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ON LINKED SURFACES IN $\mathbb{P}^4$

Abstract. We give an elementary proof of a result of Katz relating invariants of linked surfaces in $\mathbb{P}^4$. A similar result is proved for volumes in $\mathbb{P}^5$. Then we try to connect the geometry of the curve $D = S \cap S'$ to the properties of the linked surfaces, for example we show that if D is a complete intersection, then one of the surfaces is a complete intersection too.

1. Introduction

Let us suppose that S and S' are smooth surfaces in $\mathbb{P}^4$, linked by a complete intersection of type (f, g). The problem is to compute the numerical invariants of S', supposing that those of S are known. We restrict the study to a particular type of liaison, which is called *nice linkage*, but it would be possible to work under wider hypotheses.
In general if S and S' are linked by a complete intersection, it is clear that $D = S \cap S'$ is a curve, since a complete intersection is connected. It is natural then to wonder whether this curve can tell us something about the surfaces involved in the linkage.
The problem of determining invariants of linked surfaces in $\mathbb{P}^4$ also leads to think about the well known conjecture concerning the irregularity of these surfaces.

Conjecture. There exists an integer M such that if $S \subset \mathbb{P}^4$ is a smooth surface, then $q(S) \leq M$.

Indeed if it were possible to compute exactly the irregularity of a surface linked to another whose invariants are all known, this would give a tool to verify the validity of the conjecture above.
The following section concerns numerical invariants, in particular we give an elementary proof of a result by S. Katz (see Lemma 2), which states a relation between invariants of linked surfaces. The main result in the third section is Prop. 2, which links the cohomology of S and S' with that of D. Then we try to see how particular properties of D translate in terms of the surfaces. We wonder what it would mean in terms of the surfaces if D is, respectively, a. C .M., complete intersection of three hypersurfaces or degenerate (see 1, 3, 4). We conclude with some considerations about the case of linked subvarieties in $\mathbb{P}^3$ and $\mathbb{P}^5$. In particular we stress the result in Proposition 5 (and Remark 4), in which it becomes clear how the Rao module of a curve $C \subset \mathbb{P}^3$ could limit the degrees of the surfaces producing a linkage involving C.
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2. Invariants of nicely linked surfaces

DEFINITION 1. *Let S and S' be smooth surfaces in $\mathbb{P}^4$ of degrees respectively d, d'. We say that S and S' are* nicely linked *if:*
1. $S \cup S'$ is a complete intersection $G \cap F$, where F, G are hypersurfaces of degrees f, g

respectively;
2. $S \cap S'$ is a smooth curve D;
3. G may be chosen to be smooth away from D, with finitely many nodes on D.

The following result is useful in order to grant the existence of hypersurfaces of certain degrees nicely linking S to S'.

PROPOSITION 1. *Let S be a smooth surface in $\mathbb{P}^4$, if $\mathcal{I}_S(k)$ is globally generated, then for every $f, g \geq k$ we can find hypersurfaces F, G nicely linking S to a smooth surface S'.*

For a proof, see [1], Prop. 4.1. From now on, we assume that S and S' are nicely linked. The next lemma provides a formula for the degree and the genus of the curve D, in terms of the degrees of the hypersurfaces F and G and of the sectional genera of the surfaces S and S'.

LEMMA 1. *Let $S, S' \subset \mathbb{P}^4$ be smooth surfaces nicely linked by a complete intersection $F \cap G$ of type (f, g), $D = S \cap S'$, with sectional genera π, π' respectively, then:*

$$deg(D) = 2 + \frac{fg}{2}(f + g - 4) - \pi - \pi' \tag{1}$$

$$g(D) = 1 + \frac{deg(D)(f + g - 5)}{2}$$

and D is a subcanonical curve with $\omega_D = \mathcal{O}_D(f + g - 5)$.

Proof. Let H be a general hyperplane, we set $C = S \cap H$, $C' = S' \cap H$. Thus C and C' are two curves in $\mathbb{P}^3$, linked by the complete intersection $C \cup C' = (H \cap F) \cap (H \cap G)$. We have Mayer-Vietoris sequence:

$$0 \to \mathcal{O}_{C \cup C'} \to \mathcal{O}_C \oplus \mathcal{O}_{C'} \to \mathcal{O}_\Gamma \to 0$$

where $\Gamma = C \cap C'$, from which we infer: $p_a(C \cup C') = \pi + \pi' - 1 + card(\Gamma)$. Obviously $card(\Gamma) = deg(D)$ and since $C \cup C'$ is a complete intersection, its arithmetical genus can be computed easily as $p_a(C \cup C') = 1 + \frac{fg}{2}(f + g - 4)$, so we get the desired formula: $deg(D) = 2 + \frac{fg}{2}(f + g - 4) - \pi - \pi'$.
In order to compute the genus, we consider the exact sequence of liaison:

$$0 \to \mathcal{I}_U \to \mathcal{I}_S \to \omega_{S'}(5 - f - g) \to 0$$

where $U = S \cup S'$. Clearly $\omega_{S'}(5 - f - g) = \mathcal{I}_{S,U}$, the sheaf of functions on U which vanish on S. Observing that $\mathcal{I}_{S,U}$ has support S', we get $\mathcal{I}_{S,U} = \mathcal{I}_{D,S'} = \mathcal{O}_{S'}(-D)$, since D is a divisor on S'. Thus $\omega_{S'} = \mathcal{O}_{S'}(-D + f + g - 5)$ and by adjunction $\omega_D = \mathcal{O}_D(f + g - 5)$, in particular D is a subcanonical curve. Looking at the degrees we obtain: $2g(D) - 2 = deg(D)(f + g - 5)$. □

LEMMA 2. *Let $S, S' \subset \mathbb{P}^4$ be smooth surfaces nicely linked by the complete intersection $U = S \cup S' = F \cap G$, $D = S \cap S'$, then:*

$$p_g(U) = p_g(S) + p_g(S') - q(S) - q(S') + g(D) \tag{2}$$

Proof. We consider Mayer-Vietoris sequence:

$$0 \to \mathcal{O}_U \to \mathcal{O}_S \oplus \mathcal{O}_{S'} \to \mathcal{O}_D \to 0$$

and taking cohomology we have: $h^2(\mathcal{O}_U) = h^2(\mathcal{O}_S) + h^2(\mathcal{O}_{S'}) + h^1(\mathcal{O}_D) + h^1(\mathcal{O}_U) + h^0(\mathcal{O}_S) + h^0(\mathcal{O}_{S'}) - h^1(\mathcal{O}_S) - h^1(\mathcal{O}_{S'}) - h^0(\mathcal{O}_D) - h^0(\mathcal{O}_U)$.
As U is a complete intersection (f, g), its minimal free resolution is:

$$0 \to \mathcal{O}(-f-g) \to \mathcal{O}(-f) \oplus \mathcal{O}(-g) \to \mathcal{I}_U \to 0$$

so $h^1(\mathcal{I}_U) = h^2(\mathcal{I}_U) = 0$, which yields $h^0(\mathcal{O}_U) = 1$ and $h^1(\mathcal{O}_U) = 0$. Furthermore $h^0(\mathcal{O}_S) = h^0(\mathcal{O}_{S'}) = h^0(\mathcal{O}_D) = 1$, then we conclude.

□

REMARK 1. (i) This lemma was proven by S. Katz in [2], Cor. 2.4
(ii) The preceeding formula holds even if we are not in a situation of nice linkage, it is enough to have S, S' smooth and D equidimensional.
(iii) This lemma provides a relation between invariants of linked surfaces, however it does not allow us to determine such invariants completely. In fact in the general situation we are able to compute only the difference between $q(S')$ and $p_g(S')$. This impediment was to be expected if we think about the conjecture mentioned formerly. In some particular cases it is possible to determine $q(S')$ or $p_g(S')$ using different techniques and, thanks to formula (2), to compute the remaining one. For example if one of the surfaces is arithmetically Cohen-Macaulay, say S, then also the other one is a. C. M.. This implies that $q(S) = q(S') = 0$ and in such a situation all invariants of S' are determined by knowing those of S. There are also examples of non a. C. M. surfaces whose properties allow anyway to compute q and p_g for a surface linked to them.

3. The curve D

PROPOSITION 2. *With the previous notations:*

$$h^1(\mathcal{I}_D(m)) = h^1(\mathcal{I}_S(m)) + h^1(\mathcal{I}_{S'}(m)) \tag{3}$$

for every $m \in \mathbb{Z}$*.*

Proof. Let us consider the exact sequence:

$$0 \to \mathcal{I}_U(m) \to \mathcal{I}_S(m) \oplus \mathcal{I}_{S'}(m) \to \mathcal{I}_D(m) \to 0$$

taking cohomology we get: $\ldots \to H^1(\mathcal{I}_U(m)) \to H^1(\mathcal{I}_S(m)) \oplus H^1(\mathcal{I}_{S'}(m)) \to H^1(\mathcal{I}_D(m)) \to H^2(\mathcal{I}_U(m)) \to \ldots$
Since U is a complete intersection: $h^1(\mathcal{I}_U(m)) = h^2(\mathcal{I}_U(m)) = 0$ and we get the desired formula.

□

COROLLARY 1. *1. If S and S' are a. C. M., then D is a. C. M. too;*
2. if D is a. C. M., then S and S' are projectively normal and $q(S) = q(S') = 0$;
3. $h^1(\mathcal{I}_D(f+g-5)) = q(S) + q(S')$.

Proof. 1. If S and S' are a. C. M., then $h^1(\mathcal{I}_S(m)) = h^1(\mathcal{I}_{S'}(m)) = 0$ for every $m \in \mathbb{Z}$ and by Prop. 3.1 this implies that $h^1(\mathcal{I}_D(m)) = 0$.
2. If D is a. C. M. we have $h^1(\mathcal{I}_D(m)) = 0$ for every m, then $h^1(\mathcal{I}_S(m)) = h^1(\mathcal{I}_{S'}(m)) = 0$ too.
3. We recall that if S, $S' \subset \mathbb{P}^4$ are surfaces linked by a complete intersection (f, g) we have $h^2(\mathcal{I}_{S'}(m)) = h^1(\mathcal{I}_S(f+g-5-m))$. Considering formula (3) in Proposition 2 we obtain: $h^1(\mathcal{I}_D(f+g-5)) = h^2(\mathcal{I}_S) + h^2(\mathcal{I}_{S'}) = q(S) + q(S')$ using Serre duality.

□

REMARK 2. This result (part 3.) is of some interest if we consider the conjecture about bounding the irregularity. Again it is not possible to compute $q(S)$ but it becomes clear that the curve D carries informations about the cohomology of the surfaces. We have already observed that D is a subcanonical curve. We could hope to start from a subcanonical curve D on a surface S, such that $h^1(\mathcal{I}_D(f+g-5)) - q(S)$ is greater than one, and try to obtain D linking S to a smooth surface S', which would have $q > 1$. However this is probably an hopeless program. Furthermore we have to deal with the following problem: given a smooth surface S, is it possible to find surfaces S', linked to S, such that every subcanonical curve $D \subset S$ can be obtained as $S \cap S'$? The answer to this question is negative, let us consider the following counterexample.

EXAMPLE 1. Let S be Del Pezzo surface in $\mathbb{P}^4$, then S is a rational surface of degree d=4, with sectional genus $\pi = 1$, complete intersection of two hyperquadrics. One can demonstrate (see for instance [3], Theorem 10) that a divisor C on S is a smooth subcanonical curve if and only if C is one of the following:

(a) C is a line and $\omega_C = \mathcal{O}_C(-2)$;

(b) C is a smooth plane conic and $\omega_C = \mathcal{O}_C(-1)$;

(c) $C \sim (\alpha+1)H$ and $\omega_C = \mathcal{O}_C(\alpha)$, $\alpha \geq 0$, where H is an hyperplane divisor on S;

(d) $C \sim (\alpha+1)H + \sum_{j=1}^{k}(\alpha+1)L_j$ and $\omega_C = \mathcal{O}_C(\alpha)$, $\alpha \geq 0$, where $L_1, \ldots, L_k$ are $k \geq 1$ mutually skew lines.

We recall that if $C = S \cap S'$, where S and S' are nicely linked by a complete intersection (f, g), we have $\omega_C = \mathcal{O}_C(f+g-5)$.
It is easy to see that the first two types of subcanonical curves on S mentioned above cannot be realized in such a way. In fact we would have $f+g \leq 4$, so $d' = deg(S') < 1$, which is absurd. For what concerns the third class of curves, as to say multiples of hyperplane divisors, we have better hopes to find a couple of hypersurfaces producing these curves as explained before. Indeed if $C \in |mH|$, C is a. C. M. for every $m \geq 1$. Now if we consider a complete intersection $(2, m+2)$, we obtain that the intersection of S with the residual surface S' is a curve D of degree $4m$ (using the formula (1) in Lemma 1), which is the degree of $C \in |mH|$.
Now we come to the last type of subcanonical divisors on S. Let us consider $C \sim H + L$, where L is a line, $\omega_C = \mathcal{O}_C$ and C is a non degenerate elliptic quintic, then C is a. C. M.. If we suppose that C could be realized as $S \cap S'$, where S and S' are linked by a complete intersection (f, g), we obtain that $deg(C) = 4(f+g-4)$. It is clear that the quantity $4(f+g-4)$ could never be equal to five, for any $f, g \geq 1$, so $C \sim H + L$ is not one of the curves we are looking for.
We have shown with several counterexamples that not every subcanonical curve on a certain surface S is given by $S \cap S'$, with S and S' linked by a complete intersection, not even if we restrict to a. C. M. curves.

Now we examinate the case in which D is a complete intersection of three hypersurfaces F_a, F_b, F_c of degrees respectively a, b, c. Suppose $a \leq b \leq c$. For each hypersurface F_k we have to deal with the following question: does F_k contain one of the surfaces S, S'?
Let us consider: $0 \to H^0(\mathcal{I}_S(k)) \to H^0(\mathcal{I}_D(k)) \xrightarrow{\pi} H^0(\mathcal{I}_{D,S}(k)) \to \ldots$
Suppose F_k does not contain S, then F_k provides a non zero element

$$F_k' = \pi(F_k) \in H^0(\mathcal{I}_{D,S}(k)).$$

We also have the exact sequence:

$$0 \to H^0(\mathcal{I}_U(k)) \xrightarrow{i} H^0(\mathcal{I}_{S'}(k)) \xrightarrow{p} H^0(\mathcal{I}_{D,S}(k)) \to 0$$

Since p is surjective, there exists $\tilde{F}_k \in H^0(\mathcal{I}_{S'}(k))$ such that $p(\tilde{F}_k) = F_k'$. Observe that F_k and $\tilde{F}_k$ coincide over S, then $G_k = \tilde{F}_k - F_k$ belongs to $H^0(\mathcal{I}_S(k))$.

Since F_k and $\tilde{F}_k$ coincide over S, we could replace F_k with $\tilde{F}_k$ and consider D as the complete intersection $\tilde{F}_a \cap \tilde{F}_b \cap \tilde{F}_c$. We could always manage to have $D = E_a \cap E_b \cap E_c$, where the hypersurfaces E_k are such that either E_k contains S or it contains S'. In other words we can say that for $k = a, b, c$, $E_k \in H^0(\mathcal{I}_S(k))$ or $E_k \in H^0(\mathcal{I}_{S'}(k))$.

PROPOSITION 3. *With the notations above, let D be a complete intersection of three hypersurfaces of non decreasing degrees $a \le b \le c$, i.e. $D = F_a \cap F_b \cap F_c$, then one of the surfaces S, S' is a complete intersection too.*

Proof. It is clear from what said before that one of the surfaces S, S' is contained in two of the three hypersurfaces F_k, say $S \subset F_a \cap F_b$. In general we will have a residual surface $\tilde{S}$, such that $S \cup \tilde{S} = F_a \cap F_b$. However, this would imply that $D = F_c \cap (S \cup \tilde{S}) = (F_c \cap S) \cup (F_c \cap \tilde{S})$, but we recall that D is irreducible, then necessarily $\tilde{S} = \emptyset$ and $S = F_a \cap F_b$.

□

REMARK 3. The preceeding result has this consequence: if D is a complete intersection then just one of the surfaces is a complete intersection too, anyway both are a. C. M. and this implies that $q(S) = q(S') = 0$.

If we suppose D is a degenerate curve, we have the following result, which brings back to the case in which D is a complete intersection and allows us to apply Proposition 3.

PROPOSITION 4. *If D is degenerate, then D is a complete intersection.*

Proof. If D is degenerate, there exists an hyperplane H containing D, and from the previous discussion, it follows that H contains one of the surfaces S, S'. A degenerate surface S in $\mathbb{P}^4$ is a. C. M., to see it just consider the cone K over S in $\mathbb{P}^4$, S turns out to be the complete intersection of K and H. Then S and S' are a. C. M. and consequently also D is so. Moreover it is clear that if a degenerate curve is a. C. M. in $\mathbb{P}^4$, it is a. C. M. in $H \simeq \mathbb{P}^3$ too. We recall that, by Gherardelli's theorem, if $D \subset \mathbb{P}^3$ is a subcanonical, a. C. M. curve, then D is a complete intersection.

□

4. Liaison in $\mathbb{P}^3$ and $\mathbb{P}^5$

In this section we consider liaison between subvarieties in $\mathbb{P}^3$ and in $\mathbb{P}^5$.

PROPOSITION 5. *Let $C, C' \subset \mathbb{P}^3$ be curves geometrically linked by a complete intersection of type (a, b), and let D be the zerodimensional scheme $C \cap C'$, then:*

$$h^1(\mathcal{I}_C(m)) + h^1(\mathcal{I}_{C'}(m)) \le h^1(\mathcal{I}_D(m))$$

for every $m \in \mathbb{Z}$.

Proof. The proof is the same as in Proposition 2, but this time $h^2(\mathcal{I}_{C \cup C'}(m))$ is not necessarily zero, so only the inequality holds.

□

REMARK 4. The preceeding result is interesting even if it looks weaker than the one for surfaces.

We recall that for linked curves in $\mathbb{P}^3$ we have: $h^1(\mathcal{I}_{C'}(m)) = h^1(\mathcal{I}_C(a+b-4-m))$. Moreover $h^1(\mathcal{I}_D(m)) \leq deg(D)$, if D has dimension zero, thus we obtain the bound: $h^1(\mathcal{I}_C(m)) + h^1(\mathcal{I}_C(a+b-4-m)) \leq deg(D)$. It is possible to express $deg(D)$ as a function of the invariants a, b, d, g, where d, g are the degree and the genus of C, and we get: $deg(D) = 2-2g+d(a+b-4)$.
In the end we can write the formula: $h^1(\mathcal{I}_C(m))+h^1(\mathcal{I}_C(a+b-4-m)) \leq 2-2g+d(a+b-4)$. Note that just the fact of being able to make a linkage produces this bound on the cohomology of C; conversely the knowledge of the Rao function of C gives necessary conditions in order to link C.

For what concerns the liaison of threefolds in $\mathbb{P}^5$, we have the following result.

PROPOSITION 6. *Let S, $S' \subset \mathbb{P}^5$ be two threefolds, nicely linked by a complete intersection (a, b), and let D be the smooth surface $S \cap S'$, then:*

$$h^1(\mathcal{I}_S(m)) + h^1(\mathcal{I}_{S'}(m)) = h^1(\mathcal{I}_D(m))$$

$$h^2(\mathcal{I}_S(m)) + h^2(\mathcal{I}_{S'}(m)) = h^2(\mathcal{I}_D(m))$$

for every $m \in \mathbb{Z}$ and D is a subcanonical surface with $\omega_D = \mathcal{O}_D(a+b-6)$.

Proof. As in the proof of Proposition 2, we obtain the two equalities considering cohomology of the exact sequence: $0 \to \mathcal{I}_U(m) \to \mathcal{I}_S(m) \oplus \mathcal{I}_{S'}(m) \to \mathcal{I}_D(m) \to 0$. Indeed U a complete intersection and so $h^1(\mathcal{I}_U(m)) = h^2(\mathcal{I}_U(m)) = h^3(\mathcal{I}_U(m)) = 0$.
Then we look at liaison exact sequence: $0 \to \mathcal{I}_U \to \mathcal{I}_S \to \omega_{S'}(6-a-b) \to 0$, by adjunction we have again that $\omega_D = \mathcal{O}_D(a+b-6)$, so D is a subcanonical surface in $\mathbb{P}^5$.

□

LEMMA 3. *With the notations above:*

$$h^2(\mathcal{O}_{S'}) - h^3(\mathcal{O}_{S'}) = p_g(D) - q(D) - h^3(\mathcal{O}_U) - h^2(\mathcal{O}_S) + h^3(\mathcal{O}_S) \tag{4}$$

Proof. The proof is exactly the same as in Lemma 2.3, recalling that, by Barth's theorem, $h^1(\mathcal{O}_S) = 0$ for a threefold in $\mathbb{P}^5$.

□

REMARK 5. Clearly the formula (4) above still holds if S and S' are not nicely linked, it is enough for example to have S and S' smooth and D equidimensional. To have D subcanonical we only need D to be a Cartier divisor on one of the threefolds S or S'. Indeed, if so, at least one of the threefolds is smooth and we can proceed as in the proof of Proposition 6.

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