t \exp(B_s^{(\mu)})ds \tag{6.1}$$

and we need to take care of the third term on the right hand side.

Here we recall Proposition 5.2, which implies

$$E[\exp(B_s^{(\mu)})|\mathcal{Z}_s^{(\mu)}] = \left(\frac{K_{1+\mu}}{K_\mu}\right)\left(\frac{1}{z}\right), \qquad z = Z_s^{(\mu)},$$

by the integral representation for K_μ (cf. [38], p.119)

$$K_\mu(y) = \frac{1}{2}\left(\frac{y}{2}\right)^\mu \int_0^\infty e^{-t-y^2/4t} t^{-\mu-1} dt. \tag{6.2}$$

Then, by replacing in (6.1) $\exp(B_s^{(\mu)})$ by its projection on $\mathcal{Z}_s^{(\mu)}$ (cf., e.g., Liptser-Shiryaev [39], Theorem 7.17), we see that there exists a $\{\mathcal{Z}_t^{(\mu)}\}$-Brownian motion $\{\beta_t\}$ such that

$$Z_t^{(\mu)} = \int_0^t Z_s^{(\mu)} d\beta_s + \left(\frac{1}{2} - \mu\right) \int_0^t Z_s^{(\mu)} ds + \int_0^t \left(\frac{K_{1+\mu}}{K_\mu}\right)\left(\frac{1}{Z_s^{(\mu)}}\right) ds.$$

Hence, admitting Proposition 5.2 as proven, we have obtained the following.

Theorem 6.4. *Let* $\mu \in \mathbf{R}$.
(i) $Z^{(\mu)}$ *is a diffusion process on* $[0,\infty)$ *with respect to its natural filtration* $\{\mathcal{Z}_t^{(\mu)}\}$ *whose infinitesimal generator is given by*

$$\frac{1}{2}z^2\frac{d^2}{dz^2}+\left\{\left(\frac{1}{2}-\mu\right)z+\left(\frac{K_{1+\mu}}{K_\mu}\right)\left(\frac{1}{z}\right)\right\}\frac{d}{dz}.$$

(ii) *For every* $t>0$, $\mathcal{Z}_t^{(\mu)} \subsetneqq \mathcal{B}_t^{(\mu)}$.
(iii) *The diffusion processes* $Z^{(-\mu)}$ *and* $Z^{(\mu)}$ *have the same probability law.*

To compare with the original Lévy and Pitman theorems, we present the following, which can be obtained from Theorems 6.3 and 6.4 by the scaling property of Brownian motion. We call a diffusion process a Brownian motion with drift b if its generator is given by $\frac{1}{2}\frac{d^2}{dx^2}+b(x)\frac{d}{dx}$.

Theorem 6.5. (i) *For any* $\lambda>0$, *the stochastic process*

$$\left\{\frac{1}{2\lambda}\log\left(\int_0^t \exp(2\lambda B_s^{(\mu)})ds\right)-B_t^{(\mu)}+\frac{1}{2\lambda}\log\lambda^2, t>0\right\}$$

is a Brownian motion with drift

$$c^{(\mu),\lambda}(x)=-\mu+\frac{1}{2}\lambda e^{-2\lambda x}.$$

(ii) *The stochastic process*

$$\left\{\frac{1}{\lambda}\log\left(\int_0^t \exp(2\lambda B_s^{(\mu)})ds\right)-B_t^{(\mu)}+\lambda\log\lambda^2, t>0\right\}$$

is a Brownian motion with drift

$$b^{(\mu),\lambda}(x)=-\mu+\lambda e^{-\lambda x}\left(\frac{K_{1+\mu/\lambda}}{K_{\mu/\lambda}}\right)(e^{-\lambda x})=\frac{d}{dx}\big(\log K_{\mu/\lambda}(e^{-\lambda x})\big).$$

Remark 6.1. By using the integral representation for $K_\mu(x)$

$$K_\mu(x)=\frac{2^\mu\Gamma(\mu+1/2)}{x^\mu\sqrt{\pi}}\int_0^\infty \frac{\cos(xt)}{(1+t^2)^{\mu+1/2}}dt, \quad x>0, \mu>0,$$

(cf. [38], p.140), we can show

$$\lim_{\lambda\to\infty} b^{(\mu),\lambda}(x)=\mu\coth(\mu x).$$

The rest of this section is devoted to a proof of Proposition 5.2, which has played an important role not only in this section but also in the previous section and in Part I of our survey. To show the proposition, we prove an identity for an anticipative transform on path space, which may be regarded as an example of the Ramer-Kusuoka formula. For the Ramer-Kusuoka formula, see [9], [37], [52] and [58].

Another proof for the proposition which uses several properties of Bessel processes has been given in [45] and a proof based on Theorem 5.1, featuring the generalized Gaussian inverse distributions, has been given in [46].

For our purpose we consider one more transform on path space. For an **R**-valued continuous function ϕ on $[0,\infty)$, we define $T_\alpha(\phi) = \{T_\alpha(\phi)_t, t \geqq 0\}$, $\alpha \geqq 0$, by

$$T_\alpha(\phi)_t = \phi(t) - \log(1+\alpha A_t(\phi)). \tag{6.3}$$

We now summarize some properties of these transforms on path space which are easy to prove but play important roles in the following.

Proposition 6.6. *Letting A and Z be the transforms on path space defined by* (5.1) *and T be defined by* (6.3), *one obtains*

$$\text{(i)}\frac{1}{A_t(T_\alpha(\phi))} = \frac{1}{A_t(\phi)} + \alpha, \qquad \text{(ii)} Z\circ T_\alpha = Z,$$
$$\text{(iii)} T_\alpha \circ T_\beta = T_{\alpha+\beta}, \quad \alpha,\beta > 0.$$

The next theorem gives an example of the Ramer-Kusuoka formula. On the left hand side of (6.4) below, the transform $T_{\alpha/e_t^{(\mu)}}$ depends on $e_t^{(\mu)}$ and it is natural to call it anticipative. For more discussions about this transform, related topics and references, see [17].

Theorem 6.7. *Let $\mu \in \mathbf{R}, \alpha \geqq 0$ and let F be a non-negative functional on the path space $C([0,t] \to \mathbf{R})$. Then, setting $e_t^{(\mu)} = \exp(B_t^{(\mu)})$, one has*

$$E[F(T_{\alpha/e_t^{(\mu)}}(B^{(\mu)})_s, s \leqq t)] = E[F(B_s^{(\mu)}, s \leqq t)\Gamma_\alpha^{(\mu)}(e_t^{(\mu)}, Z_t^{(\mu)})] \tag{6.4}$$

for every $t > 0$, where $\Gamma_\alpha^{(\mu)}(x,z) = (1+\alpha z)^\mu \Gamma_\alpha(x,z)$ and

$$\Gamma_\alpha(x,z) = \exp\biggl(-\frac{\alpha}{2}\biggl(x - \frac{1}{(1+\alpha z)x}\biggr)\biggr).$$

Proof. We divide our proof into three steps.
Step 1. First, we prove that

$$E\biggl[\exp(-\eta e_t)F\biggl(\frac{1}{A_s}, s \leqq t\biggr)\biggr] = E\biggl[\exp(-\eta/e_t)F\biggl(\frac{1}{A_s} + 2\eta/e_t, s \leqq t\biggr)\biggr] \tag{6.5}$$

holds for every $\eta > 0$, where $e_t = e_t^{(0)}$. From Proposition 6.6, we see that this identity is equivalent to

$$E[\exp(-\eta e_t)F(B_s, s \leqq t)] = E[\exp(-\eta/e_t)F(T_{2\eta/e_t}(B)_s, s \leqq t)]. \tag{6.6}$$

Step 2. We prove (6.4) with $\mu = 0$ from (6.6).
Step 3. We prove (6.4) for general values of μ.

We start from Theorem 5.1, which says that

$$E\biggl[F\biggl(\frac{1}{A_s^{(\mu)}} + 2\gamma_\mu, s \leqq t\biggr)\biggr] = E\biggl[F\biggl(\frac{1}{A_s^{(-\mu)}}, s \leqq t\biggr)\biggr] \tag{6.7}$$

holds for any $\mu > 0$ and for any non-negative functional F on $C([0,t] \to \mathbf{R})$, where γ_μ is a gamma random variable with parameter μ independent of $B^{(\mu)}$.

We rewrite the left hand side of (6.7) into the following way:

$$
\begin{aligned}
&E\Big[F\Big(\frac{1}{A_s^{(\mu)}} + 2\gamma_\mu, s \leqq t\Big)\Big] \\
&= \frac{1}{\Gamma(\mu)} \int_0^\infty \eta^{\mu-1} e^{-\eta} E\Big[F\Big(\frac{1}{A_s^{(\mu)}} + 2\eta, s \leqq t\Big)\Big] d\eta \\
&= \frac{1}{\Gamma(\mu)} \int_0^\infty \eta^{\mu-1} e^{-\eta} E\Big[F\Big(\frac{1}{A_s} + 2\eta, s \leqq t\Big) \exp(\mu B_t - \mu^2 t/2)\Big] d\eta, \\
&= \frac{e^{-\mu^2 t/2}}{\Gamma(\mu)} \int_0^\infty e^{-\eta} E\Big[F\Big(\frac{1}{A_s} + 2\eta, s \leqq t\Big)(\eta e_t)^\mu\Big] \frac{d\eta}{\eta},
\end{aligned} \tag{6.8}
$$

where we have used the Cameron-Martin theorem for the second identity. For the right hand side of (6.7), we rewrite

$$
\begin{aligned}
&E\Big[F\Big(\frac{1}{A_s^{(-\mu)}}, s \leqq t\Big)\Big] \\
&= E\Big[F\Big(\frac{1}{A_s}, s \leqq t\Big) \exp(-\mu B_t - \mu^2 t/2)\Big] \\
&= \frac{e^{-\mu^2 t/2}}{\Gamma(\mu)} \int_0^\infty \eta^{\mu-1} e^{-\eta} E\Big[F\Big(\frac{1}{A_s}, s \leqq t\Big) \exp(-\mu B_t)\Big] d\eta \\
&= \frac{e^{-\mu^2 t/2}}{\Gamma(\mu)} \int_0^\infty e^{-\eta} E\Big[F\Big(\frac{1}{A_s}, s \leqq t\Big)(\eta/e_t)^\mu\Big] \frac{d\eta}{\eta}.
\end{aligned} \tag{6.9}
$$

Now, comparing (6.8) and (6.9), we get

$$
\begin{aligned}
&\int_0^\infty e^{-\eta} E\Big[F\Big(\frac{1}{A_s} + 2\eta, s \leqq t\Big)(\eta e_t)^\mu\Big] \frac{d\eta}{\eta} \\
&\qquad = \int_0^\infty e^{-\eta} E\Big[F\Big(\frac{1}{A_s}, s \leqq t\Big)(\eta/e_t)^\mu\Big] \frac{d\eta}{\eta}.
\end{aligned}
$$

Since this identity holds for any $\mu > 0$, we obtain

$$
\begin{aligned}
&\int_0^\infty e^{-\eta} E\Big[F\Big(\frac{1}{A_s} + 2\eta, s \leqq t\Big) f(\eta e_t)\Big] \frac{d\eta}{\eta} \\
&\qquad = \int_0^\infty e^{-\eta} E\Big[F\Big(\frac{1}{A_s}, s \leqq t\Big) f(\eta/e_t)\Big] \frac{d\eta}{\eta}
\end{aligned}
$$

for any non-negative Borel function f on $[0,\infty)$. Since the left hand side is equal to

$$
\int_0^\infty E[\exp(-\eta/e_t) F\Big(\frac{1}{A_s} + 2\eta/e_t, s \leqq t\Big)\Big] f(\eta) \frac{d\eta}{\eta}
$$

and the right hand side is

$$
\int_0^\infty E[\exp(-\eta e_t) F\Big(\frac{1}{A_s}, s \leqq t\Big)\Big] f(\eta) \frac{d\eta}{\eta},
$$

we obtain (6.5).

For Step 2, we replace $F\exp(\eta e_t)$ by F in (6.6). Then, since $\exp(T_\alpha(B)_s) = e_s/(1+\alpha A_s)$, we obtain

$$E[F(B_s, s \leqq t)] = E\bigg[\exp(-\eta/e_t)\exp\bigg(\frac{\eta e_t}{1+2\eta A_t/e_t}\bigg)F(T_{2\eta/e_t}(B)_s, s \leqq t)\bigg]$$

and, setting $\eta = \alpha/2$,

$$E[F(B_s, s \leqq t)] = E\bigg[\exp\bigg(\frac{\alpha}{2}\bigg(\frac{e_t}{1+\alpha Z_t} - \frac{1}{e_t}\bigg)\bigg)F(T_{\alpha/e_t}(B)_s, s \leqq t)\bigg]. \quad (6.10)$$

We now wish to find a function $\varphi : \mathbf{R} \times [0,\infty) \to \mathbf{R}$ such that

$$E[\varphi(B_t, A_t)F(B_s, s \leqq t)] = E[F(T_{\alpha/e_t}(B)_s, s \leqq t)].$$

With this aim in mind, we replace F by φF in (6.10). Then we have

$$\begin{aligned} &E[\varphi(B_t, A_t)F(B_s, s \leqq t)] \\ &= E\bigg[\exp\bigg(\frac{\alpha}{2}\bigg(\frac{e_t}{1+\alpha Z_t} - \frac{1}{e_t}\bigg)\bigg)\varphi\bigg(T_{\alpha/e_t}(B)_t, \frac{A_t}{1+\alpha Z_t}\bigg)F(T_{\alpha/e_t}(B)_s, s \leqq t)\bigg]. \end{aligned}$$

Hence, for our purpose, it is sufficient that

$$\exp\bigg(\frac{\alpha}{2}\bigg(\frac{e_t}{1+\alpha Z_t} - \frac{1}{e_t}\bigg)\bigg)\varphi\bigg(B_t - \log(1+\alpha Z_t), \frac{A_t}{1+\alpha Z_t}\bigg) = 1 \quad (6.11)$$

and, if we take

$$\varphi(b,u) = \exp\bigg(-\frac{\alpha}{2}\bigg(e^b - \frac{1}{e^b+\alpha u}\bigg)\bigg),$$

we get (6.11). Therefore, we obtain

$$E[F(T_{\alpha/e_t}(B)_s, s \leqq t)] = E\bigg[\exp\bigg(-\frac{\alpha}{2}\bigg(e_t - \frac{1}{e_t+\alpha A_t}\bigg)\bigg)F(B_s, s \leqq t)\bigg], \quad (6.12)$$

which is precisely (6.4) with $\mu = 0$.

For Step 3, we again use the Cameron-Martin theorem to modify the left hand side of (6.4):

$$\begin{aligned} &E[F(T_{\alpha/e_t^{(\mu)}}(B^{(\mu)})_s, s \leqq t)] \\ &= E[F(T_{\alpha/e_t}(B)_s, s \leqq t)\exp(\mu B_t - \mu^2 t/2)] \\ &= e^{-\mu^2 t/2}E[F(T_{\alpha/e_t}(B)_s, s \leqq t)\exp(\mu T_{\alpha/e_t}(B)_t)(1+\alpha Z_t)^\mu]. \end{aligned}$$

We note $Z(T_{\alpha/e_t}(B)) = Z(B)$ (Proposition 6.6) and use (6.12). Then we obtain

$$\begin{aligned} &E[F(T_{\alpha/e_t^{(\mu)}}(B^{(\mu)})_s, s \leqq t)] \\ &= e^{-\mu^2 t/2}E\bigg[F(B_s, s \leqq t)\exp(\mu B_t)(1+\alpha Z_t)^\mu \exp\bigg(-\frac{\alpha}{2}\bigg(e_t - \frac{1}{(1+\alpha Z_t)e_t}\bigg)\bigg)\bigg] \\ &= E\bigg[F(B_s^{(\mu)}, s \leqq t)(1+\alpha Z_t^{(\mu)})^\mu \exp\bigg(-\frac{\alpha}{2}\bigg(e_t^{(\mu)} - \frac{1}{(1+\alpha Z_t^{(\mu)})e_t^{(\mu)}}\bigg)\bigg)\bigg]. \end{aligned}$$

The proof is completed. □

We are now in a position to give a proof of the key proposition.

Proof of Proposition 5.2. At first we consider the case $\mu = 0$. We set $Q_t^{\omega,z}(\cdot) = P(\cdot|\mathcal{Z}_t, Z_t = z)$, the regular conditional distribution given $\mathcal{Z}_t$. Then, taking F in (6.4) as $\varphi(1/A_t)G(Z_s, s \leqq t)$ for a non-negative Borel function φ on $(0,\infty)$ and for a non-negative functional G in view of Proposition 6.6, we obtain

$$E^{Q_t^{\omega,z}}\left[\exp(-\eta e_t)\varphi\left(\frac{1}{ze_t}\right)\right] = E^{Q_t^{\omega,z}}\left[\exp(-\eta/e_t)\varphi\left(\frac{1}{ze_t} + \frac{2\eta}{e_t}\right)\right].$$

Now we assume for simplicity that the distribution of e_t under $Q_t^{\omega,z}$ has a density $g_z(x)$ with respect to the Lebesgue measure. Then we have

$$\begin{aligned} E^{Q_t^{\omega,z}}\left[\exp(-\eta e_t)\varphi\left(\frac{1}{ze_t}\right)\right] &= \int_0^\infty e^{-\eta x}\varphi\left(\frac{1}{zx}\right) g_z(x)dx \\ &= \int_0^\infty e^{-\eta/zx}\varphi(x) g_z\left(\frac{1}{zx}\right)\frac{1}{zx^2}dx \end{aligned}$$

and

$$\begin{aligned} &E^{Q_t^{\omega,z}}\left[\exp(-\eta/e_t)\varphi\left(\frac{1}{ze_t} + \frac{2\eta}{e_t}\right)\right] \\ &= \int_0^\infty e^{-\eta/x}\varphi\left(\frac{2\eta+1/z}{x}\right) g_z(x)dx \\ &= \int_0^\infty \exp\left(-\frac{\eta x}{2\eta+1/z}\right)\varphi(x) g_z\left(\frac{2\eta+1/z}{x}\right)\frac{2\eta+1/z}{x^2}dx. \end{aligned}$$

Since the function φ is arbitrary, we obtain

$$z^{-1}g_z(v/z)\exp(-\eta v/z) = (2\eta+1/z)\exp\left(-\frac{\eta}{(2\eta+1/z)v}\right) g_z((2\eta+1/z)v),$$

where we have set $v = 1/x$. From the last identity, we obtain

$$g_z(x) = \text{const.}\ x^{-1}\exp\left(-\frac{1}{2z}\left(x + \frac{1}{x}\right)\right)$$

by simple algebra and, by using the integral representation (6.2) for the Macdonald function, we obtain (5.6) when $\mu = 0$.

For a general value of μ, a standard argument with the Cameron-Martin theorem leads us to the result. □

We finally show that the semigroups of the diffusion processes $e^{(\mu)}$, $\xi^{(\mu)}$ and $Z^{(\mu)}$ satisfy some intertwining properties. For some general discussions and examples of intertwinings between Markov semigroups, see [6], [10], [21], [54] and [61] among others.

To present these results, we denote by $I_z^{(\mu)}$ a generalized inverse Gaussian random variable whose density is given by (5.6) and define the Markov kernels $\mathbb{K}_1^{(\mu)}$ and $\mathbb{K}_2^{(\mu)}$ by

$$\mathbb{K}_1^{(\mu)}\varphi(z) = E[\varphi(I_z^{(\mu)})] \quad \text{and} \quad \mathbb{K}_2^{(\mu)}\varphi(z) = E\left[\varphi\left(\frac{z}{I_z^{(\mu)}}\right)\right]$$

for a generic function φ on $\mathbf{R}_+$.

Theorem 6.8. *Let* $\{P_t^{(\mu)}\}$, $\{Q_t^{(\mu)}\}$ *and* $\{R_t^{(\mu)}\}$ *be the semigroup of the diffusion processes* $e^{(\mu)}$, $\xi^{(\mu)}$ *and* $Z^{(\mu)}$, *respectively. Then we have*

$$R_t^{(\mu)}\mathbb{K}_1^{(\mu)} = \mathbb{K}_1^{(\mu)}P_t^{(\mu)} \quad \textit{and} \quad R_t^{(\mu)}\mathbb{K}_2^{(\mu)} = \mathbb{K}_2^{(\mu)}Q_t^{(\mu)}.$$

Proof. The key proposition (Proposition 5.2) again plays an important role. We give a proof for the first identity. The second one is proven in a similar way.

For $s, t > 0$, we compute $E[\varphi(e_{s+t}^{(\mu)})|\mathcal{Z}_s^{(\mu)}, Z_s^{(\mu)} = z]$ in two ways. First, we use the Markov property of $e^{(\mu)}$ to obtain

$$\begin{aligned} E[\varphi(e_{s+t}^{(\mu)})|\mathcal{Z}_s^{(\mu)}, Z_s^{(\mu)} = z] &= E[E[\varphi(e_{s+t}^{(\mu)})|\mathcal{B}_s^{(\mu)}]|\mathcal{Z}_s^{(\mu)}, Z_s^{(\mu)} = z] \\ &= E[(P_t^{(\mu)}\varphi)(e_s^{(\mu)})|\mathcal{Z}_s^{(\mu)}, Z_s^{(\mu)} = z] \\ &= E[(P_t^{(\mu)}\varphi)(I_z^{(\mu)})] \\ &= (\mathbb{K}_1^{(\mu)}P_t^{(\mu)}\varphi)(z), \end{aligned}$$

where $\mathcal{B}_s^{(\mu)} = \sigma\{e_u^{(\mu)}, u \leqq s\} = \sigma\{B_u^{(\mu)}, u \leqq s\}$ and we have used Proposition 5.2 for the third equality.

Next, we note $E[\varphi(e_{s+t}^{(\mu)})|\mathcal{Z}_{s+t}^{(\mu)}] = (\mathbb{K}_1^{(\mu)}\varphi)(Z_{s+t}^{(\mu)})$. Then, by the Markov property of $Z^{(\mu)}$, we obtain

$$\begin{aligned} E[\varphi(e_{s+t}^{(\mu)})|\mathcal{Z}_s^{(\mu)}, Z_s^{(\mu)} = z] &= E[\varphi(e_{s+t}^{(\mu)})|\mathcal{Z}_{s+t}^{(\mu)}]|\mathcal{Z}_s^{(\mu)}, Z_s^{(\mu)} = z] \\ &= E[(\mathbb{K}_1^{(\mu)}\varphi)(Z_{s+t}^{(\mu)})|\mathcal{Z}_s^{(\mu)}, Z_s^{(\mu)} = z] \\ &= (R_t^{(\mu)}\mathbb{K}_1^{(\mu)}\varphi)(z) \end{aligned}$$

and the desired identity $R_t^{(\mu)}\mathbb{K}_1^{(\mu)} = \mathbb{K}_1^{(\mu)}P_t^{(\mu)}$. □

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