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Around Tsirelson's equation, or: The evolution process may not explain everything

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A survey dedicated to Jean-Paul Thouvenot, with deep thanks for his constant willingness to listen and help

Abstract: We present a synthesis of a number of developments which have been made around the celebrated Tsirelson's equation (1975), conveniently modified in the framework of a Markov chain taking values in a compact group G, and indexed by negative time. To illustrate, we discuss in detail the case of the one-dimensional torus $G = \mathbb{T}$.

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Contents

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1. Introduction

1°). The contents of this paper, which were presented at the Meeting "Dynamical Systems and Randomness" at IHP, Paris (May 15th, 2009), were motivated mainly by the authors' desire ([23, 1] and [13]) to understand deeply Tsirelson's equation [3]. This equation shall be discussed in Section 4, while, as a preparation for the main part of the paper, we shall discuss the stochastic equation:

$$\eta_k = \xi_k \eta_{k-1} \tag{1}$$

on a compact group G, where k varies in $-\mathbb{N}$, $(\xi_k)_{k\le 0}$ is the "evolution process", and $(\eta_k)_{k\le 0}$ is the unknown process, both taking values in G. We believe that equation (1) is the "right" abstraction of Tsirelson's equation, as shown in Section 4.

More precisely, for every $k \le 0$, the law of ξ_k is a given probability μ_k on G, and the ξ_k is assumed independent of the past $\{\eta_j, \xi_j : j \le k-1\}$. We shall denote the sequence $(\mu_k)_{k\le 0}$ simply by μ. It is immediate that $(\eta_k)_{k\le 0}$ is a Markov chain, the transitions of which are given by

$$P(\eta_k \in A | \mathcal{F}^\eta_{k-1}) = \mu_k(A\eta^{-1}_{k-1}(\omega)), \quad A \in \mathcal{B}(G). \tag{2}$$

We may thus define $\mathscr{P}_\mu$, which may be regarded as the set of all solutions of (1), as follows: $P \in \mathscr{P}_\mu$ iff P is a probability on the product space $G^{-\mathbb{N}}$ such that (2) holds for every $k \le 0$, where the η_k's are interpreted as the coordinate process.

The problem is that, as k varies in $-\mathbb{N}$, there is no initial state ("at time $-\infty$"), and the study of $\mathscr{P}_\mu$ necessitates some care.

Let $\mathrm{ex}(\mathscr{P}_\mu)$ denote the set of all extremal points of the compact convex set $\mathscr{P}_\mu$. Let $\mathscr{S}_\mu$ denote the set of laws P of "strong" solutions, i.e.: under P, $\mathcal{F}^\eta_k \subset \mathcal{F}^\xi_k$, for every k. Note that $P \in \mathscr{S}_\mu$ iff under P, $\mathcal{F}^\eta_k = \mathcal{F}^\xi_k$, for every k, since, from (1), there is the identity:

$$\xi_k = \eta_k \eta^{-1}_{k-1}. \tag{3}$$

We shall also be interested in $\mathrm{ex}(\mathscr{P}_\mu)$ as well as in $\mathscr{S}_\mu$.

A number of natural questions now arise: given $\mu = (\mu_k)_{k\le 0}$,

a) is there existence for (1)?, i.e.: $\mathscr{P}_\mu \ne \emptyset$.

b) is there uniqueness?, i.e., $\sharp(\mathscr{P}_\mu) = 1$.

c) is there a strong solution?, i.e.: $\mathscr{S}_\mu \ne \emptyset$.

We shall see that these different questions may be answered very precisely, in particular if $G = \mathbb{T} \simeq [0, 1)$ is the one-dimensional torus, in terms of criteria on μ.

2°). Interpreting Tsirelson's equation:

• Before proceeding, we would like to give a "light" interpretation of equation (1), we mean one not to be taken too seriously!: η_k describes the "state of the universe" at time k; this state is "created" by the state at time $k-1$, followed by the action of the "evolution" ξ_k. The main question is: can today's state of the universe, i.e.: η_0, be explained solely from the evolution process?, an almost metaphysical question... We shall see that the answer(s), in terms of μ, are somewhat paradoxical...

• Then our "interpretation" also allows us to justify our quite general choice of the probabilities $(\mu_k)_{k\le 0}$. Indeed, today's "historians of the universe" see the evolution process at work, say, for times $j \in [K, 0]$, for some large negative K (K decreases as "today" increases...). But, they have no knowledge beyond that K, hence, we need to make the most general assumptions on the (μ_k)'s to understand all possible cases...

Thus, depending on the choice of μ, mathematicians give an answer as to how much the evolution process determines the present state, but this choice remains to be made!

3°). Plan of the remainder of the paper:

Section 2. A simple example: Wrapping Gaussians on the circle
Section 3. The general group framework—Questions and facts
Section 4. The motivation for this study: Tsirelson's equation
Section 5. Some related questions and final comments

2. A simple example: Wrapping Gaussians on the circle

Here, $G = \mathbb{T} \simeq [0, 1)$; $\eta_k = \exp(2i\pi\theta_k)$; $\xi_k = \exp(2i\pi g_k)$, with $\theta_k \in [0, 1)$, and g_k Gaussian, centered, $E[g_k^2] = \sigma_k^2$.

The answers to our previous questions are radically different depending on whether

$$\sum_{k\le 0} \sigma_k^2 = \infty \quad \text{or} \quad \sum_{k\le 0} \sigma_k^2 < \infty. \tag{4}$$

The case: $\sum_{k\le 0} \sigma_k^2 = \infty$.

- For any fixed k, η_k is uniformly distributed on the torus, (or θ_k is uniform on $[0, 1)$), independent from the evolution sequence $(\xi_j)_{j\le 0}$.
- $\mathscr{P}_\mu$ consists of exactly one solution: P_μ^*, "the uniform solution".
- $\mathcal{F}_k^\eta = \sigma(\eta_k) \vee \mathcal{F}_k^\xi = \sigma(\eta_j) \vee \mathcal{F}_k^\xi$ for $j \le k$ where η_j is independent from $\mathcal{F}_k^\xi$ and, in fact, even from $\mathcal{F}_0^\xi$.
- $\mathcal{F}_{-\infty}^\eta$ is trivial (under P_μ^*).

The case: $\sum_{k\le 0} \sigma_k^2 < \infty$.

- $\prod_{-N\le j\le 0} \xi_j \to \prod_{-\infty}^0 \xi_j$ as $N \to \infty$ a.s.
- Any solution (η_k) satisfies: $\eta_k \to V$ as $k \to -\infty$ a.s. with V independent of the evolution (ξ_j),

$$\eta_k = \left(\prod_{-\infty}^{k} \xi_j\right) V. \tag{5}$$

- $\mathcal{F}_k^\eta = \mathcal{F}_{-\infty}^\eta \vee \mathcal{F}_k^\xi$ and $\mathcal{F}_{-\infty}^\eta = \sigma(V)$.
- $P \in \mathrm{ex}(\mathscr{P}_\mu) \iff P \in \mathscr{S}_\mu \iff V = v$, under P for some constant $v \in G$.

Thus, here in the framework of (4), there is nonuniqueness iff there is a strong solution, which is indeed a puzzling result.

We give the main arguments of proof when:

$$\sum_{k \le 0} \sigma_k^2 = \infty. \tag{6}$$

We first show that: $\forall k$, η_k is uniformly distributed, i.e.: for $p \in \mathbb{Z}$, $p \neq 0$, we obtain:

$$\varphi_k(p) := E[\exp(2i\pi p\theta_k)] \tag{7}$$

$$= E[\exp(2i\pi p g_k)]\varphi_{k-1}(p) \tag{8}$$

$$= \exp\left(-2\pi^2 p^2 \left(\sum_{-N \le j \le k} \sigma_j^2\right)\right)\varphi_{-N-1}(p) \tag{9}$$

$$\to 0 \quad (N \to \infty). \tag{10}$$

Thus, θ_k is uniform on $[0,1)$, i.e.: η_k is uniform on the torus. This reinforces, as we can show, likewise:

$$E[\exp(2i\pi p\theta_k)|\xi_k, \xi_{k-1}, \dots, \xi_{-N}] = 0. \tag{11}$$

Then, letting $N \to \infty$:

$$E[\exp(2i\pi p\theta_k)|(\xi_j)_{j\le 0}] = 0. \tag{12}$$

Hence, the independence of η_k, for any fixed k, from the evolution process $(\xi_j)_{j\le 0}$. In consequence, the law of $(\eta_k)_{k\le 0}$ is uniquely determined by this independence property.

The triviality of $\mathcal{F}_{-\infty}^\eta$ will be explained (in the next section, Theorem 3.2) by a general result, i.e.: the triviality of $\mathcal{F}_{-\infty}^\eta$ under any $P \in \mathrm{ex}(\mathscr{P}_\mu)$, but, here, there is only one solution!!

3. The general group framework—Questions and facts

Let G be a general compact group; there is the uniform distribution (= Haar measure), and we now take up the discussion of the general questions a), b), c) stated in the Introduction.

Recall that, under $P \in \mathscr{P}_\mu$, $(\eta_k)_{k\le 0}$ is a Markov chain, i.e.:

$$P(\eta_k \in A|\mathcal{F}_{k-1}^\eta) = \mu_k(A\eta_{k-1}^{-1}(\omega)). \tag{13}$$

Theorem 3.1 (Yor [23])**.** *For any $\mu = (\mu_k)_{k\le 0}$, there exists the "uniform solution" P_μ^* which may be characterized by: $\forall k$, η_k is uniform, independent from the $(\xi_j)_{j\le 0}$.*

Proof. It follows from Kolmogorov's extension theorem, since for any k, we may consider the law $U_k^{(\mu)}$ on $G^{(-k+1)}$ that of $[\eta_k, \eta_{k+1}, \ldots, \eta_0]$; then, for $k < l$ and $p_{k,l}$, the obvious projection, we find that: $p_{k,l}(U_k^{(\mu)}) = U_l^{(\mu)}$. So, the laws $(U_k^{(\mu)})$ are consistent, and P_μ^* exists. □

Thus, there is always existence; and, we may ask the 2 questions: b) is there uniqueness?, i.e.: $\sharp(\mathscr{P}_\mu) = 1$?; c) does a strong solution exist?, i.e.: $\mathscr{S}_\mu \neq \emptyset$?

Let us make the following table:

strong solution	uniqueness holds	uniqueness fails
exists	C_0	C_2
does not exist	C_1	C_3

<u>Discussion</u>: We immediately rule out C_0, since, from Theorem 3.1, under C_0, the unique solution P_μ^* is not strong. Thus, there remains to discuss the trichotomy: $\boxed{C_1\text{-}C_2\text{-}C_3}$.

For $g \in G$ and $P \in \mathscr{P}_\mu$, we write $\tau_g(P)$ for the law of $(\eta_k g)_k$ under P. It is obvious that $\tau_g(P)$ also belongs to $\mathscr{P}_\mu$, since $(\eta_k g)(\eta_{k-1} g)^{-1} = \eta_k \eta_{k-1}^{-1}$.

We now state several general results:

Theorem 3.2 (Akahori–Uenishi–Yano [1]).
i). *$P(\in \mathscr{P}_\mu)$ is in fact in $\mathrm{ex}(\mathscr{P}_\mu)$ iff $\mathcal{F}_{-\infty}^\eta$ is trivial under P.*
ii). *The group G acts transitively over $\mathrm{ex}(\mathscr{P}_\mu)$ by $g \mapsto \tau_g$, i.e.: for any $P^0 \in \mathrm{ex}(\mathscr{P}_\mu)$, the orbit $\{\tau_g(P^0) : g \in G\}$ is equal to $\mathrm{ex}(\mathscr{P}_\mu)$.*
iii). *Any solution $(\eta_k)_{k \le 0}$ may be represented as:*

$$(\eta_k)_k \stackrel{\text{law}}{=} (\eta_k^0 V)_k \tag{14}$$

with V G-valued and independent of $(\eta_k^0)_k$. This yields directly the Krein–Milman integral representation:

$$(\mathscr{P}_\mu \ni) P = \int P(V \in \mathrm{d}v) P^{(\eta_k^0 v)_k} \tag{15}$$

for any given extremal solution $(\eta_k^0)_k$.

To get a good feeling /introduction/ for the following discussion, we recall a result of Csiszár [5]: i.e., the "almost" convergence in law of infinite products of independent random variables.

Theorem 3.3 (Csiszár [5]). *Let $(\xi_j)_{j \le 0}$ be our evolution sequence. There exists a sequence $(\alpha_l, l \in -\mathbb{N})$ of deterministic elements of G such that, for any fixed $k \in -\mathbb{N}$, the products*

$$\xi_k \xi_{k-1} \cdots \xi_l \alpha_l \tag{16}$$

converge in law, as $l \to -\infty$, and, their natural projections on the quotient space G/H converge a.s. as $l \to -\infty$.

We give two illustrations of Theorem 3.3.

1°). As a first illustration of Theorem 3.3, let us go back to the Gaussian set-up of Section 2, where we now consider more generally

$$\xi_k = \exp(2i\pi g_k), \tag{17}$$

with g_k Gaussian, with variance σ_k^2, and mean m_k. Then, we may choose

$$\alpha_l = \exp\left(-2i\pi \sum_{l\le j\le 0} m_j\right) \tag{18}$$

as "centering sequence".

2°). To illustrate further Theorem 3.3, or may be, more accurately, point 1) of Theorem 3.4 below, we may consider the case where all the laws μ_k are the same; then, Stromberg [18] (see also Collins [4]) showed that, for ν a given probability on G, ν^{*n} converges to Haar measure as soon as the smallest subgroup which contains the support of ν is equal to G.

We may now present a characterization of C_1 and C_2.

Theorem 3.4 (Hirayama–Yano [13]). *The following statements hold:*

1). *Uniqueness holds iff, for each $k \in -\mathbb{N}$, the products $\xi_k\xi_{k-1}\cdots\xi_l$ converge in law as $l \to -\infty$, to the uniform law on G.*

2). *There exists a strong solution iff there exists a sequence (α_l) of deterministic elements of G such that the products*

$$\xi_k\xi_{k-1}\cdots\xi_l\alpha_l \quad \text{converge a.s. as } l \to -\infty. \tag{19}$$

Then, every extremal solution is strong and is the law of the a.s. limit of $(\xi_k\xi_{k-1}\cdots\xi_l\alpha_l g)$, for some $g \in G$.

Again, to illustrate Theorem 3.4, we may consider the general Gaussian hypothesis made after Theorem 3.3; clearly, uniqueness holds iff $\sum_k \sigma_k^2 = \infty$, whereas C_2 holds iff $\sum_k \sigma_k^2 < \infty$. Note that C_3 never occurs in this set-up.

Theorem 3.4 is obtained as a corollary of the following theorem, whose assertion is stronger than iii) of Theorem 3.2.

Theorem 3.5 (Hirayama–Yano [13]). *Any solution $(\eta_k)_{k\le 0}$ may be represented as:*

$$\eta_k = \phi_k U_k V \tag{20}$$

with G-valued random variables ϕ_k, U_k $(k \le 0)$ and V such that, for each $k \le 0$, ϕ_k is $\mathcal{F}_k^\xi$-measurable, V is $\mathcal{F}_{-\infty}^\eta$-measurable and U_k is independent of $\sigma(V)\vee\mathcal{F}_0^\xi$.

To give a full discussion of the trichotomy, we go back to $G = \mathbb{T} \simeq [0,1)$. We introduce:

$$\mathbb{Z}_\mu = \left\{ p \in \mathbb{Z} : \text{for some } k,\ \prod_{j\le k} \left| \int_{[0,1)} \mathrm{e}^{2i\pi px} \mu_j(\mathrm{d}x) \right| > 0 \right\}. \tag{21}$$

Then, there is the

Proposition 3.1 (Yor [23])**.** *$\mathbb{Z}_\mu$ is a subgroup of $\mathbb{Z}$; hence, there exists a unique integer $p_\mu \geq 0$ such that $\mathbb{Z}_\mu = p_\mu \mathbb{Z}$.*

This Proposition now allows us to discuss fully the trichotomy C_1-C_2-C_3. We use the notation of Theorem 3.5.

Theorem 3.6 (Yor [23] and Hirayama–Yano [13])**.** *The trichotomy C_1-C_2-C_3 may be described as follows, in terms of p_μ:*

1). *Uniqueness in law iff $p_\mu = 0$, i.e., $\mathbb{Z}_\mu = \{0\}$. Then,*
- *$\mathcal{F}^\eta_{-\infty}$ is trivial;*
- *$\forall k$, η_k is uniform;*
- *$\mathcal{F}^\eta_k = \sigma(\eta_k) \vee \mathcal{F}^\xi_k$, with independence of η_k and $\mathcal{F}^\xi_k$.*

2). *Existence of a strong solution iff $p_\mu = 1$, i.e., $\mathbb{Z}_\mu = \mathbb{Z}$. Then,*
- *$\mathcal{F}^\eta_{-\infty} = \sigma(V)$*
- *for any $k \in -\mathbb{N}$, $\mathcal{F}^\eta_k = \sigma(V) \vee \mathcal{F}^\xi_k$.*

3). *No strong solution, no uniqueness iff $p_\mu \geq 2$. Let θ_k $(k \leq 0)$ and v denote $[0,1)$-valued random variables such that $\eta_k = \exp(2\pi i \theta_k)$ and $V = \exp(2\pi i v)$. Then,*
- *$\forall k \in -\mathbb{N}$, $\{p_\mu \theta_k\}$ is $\mathcal{F}^\xi_k$-measurable;*
- *$[p_\mu \theta_k]/p_\mu$ is uniform on $\left(0, \frac{1}{p_\mu}, \frac{2}{p_\mu}, \ldots, \frac{p_\mu - 1}{p_\mu}\right)$;*
- *$\mathcal{F}^\eta_k = \sigma([p_\mu \theta_k]) \vee \sigma(\{p_\mu v\}) \vee \mathcal{F}^\xi_k$ with independence of the three σ-fields.*

Here, for $x \in \mathbb{R}$, we write $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$ and $\{x\} = x - [x]$.

We may give (many!) sufficient conditions on μ which ensure either C_1, or C_2, or C_3:

a). There is uniqueness in law (i.e.: $p_\mu = 0$) as soon as $\xi_{n_j} \overset{\text{law}}{=} \exp(2\pi i \varepsilon_j \gamma)$ for some subsequence (n_j), some $\varepsilon_j \in \mathbb{R}$ with $|\varepsilon_j| \to \infty$ as $j \to \infty$, and some random variable γ with absolutely continuous density.

b). Assume $\mu_j = \nu$, for all j. Then:
- i). $p_\mu = 0$ iff ν is not arithmetic;
- ii). $p_\mu = 1$ iff $\exists x \in \mathbb{R}$, $\nu(x + \mathbb{Z}) = 1$;
- iii). $p_\mu = p \geq 2$ if ν charges precisely $(0, \frac{1}{p}, \frac{2}{p}, \ldots, \frac{p-1}{p})$.

Remark. A special case of a) above is when $\varepsilon_j = j$ for some random variable γ with absolutely continuous density. This is called "Poincaré roulette wheel, leading to equidistribution"; see [8, Theorem 3.2] for detail.

4. The motivation for this study: Tsirelson's equation

A result of Zvonkin [24] (see also Zvonkin–Krylov [25]) asserts that the stochastic differential equation driven by BM:

$$X_t = B_t + \int_0^t \mathrm{d}s\ b(X_s), \tag{22}$$

where $b(\cdot)$ is only assumed to be bounded and Borel enjoys *strong uniqueness*. (For h such that $\frac{1}{2}h'' + bh' = 0$, the process $Y_t = h(X_t)$ solves $Y_t = \int_0^t (h' \circ h^{-1})(Y_s)\mathrm{d}B_s$ where $h' \circ h^{-1}$ is Lipschitz.)

Then, the question arose whether the same strong uniqueness result might still be true with a bounded Borel drift depending more generally on the past of X, i.e.:

$$X_t = B_t + \int_0^t \mathrm{d}s \; b(X_u, u \le s) \tag{23}$$

(Uniqueness in law is ensured by Girsanov's theorem). Tsirelson gave a negative answer to this question by producing the drift:

$$b(X_u, u \le s) = \sum_{k \in -\mathbb{N}} \left\{ \frac{X_{t_k} - X_{t_{k-1}}}{t_k - t_{k-1}} \right\} 1_{(t_k, t_{k+1}]}(s) \tag{24}$$

for any sequence $t_k \downarrow 0$ as $k \downarrow -\infty$.

To prove that the solution is non-strong, it suffices to study the discrete time skeleton equation:

$$\frac{X_{t_{k+1}} - X_{t_k}}{t_{k+1} - t_k} = \frac{B_{t_{k+1}} - B_{t_k}}{t_{k+1} - t_k} + \left\{ \frac{X_{t_k} - X_{t_{k-1}}}{t_k - t_{k-1}} \right\} \tag{25}$$

i.e.:

$$\eta_k = \xi_k + \{\eta_{k-1}\}. \tag{26}$$

Slight modifications of our previous arguments show that: $\forall k$, $\{\eta_k\}$ is independent from the BM, and uniformly distributed on $[0, 1)$; moreover, $\forall t$, $\forall t_k \le t$,

$$\mathcal{F}_t^X = \sigma(\{\eta_{k-1}\}) \vee \mathcal{F}_t^B. \tag{27}$$

For many further references, see Tsirelson's web page [20].

5. Some related questions and final comments

a). The case C_1 [only P_μ^* solution] gives a beautiful example where:

$$\mathcal{F}_k^\eta = \mathcal{F}_j^\eta \vee \mathcal{F}_k^\xi, \quad j \le k \tag{28}$$

$$= \bigcap_j (\mathcal{F}_j^\eta \vee \mathcal{F}_k^\xi) \ne \left(\bigcap_j \mathcal{F}_j^\eta \right) \vee \mathcal{F}_k^\xi \tag{29}$$

since in the C_1 case, $\bigcap_j \mathcal{F}_j^\eta = \mathcal{F}_{-\infty}^\eta$ is trivial. This is a discussion which has been a trap for a number of very distinguished mathematicians… See, e.g., N. Wiener ([22], Chap 2) where he assumes that $\cap$ and $\vee$ may be interverted for σ-fields; if so, from Wiener's set-up, this would lead to: K-automorphism is always Bernoulli!

b). As (27) above clearly shows, any solution to Tsirelson's equation (23)–(24) is not strong, i.e., (X_t) cannot be recovered from the Brownian motion (B_t) in (23)–(24). Nevertheless, as shown in Emery–Schachermayer [7], the natural filtration of X, that is: $(\mathcal{F}_t^X)_{t\geq 0}$ is generated by some Brownian motion $(\beta_t)_{t\geq 0}$; thus, in that sense, the filtration $(\mathcal{F}_t^X)_{t\geq 0}$ is a strong Brownian filtration. The question then arose naturally whether under any probability Q on $C([0,1],\mathbb{R})$, equivalent to Wiener measure, the natural filtration of the canonical coordinate process is always a strong Brownian filtration. This is definitely not the case, as shown, e.g., in Dubins–Feldman–Smorodinsky–Tsirelson [6]. The paper [7] contains a number of important references around the topic treated in [6], where again the role of B. Tsirelson has been crucial. In particular, it is another beautiful result of B. Tsirelson [19] that the natural filtration of the Brownian spider with $N(\geq 3)$ legs is not strongly Brownian.

c). Related to the problem b), we may ask the following question: For a solution $(\eta_k)_{k\leq 0}$ of equation (1), when does there exist some sequence $\theta = (\theta_k)_{k\leq 0}$ of independent random variables such that $\eta_k \in \mathcal{F}_k^\theta$ for any $k \leq 0$? There are lots of studies in the case where η is a stationary process; some positive answers to this question are found in [17] and [12]. See [16] for historical remarks and related references.

d). For stationary processes with common law μ on a state space S with a continuous group action of a locally compact group G, Furstenberg [10, Definition 8.1] introduced the notion of a "μ-boundary". When we confine ourselves to the case where $S = G$ with the canonical group action and where the noise process is assumed to be identically distributed, Furstenberg's μ-boundary is essentially the same as a strong solution in our terminology, and Theorem 14.1 of [10] coincides with our Theorem 3.4.

Let G be a compact group with a countable base endowed with metric d and let S be a compact space with continuous G-action. Furstenberg [9], in his study of stationary measures, proved the following: If the action is *distal*, i.e., for any $x, y \in S$ with $x \neq y$, it holds that $\inf_{g\in G} d(gx, gy) > 0$, then the action is *stiff*, i.e., for any probability law μ on G whose support generates G, any μ-invariant probability measure on S is G-invariant. Coming back to our setting, we assume $S = G$ with the canonical group action. Then the well-known theorem by Birkhoff [2] and Kakutani [15] shows that we may choose the metric d so that it remains invariant under the action of G, i.e., $d(gx, gy) = d(x, y)$ for any $g \in G$. Thus the canonical action of G may be assumed to be distal, and we see that, then, the canonical action is stiff.

e). We would also like to point out the relevance of the Itô–Nisio [14] study of all stationary solutions $(X_t)_{-\infty<t<\infty}$ of some stochastic differential equations driven by Brownian motion $(B_t)_{-\infty<t<\infty}$. They discuss whether either of the following properties holds:

(i) $\mathcal{F}_{-\infty,t}^X \subset \mathcal{F}_{-\infty,t}^B$;
(ii) $\cap_t \mathcal{F}_{-\infty,t}^X$ is trivial;
(iii) $\mathcal{F}_{-\infty,t}^X \subset \mathcal{F}_{-\infty,s}^X \vee \mathcal{F}_{s,t}^B$ for $s < t$,

where we have used obvious notations for σ-fields. Itô–Nisio [14, Section 13] identify cases where (ii) and (iii) hold, but not (i); this is similar to the situation in Tsirelson's original equation (23)–(24), or more generally the C_1 case. But, even worse, Itô–Nisio [14, Section 14] also discuss a case where (iii) does not hold. This case originates from Girsanov's equation [11] with diffusion coefficient $|x|^\alpha$ for $\alpha < 1/2$, which is well-known to generate non-uniqueness in law.

Let us give a comment as a word of conclusion, which may essentially be the same as the last footnote of Vershik [21]. Although we do not claim that equation (1) has a deep "cosmological value", it would probably never come to the mind of "universe historians" that, in some cases, despite "the emptiness of the beginning", today's state may be independent of the evolution mechanism. Thus, seen in this light, Tsirelson's equation, and its abstraction (1) provide us with a beautiful and mind-boggling statement.

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References

[1] Akahori, J., Uenishi, C., and Yano, K., Stochastic equations on compact groups in discrete negative time. *Probab. Theory Related Fields*, 140(3–4):569–593, 2008. MR2365485

[2] Birkhoff, G., A note on topological groups. *Compositio Math.*, 3:427–430, 1936. MR1556955

[3] Cirel'son, B. S., An example of a stochastic differential equation that has no strong solution. *Teor. Verojatnost. i Primenen.*, 20(2):427–430, 1975. MR0375461

[4] Collins, H. S., Convergence of convolution iterates of measures. *Duke Math. J.*, 29:259–264, 1962. MR0137789

[5] Csiszár, I., On infinite products of random elements and infinite convolutions of probability distributions on locally compact groups. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 5:279–295, 1966. MR0205306

[6] Dubins, L., Feldman, J., Smorodinsky, M., and Tsirelson, B., Decreasing sequences of σ-fields and a measure change for Brownian motion. *Ann. Probab.*, 24(2):882–904, 1996. MR1404533

[7] Émery, M. and Schachermayer, W., A remark on Tsirelson's stochastic differential equation. In *Séminaire de Probabilités, XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 291–303. Springer, Berlin, 1999. MR1768003

[8] Engel, E. M. R. A., *A road to randomness in physical systems*, volume 71 of *Lecture Notes in Statistics*. Springer-Verlag, Berlin, 1992. MR1177596

[9] Furstenberg, H., Stiffness of group actions. *Lie groups and ergodic theory (Mumbai, 1996)*, 105–117, Tata Inst. Fund. Res. Stud. Math., **14**, Tata Inst. Fund. Res., Bombay, 1998. MR1699360

[10] Furstenberg, H., Boundary theory and stochastic processes on homogeneous spaces. *Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972)*, 193–229. Amer. Math. Soc., Providence, R.I., 1973. MR0352328

[11] Girsanov, I. V., An example of non-uniqueness of the solution of the stochastic equation of K. Itô. *Theory Probab. Appl.*, 7:325–331, 1962.

[12] Hanson, D. L., On the representation problem for stationary stochastic processes with trivial tail field. *J. Math. Mech.*, 12:293–301, 1963. MR0146896

[13] Hirayama, T. and Yano, K., Extremal solutions for stochastic equations indexed by negative integers and taking values in compact groups. *Stochastic Process. Appl.*, 120(8):1404–1423, 2010. MR2653259

[14] Itô, K. and Nisio, M., On stationary solutions of a stochastic differential equation. *J. Math. Kyoto Univ.*, 4:1–75, 1964. MR0177456

[15] Kakutani, S., Über die Metrisation der topologischen Gruppen. *Proc. Imp. Acad.*, 12(4):82–84, 1936. MR1568424

[16] Laurent, S., Further comments on the representation problem for stationary processes. *Statist. Probab. Lett.*, to appear. MR2595135

[17] Rosenblatt, M., Stationary Markov chains and independent random variables. *J. Math. Mech.*, 9:945–949, 1960. MR0166839

[18] Stromberg, K., Probabilities on a compact group. *Trans. Amer. Math. Soc.*, 94:295–309, 1960. MR0114874

[19] Tsirelson, B., Triple points: from non-Brownian filtrations to harmonic measures. *Geom. Funct. Anal.*, 7(6):1096–1142, 1997. MR1487755

[20] Tsirelson, B., My drift, Citing works. `http://www.math.tau.ac.il/~tsirel/Research/mydrift/citing.html`.

[21] Vershik, A. M., Decreasing sequences of measurable partitions and their applications. *Dokl. Akad. Nauk SSSR*, 193(4):748–751, 1970; English transl. in *Soviet Math. Dokl.* 11(4):1007–1011, 1970. MR1819177

[22] Wiener, N., *Extrapolation, Interpolation, and Smoothing of Stationary Time Series. With Engineering Applications.* The Technology Press of the Massachusetts Institute of Technology, Cambridge, Mass, 1949. MR0031213

[23] Yor, M., Tsirel'son's equation in discrete time. *Probab. Theory Related Fields*, 91(2):135–152, 1992. MR1147613

[24] Zvonkin, A. K., A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974. MR0336813

[25] Zvonkin, A. K. and Krylov, N. V., Strong solutions of stochastic differential equations. In *Proceedings of the School and Seminar on the Theory of Random Processes (Druskininkai, 1974), Part II (Russian)*, pages 9–88. Inst. Fiz. i Mat. Akad. Nauk Litovsk. SSR, Vilnius, 1975. MR0426154