Rend. Sem. Mat. Univ. Pol. Torino
Vol. 57, 1 (1999)

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LOCAL SOLVABILITY OF SOME CLASSES OF LINEAR AND NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

Abstract.

The paper deals with the local nonsolvability of several examples of linear and nonlinear partial differential equations. In the linear case we prove nonsolvability in Schwartz distribution space while in the nonlinear case we prove the nonexistence of classical solutions as well as the nonexistence of $L^\infty \cap H^s$, $s > 0$, solutions.

1. This paper deals with the local nonsolvability of several examples of linear and nonlinear partial differential equations (PDE). In the linear case we prove nonsolvability in Schwartz distribution space D' while in the nonlinear case we prove the nonexistence of classical solutions as well as the nonexistence of $L^\infty \cap H^s$, $s > 0$ solutions. We hope that some illustrative examples in the nonlinear case could be useful in a further development of the theory of the local nonsolvability. Y.V. Egorov stated the problem of finding necessary conditions for the local solvability of nonlinear PDE having in mind the well known Hormander's necessary condition for the local solvability of linear PDE in D' [2]. We analyse in this paper several examples in order to stress some difficulties arising in the nonlinear situation.

2. We shall propose at first some results on nonsolvability (nonhypoellipticity) of several examples of linear PDE in D'. So consider the following class of PDE with C^∞ coefficients

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha, \quad a_\alpha(x) \in C^\infty\left(\mathbb{R}^n\right) . \tag{1}$$

DEFINITION 2.1. *The operator (1) is quasihomogeneous if and only if* $P\big(t^\mu x, t^{-\mu}\xi\big) = t^\gamma P(x, \xi)$, $\forall t > 0$, $\forall (x, \xi) \in \mathbb{R}^{2n}$, $\gamma = const$.

As usual, $\mu = (\mu_1, \dots, \mu_n)$, $\mu_j > 0$, $1 \leq j \leq n$, $t^\mu x = \big(t^{\mu_1} x_1, \dots, t^{\mu_n} x_n\big)$. Without loss of generality we assume that $0 < \mu_1 \leq \mu_2 \leq \mu_2 \leq \dots \leq \mu_n$.

REMARK 2.1. P quasihomogeneous implies that its formal adjoint operator ${}^t P$ is quasihomogeneous too.

Assume that

(i) $Ker\,{}^t P \cap S\big(\mathbb{R}^n\big) \neq 0$ and $S\big(\mathbb{R}^n\big)$ is the Schwartz space of the rapidly decreasing functions at infinity.

THEOREM 2.1. *The operator* $R = P + Q$, *where the quasihomogeneous operator* P *satisfies (i) and* R, $ord\, R = s$ *is an arbitrary differential operator with coefficients flat at 0, is locally nonsolvable at 0 in* D'.

The proof of this theorem is a modification of the proof of the central result in [7] and we omit the details.

We will illustrate Theorem 2.1 with several examples. Some generalizations will be considered too.

EXAMPLE 2.1. Let $P = x\partial_y - y\partial_x + h\big(x^2+y^2\big)$, $h\big(x^2+y^2\big) \in C^\infty$ near $(0,0)$. This is an operator of the real principal type. We claim that P is locally nonsolvable at the origin in D' if and only if $h(0) \in i\{0, \pm 1, \pm 2, \ldots\}$ and $h\big(x^2+y^2\big) - h(0)$ is flat at 0.

EXAMPLE 2.2. Let $P(t, D_t, D_x) = D_t^m + atD_x^p$, $p \geq 1$, m -odd. Then P is quasihomogeneous with $\mu_1 = 1$, $\mu_2 = \frac{m+1}{p}$. The operator P is locally nonsolvable at 0 in D' if: 1) p - even, $Ima > 0$, 2) p - odd, $Ima \neq 0$.

EXAMPLE 2.3 (M. CHRIST, G. KARADZHOV [8]). Let $L = -X^2 - Y^2 + i\,a(x)\,[X, Y]$, where $X = \partial_x$, $Y = \partial_y + x^k\partial_t$, $[X, Y] = kx^{k-1}\partial_t$.

The operator L is not locally solvable at 0 in D' if and only if
1) $k = 1$, $a(0) \in \{\pm 1, \pm 3, \pm 5, \ldots\}$, $a^{(m)}(0) = 0$, $\forall m \geq 1$,
2) $k \geq 2$, $a(0) \in \{\pm 1\}$, $a^{(m)}(0) = 0$, $\forall m \geq 1$.

We shall propose a new and elementary proof of the sufficiency of this result in the case k-odd.

The approach used in studying Examples 2.1 and 2.3 enables us to investigate the local nonsolvability of some operators with coefficients flat at 0. So consider the following model example from [6]:

$$L = D_t^2 + \lambda^2(t)D_x^2 - a(t)\frac{\lambda^2(t)}{\Lambda(t)}D_x\,, \tag{2}$$

where $\Lambda(t) = e^{i\Phi}e^{-|t|^{-1}}$, $\Phi \in \big[0, \frac{\pi}{2}\big)$, $a(t) = \begin{cases} a_-, t < 0 \\ a_+, t \geq 0, \end{cases}$, $a_\pm = const.$, $\lambda(t) = \Lambda'(t) = e^{i\Phi}t^{-2}sgnte^{-|t|^{-1}}$, i.e. if we put $\Lambda_0(t) = e^{-|t|^{-1}} \Rightarrow \Lambda(t) = e^{i\Phi}\Lambda_0$, $\lambda(t) = e^{i\Phi}t^{-2}sgnt\Lambda_0$. Certainly, Λ and λ are flat at 0. It is proved in Theorem 2.4.32 from [6] that if $a_- = -2n-1$, $a_+ = -2l-1$ or $a_- = 2n+1$, $a_+ = 2l+1$, l, n being nonnegative integers then the operator (2) is not locally solvable at $(0, 0)$ in D'. The proof in [6] is based on violation of the well known Hörmander necessary condition for local solvability from [2]. We give here a rather different proof of the same result explaining the local nonsolvability of (2) by the existence of infinitely many compatibility conditions to be satisfied by the right hand side f of $Lu = f$, $u \in D'$.

EXAMPLE 2.4. Consider now the operator of real principal type

$$P_c = x\partial_y - y\partial_x + c\,, \quad c = const. \tag{3}$$

According to Example 2.1 P_c is locally nonsolvable at the origin in D' if $c \in i\mathbb{Z}$, while it is locally solvable at 0 if $c \notin i\mathbb{Z}$.

After the polar change of the variables $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ we get that $P_c \to \frac{\partial}{\partial\varphi} + c$ and if $c \in \mathbb{R}^1$ we reduce the solvability of (3) in $C\big(x^2+y^2 < \varepsilon^2\big)$ to the solvability of the next ODE:

$$\frac{\partial u}{\partial\varphi} + cu = f(\rho, \varphi)\,, \quad u(\rho, \varphi + 2\pi) \equiv u(\rho, \varphi)\,,$$

$\forall \rho \in [0, \varepsilon]$, $\forall \varphi \in [0, 2\pi]$.

Thus $u(\rho, 0)\left(1 - e^{-2\pi c}\right) = e^{-2\pi c} \int_0^{2\pi} e^{cs} f(\rho, s)\, ds$.

As $e^{-2\pi c} \neq 1$ for $c \neq 0$ a periodic solution always exists. In the case $c = 0 \Rightarrow \int_0^{2\pi} f(\rho, s)\, ds = 0$, $\forall \rho \in [0, \varepsilon]$ and this is the explanation of the local nonsolvability of P_0 at the origin.

EXAMPLE 2.5. Consider now the nonlinear equation

$$P_0 u = f(x, y) + g(u), \quad g \in C^1\left(\mathbb{R}^1\right), \quad g(u) \geq 0, \quad g(u) = 0, \quad \Leftrightarrow u = 0 \tag{4}$$

and suppose that the necessary condition for the local solvability of $P_0 v = f \in C^1$ near the origin is fulfilled:

$$\int_0^{2\pi} f(\rho, \varphi)\, d\varphi = 0, \quad \forall \rho \in [0, \varepsilon]\,. \tag{5}$$

Moreover, let the function f be nontrivial, i.e.

$$\exists (\rho_\nu, \varphi_\nu) \to (0, \varphi_0), \quad f(\rho_\nu, \varphi_\nu) \neq 0 \tag{6}$$

Then we claim that the equation (4) is locally nonsolvable in C^1 near the origin.

Assume now that (5) is violated, i.e. $\exists \rho_\nu \to 0$ such that $\int_0^{2\pi} f(\rho_\nu, \varphi)\, d\varphi > 0$ (< 0), $g(u) \geq 0$ (≤ 0).

Then (4) is locally nonsolvable near 0 in C^1. Thus even small nonlinear perturbations $g(u)$ of the locally solvable equation $P_0 v = f$ lead to nonsolvability. The effect just observed is not only due to the fact that P_0 is locally nonsolvable at 0 in D'. In fact, consider the locally solvable in L^2 operator P_c, c-real valued constant, $c \neq 0$. We shall investigate the local nonsolvability of

$$P_c u = f(u), \quad f \in C^1\left(\mathbb{R}^1\right) \tag{7}$$

EXAMPLE 2.6. The equation (7) is locally nonsolvable near the origin in the class C^1 if and only if $f(\lambda) \neq c\lambda$, $\forall \lambda \in \mathbb{R}^1$.

Thus each nonlinear perturbation $f(\lambda)$ of P_c located above (below) the straight line $y = c\lambda$ leads to nonexistence of a classical solution (even locally) of the equation (7).

In our previous Examples 2.5, 2.6 local nonsolvability in C^1 was shown. Here we study nonsolvability in $H^s \cap L^\infty$, $s > 0$ as well. For the sake of simplicity we shall investigate second order PDE with real valued C^∞ coefficients and only real valued solutions will be checked. Thus assume that the operator L_2 is locally nonsolvable at the origin in D'. More precisely, we suppose that for the real valued function $f \in C^\infty(\omega)$, $\omega \ni 0$ there does not exist a distribution solution $u \in D'$ of $L_2 u = f$ in ω. Put $u_1 \in C^\infty(\omega)$, $L_2 u_1 = f_1 \in C^\infty(\omega)$. Then the operator P is nonsolvable in ω for the right hand side $f + f_1$.

Let us make a change of the unknown function u in the operator L_2: $u = \varphi(v)$, $\varphi \in C^2\left(\mathbb{R}^1\right)$, $\varphi' > (<) 0$. Then $u_{x_i} = \varphi'(v) v_{x_i}$, $u_{x_i x_j} = \varphi'(v) v_{x_i x_j} + \varphi''(v) v_{x_i} v_{x_j}$.

Putting $\frac{\varphi''(v)}{\varphi'(v)} = g(v) \in C(\mathbb{R})$ we have $\varphi(v) = \int_0^v e^{\int_0^s g(\lambda)\, d\lambda} ds$, $\varphi'(v) = e^{\int_0^v g(\lambda)\, d\lambda}$, $\varphi''(v) = g(v) e^{\int_0^v g(\lambda)\, d\lambda}$.

EXAMPLE 2.7. (a) $L_2 = \partial_t^2 - a^2(t)\partial_x^2 + b(t)\partial_x$ (Egorov [5]).

Suppose that the equation $L_2 u = f \in C^\infty(\omega)$ is nonsolvable in $C^2(\omega)$. Then the nonlinear equation

$$\tilde{L}_2(v) = L_2(v) + g(v)\left(v_t^2 - a^2(t)v_x^2\right) = f e^{-\int_0^{v(t,x)} g(\lambda)\,d\lambda} \tag{8}$$

is nonsolvable in $C^2(\omega)$.

(b) Let $L_2 u_1 = f_1, u_1 \in C^\infty(\omega)$ and $L_2 u = f \in C^\infty(\omega)$ be nonsolvable in $C^2(\omega)$. Then the nonlinear equation

$$\tilde{\tilde{L}}_2(v) = L_2(v) + g(v)\left(v_t^2 - a^2 v_x^2\right) - f e^{-\int_0^v g(\lambda)\,d\lambda} = f_1 e^{-\int_0^v g(\lambda)\,d\lambda}. \tag{9}$$

is nonsolvable in $C^2(\omega)$.

Thus for each function $f_2 \in C^\infty(\omega)$, $\omega \ni 0$ we can find a nonlinear perturbation of the locally nonsolvable at 0 in D' operator L_2 and such that the corresponding nonlinear equation $\tilde{L}_2(v) = \tilde{f}_2 \left(\tilde{\tilde{L}}_2(v) = \tilde{\tilde{f}}_2\right)$ is nonsolvable in $C^2(\omega)$.

If the function $g \in C^\infty(\mathbb{R}^1)$ then the nonlinear operator $f(u)$ is well defined for each $u \in L^\infty \cap H^s_{\mathrm{loc}}$, $s > 0$. This way we prove nonsolvability of the equations considered in Example 2.7 not only in $C^2(\omega)$ but in the Sobolev spaces as well.

Proof of Example 2.1. After the polar change $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ we have that $P \to \frac{\partial}{\partial\varphi} + h(\rho^2)$ and $z^n = (x+iy)^n$, $x^2 + y^2 = 1$, $\forall n \in \mathbb{Z}$ are the eigenfunctions on the torus T^1 of the differential operator $\frac{d}{d\varphi}$; $Lz^n = inz^n$, $L = x\partial_y - y\partial_x$, $z = x + iy$.

Let $Pu = f \in C_0^\infty(\omega)$ and ω is a circular neighbourhood of the origin. Then $f(x,y) = f(\rho\cos\varphi, \rho\sin\varphi) = \sum_{-\infty}^{+\infty} f_n(\rho)e^{in\varphi}$, $f_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} f(\rho\cos\varphi, \rho\sin\varphi)e^{-in\varphi}\,d\varphi$, $f(0,0) = f_0(0)$, $f_n(0) = 0, n \neq 0$. Moreover, $|f_n(\rho)| \le \frac{C_k}{1+|n|^k}$, $\forall k \in \mathbb{Z}_+$, $C_k = \text{const.}$, $\|f\|^2_{L^2(\omega)} = 2\pi\sum_{-\infty}^{+\infty}\int_0^{\varepsilon_0}|f_n(\rho)|^2\rho\,d\rho$, $\varepsilon_0 = \operatorname{diam}\omega$.

We are looking for a solution u which is a vector valued distribution with values in $D'(T^1)$. Thus

$$u(x,y) = u(\rho,\varphi) = \sum_{-\infty}^{+\infty} u_n(\rho)e^{in\varphi},\ u_n(\rho) \in D'(0,\varepsilon_0),\ u \in D'\left((0,\varepsilon_0)\otimes T^1\right).$$

So $Pu = f$ implies that

$$u_n(\rho)\big(in + h(\rho^2)\big) = f_n(\rho),$$

as $\{e^{in\varphi}\}$ forms a basis in $D'(T^1)$.

We shall study several cases:

1. $h \equiv in_0$ for some $n_0 \in \mathbb{Z}$ ($\Rightarrow h \neq n, \forall n \neq n_0$)
2. $h(0) \notin i\mathbb{Z}$
3. $h(\rho^2) = in_0 + c_0\rho^{2k} + O(\rho^{2k+2})$, $k \in \mathbb{Z}_+$, $k \ge 1$, $c_0 = \text{const.} \neq 0$
4. $h(\rho^2) = in_0 + e(\rho^2)$ and $e(\rho^2)$ is flat at 0.

Case 1. $f_{-n_0}(\rho) = 0$ in $(0, \varepsilon_0)$, i.e. $\int_0^{2\pi} f(\rho\cos\varphi, \rho\sin\varphi)e^{in_0\varphi}\,d\varphi = 0\ \forall\rho \in (0, \varepsilon_0)$. Thus we have infinitely many compatibility conditions to be satisfied by the right handside f. So P is locally nonsolvable at 0 in D'.

Case 2. Let $P = L + c(x, y)$, $c \in C^\infty$ near 0, $c(0, 0) \notin i\mathbb{Z}$. Therefore, a more general case will be treated. Obviously $P = (L + c(0, 0)) + c(x, y) - c(0, 0) \equiv L_0 + d(x, y)$. Simple computations show that

$$\begin{aligned} \|Pu\|_{L^2(\omega)} &\geq \|L_0u\|_{L^2(\omega)} - \|d(x, y)u\|_{L^2(\omega)} \\ &\geq \|L_0u\|_{L^2(\omega)} + O(\operatorname{diam}\omega)\|u\|_{L^2(\omega)}, \forall u \in C_0^\infty(\omega)\,. \end{aligned}$$

Having in mind that $|in + c(0, 0)| \geq c_1 = \text{const.} > 0$ we conclude that

$$\|L_0u\|_0^2 = 2\pi\sum_{-\infty}^{+\infty}\int_0^{\varepsilon_0}\rho|u_n(\rho)|^2|in + c(0, 0)|^2\,d\rho \geq c_1^2\|u\|_0^2\,,$$

i.e. $\|Pu\|_0 \geq \frac{c_1}{2}\|u\|_0$, $\forall u \in C_0^\infty(\omega)$ taking $\operatorname{diam}\omega$ to be sufficiently small.

So, ${}^tP$ is locally solvable at 0. The same result is valid for P.

Case 3. Let $n \neq -n_0$ and $0 < \varepsilon_0 \ll 1$. Then $|i(n + n_0) + c_0\rho^{2k} + O(\rho^{2k+2})| \geq \frac{1}{2}$, while $u_{-n_0}(\rho) = \frac{f_{-n_0}(\rho)}{c_0\rho^{2k}(1+O(\rho^2))} \in D'(0, \varepsilon_0)$.

If we are looking for $L^2(\omega)$ solution of our problem we must impose the next additional requirements: $f_{-n_0}(0) = \ldots = f_{-n_0}^{(2k-1)}(0) = 0 \Rightarrow u_{-n_0}(\rho)$ will be smooth in $(0, \varepsilon_0)$.

We point out that in case 3.

$\left\|\rho^{2k}u\right\|_0 \leq d_0\|f\|_0$, $\forall u \in C_0^\infty(\omega)$, i.e. we have local solvability near the origin for each $f \in C_0^\infty(\omega)$ and the corresponding solution is such that $\left\|(x^2 + y^2)^k u\right\|_{L^2(\omega)} < \infty$.

Assume now that $\left\|\rho^{-2k}f\right\|_0 < \infty$. Then for each $u \in C_0^\infty(\omega)$: $|f(u)| \leq d_0\|Pu\|_0$ $\left(f(u) = \int_\omega \rho^{-2k}f\rho^{2k}u\right)$. According to Riesz representation theorem there exists a function $w \in L^2(\omega)$ such that $f(u) = (w, Pu)$, i.e. ${}^tPw = f$. Therefore, a local solvability result in L^2 is valid under finitely many compatibility conditions on f, namely $\left\|\rho^{-2k}f\right\|_{L^2(\omega)} < \infty$.

Case 4. Consider now the functions $\Theta(\rho^2)(x_1 + ix_2)^{-n_0}$, $\Theta \in C_0^\infty$, Θ flat at 0, $\Theta \not\equiv 0$, $0 \leq \Theta \leq 1$ for $0 \leq \rho \leq \frac{\varepsilon_0}{2}$ and $\Theta(\rho^2)(x_1+ix_2)^{n_0}$. Obviously $\Theta(\rho^2)(x_1+ix_2)^{-n_0} \in \operatorname{Ker} L_1 \cap S(\mathbb{R}^2)$, $L_1 = x\partial_y - y\partial_x + in_0$. As the operator L_1 is quasihomogeneous with $\mu_1 = \mu_2 = 1$ we apply theorem 2.1 and conclude that the operator P is locally nonsolvable at 0 in case 4.

□

Proof of Example 2.2. In case 1 we find a rapidly decreasing exponent in the kernel of $D_t^m - at$ and in case 2 in the kernel of $D_t^m \pm at$ by using Fourier transformation in t.

□

Proof of Example 2.3. Case 2. Then $L = D_x^2 + \left(D_y + x^kD_t\right)^2 - ka(0)x^{k-1}D_t - kx^{k-1}(a(x) - a(0))D_t = P + Q$ and Q has flat coefficients at 0. Put $P = \xi^2 + \left(\eta + x^k\tau\right)^2 - ka(0)\tau x^{k-1}$. Obviously P is quasihomogeneous with $\mu_1 = \mu_2 = 1$, $\mu_3 = k + 1$, $\gamma = -2$. In order to apply Theorem 2.1 we are seeking for a nontrivial solution $\varphi \in S(\mathbb{R}^3)$, $P(\varphi) = 0$. A partial Fourier transformation with respect to (y, t) gives us:

$$\hat{P} = D_x^2 + (\eta + x^k\tau)^2 - kx^{k-1}\tau = \left(D_x + i(\eta + x^k\tau)\right)\left(D_x - i(\eta + x^k\tau)\right)$$

(To fix the ideas we assume that $a(0) = 1$)

Evidently $\hat{u} = e^{-\eta x - \tau\frac{x^{k+1}}{k+1}} \in \operatorname{Ker}\left(D_x - i\left(\eta + x^k\tau\right)\right)$. We point out that $\hat{u}$ depends on two parameters η, τ and $\hat{u} \in S\left(\mathbb{R}^1_x\right)$ if (η, τ) belongs to a compact set in $\tau > 0$. So let

$$\varphi(x, y, t) = \iint_{\mathbb{R}^2} e^{i(\eta y + t\tau)} e^{-\left(\eta x + \frac{\tau x^{k+1}}{k+1}\right)} h_1(\tau) h_2(\eta)\, d\tau\, d\eta\,,$$

where $h_{1,2}$ are cut off functions, $\operatorname{supp} h_1 \in [1, 2]$, $0 < h_1(\tau) < 1$ if $\tau \in (1, 2)$, $\operatorname{supp} h_2 \in [0, 1]$, $0 < h_2(\eta) < 1$, if $\eta \in (0, 1)$.

We claim that $\varphi \in S\left(\mathbb{R}^3\right)$, $\varphi \not\equiv 0$. This fact can be verified by integration by parts, namely

$$\begin{aligned} \partial_\eta \left(e^{i(\eta y + t\tau)}\right) &= e^{i(\eta y + t\tau)} iy\,,\\ \partial_\tau \left(e^{i(\eta y + t\tau)}\right) &= e^{i(\eta y + t\tau)} it\,. \end{aligned}$$

The more complicated cases are (i) $|y| \geq A = \text{const.} > 0$, $|t| \leq A$, (ii) $|t| \geq A$, $|y| \leq A$, (iii) $|t| \geq A$, $|y| \geq A$. In case (i) we use the identity

$$\varphi = \frac{(-1)^N}{(iy)^N} \iint_{\mathbb{R}^2} e^{i(\eta y + t\tau)} \partial_\eta^N \left(e^{-\eta x - \tau\frac{x^{k+1}}{k+1}} h_2(\eta)\right) h_1(\tau)\, d\tau\, d\eta$$

and the fact that N is an arbitrary integer and $e^{-\eta x - \tau\frac{x^{k+1}}{k+1}} \in S\left(\mathbb{R}^1_x\right)$ form a bounded family in $S\left(\mathbb{R}^1_x\right)$ for $\{0 \leq \eta \leq 1, 1 \leq \tau \leq 2\}$.

The case (ii) is treated similarly as (i). In case (iii) we apply the identity

$$\varphi = \frac{1}{(it)^N (iy)^N} \iint_{\mathbb{R}^2} e^{i(\eta y + t\tau)} \partial_\tau^N \partial_\eta^N \left(h_1(\tau) h_2(\eta) e^{-\eta x - \tau\frac{x^{k+1}}{k+1}}\right) d\eta\, d\tau\,.$$

There are no difficulties to see that both the operators P, ${}^tP$ are locally nonsolvable at 0 which implies the nonsolvability of L, ${}^tL$.

Case 1. We make the change $z = \eta + x\tau$, $\tau \neq 0$ in the equation

$$\left[D_x^2 + (\eta + x\tau)^2 - a(0)\tau\right] \hat{u} = 0,\ \ a(0) \in \{\pm 1, \pm 3, \ldots\}$$

and we obtain

$$\left[D_z^2 \tau + \frac{z^2}{\tau} - a(0)\right] \hat{u}(z) = 0\,.$$

The change $z = \sqrt{\tau} y$, $\tau > 0$ leads us to the equation

$$\left(D_y^2 + y^2 - a(0)\right) v(y) = 0\,.$$

This is the harmonic oscillator equation if $a(0) \in \{1, 3, \ldots\}$. We remind to the reader that $v_n(x) = (-1)^n e^{\frac{x^2}{2}} \left(e^{-x^2}\right)^{(n)} \in S(\mathbb{R})$ are the solutions of $\left(D_x^2 + x^2\right) v_n = (2n + 1) v_n$, $n \in \{0, 1, 2, \ldots\}$. So we take $\hat{u} = v_n\left(\tau^{-1/2}\eta + x\tau^{1/2}\right)$ and then $\hat{u}$ form a bounded family in $S\left(\mathbb{R}^1_x\right)$ for $\{1 \leq \tau \leq 2, 0 \leq \eta \leq 1\}$.

There are no difficulties to verify that the function

$$\varphi = \iint_{\mathbb{R}^2} e^{i(\eta y + t\tau)} v_n \left(\tau^{-1/2}\eta + \tau^{1/2}x\right) h_1(\tau) h_2(\eta)\, d\eta\, d\tau$$

is nontrivial and belongs to $S(\mathbb{R}^3)$.

According to Theorem 2.1 both the operators $L, {}^tL$ are locally nonsolvable at 0 and are not C^∞ hypoelliptic too.

$\square$

Proof of Example 2.4. Let us make a partial Fourier transformation with respect to x in $Lu = 0$ and put $x \to \xi$, $\hat{u} = \hat{u}(t, \xi)$. So we have

$$\hat{u}_{tt} - \lambda^2(t)\xi^2\hat{u} + \tilde{a}\frac{\lambda^2(t)}{\Lambda(t)}\xi\hat{u} = 0, \;\; \tilde{a} = \text{const.}$$

The change $\hat{u} = t\hat{w}(t, \xi)$ in the previous equation leads to

$$\hat{w}_{tt} + \frac{2}{t}\hat{w}_t - \lambda^2\xi^2\left(1 - \frac{\tilde{a}}{\xi\Lambda}\right)\hat{w} = 0\,.$$

Entering in the complex domain by the change $\tau = i\Lambda(t)\xi$ we get

$$\hat{w}_{\tau\tau} + \frac{1}{\tau}\hat{w}_\tau + \left(1 - i\frac{\tilde{a}}{\tau}\right)\hat{w} = 0\,,$$

i.e.

$$\tau\hat{w}_{\tau\tau} + \hat{w}_\tau + (\tau - i\tilde{a})\hat{w} = 0\,.$$

Another change of the unknown function

$$\hat{w}\left(\frac{z}{2i}\right) = e^{-\frac{z}{2}} f(z)\,, \;\; z = 2i\tau = -2\Lambda(t)\xi$$

enables us to conclude that

$$z f_{zz} + (1 - z) f_z - \alpha f = 0\,, \;\; \alpha = \frac{1 + \tilde{a}}{2}\,. \tag{10}$$

But (10) is the confluent hypergeometric equation (Kummer's equation). As it is well known [1] the ODE (10) has two linearly independent solutions

$$f_1(z) = \psi(\alpha, 1, z)\,, \;\; f_2(z) = \psi(1 - \alpha, 1, -z)\,,$$

the function ψ being given by a rather complicated integral formula. In our special case $\tilde{a} = \pm(2n + 1)$, n nonnegative integer, or $\tilde{a} = \pm(2l + 1)$, l - nonnegative integer $\Rightarrow \alpha = -n$ if $\tilde{a} = -2n - 1$, and $1 - \alpha = -n$ if $\tilde{a} = 2n + 1$, $\alpha = -l$ if $\tilde{a} = -2l - 1$, and $1 - \alpha = -l$ if $\tilde{a} = 2l + 1$.

To fix the things let $\left\{ \begin{array}{l} a_+ = -2l - 1 \\ a_- = -2n - 1 \end{array} \right.$. According to the theory of special functions [1]

$$\psi(-n, 1, z) = (-1)^n n! L_n^0(z)\,, \;\; \psi(-l, 1, z) = (-1)^l l! L_l^0(z)\,,$$

and $L_n^0(z) = \frac{1}{n!}e^z\frac{d^n}{dz^n}\left(e^{-z}z^n\right)$ are the famous Laguerre polynomials, $z\left(L_n^0\right)'' + (1 - z)\left(L_n^0\right)' + nL_n^0 = 0$. Obviously, $L_n^0(0) = L_l^0(0) = 1$.

So $\hat{u} = t\hat{w} = \begin{cases} te^{\Lambda(t)\xi} L_n^0(-2\Lambda(t)\xi)\,, & t \le 0, \\ te^{\Lambda(t)\xi} L_l^0(-2\Lambda(t)\xi)\,, & t \ge 0. \end{cases}$ is a C^∞ solution of (2) with right-hand side 0.

Consider now the Fourier integral operator (FIO)

$$Ew_1(t,x) = u(t,x) = \int_{-\infty}^{\infty} h(\xi) e^{ix\xi} \hat{u}(t,\xi) \hat{w}_1(\xi)\, d\xi\,, \tag{11}$$

where $h \in C^\infty(\mathbb{R}^1)$, $h(\xi) = 1$, $\xi \le -1$, $0 \le h(\xi) \le 1$, $h = 0$ for $\xi \ge -1/2$, $w_1 \in \mathcal{E}'(\mathbb{R}^1)$.

Our investigations are microlocal in the cone $\Gamma : -\xi \ge c_0|\tau|$, $c_0 > 0$, $(t,x) \in \omega$, $(0,0) \in \omega$.

Obviously, $\mathrm{Re}\Lambda > 0$ for $t \ne 0$.

The kernel of (11) is given by

$$\int_{-\infty}^{0} h(\xi) e^{i(x-y)\xi} te^{\Lambda(t)\xi} L_n^0(-2\Lambda(t)\xi)\, d\xi \text{ for } t \le 0$$

and by

$$\int_{-\infty}^{0} h(\xi) e^{i(x-y)\xi} te^{\Lambda(t)\xi} L_l^0(-2\Lambda(t)\xi)\, d\xi \text{ for } t \ge 0$$

i.e. the phase function is $(x-y)\xi$ while the amplitude

$$a(x,y,t,\xi) = th(\xi) L_n^0(-2\Lambda(t)\xi) e^{\Lambda(t)\xi} \text{ for } t \le 0\,.$$

We shall prove that $a \in S^n_{1,1/2}(\Gamma)$. The same results are valid for $t \ge 0$.

So we have to show that $e^{\Lambda(t)\xi} \in S^0_{1,1/2}(\Gamma)$.

In fact, $\frac{\partial^k}{\partial t^k} \frac{\partial^l}{\partial \xi^l} e^{\Lambda\xi} = \frac{\partial^k}{\partial t^k}\left(\Lambda^l e^{\Lambda\xi}\right)$ and we have to prove at first inductively that

$$\left| \frac{\partial^k}{\partial t^k} e^{\Lambda(t)\xi} \right| \le c_k |\xi|^{\frac{k}{2}} e^{\mathrm{Re}\Lambda \frac{\xi}{2^k}}\,, \quad k = 0, 1, \ldots. \tag{12}$$

The observations that $\left(\Lambda_0'\right)^2 \le \mathrm{const}\Lambda_0$ and $e^{\cos\Phi \Lambda_0 \frac{\xi}{2^k}} \Lambda_0 \le \mathrm{const}|\xi|$ complete the proof of (12).

The estimation

$$\left| \frac{\partial^k}{\partial t^k} \frac{\partial^l}{\partial \xi^l} e^{\Lambda\xi} \right| \le d_{k,l} |\xi|^{-l+\frac{k}{2}} e^{\mathrm{Re}\Lambda \frac{\xi}{2^k}}$$

is proved inductively too with respect to k having in mind that

$$|\Lambda^l e^{\Lambda\xi}| \le c_l |\xi|^{-l} e^{\mathrm{Re}\Lambda \frac{\xi}{2}}\,, \quad l \ge 0$$

and that

$$|\Lambda' e^{\Lambda\xi}| \le \mathrm{const}|\xi|^{-\frac{1}{2}} e^{\mathrm{Re}\Lambda \frac{\xi}{2}}$$

Thus according to [3]

$$\Gamma \bigcap WF'(E) \subseteq \{(t,x,y;\tau,\xi,\eta) : x = y, \tau = 0, \xi = \eta < 0, t = 0\}$$

as $a \in S^{-\infty}_{1,1/2}$ for $t \ne 0$.

Obviously $LEw_1 = 0$. The restriction $Ew_1|_{t=0}$ is well defined as $WF'(E)\bigcap\{\tau \neq 0\} = \emptyset$ and

$$\frac{\partial}{\partial t}u(0,x) = \int_{-\infty}^{0} h(\xi)e^{ix\xi}\hat{w}_1(\xi)\,d\xi\,.$$

The ψ.d.o. just obtained is microlocally hypoelliptic for $\xi < 0 \Rightarrow WF(\partial_t u(0,x))\bigcap\Gamma = WF(u)\bigcap\Gamma$.

Taking $w_1 \in \mathcal{E}'(\mathbb{R}^1)$ s.t. $WF(w_1) = \{(0,\xi), \xi < 0\}$ we conclude that $\Gamma\bigcap WF(u) = \{(0,0,0,\xi) in -\xi \geq c_0|\tau|\}$.

The well known properties of the FIO E [3] enable us to define its formally adjoint operator ${}^tE$ by the formula

$$\langle {}^tEw(t,x), v(x)\rangle_{D'(\mathbb{R}^1)} = \langle w(t,x), Ev(x)\rangle_{D'(\mathbb{R}^2)}\,.$$

Let $v \in \mathcal{E}'(\mathbb{R}^2)$ and ${}^tLv = f\ \big(\Rightarrow f \in \mathcal{E}'(\mathbb{R}^2)\big) \Rightarrow {}^tE\,{}^tLv = {}^tEf \Rightarrow$

$$ {}^tEf = 0 \tag{13}$$

as $LE = 0$.

So the solvability of the equation ${}^tLv = f$ in $\mathcal{E}'$ leads to the fulfillment of infinitely many compatibility conditions by the right-hand side f.

Obviously, for each $g \in D'(\mathbb{R}^2)\ \big(\mathcal{E}'(\mathbb{R}^2)\big)$

$${}^tEg(y) = \iint_{\mathbb{R}^2} e^{-iy\xi}h(\xi)\hat{u}(t,\xi)\hat{g}(t,-\xi)\,d\xi\,dt$$

and $\hat{g}(t,-\xi)$ is the partial Fourier transformation of g with respect to x.

The necessary condition (13) on the right-hand side f of ${}^tLv = f$ for local solvability at the origin can be rewritten in the next form:

$$\begin{aligned}&\iint_{t\geq 0} e^{-iy\xi}h(\xi)te^{\Lambda(t)\xi}L_l^0(-2\Lambda(t)\xi)\,\hat{f}(t,-\xi)\,dt\,d\xi +\\ &\iint_{t\leq 0} e^{-iy\xi}h(\xi)te^{\Lambda(t)\xi}L_n^0(-2\Lambda(t)\xi)\,\hat{f}(t,-\xi)\,dt\,d\xi = 0\,,\end{aligned} \tag{14}$$

$\forall y \in \mathbb{R}^1$.

This way we proved the local nonsolvability of ${}^tL$ and the existence of a solution of $Lu = f \in C^\infty$ having $WF(u) = \{(0,0,0,\xi), \xi < 0\}$, i.e. a solution with an isolated singularity along a conic ray.

□

REMARK 2.2. The coefficients of (2) belong to Gevrey class G_2 and the projector on the kernel (11) can be estimated in the ultradistribution spaces G'_θ, $\theta \geq 2$. This way we have results on the existence of a solution with a prescribed Gevrey singularity along a conic ray as well we can prove a theorem on local nonsolvability in the corresponding ultradistribution spaces. To do this we use several results from [3] and the fact that the cutoff symbol $h(\xi)$ can be chosen in G_θ, for each $\theta > 1$.

Proof of Example 2.5. According to the necessary condition for local solvability established in the linear case we have that under the assumptions (5), (6)

$$\int_0^{2\pi} f(\rho,\varphi)\,d\varphi + \int_0^{2\pi} g(u(\rho,\varphi))\,d\varphi = 0\,,\ \ \forall\rho \in [0,\varepsilon_0]\,,\ \ \varepsilon_0 > 0$$
$$\Rightarrow g(u(\rho,\varphi)) = 0\,,\ \ \forall\varphi \in [0,2\pi]\,,\ \ \forall\rho \in [0,\varepsilon_0] \Rightarrow u(\rho,\varphi) \equiv 0 \Rightarrow$$

$\Rightarrow$ contradiction with (6).

□

Proof of Example 2.6. Assuming the existence of a solution $u \in C^1\big(x^2+y^2 < \rho_0^2\big)$ we get from (7)

$$\frac{\partial u}{\partial\varphi} + cu = f(u)\,,\ \ u(\rho,2\pi) = u(\rho,0)\,,$$

$\forall\rho \in [0,\rho_0]$, $\rho_0 > 0$. Thus for each $\rho \in (0,\rho_0]$ there exists $\varphi(\rho)$, $0 < \varphi(\rho) < 2\pi$ s.t. $\frac{\partial u}{\partial\varphi}(\rho,\varphi(\rho)) = 0 \Rightarrow$

$$\Rightarrow cu(\rho,\varphi(\rho)) = f(u(\rho,\varphi(\rho)))\,.$$

So the equation $c\lambda = f(\lambda)$ possesses a real root. Let λ_0 be a real root of the equation $c\lambda = f(\lambda)$. Then $u \equiv \lambda_0$ is a solution of (7).

□

Proof of Example 2.7 (a). Let $v \in C^2(\omega)$ be a solution of (8) and make the change $u = \varphi(v)$, $\varphi \in C^2\big(\mathbb{R}^1\big)$, $\varphi'(v) > 0$, $g = \frac{\varphi''(v)}{\varphi'(v)}$. Then the function u will satisfy in ω the equation $L_2 u = f \Rightarrow$ contradiction.

The case 2.7 (b) is obvious.

□

Acknowledgment. This paper was partially carried out during the visit of the author at the Department of Mathematics of the University of Torino and the Department of Mathematics of the University of Ferrara. Acknowledgments are due to Prof. Zanghirati and to Prof. Rodino for the useful discussions.

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AMS Subject Classification: 35A07, 35A20.

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Lavoro pervenuto in redazione il 3.12.1997 e, in forma definitiva, il 25.3.1998.