Coxeter Structure and Finite Group Action

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Abstract

Let $U(\mathfrak{g})$ be the enveloping algebra of a semi-simple Lie algebra $\mathfrak{g}$. Very little is known about the nature of $Aut\, U(\mathfrak{g})$. However, if G is a finite subgroup of $Aut\, U(\mathfrak{g})$ then very general results of Lorenz-Passman and of Montgomery can be used to relate $Spec\, U(\mathfrak{g})$ to $Spec\, U(\mathfrak{g})^G$. As noted by Alev-Polo one may read off the Dynkin diagram of $\mathfrak{g}$ from $Spec\, U(\mathfrak{g})$ and they used this to show that $U(\mathfrak{g})^G$ could not be again the enveloping algebra of a semi-simple Lie algebra unless G is trivial. Again let U be the minimal primitive quotient of $U(\mathfrak{g})$ admitting the trivial representation of $\mathfrak{g}$. A theorem of Polo asserts that if U^G is isomorphic to a similarly defined quotient of $U(\mathfrak{g}') : \mathfrak{g}'$ semi-simple, then $\mathfrak{g} \cong \mathfrak{g}'$. However in this case one cannot say that G is trivial.

The main content of this paper is the possible generalization of Polo's theorem to other minimal primitive quotients. A very significant technical difficulty arises from the Goldie rank of the almost minimal primitive quotients being > 1. Even under relatively strong hypotheses (regularity and integrality of the central character) one is only able to say that the Coxeter diagrams of $\mathfrak{g}$ and $\mathfrak{g}'$ coincide. The main thrust of the proofs is a systematic use of the Lorenz-Passman-Montgomery theory and the known very detailed description of $Prim\, U(\mathfrak{g})$. Unfortunately there is a severe lack of good examples. During this work some purely ring theoretic results involving Goldie rank comparisons and skew-field extensions are presented. A new inequality for Gelfand-Kirillov dimension is obtained and this leads to an interesting question involving a possible application of the intersection theorem.

Résumé

Soit $U(\mathfrak{g})$ l'algèbre enveloppante d'une algèbre de Lie semi-simple $\mathfrak{g}$. On sait très peu de choses sur $Aut\, U(\mathfrak{g})$. Néanmoins, si G désigne un sous-groupe fini de $Aut\, U(\mathfrak{g})$, alors des résultats généraux de Lorenz-Passman et Montgomery relient $Spec\, U(\mathfrak{g})$ à $Spec\, U(\mathfrak{g})^G$. Alev et Polo ont observé qu'on peut lire le diagramme de Dynkin de $\mathfrak{g}$ sur $Spec\, U(\mathfrak{g})$ et ils en ont déduit que $U(\mathfrak{g})^G$ ne peut être isomorphe à l'algèbre enveloppante d'une algèbre de Lie que si G est trivial. Soit U le quotient primitif minimal de $U(\mathfrak{g})$ admettant la représentation triviale de $\mathfrak{g}$. D'après un théorème de Polo, si U^G est isomorphe à un quotient

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de $U(\mathfrak{g}')$: $\mathfrak{g}'$ semi-simple, alors $\mathfrak{g} \cong \mathfrak{g}'$. Mais dans ce cas on ne peut affirmer que G est trivial.

Le contenu principal de ce papier est une possible généralisation du résultat de Polo à d'autres quotients primitifs minimaux. Une difficulté technique significative provient du fait que la dimension de Goldie peut alors être > 1. Même sous des hypothèses relativement fortes (régularité et intégralité du caractère central) on peut seulement dire que les diagrammes de Coxeter de $\mathfrak{g}$ et $\mathfrak{g}'$ coincident. Les preuves sont basées sur une utilisation systématique de la théorie de Lorenz-Passman et Montgomery et la connaissance très détaillée de $Prim\, U(\mathfrak{g})$. Malheureusement, il y a un manque sévère d'exemples. Dans ce travail, on présente quelques résultats de théorie des anneaux concernant des comparaisons de rangs de Goldie et des extensions de corps gauches. On obtient une nouvelle inégalité pour la dimension de Gelfand-Kirillov qui conduit à une question intéressante concernant une application du théorème d'intersection.

1 Introduction

1.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $U(\mathfrak{g})$ its enveloping algebra. Let G be a finite subgroup of Aut$U(\mathfrak{g})$. A remarkable recent result of J. Alev and P. Polo [AP, Thm.1] shows that $U(\mathfrak{g})^G$ cannot be again the enveloping algebra of some possibly different semisimple Lie algebra $\mathfrak{g}'$ unless G is trivial. Again let U_ρ (resp. V_ρ) be the minimal primitive quotient of $U(\mathfrak{g})$ (resp. $U(\mathfrak{g}')$) admitting the trivial representation of $\mathfrak{g}$ (resp. $\mathfrak{g}'$) and G a finite subgroup of AutU_ρ. Polo [P, Thm.7.1] has shown that if $U_\rho^G \cong V_\rho$ then $\mathfrak{g} \cong \mathfrak{g}'$.

1.2. The proof of the above results uses some general results on finite group actions (see Section 2) and some knowledge of Prim$U(\mathfrak{g})$. However the proofs are not particularly difficult and need relatively little from these two theories.

1.3. The aim of this paper is to generalize Polo's theorem to arbitrary (regular) central characters. At present the only interest for doing this is that the problem becomes very significantly harder and we need practically all that is known on the two theories discussed in 1.2. The obvious critique is that we know of no non-trivial examples of such finite group actions. Yet for example take $\mathfrak{g}$ of type B_2 (resp. G_2) with $\mathbb{Z}_2$ (resp. $\mathbb{Z}_3$) $\subset$ Aut$U(\mathfrak{g})$ acting via scalar multiplication on short root vectors. Then the maximal completely prime ideal P associated to the 4 (resp. 6) dimensional coadjoint orbit [J1] is $\mathbb{Z}_2$ (resp. is $\mathbb{Z}_3$) stable and "accidentally" the fixed subalgebra is isomorphic to a minimal primitive quotient of $U(\mathfrak{sl}(2) \times \mathfrak{sl}(2))$ (resp. $U(\mathfrak{sl}(3))$).

1.4. In Section 2 we review some general results on finite group actions and in particular the Montgomery bijection. In Section 3 we develop some comparison results on Goldie rank, particularly with respect to the additivity principle. In Section 4 we show that the isomorphism $U_\lambda^G \cong V_\mu$ (where λ, μ are dominant, regular elements of the appropriate Cartan subalgebras) implies that the (relative) Coxeter diagrams pertaining to U_λ and V_μ are isomorphic (Theorem 4.20). Unlike the situation encountered in the special case of Polo's theorem we are not able to say that G orbits in SpecU_λ are trivial (which also "trivializes" Montgomery's bijection). In Section 5 we relate λ, μ through an additivity principle (Theorem 5.16). However we are *not* able to say that the (relative) Dynkin diagrams pertaining to U_λ and V_μ are isomorphic. This question is examined in Section 6 where we show that it cannot be resolved by passage to rings of fractions and analysis of Goldie rank except in what we call the indivisible Goldie rank case (Theorem 6.7). This occurs for example in Polo's situation and leads to a refinement of that result. I would like to thank the referee for some remarks and corrections.

2 Finite Group Actions on Rings

2.1. Let B be a ring, G a finite subgroup of Aut$B$ and $A := B^G$ the fixed ring. A number of remarkable very general results relating Spec$B$ to Spec$A$ derive from the work of G. Bergman and I.M. Isaacs [BI], M. Lorenz and D.S. Passman [LP] and S. Montgomery [M2]. We detail what we need of this theory below. It will be assumed here and throughout this paper that $|G| \neq 0$ in B. We remark that in applications A, B are assumed noetherian and then the resulting weaker versions of these results partly go further back.

2.2. It is clear that G acts on Spec$B$ which is hence a disjoint union of G orbits which are in turn finite sets. Given $P \in$ Spec$B$ we denote by $O(P)$ the G orbit containing P. Then $I(P) := \bigcap_{P_i \in O(P)} P_i$, or simply I, is G invariant and so it is natural to consider I^G which is a semiprime ideal [BI] of A. Note however that $I^G = P_i^G = P_i \cap A$ for all $P_i \in O(P)$. If $P, P' \in$ Spec$B$ lie in the same G orbit we write $P \sim P'$. Obviously

Lemma — *The following are equivalent*

(i) $I(P) \supset I(P')$.

(ii) *For all* $P_i \in O(P)$ *there exists* $P'_j \in O(P')$ *such that* $P_i \supset P'_j$.

(iii) *For all* $P'_i \in O(P')$ *there exists* $P_j \in O(P)$ *such that* $P_j \supset P'_i$.

We write $O(P) \geq O(P')$ when one of these hold.

2.3. Define the group ring BG to be the free B module on generators $g \in G$ with multiplication $(bg, b'g') = (bg(b'), gg')$ where $b' \mapsto g(b')$ denotes the action of G on B.

Set $e = \frac{1}{|G|} \sum_{g \in G} g$ which is an idempotent of BG. After a classical result of Jacobson (see [LP, Lemma 4.5] for example) the map $\varphi : Q \mapsto eQe$ is an order isomorphism of $\{Q \in \text{Spec}BG \mid e \notin Q\}$ onto Spec$A$. Extend φ to a bijection of semiprime ideals. Define an equivalence relation $\sim$ on Spec$A$ by $p \sim p'$ if $\varphi^{-1}(p) \cap B = \varphi^{-1}(p') \cap B$ and let $C(p)$ denote the equivalence class containing p. Set $I(p) = \bigcap_{p' \in C(p)} p'$.

2.4. There are three key facts which lead to the Montgomery isomorphism [M2, Sect.3], namely

(i) For all $Q \in \text{Spec}BG$ there exists $P \in \text{Spec}B$ such that $Q \cap B = I(P)$.

(ii) $Q_i \in \text{Spec}BG$ is minimal over $I(P)G \iff Q_i \cap B = I(P)$. Moreover there are finitely many such Q_i and $\bigcap_{i=1}^n Q_i = I(P)G$.

(iii) If an ideal J of BG strictly contains a prime Q then $J \cap B \supsetneq Q \cap B$.

With the exception of the very last statement of (ii), these are due in their most general form to Lorenz and Passman [LP, Lemma 4.2, Thm. 1.3, Thm. 1.2].

In (ii) choose $m \in \mathbb{N}$ and order the Q_i so that $e \in Q_i \iff i > m$. Taking $Q = \varphi^{-1}(p) : p \in \text{Spec}A$ it follows from (ii) that $C(p) = \{\varphi(Q_i) : i \leq m\}$ and $\varphi^{-1}(p) \cap A = \bigcap_{i=1}^m \varphi(Q_i)$. In particular we note the

Lemma — *$C(p)$ is the set of minimal primes over $\varphi^{-1}(p) \cap A$.*

2.5. The above result immediately leads [M2, 3.5 (3)] to a partial analogue of 2.2 namely

Lemma — *The following are equivalent*

(i) $\varphi^{-1}(p) \cap B \supset \varphi^{-1}(p') \cap B$.

(ii) *For each $p_i \in C(p)$ there exists $p'_j \in C(p')$ such that $p_i \supset p'_j$.*

Proof. Assume (i). Then $\varphi^{-1}(p_i) \supset \bigcap_{j=1}^{n'} Q'_j$ and so p_i contains some $\varphi(Q'_j)$. □

We write $C(p) \geq C(p')$ when one of these hold. For our purposes it is a significant technical difficulty that we have no analogue of 2.2 (iii).

2.6. From 2.4 (i) and 2.4 (ii) one immediately obtains [M2, Thm. 3.4] the

Theorem — *The map $p \mapsto O(P)$, where $I(P) = \varphi^{-1}(p) \cap B$ factors to an order isomorphism Φ of* Spec$A/\sim$ *onto* Spec$B/\sim$.

2.7. It is clear that primes of Spec$B$ in the same G orbit have the same height. By 2.4 (iii) it follows [M2, Prop. 3.5] that equivalent primes of Spec$A$ are incomparable and have the same height. Moreover

Lemma — *One has $htp = htP$ given $P \in \Phi(p)$.*

2.8. Whilst $\varphi^{-1}(I(P)^G) = I(P)G$ it is not quite obvious if this implies that the inclusion $BI(P)^G B \subset I(P)$ is an equality. Fortunately we shall only need the

Lemma — *The minimal primes over $BI(P)^G B$ are the $P_i \in O(P)$.*

Proof. If P' is a minimal prime over $BI(P)^G B$ then so are its G translates and $I(P') \supset BI(P)^G B$. Consequently $I(P')^G \supset I(P)^G$. Then $I(P') \supset I(P)$ from 2.6 or directly from 2.4. Then P' contains some $P_i \in O(P)$. Conversely $BI(P)^G B \subset I(P)$ so P' is contained in some $P_j \in O(P)$. ☐

3 Goldie Rank Comparison

3.1. We use some fairly standard methods to compare various Goldie ranks. In this we retain the hypotheses and notation of Section 2 except that B is assumed semisimple artinian. This implies (see for example [M1, Cor. 0.2 and Thm. 1.15]) the correspond ing properties for BG and then for A.

3.2. Observe that a left BG submodule of B is just a left ideal which is G stable.

Lemma — *A left ideal L of B is G stable (resp. and minimal) if and only if it takes the form Be with $e \in A$ idempotent (resp. and minimal).*

Proof. Let L' be a BG stable complement of L in B. Then e is obtained as the image of $1 \in B$ under the projection onto L defined by the decomposition $B = L \oplus L'$. Conversely right multiplication gives an algebra anti-isomorphism $A = B^G \xrightarrow{\sim} \mathrm{End}_{BG}B$ which restricts to an anti-isomorphism $K := eAe \xrightarrow{\sim} \mathrm{End}_{BG}Be$. Yet K is a skew-field if and only if e is minimal. ☐

3.3. Let M be a left BG module.

Lemma — *Suppose $Ann_A M \in \mathrm{Spec}A$. Then the multiplication map $\theta : B \otimes_A M^G \to M$ is injective.*

Proof. Let e be a minimal idempotent of A such that $eM \neq 0$ and set $K = eAe$. The hypothesis on $Ann_A M$ implies that $B \otimes_A M^G = Be \otimes_K eM^G$.

Suppose $ker\theta \neq 0$. Since Be is a simple BG module by 3.2 and $\mathrm{End}_{BG}Be = K$ one may apply the Jacobson density theorem [H, Thm. 2.1.4] to obtain $m \in eM^G \setminus \{0\}$ for which $Be \otimes m \subset ker\ \theta$. Then $em = 0$ which is absurd. ☐

3.4. Let M be a left BG module. One may give $B' := \mathrm{End}_B M$ a G-algebra structure through the action $\psi \mapsto g.\psi, \forall\ g \in G, \psi \in \mathrm{End}_B M$ by $(g.\psi)(m) = g(\psi(g^{-1}m)), \forall m \in M$. Then $g(\psi(bm)) = (g.\psi)(g(b)(gm))$. Set $A' = B'^G$. Then A', B' are also semisimple artinian rings.

Lemma — *Assume that A, B, A', B' are all simple and that $M^G \neq 0$. Then $rkB/rkA = rkB'/rkA'$.*

Proof. Take $m \in M^G \setminus \{0\}$. Then $Ann_B m$ is a BG submodule of B and so of the form Be' for some idempotent $e' \in A \setminus \{1\}$. Let $e \leq 1 - e'$ be a minimal idempotent of A. Since Be is a simple BG module by 3.2 we obtain an isomorphism $Be \xrightarrow{\sim} Bem$. In particular Bem is a simple BG submodule of M. Now $\mathrm{End}_B M = B'$ so $\mathrm{End}_{BG}M = B'^G = A'$. Since A' is assumed simple, it follows that M is an isotypical BG module and moreover rkA' copies of Be. Let $f \leq e$ be a minimal idempotent of B. Then Be is rkB/rkA copies of Bf. Finally $\text{End}_B M = B'$ and since B' is assumed simple, it follows that M is $rk\ B'$ copies of Bf. Thus $(rkA')(rkB/rkA) = rkB'$ as required. □

3.5. It is easy to see that 3.4 fails if $M^G = 0$. Indeed let k be an algebraically closed field and S, S' simple kG module say of dimensions n and n' respectively. Set $B = \text{End}_k S, B' = \text{End}_k S'$. Then $A := BG = k, A' := B'^G = k$ and $M := S \otimes S'$ is a $B - B'$ module satisfying the hypotheses of 3.4 except for the condition on M^G; yet n and n' are in general unrelated. The condition $M^G \neq 0$ is equivalent to S' being the contragredient module to S forcing $n = n'$.

3.6. The above example is in some sense generic if B is finite dimensional over its centre F and G is assumed to fix F pointwise. In this case it is easy to check (and well-known) that rkB/rkA is just the dimension of a projective representation of FG. This intrinsic classification of the Goldie rank ratio may lead one to hope that the conclusion of 3.4 also results even if B' is not the full endomorphism ring $\text{End}_B M$; but just a simple G stable subring with $A' = B'^G$ also simple. This already fails if F is not algebraically closed;but that is not a case of interest to us. We shall construct a more relevant counterexample in Section 6.

3.7. It is necessary to generalize 3.4 to the following situation. Define B, G, A, M, B', A' as in 3.1, 3.3 and 3.4. Assume B, B' simple. Let $\{z_i\}_{i=1}^t, \{z_i'\}_{i=1}^{t'}$ be the minimal central idempotents of A, A'. Set $A_i = Az_i, A_i' = A'z_i', B_i = z_i B z_i, B_i' = z_i' B' z_i'$ which are all simple, artinian rings. Set $M_{i,j} = z_i M z_j'$.

Corollary — *There exist integers $m_i, m_j' > 0$ such that*

$$rkB = \sum_{i=1}^{t} m_i rkA_i, \quad rkB' = \sum_{i=1}^{t'} m_j' rkA_j'$$

with $m_i = m_j'$ whenever $M_{i,j}^G \neq 0$.

Proof. Clearly $\text{End}_{B_i} M_i = B_i'$ and $B_i^G = A_i, B_i'^G = A_i'$. Suppose $M_{i,j}^G \neq 0$. Then by 3.4 one has $m_i := rkB_i/rkA_i = rkB_j'/rkA_j' =: m_j'$. Finally B is a direct sum of the Bz_i and the rank of Bz_i as a left B module is the rank of its endomorphism ring which is $z_i B z_i$. Hence $rkB = \sum rkB_i$. Similarly $rkB' = \sum rkB_i'$. □

3.8. Retain the notation and hypotheses of 3.7. In particular B, B' are simple.

Proposition — *Assume $t = t'$ and $M_{i,i}^G \neq 0$ for all i. Then $BM^G = M = M^G B$.*

Proof. One may write $N_i := M_{i,i}^G$ as a direct sum of say n_i simple left A_i modules. Then $n_i = rk\text{End}_{A_i} N_i = rkA_i'$. Choosing a minimal idempotent e_i for A_i we have $B \otimes_A A_i e_i = B \otimes_A Ae_i \xrightarrow{\sim} Be_i$ by 3.3. Furthermore by 3.3 again $B \otimes_A N_i \xrightarrow{\sim} BN_i$

and is just n_i copies of Be_i as a left B module. Yet Bz_i is just rkA_i copies of Be_i and rkB_i copies of the simple B module S. Finally the $M_{i,i}$ form a direct sum M' in M and $N' := M'^G = \oplus N_i$. Moreover this direct sum arises from right multiplication by the central idempotents z_i' and so $BN' = \oplus BN_i$. From the above we conclude that BN' is exactly $s := \sum_{i=1}^t n_i(rkB_i/rkA_i) = \sum_{i=1}^t n_i m_i$ copies of S.

On the other hand $rk\mathrm{End}_B M = rkB' = \sum_{i=1}^t rkB_i' \overset{3.7}{=} \sum_{i=1}^t m_i rkA_i' = \sum_{i=1}^t m_i n_i = s$. Consequently M is also s copies of S and so the inclusion $BN' \subset M$ is an equality. A fortiori $BM^G = M$. Similarly $M^G B = M$. □

Remark. Consider the example of 3.5 with $M = S \otimes S^*$. Then $M = \sum s_i \otimes s_i^*$ where $\{s_i\}$ is a basis for S and $\{s_i^*\}$ a dual basis for the contragredient module S^*. Then $BM^G = M = M^G B$ follows directly from the Jacobson density theorem (which was also used in 3.3).

4 Coxeter Structure

4.1. Let $\mathfrak{g}$ be a complex semisimple Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. Let Δ (resp. Δ^+) be the corresponding set of non-zero, (resp. positive) roots and ρ the half-sum of the positive roots. Fix $\lambda \in \mathfrak{h}^*$ dominant and regular (that is $\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \notin \{0,-1,-2,\cdots\}$ for all $\alpha \in \Delta^+$) and let $M(\lambda)$ denote the Verma module with highest weight $\lambda - \rho$. Set $U_\lambda = U(\mathfrak{g})/AnnM(\lambda)$ which is a minimal primitive quotient of $U(\mathfrak{g})$. Set $\Delta_\lambda = \{\alpha \in \Delta \mid 2(\alpha,\lambda)/(\alpha,\alpha) \in \mathbb{Z}\}$ which is a root subsystem [Ja1, 1.3] of Δ with Weyl group W_λ generated by the reflections $s_\alpha : \alpha \in \Delta_\lambda$. Set $\Delta_\lambda^+ = \Delta_\lambda \cap \Delta^+$ and let $\pi_\lambda \subset \Delta_\lambda^+$ be the corresponding set of simple roots. One knows that to a large extent the structure of SpecU_λ depends just on the Coxeter diagram assigned to π_λ, a fact made even more precise by the truth of the Kazhdan-Lusztig conjectures. Some finer points involving Goldie rank ratios for certain Dixmier algebras depend also on the Dynkin diagram of π_λ (where root lengths are also considered).

4.2. The above considerations of course apply to a second complex semisimple Lie algebra and we denote the corresponding minimal primitive quotient by V_μ with Δ_μ^0 (resp. π_μ^0) the corresponding set of non-zero (resp. simple) roots. Now let G be a finite subgroup of AutU_λ. Our assumption throughout the rest of this paper is that U_λ^G is isomorphic as an algebra to V_μ. The main result of this section is that π_λ has the same Coxeter diagram as π_μ^0. We generally omit the λ, μ subscripts.

4.3. Given a finitely generated algebra B with identity and M a finitely generated left B module let $d_B(M)$ denote the Gelfand-Kirillov dimension [Ja2, 8.3] of M over B. If M is a finitely generated right B module we shall write $d_B'(M)$ for $d_{B^{op}}(M)$ for M viewed as a left B^{op} module. One has $d_B(B) = d_B'(B)$ which we denote simply as

$d(B)$. If $G \subset \text{Aut}B$ is a finite group, then B is a finitely generated [M1, 5.9] left or right $A := B^G$ module. Now let M be a $B - A$ module which is a finitely generated left B (resp. right A) module. A simple argument (due to W. Borho [B, 2.3]) shows that

$$(*) \qquad d_B(M) = d'_A(M)$$

(here A, B need not be related). In particular taking $M = B$ one obtains $d(B) = d_B(B) = d'_A(B) = d'_A(A) = d(A) \leq d(B)$. Thus $d(A) = d(B)$.

4.4. Applying 4.3 to 4.2 we conclude that $d(U) = d(V)$. Recall [Ja2, Kap.10] that $d(U/I)$ is an even integer for any ideal I of U which is $|\Delta|$ if and only if $I = 0$. In particular $|\Delta| = |\Delta^0|$. Apply the correspondence of 2.6 to the pair $U, V = U^G$. For all $P \in \text{Spec}U$, set $C(P) = C(\Phi^{-1}(O(P)))$.

Lemma — *For each* $P \in \text{Spec}U$ *and each* $p \in C(P)$ *one has* $d_V(V/p) = d_U(U/P)$.

Proof. This is a slight extension of [JS, 3.9]. We sketch the details for completion. Obviously $d(U/P_i)$ is independent of $P_i \in O(P)$ and this common value is $d(U/I(P))$. Let $\bar{L}$ be a non-zero left ideal of $\bar{U} := U/I(P)$. Since U is left noetherian one may write $\bar{L}U$ as a finite sum $\sum_i \bar{L}u_i : u_i \in U$ and then $d_U(\bar{L}) = d_U(\bar{L}U)$. Moreover the image of $\bar{L}U$ in some prime quotient $U/g(P) : g \in G$ must be non-zero and so $d_U(\bar{L}U) = d_U(U/g(P))$.

Now let $\bar{K}$ be a non-zero ideal of $\bar{V} := V/I(P)^G \hookrightarrow U/I(P) = \bar{U}$ and set $\bar{L} = U\bar{K}$. Then

$$\begin{aligned} d(U/I(P)) &= d_U(\bar{L}), \text{by the above,} \\ &= d'_V(U\bar{K}), \text{by 4.3 } (*), \\ &= d'_V(\bar{K}), \text{since } U_V \text{ is finitely generated,} \\ &= d_V(\bar{K}), \text{by 4.3 } (*). \end{aligned}$$

Thus $d_V(\bar{K}) = d_V(\bar{V})$, by 4.3. In view of the noetherianity of V and 4.3 $(*)$ (applied to the case $A = B = \bar{V}$) it follows that $\bar{V}$ satisfies the hypotheses of [JS, 2.6 (i)]. From its conclusion $d_V(\bar{V}/\bar{p}) = d_V(\bar{V}) = d_U(\bar{U})$ for every minimal prime $\bar{p}$ over $\bar{V}$. Recalling 2.4 the required conclusion follows. □

Remark. It is convenient to define $\text{cod}(U/P) = d(U) - d(U/P)$ for all $P \in \text{Spec}U$ with a similar definition in $\text{Spec}V$.

4.5. Let $\{P_\alpha : \alpha \in \pi\}$ (resp. $\{p_\alpha : \alpha \in \pi^0\}$) denote the set of almost minimal prime ideals of U (resp. V). This set was introduced in [BJ], [D] and studied in particular detail in [J2], [GJ]. Each such prime is characterized by the property $\text{cod}(U/P) = 2$. It is then immediate that the action of G permutes the $P_\alpha : \alpha \in \pi$. More precisely it follows from [P, Thm. 3.1 (b)] that G induces a group of Dynkin diagram automorphisms of π which we shall also denote by G. For each $\alpha \in \pi$ let $o(\alpha)$ denote the G orbit containing α. The following is immediate from 2.6, 2.8 and 4.4. Recall the notation of 2.2 and 2.3 concerning $I(\cdot)$.

Lemma — *For each $\alpha \in \pi$ there exists a subset $c(\alpha) \subset \pi^0$ such that $C(P_\alpha) = \{p_\beta \mid \beta \in c(\alpha)\}$. Moreover the $c(\alpha) : \alpha \in \pi/G$ form a disjoint union of π^0 and*

(i) *$I(P_\alpha)^G = \bigcap_{\beta \in c(\alpha)} p_\beta = I(p_\alpha)$.*

(ii) *The $P_\beta : \beta \in o(\alpha)$ are the minimal primes over $UI(p_\alpha)U$.*

Remark. If the P_α are completely prime, the $c(\alpha)$ automatically are singletons and this makes the subsequent analysis far easier [P, Thm. 7.1].

4.6. Recall [BJ, Sect.2; D, Sect.3] that the τ invariant on Spec$U$ is defined by $\tau(P) = \{\alpha \in \pi \mid P \supset P_\alpha\}$. For each subset $\pi' \subset \pi$ there is a unique minimal (resp. maximal) ideal $P_{\pi'}$ (resp. $P^{\pi'}$) whose τ value is π'. We denote $P^{\pi \setminus \{\alpha\}}$ simply by P^α. By [D, Prop. 12; GJ, Cor. 5.2] one has

$$(*) \qquad P_{\pi'} = \sum_{\alpha \in \pi'} P_\alpha$$

Again $P_\pi = P^\pi$ and is the unique maximal ideal $P_{\max}$ of U, whilst $P_\phi = P^\phi = 0$.

More generally for each pair $\pi'' \subset \pi' \subset \pi$ there is a unique maximal ideal $P_{\pi'}^{\pi''}$ contained in $P_{\pi'}$ on which τ takes the value π''. We denote $P_{\pi'}^{\pi' \setminus \{\beta\}}$ simply by $P_{\pi'}^\beta : \beta \in \pi'$. It is the unique maximal ideal of U contained in $P_{\pi'}$ and not containing P_β. Similar definitions apply to Spec$V$ for which we replace P by p.

Lemma — *For all $\pi' \subset \pi$,*

(i) *$P_{\pi'}^2 = P_{\pi'}$.*

(ii) *If $\pi'' \subset \pi'$ are both G stable, then so is $P_{\pi'}^{\pi''}$.*

Proof. (i) is just [J2, 4.5] combined with [GJ, Cor. 5.2]. (ii) is clear. □

4.7. Recall the notation of 4.5.

Proposition — *For all $\alpha \in \pi/G$ one has*

(i) *$P_{o(\alpha)}^G \supset p_{c(\alpha)}$.*

(ii) *$P_{o(\alpha)} = Up_{c(\alpha)}U$.*

Proof. Set $p'_{c(\alpha)} = P_{o(\alpha)}^G = V \cap P_{o(\alpha)}$. Since the $P_{o(\alpha)}$ are G stable (4.6 (ii)) we have

$$(*) \qquad V \cap P_{\max} = V \cap \sum_{\alpha \in \pi/G} P_{o(\alpha)} = \sum_{\alpha \in \pi/G} p'_{c(\alpha)},$$

and by 2.6 that $V \cap P_{\max} = p_{\max}$. Now suppose there exists $\beta \in \pi^0$ such that $p'_{c(\alpha)} \not\supset p_\beta$. Then recalling 4.6 (applied to V) we obtain $p'_{c(\alpha)} \subset p^\beta$. This cannot hold for all $\alpha \in \pi/G$ since otherwise $p_{\max} \subset p^{\beta}$ by $(*)$. Thus defining $\pi^0_\alpha = \{\beta \in \pi^0 \mid p_\beta \subset p'_{c(\alpha)}\}$ one obtains

$$(**) \qquad \bigcup_{\alpha\in\pi/G} \pi^0_\alpha = \pi^0.$$

Let us show that

$$(***) \qquad \pi^0_\alpha = c(\alpha), \forall \alpha \in \pi/G.$$

Suppose that $\pi^0_\alpha \cap c(\beta) \neq \emptyset$ and take γ in this intersection. Then $I(p_{c(\beta)}) \subset p_\gamma \subset p'_{c(\alpha)}$ and so $UI(p_\beta)U \subset Up'_{c(\alpha)}U \subset P_{o(\alpha)}$. Then 4.5 (ii) implies that $P_\delta \subset P_{o(\alpha)}$ for some $\delta \in o(\beta)$. Consequently $\delta \in o(\alpha)$ which forces $o(\alpha) = o(\beta)$. We conclude that $\pi^0_\alpha \cap \bigcup_{\beta\in\pi/G\setminus o(\alpha)} c(\beta) = \emptyset$ and so $\pi^0_\alpha \subset c(\alpha)$ by 4.5. By $(**)$ this gives $(***)$.

By 4.6 $(*)$ (applied to V) and $(***)$ conclusion (i) results.

By 2.8 and (i) we obtain $P_{o(\alpha)} \supset Up_{c(\alpha)}U$. Let P be a minimal prime over $Up_{c(\alpha)}U$. Suppose $P \not\supset P_\gamma$ for some $\gamma \in o(\alpha)$. Then $Up_{c(\alpha)}U \subset P \subset P^\gamma$, whilst if $o(\beta) \neq o(\alpha)$ one also has $Up_{c(\beta)}U \subset P_{o(\beta)} \subset P^\gamma$. Then $Up_{\max}U = \sum_{\beta\in\pi/G} Up_{c(\beta)}U \subset P^\gamma$ which contradicts 2.8. We conclude that $P_{o(\alpha)}$ is the unique minimal prime over $Up_{c(\alpha)}U$. Since U is noetherian the latter contains a power of $P_{o(\alpha)}$ and so (ii) results from 4.6 (i). □

4.8. The following is an obvious consequence of 2.2 (ii) and 2.5 (ii).

Lemma — *For all* $P, P' \in \operatorname{Spec} U; p, p' \in \operatorname{Spec} V$ *one has*

(i) $P \sim P' \Longrightarrow (\tau(P) \cap o(\alpha) \neq \emptyset \Longrightarrow \tau(P') \cap o(\alpha) \neq \emptyset)$

(ii) $p \sim p' \Longrightarrow (\tau(p') \cap c(\alpha) \neq \emptyset \Longrightarrow \tau(p') \cap c(\alpha) \neq \emptyset)$.

4.9. Each G stable subset $\hat{\pi}$ of π is a union of G orbits $o(\alpha)$ and we let $c(\hat{\pi}) \subset \pi^0$ denote the corresponding union of the $c(\alpha)$.

Proposition — *For each G stable subset $\hat{\pi}$ of π one has* $(P^{\hat{\pi}})^G = p^{c(\hat{\pi})}$.

Proof. Suppose $p' \sim p^{c(\hat{\pi})}$. Then by 4.8 (ii) we have $\tau(p') \subset c(\hat{\pi})$ and so $p' \subset p^{c(\hat{\pi})}$. Then by incomparability (2.7) we obtain $p' = p^{c(\hat{\pi})}$. Let $\{P_i\}$ be the G orbit corresponding (2.6) to $p^{c(\hat{\pi})}$. Let us show that $\tau(P_i) = \hat{\pi}$ for all i.

Suppose $\beta \in \tau(p_i)$. Then $I(p_\beta) = P^G_\beta \subset P^G_i = p^{c(\hat{\pi})}$ and so $p_{\beta'} \subset p^{c(\hat{\pi})}$ for some $\beta' \in c(\beta)$. This forces $\tau(P_i) \subset \hat{\pi}$. On the other hand $p^{c(\hat{\pi})} \supset p_{c(\alpha)}$ for all $c(\alpha) \subset c(\hat{\pi})$ and so by 4.7 (ii) and 2.8 we obtain $P_i \supset Up^{c(\hat{\pi})}U \supset Up_{c(\alpha)}U = P_{o(\alpha)}$ which gives the opposite inclusion.

Recalling 2.8 and 4.6 we conclude that $Up^{c(\hat{\pi})}U \subset P^{\hat{\pi}}$. Thus $(P^{\hat{\pi}})^G \supset p^{c(\hat{\pi})}$. For the opposite inclusion set $p' = (P^{\hat{\pi}})^G$. If $p' \supset p_\alpha$ then $P^{\hat{\pi}} \supset Up'U \supset UI(p_\alpha)U$. By 4.5 (ii) this forces $\tau(P^{\hat{\pi}}) \cap o(\alpha) \neq \emptyset$. Thus $o(\alpha) \subset \hat{\pi}$ and then $\tau(p') \subset c(\hat{\pi})$. Consequently $p' \subset p^{c(\hat{\pi})}$ as required. □

4.10. To prove equality in 4.7 (i) it would be useful to have the analogue of 2.2 (iii) to hold in 2.5. To circumvent this lacuna we combine 4.9 with the following well-known [J4, III, 4.7] consequence of the truth of the Kazhdan-Lusztig conjectures and [V]. Consider Spec$U$ as an ordered set for inclusion.

Proposition — *There exists an order reversing involution $\sigma \in$ Spec$U$ such that $\sigma(P_\alpha) = P^\alpha$ for all $\alpha \in \pi$. In particular $\sigma(P_{\pi'}) = P^{\pi \setminus \pi'}$ for all $\pi' \subset \pi$.*

Remark. Of course a corresponding result holds for Spec$V$.

4.11. Let $\hat{\pi}$ be a G stable subset of π. Then by 2.2 or directly $\mathrm{Spec}^{\hat{\pi}}U := \{P \in \mathrm{Spec}U \mid P \supset P^{\hat{\pi}}\}$ is a union of G orbits. Again by 2.5 and 4.9 we may also conclude that $\mathrm{Spec}^{c(\hat{\pi})}V := \{p \in \mathrm{Spec}V \mid p \supset p^{c(\hat{\pi})}\}$ is a union of equivalence classes in the sense of 2.3. Moreover it is clear that Φ of 2.6 restricts to an order preserving isomorphism of $\mathrm{Spec}^{c(\hat{\pi})}V/\sim$ onto $\mathrm{Spec}^{\hat{\pi}}U/\sim$. Given $P \in \mathrm{Spec}^{\hat{\pi}}U$ we set $ht_{\hat{\pi}}P = ht(P/P^{\hat{\pi}})$ with a similar definition for V. Given $P \in \mathrm{Spec}^{\hat{\pi}}U$ and p minimal over P^G then by 2.7 it follows that $ht_{\hat{\pi}}P = ht_{c(\hat{\pi})}p$.

Theorem — *For all $\alpha \in \pi/G$ one has $P^G_{o(\alpha)} = p_{c(\alpha)}$.*

Proof. Take $p \in C(P_{o(\alpha)})$. By 4.7 (i) one obtains $p \supset p_{c(\alpha)}$ and so it is enough to show that $htp = htp_{c(\alpha)}$. Set $\hat{\pi} = \pi \setminus o(\alpha)$. By 2.7, 4.10 and the above remarks $htp = htP_{o(\alpha)} = ht_{\hat{\pi}}(P_{\max}) = ht_{c(\hat{\pi})}(p_{\max}) = htp_{c(\alpha)}$, as required. □

4.12. For each G stable subset $\hat{\pi}$ we may define $\mathrm{Spec}_{\hat{\pi}}U = \{P \in \mathrm{Spec}U \mid P \subset P_{\hat{\pi}}\}$ which by 2.2 (iii) or directly is clearly a union of G orbits. Again one may define $\mathrm{Spec}_{c(\hat{\pi})}V = \{p \in \mathrm{Spec}V \mid p \subset p_{c(\hat{\pi})}\}$. Unfortunately it is *not* obvious that $\mathrm{Spec}_{\hat{\pi}}V$ is a union of classes. Nevertheless one has the

Lemma — *Let $\hat{\pi}$ be a G stable subset of π. Then*

(i) $C(P) \cap \mathrm{Spec}_{c(\hat{\pi})}V \neq \emptyset, \forall P \in \mathrm{Spec}_{\hat{\pi}}U$.

(ii) $\mathrm{Spec}_{c(\hat{\pi})}V = \bigcup_{P \in \mathrm{Spec}_{\hat{\pi}}U}(C(P) \cap \mathrm{Spec}_{c(\hat{\pi})}V)$.

(iii) *$\tau(p) \subset c(\hat{\pi})$ for all $p \in C(P)$ with $P \in \mathrm{Spec}_{\hat{\pi}}U$.*

Proof. Given $P \subset P_{\hat{\pi}}$, one has $P^G \subset P^G_{\hat{\pi}} = p_{c(\hat{\pi})}$ by 4.11. Then (i) results from 2.4 or 2.5 (ii). Conversely given $p \in \mathrm{Spec}_{c(\hat{\pi})}V$ one has $UI(p)U \subset Up_{c(\hat{\pi})}U = P_{\hat{\pi}}$ by 4.7 (ii). Let P be a minimal prime over $UI(p)U$. Then $P \subset P_{\hat{\pi}}$ and $p \in C(P)$ by 2.8. Finally (iii) follows from (i) and 4.8 (ii). □

4.13. Let $\hat{\pi}$ be a G stable subset of π. For each $\alpha \in \hat{\pi}$ it is clear that $O^{o(\alpha)}_{\hat{\pi}} := \{P^\beta_{\hat{\pi}} \mid \beta \in o(\alpha)\}$ is a single G orbit. Set $C^{c(\alpha)}_{c(\hat{\pi})} = \Phi^{-1}(O^{o(\alpha)}_{\hat{\pi}})$.

Lemma — $\mathrm{Spec}_{c(\hat{\pi})}V \cap C^{c(\alpha)}_{c(\hat{\pi})} = \{p^\beta_{c(\hat{\pi})} \mid \beta \in c(\alpha)\}$.

Proof. It is clear from 4.11 and 2.6 that $C^{c(\alpha)}_{c(\hat{\pi})} \not\leq \{p_{c(\hat{\pi})}\}$ and are exactly the maximal classes with this property. Then by 2.5 and 4.12 it follows that

$$\operatorname{Spec}_{c(\hat{\pi})}V \cap \bigcup_{\alpha\in\hat{\pi}/G} C_{c(\hat{\pi})}^{c(\alpha)} = \{p_{c(\hat{\pi})}^{\beta} \mid \beta \in c(\hat{\pi})\}.$$

Finally take $\beta \in \hat{\pi}$. Then $(P_{\hat{\pi}}^{\beta})^G \supset P_{o(\alpha)}^G \overset{4.11}{=} p_{c(\alpha)}$, for all $\alpha \in \hat{\pi} \setminus o(\beta)$. Thus a minimal prime p over $(P_{\hat{\pi}}^{\beta})^G$ contains $p_{c(\hat{\pi})\setminus c(\beta)}$. Recalling 2.4 this completes the proof. □

4.14. Let $\pi = \coprod_{i=1}^n \hat{\pi}_i$ be a decomposition into G orbits of connected components (of the Dynkin diagram).

For any subset $\pi' \subset \pi$ set $\Delta_{\pi'} = \mathbb{Z}\pi' \cap \Delta$. It is well-known (combine [J4, II, Thm. 5.1; Ja1, 2.16; Ja2, 10.9]) that

$$(*) \qquad \operatorname{cod}(U/P_{\pi'}) = |\Delta_{\pi'}|.$$

This implies for any two disjoint subsets π', π'' of π that

$$(**) \qquad \operatorname{cod}(U/P_{\pi'}) + \operatorname{cod}(U/P_{\pi''}) \leq \operatorname{cod}(U/P_{\pi'\cup\pi''})$$

with equality if and only if π', π'' are components of π. Of course a similar assertion holds for V.

Lemma — *The $c(\hat{\pi}_i)$ are components of π^0.*

Proof. This follows from 4.4, 4.6($*$), 4.10 and ($**$) above. □

4.15. It is immediate from Duflo's theorem [D, Thm.1] describing Spec$U$ that if π' is a component of π and $P \in \operatorname{Spec}U$ satisfies $\tau(P) \subset \pi'$ then $P \subset P^{\tau(P)} \subset P_{\pi'}$. Applying the analogous result for V to the conclusion of 4.14 it follows from 4.12 that

Corollary — *Take $i \in \{1, 2, \cdots, n\}$. Then $\operatorname{Spec}_{c(\hat{\pi}_i)}V$ is a union of classes and Φ (of (2.6)) restricts to an order isomorphism of $\operatorname{Spec}_{c(\hat{\pi}_i)}V/\sim$ onto $\operatorname{Spec}_{\hat{\pi}_i}U/\sim$.*

4.16. In virtue of 4.15 we may assume that $n = 1$ in 4.14 from now on without loss of generality. Consequently π is a union of connected Dynkin diagrams of the same type, permuted transitively by G. We would like to show that then π^0 is a union of Dynkin diagrams of this same type. However we are only able to show that the Coxeter structure is preserved. This involves a rather messy case by case analysis.

4.17. Let $D_{\pi'}$ denote the Dynkin diagram of π'.

Lemma — *For each $\alpha \in \pi/G$ there is an isomorphism $\theta_\alpha : D_{o(\alpha)} \xrightarrow{\sim} D_{c(\alpha)}$ of Dynkin diagrams.*

Proof. Since G acts by Dynkin diagram automorphisms it follows (from the well-known classification of such diagrams) that one of the following hold

$$D_{o(\alpha)} = \begin{cases} A_1^r & : \text{case 1)}, \\ A_2^s & : \text{case 2)}. \end{cases}$$

By 4.4, 4.11 and 4.13 we obtain

$$(*) \qquad d(U/P^{\beta}_{o(\alpha)}) - d(U/P_{o(\alpha)}) = d(V/p^{\gamma}_{c(\alpha)}) - d(V/p_{c(\alpha)}), \forall \beta \in o(\alpha), \gamma \in c(\alpha).$$

Now the left hand side equals 2 in case 1) (resp. 4 in case 2)). From Table 1 (see 4.19) this forces $\gamma \in c(\alpha)$ to belong to a component of $c(\alpha)$ of type A_1 (resp. type A_2). It then remains to apply 4.14 $(*)$ with $\pi' = o(\alpha)$. □

4.18. Take $\alpha, \beta \in \pi/G$. Set $o(\alpha,\beta) = o(\alpha) \amalg o(\beta)$ and $c(\alpha,\beta) = c(\alpha) \amalg c(\beta)$. Set $S^{o(\beta)}_{o(\alpha,\beta)} = \{P \in \operatorname{Spec}_{o(\alpha,\beta)} U \mid P \supset P_{o(\beta)}\}$ and $T^{c(\beta)}_{c(\alpha,\beta)} = \Phi^{-1}(S^{o(\beta)}_{o(\alpha,\beta)})$.

Lemma — (i) *Every class in $T^{c(\beta)}_{c(\alpha,\beta)}$ has a non-empty intersection with $\operatorname{Spec}_{c(\alpha,\beta)} V$.*

(ii) $\operatorname{Spec}_{c(\alpha,\beta)} V \cap T^{c(\beta)}_{c(\alpha,\beta)} = \{p \in \operatorname{Spec}_{c(\alpha,\beta)} V \mid p \supset p_{c(\beta)}\}$.

Proof. (i) follows from 4.12(i). (ii) follows from 2.4 and 4.11 as in the proof of 4.13. □

4.19. Take $\alpha, \beta \in \pi/G$. By 4.17 we have a bijection $\theta_\alpha \times \theta_\beta : o(\alpha,\beta) \xrightarrow{\sim} c(\alpha,\beta)$ defined up to permutation.

Proposition — *$\theta_\alpha \times \theta_\beta$ can be chosen to be an isomorphism of Coxeter diagrams.*

Proof. Suppose first that both $o(\alpha)$ and $o(\beta)$ satisfy 1) of 4.17. Since π is assumed to be an orbit of a connected component it follows that both must be of type A_1^k for some k and moreover one of the following hold

$$D_{o(\alpha,\beta)} = \begin{cases} (A_1 \times A_1)^k & : \text{case } 1_1, \\ A_2^k & : \text{case } 1_2, \\ B_2^k & : \text{case } 1_3, \\ G_2^k & : \text{case } 1_4, \\ A_3^k & : \text{case } 1_5, \\ D_4^k & : \text{case } 1_6, \end{cases}$$

We must show that a similar conclusion holds for $D_{c(\alpha,\beta)}$. By 4.10 and 4.14 $(*)$ we have

$$(*) \qquad |\Delta_{o(\alpha,\beta)}| = |\Delta^0_{c(\alpha,\beta)}|.$$

This dispenses with case 1_1. For the remaining cases we need to have for each simple Lie algebra $\mathfrak{g}_{\pi'}$ (with simple root system π') some information on the common value of $d(U(\mathfrak{g}_{\pi'})/P^\beta) : \beta \in \pi'$. (These are the same as the values of $d(U/P^\beta_{\pi'}) - d(U/P_{\pi'}) : \beta \in \pi'$). The relevant information is given by the following table.

$D_{\pi'}$	$1/2\, d(U(\mathfrak{g}_{\pi'})/P^\beta) : \beta \in \pi'$
A_n	n
B_2	3
G_2	5
B_3	5
D_4	5

Table 1.

In all other cases the second entry is > 5.

By 4.4, 4.10 and 4.13, the analogue of 4.17 $(*)$ holds with $o(\alpha)$ (resp. $c(\alpha)$) replaced by $o(\alpha,\beta)$ (resp. $c(\alpha,\beta)$). In view of Table 1 this dispenses with case 1_2.

Recall the notation of 4.18. Take $P \in S^{o(\alpha)}_{o(\alpha,\beta)}$. Then by Table 1 we have

$$(**) \qquad 1/2\left[d(U/P) - d(U/P_{o(\alpha,\beta)})\right] = \begin{cases} 3i & : i = 0,1,\dots,k \text{ in case } 1_3, \\ 5i & : i = 0,1,\dots,k \text{ in case } 1_4. \end{cases}$$

Thus from 4.4 and 4.18 the connected components of $c(\alpha,\beta)$ must be of types A_3, B_2. Suppose $\gamma_1,\gamma_2,\gamma_3$ form a system of type A_3. Then up to permutation of α,β we must have $\gamma_1,\gamma_3 \in c(\alpha)$ and $\gamma_2 \in c(\beta)$. Consider $p = p_{\gamma_1} + p_{\gamma_3} + p_{c(\alpha,\beta)\setminus\{\gamma_1,\gamma_3\}}$. Then $p \subsetneq p^{\gamma_2}_{c(\alpha,\beta)}$ and one further sees that $1/2[d(V/p) - d(V/p_{c(\alpha,\beta)})] = 1/2(|\Delta^o_{\{\gamma_1,\gamma_2,\gamma_3\}}| - |\Delta^o_{\{\gamma_1,\gamma_3\}}|) = 1/2(12-4) = 4$. This is incompatible with 4.4, 4.18 and $(**)$. This dispenses with case 1_3.

In case 1_4 we must exclude types B_3, D_4, A_5. It is convenient to present the calculation diagrammatically. In this we can assume without loss of generality that there are just 2 diagrams of each of the above types in $c(\alpha,\beta)$. This gives rise to the following possibilities where we have included the A_3 case from 1_3 for illustration.

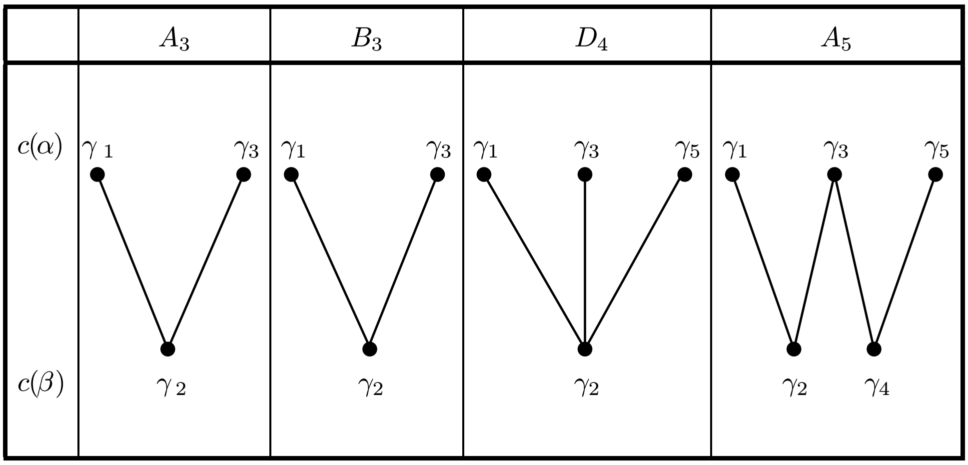

Table 2.

In type B_3 we take $p = p_{\gamma_1} + p_{\gamma_3} + p_{c(\beta)}$. Then $1/2[d(V/p) - d(V/p_{c(\alpha,\beta)})] = 1/2(|\Delta^o_{\{\gamma_1,\gamma_2,\gamma_3\}}| - |\Delta^o_{\{\gamma_1,\gamma_3\}}|) = 7$. In types D_4, A_5 we take $p = p_{\gamma_1} + p_{\gamma_3} + p_{\gamma_5} + p_{c(\beta)}$ and then $1/2[d(V/p) - d(V/p_{c(\alpha,\beta)})] = 12 - 3 = 9$ in the first case and 15-3=12 in the second. None of the above are multiples of 5 so this dispenses with case 1_4.

Now consider case 1_5. As in case 1_2 it follows from Table 1 that $c(\alpha,\beta)$ must be a union of k_1 diagrams of type B_2 and k_2 diagrams of type A_3. Now we can take $o(\alpha)$ to be the orbit containing the central roots of the A_3 diagrams. If $\{\gamma_1,\gamma_2,\gamma_3\} \subset c(\alpha,\beta)$ is a system of type A_3 then by 4.17 either $\{\gamma_1,\gamma_3) \subset c(\alpha)$, $\gamma_2 \in c(\beta)$ or vise versa. Hence $k = |o(\alpha)| = |c(\alpha)| \leq k_1 + k_2$ whilst $2k = |o(\beta)| = |c(\beta)| \geq k_1 + 2k_2$ which forces $k_1 = 0$ and dispenses with case 1_5. The case 1_6 is similar. By the table $c(\alpha,\beta)$ must be a union k_1 diagrams of type A_5, k_2 diagrams of type G_2, k_3 diagrams of type B_3, k_4 diagrams of type D_4. Take $o(\alpha)$ to be the orbit containing the central roots of the D_4 diagrams. Then as above $k = |o(\alpha)| \geq 2k_1 + k_2 + k_3 + k_4$ whilst $3k = |o(\beta)| \leq 3k_1 + k_2 + 2k_3 + 3k_4$ which forces $k_1 = k_2 = k_3 = 0$ and dispenses with case 1_6.

Finally suppose that $o(\alpha)$ or $o(\beta)$ satisfies 2) of 4.17. Then $o(\alpha,\beta)$ must be of type A_4^k and as in case 1_2 it follows from the table that $c(\alpha,\beta)$ has only components of type A_4 and then by cardinality must be of type A_4^k also. □

4.20. Let π' be a connected component of π (which we can assume equals $G\pi'$).

Theorem — *The bijection $\{\theta_\alpha\}_{\alpha\in\pi'}$ (defined up to permutations) can be chosen to be an isomorphism of Coxeter diagrams.*

Proof. Suppose $Stab_G\pi'$ is the identity of G. Then by 4.17 there exists $k \in \mathbb{N}^+$ such that $G\alpha$ is of type A_1^k for all $\alpha \in \pi'$. Let α_i, α_j be elements of π'. By 4.19 we can label $c(\alpha_i) = \{\beta_i^1, \beta_i^2, \cdots, \beta_i^k\}$ and $c(\alpha_j) = \{\beta_j^1, \beta_j^2, \cdots, \beta_j^k\}$ so that $\{\beta_i^\ell, \beta_j^\ell\}$ has the same Coxeter diagram as $\{\alpha_i, \alpha_j\}$ for $\ell = 1, 2, \cdots, k$. Then $\pi_\ell := \{\beta_1^\ell, \cdots, \beta_{|\pi'|}^\ell\}$ has the same Coxeter diagram as π' and is a (connected) component of π^o.

If $Stab_G \pi'$ is not the identity of G, then $D_{\pi'}$ is of type A_{2n+1}, D_n, E_6 or A_{2n}. Case D_4 (and when G permutes transitivity the three neighbours of the central roots) is just case 1_6 of 4.19. Otherwise in the first three types above there is a unique central root with two neighbours permuted by G. This gives a subsystem of π' of type A_3 and the result obtains from case 1_5 of 4.19 and an analysis similar to the trivial stabilizer case. Finally the assertion for π' of type A_{2n} results similarly from the type A_4 case in 4.19. □

4.21. The above result obtains far more easily (see [P, Thm. 7.1]) when G acts trivially on D_π, equivalently when G stabilizes each $P_\alpha : \alpha \in \pi$. As in [P, Thm. 5.5] it further follows that each $P \in \text{Spec}U$ is G stable. Now from the truth of Kazhdan-Lusztig conjectures $|\text{Spec}U|$, which is finite, depends only on the Coxeter diagram of π. Thus $|\text{Spec}U| = |\text{Spec}V|$ in this case and this further forces each class in $\text{Spec}V$ to be a singleton. Then 4.12 implies that if $P \in \text{Spec}_{\hat{\pi}}U$, then $P^G \in \text{Spec}_{c(\hat{\pi})}V$ whilst *in general* it is not at all obvious if $C(P) \subset \text{Spec}_{c(\hat{\pi})}V$.

5 Comparison of Weights—Additivity Principle

5.1. Suppose for the moment that π_λ is connected. Then by 4.20 so is π_μ and moreover has the same Coxeter diagram as π_λ. Then there is a natural sense in which λ, μ can be said to be proportional and we show that μ divides λ. In the general case we show that restriction to each connected component can be defined and that the restricted values of μ satisfy an additivity principle. The case when G acts trivially on π_λ is significantly easier and illustrates the main technique. In this case $P_\alpha^G = p_\alpha$, for all $\alpha \in \pi$ by 4.5. Then (see 5.10) one has $P_w^G = p_w$ for all $w \in W$. Then $Up_w \subset P_w$ and suppose we can show that equality holds. Taking successive quotients for appropriate choices of w, namely $w = s_\alpha s_\beta w_\pi, w = s_\alpha w_\pi$ one may relate (see remark following Thm. 5.16) Goldie rank ratios for the almost minimal primitive ideals in U, to those in V using the results in Section 3. This gives our main result (Theorem 5.16). Sections 5.2-5.11 allow one to avoid assuming that the action of G on π_λ is trivial. Unfortunately they are rather technical. Sections 5.12-5.15 analyse to what extent the above equality can be established. They simplify significantly when G acts trivially on π_λ.

5.2. Let us start with a result which is a strengthening of a special case of 4.14 $(**)$ and which does not seem to have been noticed previously.

Lemma — *Let $\coprod_{i=1}^n \pi_i$ form a disjoint union of π_λ. Then*

$$\sum_{i=1}^n \operatorname{cod}(U/P^{\pi_i}) \leq \operatorname{cod}(U/P^{\pi_\lambda})$$

with equality if and only if the π_i are components of π_λ.

Proof. By the truth of the Kazhdan-Lusztig conjectures and [J4, II, Thm. 5.1; Ja2, 10.9] it is enough to establish this for λ integral and then we omit the λ subscript. In this case P^π has finite codimension and so $d(U/P^\pi) = |\Delta|$.

Recall the canonical filtration $\mathscr{F}$ of $U(\mathfrak{g})$. Then $S(\mathfrak{g}) = gr_{\mathscr{F}} U(\mathfrak{g})$ identifies with the algebra of polynomial functions on $\mathfrak{g}^*$. View P^{π_i} as an ideal of $U(\mathfrak{g})$ by inverse image. An elementary calculation gives $d(U/P^{\pi_i}) = d(S(\mathfrak{g})/\sqrt{gr_{\mathscr{F}} P^{\pi_i}}) = \dim(\mathscr{V}(gr_{\mathscr{F}} P^{\pi_i}))$, where $\mathscr{V}(Q) \subset \mathfrak{g}^*$ denotes the zero variety of an ideal Q of $S(\mathfrak{g})$.

For each $\alpha \in \pi$ let $x_\alpha \in \mathfrak{g}$ be a root vector of weight α. For each subset $\pi' \subset \pi$ set $x_{\pi'} := \sum_{\alpha \in \pi'} x_\alpha$. Identity $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form. By [J5, 8.15] one has $x_{\pi \setminus \pi'} \in \mathscr{V}(gr_{\mathscr{F}} P^{\pi'})$. Though we don't need this $\mathscr{V}(gr_{\mathscr{F}} P^{\pi'})$ is a nilpotent orbit closure [J6]; but may be strictly bigger than that given by $x_{\pi\setminus\pi'}$, except for example if π' is a component of π.

By the above observations the lemma is reduced to the purely geometric problem of estimating certain orbit dimensions and this in turn reduces to the linear algebra inequality

$$(*) \qquad \sum_{i=1}^n (|\Delta| - \dim\,[\mathfrak{g}, x_{\pi\setminus\pi_i}]) \leq |\Delta|,$$

with equality whenever the π_i are components of π. This is established below.

Set $V_j = [\mathfrak{g}, x_{\pi_j}]$ which is a subspace of $V := [\mathfrak{g}, x_\pi]$. For all $i \geq 2$ one has

$$\begin{aligned}
\dim\left(\sum_{j(\neq i)=1}^n V_j\right) &+ \dim\left(\sum_{k=2}^i V_k\right) \\
&= \dim V + \dim\left(\sum_{k=2}^{i-1} V_k\right) + \dim\left(V_i \cap \sum_{j(\neq i)=1}^n V_j\right) - \dim\left(V_i \cap \sum_{k=2}^{i-1} V_k\right) \\
&\geq \dim V + \dim\left(\sum_{k=2}^{i-1} V_k\right)
\end{aligned}$$

and so

$$\sum_{i=1}^n \dim\left(\sum_{j(\neq i)=1}^n V_j\right) \geq (n-1)\dim V = (n-1)|\Delta|,$$

which gives $(*)$. Equality implies in particular that $[\mathfrak{g}, x_{\pi_i}] \cap [\mathfrak{g}, x_{\pi\setminus\pi_i}] = 0$ for $i = 2$; but as the ordering is arbitrary this holds for all i. It implies that π_i and $\pi \setminus \pi_i$ form separate components, as required. $\square$

Remark. Suppose π_1, π_2 are disjoint subsets of π. One can ask if $\operatorname{cod}(U/P^{\pi_1}) + \operatorname{cod}(U/P^{\pi_2}) \leq \operatorname{cod}(U/P^{\pi_1 \amalg \pi_2})$. This apparently requires a more sophisticated analysis of tangent spaces. Indeed although the dimension of the intersection of the tangent spaces $[\mathfrak{g}, x_{\pi\setminus\pi_i}] : i = 1, 2$ has the correct dimension we cannot immediately apply the intersection theorem since in each orbit $Gx_{\pi\setminus\pi_i}$ the tangent space is taken at a different point, namely $x_{\pi\setminus\pi_i}$.

5.3. Assume that π is connected and let Γ denote the automorphism group of the Coxeter diagram of π. When Γ is non-trivial we shall need a technical modification of the above result. Namely we relax the condition that $\pi_i : i = 1, 2, \cdots, n$ form a disjoint union and require only for each $\alpha \in \pi$ that

$$(*) \qquad \sum_{i=1}^{n} |\Gamma\alpha \cap \pi_i| = |\Gamma\alpha|.$$

It is easy to see that this change only affects types A_{m-1} and E_6. In type A_{m-1} we establish that a similar conclusion holds as a corollary of the lemma below. Given $D_\pi \cong A_{m-1}$, view $\pi' \subset \pi$ as a partition of m and let $S(\pi')$ denote the corresponding Young diagram.

Lemma — *Assume $D_\pi \cong A_{m-1}$. Let $\pi_i \subset \pi : i = 1, 2, \cdots, n$ be such that $\sum_{i=1}^n |\pi_i| \leq m - 1$. Then $\sum_{i=1}^n \operatorname{cod}(U/P^{\pi_i}) \leq \operatorname{cod}(U)$ with equality if and only if there is an equality in the first sum and the $S(\pi \setminus \pi_i)$ are all rectangular.*

Proof. In type A_{m-1} our previous inequality $\dim \mathscr{V}(gr_{\mathscr{F}} P^{\pi'}) \geq \dim [\mathfrak{g}, x_{\pi\setminus\pi'}]$ becomes an equality [J5, 9.14]. It thus suffices to show that $|\Delta| - \dim [\mathfrak{g}, x_{\pi\setminus\pi'}] \leq m|\pi'|$ with equality if and only if $S(\pi \setminus \pi')$ is rectangular. Set $k = |\pi'|$. Let $\{m_i\}_{i=1}^{k+1}$ be the partition defined by the Jordan blocks of $x_{\pi\setminus\pi'}$. Then the left hand side above equals $2\sum_{i=1}^{k+1}(i-1)m_i$. It is easy to check (and well-known) that this sum takes its maximal value exactly when the m_i are equal (necessarily to $m/k+1$) and this value is km. $\square$

Remark (1). If $S(\pi \setminus \pi_i)$ is rectangular, then π_i is Γ stable. If this holds for all i, then $(*)$ is equivalent to the union $\bigcup \pi_i$ being disjoint, so in this case the required conclusion (of strict inequality unless $n = 1$) results from 5.2. For example if $m = 6$ then $\pi_1 = \{\alpha_2, \alpha_4\}, \pi_2 = \{\alpha_3\}$ are the unique diagrams for which $\pi \setminus \pi_i$ is proper and $S(\pi \setminus \pi_i)$ is rectangular; but they are insufficient to cover π.

Remark (2). In type E_6 there is exactly one bad configuration not covered by 5.2, namely when $\pi_1 \supset \{\alpha_1, \alpha_3\}, \pi_2 \supset \{\alpha_1, \alpha_5\}$. In Table 3 we list all possible choices of the π_i writing α_j simply as j. We designate the orbit $\mathbb{O}_i$ generated by $\alpha_{\pi\setminus\pi_i}$ using its Dynkin data.The values of $d_i := {}^1\!/_2(|\Delta| - \dim [\mathfrak{g}, x_{\pi\setminus\pi_i}])$ obtain from the column labelled $\dim \mathcal{B}_u$ in [C, p.402]. The last column gives the left hand side of 5.2($*$) and is always strictly less than ${}^1\!/_2|\Delta| = 36$.

π_1	π_2	π_3	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{O}_3$	d_1	d_2	d_3	$\sum d_i$
1,3	1,5	2,4	20002 2	01010 1	20002 0	6	8	12	26
1,2,3	1,5	4	10001 2	01010 0	20202 0	10	8	3	21
1,3,4	1,5	2	10001 1	01010 1	21012 1	13	8	4	25
1,3	1,5,2	4	20002 2	10001 1	20202 0	6	13	3	22
1,3	1,5,4	2	20002 2	10001 1	21012 1	6	10	4	20
1,2,3	1,5,4	-	10001 2	10001 2	-	10	10	-	20
1,3,4	1,5,2	-	10001 1	10001 1	-	13	13	-	26
1,2,3,4	1,5	-	10001 0	01010 1	-	20	8	-	28
1,3	1,2,4,5	-	20002 2	10001 1	-	6	20	-	26

Table 3.

In the remaining case $\pi_1 = \{\alpha_1, \alpha_3\}, \pi_2 = \{\alpha_1, \alpha_5\}, \pi_3 = \{\alpha_2\}, \pi_4 = \{\alpha_4\}$ we obtain $\sum d_i = 21$ from rows 1-3 above.

5.4. Let π_1 be a connected component of π. We wish to calculate $C(P_{\pi_1})$ which should be some analogue of a combination of 4.5 and 4.11. This would have been easy had we an analogue in 2.5 of 2.2 (iii). By 4.15 we can assume that $G\pi_1 = \pi$ without loss of generality.

Lemma — *Suppose $o(\alpha) \subset \pi$ has cardinality ≥ 2. Let P be a minimal prime over $U\left(\bigcap_{\beta\in c(\alpha)\setminus\{\alpha\}} p_\beta\right)U$. Then there exist $\gamma, \delta \in o(\alpha)$ distinct such that $P \supset P_\gamma + P_\delta$.*

Proof. Clearly $U\left(\bigcap_{\beta\in c(\alpha)\setminus\{\alpha\}} p_\beta\right)U \supset U\left(\bigcap_{\beta\in c(\alpha)} p_\beta\right)U$ and so by 4.5 (ii) one has $P \supset P_\gamma$ for some $\gamma \in o(\alpha)$. If equality holds then $\left(\bigcap_{\beta\in c(\alpha)\setminus\{\alpha\}} p_\beta\right) \subset P^G = P_\gamma^G = \bigcap_{\beta\in c(\alpha)} p_\beta$, by 4.5 (i) which is impossible. Hence $P \supsetneq P_\gamma$. On the other hand $P \subset P_{o(\alpha)}$ by 4.7 (ii). Since $D_{o(\alpha)}$ is of type A_1^r or A_2^s every such prime [BJ, 2.20; D, Prop.12] is the sum of some $P_\gamma : \gamma \in o(\alpha)$. Hence the assertion. ☐

5.5. Take $\alpha \in \pi$. As noted in the proof of 5.4 every $P \in \mathrm{Spec}_{o(\alpha)}U$ takes the form $P = P_{\tau(P)}$. By 4.17 the same holds for $\mathrm{Spec}_{c(\alpha)}V$.

Lemma — *For all $P \in \mathrm{Spec}_{o(\alpha)}U$ and $p \in \mathrm{Spec}_{c(\alpha)}V \cap C(P)$ one has $D_{\tau(P)} \cong D_{\tau(p)}$.*

Proof. If $D_{o(\alpha)} \cong A_1^r$ then the assertion is immediate from 4.4 and 4.14($*$). Now suppose $D_{o(\alpha)} \cong A_2^s$ and suppose $D_{\tau(P)} \cong A_2^u \times A_1^v$ and $D_{\tau(p)} \cong A_2^{u'} \times A_1^{v'}$. Then

$3u+v = \text{cod}(U/P) = \text{cod}(V/p) = 3u'+v'$ by 4.4 and $2u+v = htP = htp = 2u'+v'$ by 2.7. Hence $u = u', v = v'$ as required. □

5.6. Let $\pi_i : i = 1, 2, \cdots, m$ be the G translates of (the connected component) π_1. Obviously the $P_i := P_{\pi_i} : i = 1, 2, \cdots, m$ form a single G orbit in Spec$U$. By 4.20, π^0 is a disjoint union $\amalg c(\pi_i)$ where each $c(\pi_i)$ has the same Coxeter diagram as π_1. Let Γ denote the group of Dynkin diagram automorphisms of π, equivalently of π^0. For each $\alpha \in \pi_1$ set $r(\alpha) = \Gamma\alpha \cap \pi_1$ and let $r_i(\alpha)$ denote the corresponding subset of $c(\pi_i)$. (This makes sense though we cannot identify an individual element $\alpha \in \pi_1$ with an element of $c(\pi_i)$. In general $G\alpha \cap \pi_1 \subset r(\alpha)$ and the inclusion may be strict;but this will not matter).

Theorem — *Let π_1 be a connected component of π. Then $C(P_{\pi_1}) = \{p_{c(\pi_i)} : i = 1, 2, \cdots, m\}$.*

Proof. Take $\alpha \in \pi_1$ and $p \in C(P_1)$. Suppose $s := \sum_i |\tau(p) \cap r_i(\alpha)| > |r(\alpha)| =: r$. We may write $p \supset p' := \sum_{i=1}^s p_{\gamma_i}$ for some $\gamma_i \in c(\alpha)$. Then $C(p) \geq C(p')$ by 2.5 and so $C(P_1) = \Phi(C(p)) \geq \Phi(C(p'))$ by 2.6. By 5.4 we obtain $P_1 \supset \sum_{i=1}^s P_{\delta_i}$ for some $\delta_i \in o(\alpha)$. Since $s > r$ this is impossible.

We conclude that $\sum_i |\tau(p) \cap r_i(\alpha)| \leq |r(\alpha)|$ for every $\alpha \in \pi_1$. Identify the Coxeter diagram of each $c(\pi_i)$ with that π_1 via 4.20. For simplicity consider first the case when Γ is trivial. Then this inequality can be interpreted through the above identification as saying that the $\pi_i(p) := \tau(p) \cap c(\pi_i)$ are disjoint and their union is contained in π_1. By the truth of the Kazhdan-Lusztig conjectures and [J4, Thm. 5.1; Ja2, 10.9] the $d(U/P) : P \in \text{Spec}U$ depends only on the Coxeter diagram of π (and not on the more refined Dynkin diagram). Thus we may apply 5.2 to conclude that $\text{cod}(V/p) < \text{cod}(U/P_1)$ with equality if and only if $\tau(p) = c(\pi_i)$ for some i. The case when Γ is non-trivial (which only arises in types A_n, E_6, D_n) is essentially the same as the above, in type D_n ; but in types A_n and E_6 we also need 5.3.

From the above we conclude that there exists i such that $p = p_{c(\pi_i)}$ (which also equals $p^{c(\pi_i)}$ since π_i is a component of $c(\pi)$). Finally suppose some $p_{c(\pi_j)}$ is absent from $C(P_1)$. Then by 2.7 and 5.4 we may conclude that for any $\alpha \in \pi_1$ there exists $\gamma, \delta \in o(\alpha)$ *distinct* such that $P_{\pi_1} \supset P_\gamma + P_\delta$. This is clearly impossible unless $|o(\alpha) \cap \pi_1| \geq 2$. Except in type A_{2n} we can always choose α so that this is not so and this gives the required contradiction. In type A_{2n} one may choose α so that $o(\alpha) \cap \pi_1$ is a system of type A_2. We write $o(\alpha) \cap \pi_i = \{\alpha_i, \alpha_i'\}$. It is clear that the set $\{P_{\alpha_i} + P_{\alpha_i'} : i = 1, 2, \cdots, m\}$ forms a single G orbit in $\text{Spec}_{o(\alpha)}U$. Since it is the only orbit in $\text{Spec}_{o(\alpha)}U$ of primes P for which $D_{\tau(P)} \cong A_2$ it follows from 4.12 (ii) and 5.5 that $C(P_{\alpha_1} + P_{\alpha_1'})$ contains $\{p_{c(\alpha_i)} + p_{c(\alpha_i')} : i = 1, 2, \cdots, m\}$. Now let P be a minimal prime over $U\left(\bigcap_{i(\neq j)=1}^m (p_{c(\alpha_i)} + p_{c(\alpha_i')})\right)U$. Then $P \supset P_{\alpha_i} + P_{\alpha_i'}$ for some i. Equality implies $P^G \subset \bigcap_{i(\neq j)=1}^m (p_{c(\alpha_i)} + p_{c(\alpha_i')})$ which is absurd. Yet $P \subset P_{o(\alpha)}$

and so we can assume $P \supset P_{\alpha_i} + P_{\alpha'_i} + P_\beta$ for some $\beta \in \{\alpha_s, \alpha'_s\} : s \neq i$. Then the required contradiction obtains as before. □

5.7. Apart from its own interest our main need for the above theorem is to establish the following "triviality". Retain the notation 5.6.

Corollary — *Fix $\alpha \in \pi$ and $\beta, \gamma \in o(\alpha)$ in distinct components of π. Then $C(P_\beta + P_\gamma) \subset \mathrm{Spec}_{c(\alpha)} V$.*

Proof. Consider $p \in C(P_\beta + P_\gamma) \setminus \mathrm{Spec}_{c(\alpha)} V$. Then $\mathrm{cod}(V/p) = \mathrm{cod}(U/P_\beta + P_\gamma) = 4$ and so $D_{\tau(p)} = A_1 \times A_1$ or A_1. Yet $\tau(p) \subset c(\alpha)$ by 4.8 (ii) and so $p \in \mathrm{Spec}_{c(\alpha)} V$ if $|\tau(p)| = 2$. If not, $\tau(p) \in c(\pi_i)$ for some i and so $p \subset p^{c(\pi_i)} = p_{c(\pi_i)}$. Then $\varphi^{-1}(p) \cap U \subset \varphi^{-1}(p_{c(\pi_i)}) \cap U$ and so by 2.6, 5.6 we obtain $O(P_\beta + P_\gamma) \leq O(P_{\pi_1})$. Then $P_\beta + P_\gamma \subset P_{\pi_j}$ for some j, which is excluded by the hypothesis. □

5.8. One also has a "dual" version of 5.7 with a quite different proof.

Lemma — *Fix $\alpha \in \pi$ and $\beta, \gamma \in o(\alpha)$ distinct and in the same connected component of π. Then $C(P_\beta + P_\gamma) \subset \mathrm{Spec}_{c(\alpha)} V$.*

Proof. Consider $p \in C(P_\beta + P_\gamma) \setminus \mathrm{Spec}_{c(\alpha)} V$. Then $htp = ht(P_\beta + P_\gamma) = 2$ by 2.7 and $\tau(p) \subset c(\alpha)$ by 2.8 (ii). This forces $|\tau(p)| = 1$, whilst $\mathrm{cod}(V/p) = \mathrm{cod}(U/P_\beta + P_\gamma)$ which is 4 or 6. Let $\wp$ denote the set of all such primes in Spec$V$ and $\mathscr{P}$ the set of all primes of Spec$U$ satisfying analogous conditions. Consider $P \in \mathscr{P}$. One checks that $P \notin \mathrm{Spec}_{o(\alpha)} U$ and so $C(P) \cap \mathrm{Spec}_{c(\alpha)} V = \emptyset$. This forces $C(P) \subset \wp$ and equality would contradict 2.6. Now by 5.4 there exists for each $\gamma \in c(\alpha)$ some $p' \in \wp$ with $\tau(p') = \gamma$. Thus it suffices to show that $p \in \wp$ is uniquely determined by $\tau(p)$.

Suppose first that $D_{\{\beta,\gamma\}} \cong A_1 \times A_1$. We claim that $p \in \wp$ exists if and only if $\tau(p)$ has at least two neighbours in D_{π^0}, is unique if it has exactly two neighbours;whilst there are exactly three solutions if $\tau(p)$ has three neighbours. (This last possibility is excluded by β, γ being in the same Γ orbit. It is the reason why 5.7 cannot be likewise proved).

The proof of the above claim is an easy exercise with the Goldie rank polynomial of V/p which by [J4, Thm. 5.4] must belong to the W module M generated by any $\alpha'\gamma' : \alpha', \gamma' \in \pi^0$ that are not neighbours. Moreover if β' has α', γ' as neighbours then $s_{\beta'}(\alpha'\gamma') - \alpha'\gamma'$ is proportional to a Goldie rank polynomial of some $p \in \wp$ with $\tau(p) = \beta'$. No other Goldie rank polynomials can appear in virtue of their linear independence [J4, Thm. 5.5].

Finally suppose $D_{\{\beta,\gamma\}} \cong A_2$. Then any $c(\pi_i)$ is of type A_{2n}. In this case $\mathrm{Spec}_{c(\pi_i)} V$ is classified by standard Young tableaux [J3]. The partitions (of $2n+1$) concerning $p \in \wp$ are either $2n-1, 1, 1$ or $2n-2, 3$. Let T be the tableaux assigned to $p \in \mathrm{Spec}_{c(\pi_i)} V$. Let $r(j)$ denote the row containing j. Then (see for example) [Me, 2.2.12] $\tau(p) = \{\alpha_j | r(j+1) > r(j)\}$. It follows that $\tau(p) = \{\alpha_j\}$ if and only if T takes the form

1	2	3	4	. . . . . .	j	j+4	j+5	. . . . . .	2n+1
j+1	j+2	j+3							

This proves uniqueness. (However we do not claim existence, in fact probably $htp = 3$ in this case). $\square$

5.9. For each subset $\pi' \subset \pi$ there is a unique [CD, 5.5; D, II] minimal ideal $P_{\pi',\min}$ whose radical is $\bigcap_{\alpha\in\pi'} P_\alpha$. It is a power of its radical and can also be written [J2, Thm. 5.1] as a suitable product of the $P_\alpha : \alpha \in \pi'$. We denote $P_{\pi,\min}$ simply by $P_{\min}$. As before we replace P by p to denote the corresponding ideal of V.

Lemma — *For each G stable subset $\hat{\pi} \subset \pi$ one has*

(i) $P^G_{\hat{\pi},\min} \supset p_{\hat{\pi},\min}$.

(ii) $P_{\min} = Up_{\min} = p_{\min}U$.

(iii) $P^G_{\min} = p_{\min}$.

Proof. (i) follows from 4.5 (i) and the above remarks. For (ii) consider $M := P_{\min}/Up_{\min}$ as a $U - V$ module. Clearly $Ann_V M \supset p_{\min}$. Since U_V is finitely generated and V is noetherian it follows that M_V is finitely generated and so $d'_V(M) \leq d'_V(V/p_{\min}) < d(V)$. Since U is noetherian, ${}_UM$ is finitely generated and so by 4.3(∗) we obtain $d_U(M) < d(V) = d(U)$. Now M_V is finitely generated so by [JS, 2.1] we obtain $d_U(U/Ann_U M) \leq d_U(M)$ and this forces $Ann_U M \neq 0$. Then $Ann_U M \supset P_{\min}$ and since $P^2_{\min} = P_{\min}$ we conclude that $M = 0$ so $P_{\min} = Up_{\min}$. Similarly $p_{\min}U = P_{\min}$. Obviously (ii) implies (iii). $\square$

5.10. Fix $w \in W$ and let $w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_r} : \alpha_i \in \pi$ be a reduced decomposition and $\ell(w) = r$ its reduced length. By [J2, 4.11] $P_w := P_{\alpha_1}P_{\alpha_2}\cdots P_{\alpha_r}$ is independent of the reduced decomposition chosen. We remark [J2, 5.3] that $P_{\pi',\min} = P_{w_{\pi'}}$ where $w_{\pi'}$ is the unique longest element in the subgroup $W_{\pi'}$ of W generated by the $s_\alpha : \alpha \in \pi'$.

Lemma — *For each G orbit $o(\alpha)$ in π one has*

$$\sum_{\beta\in o(\alpha)} P_{s_\beta w_{o(\alpha)}} = \left(\bigcap_{\substack{\beta,\gamma\in o(\alpha)\\ \text{of type } A_2}} (P_\beta \cap P_\gamma) \right) \left(\bigcap_{\substack{\beta,\gamma\in o(\alpha)\\ \text{of type } A_1^2}} (P_\beta + P_\gamma) \right).$$

Proof. Assume we have proved the inclusion $\subset$. For the opposite inclusion we need to know that the $P_w : w \in W$ generate a distributive lattice of subspaces of U. Since $P_w M(\lambda) = M(w\lambda)$ and the map $I \mapsto IM(\lambda)$ is a bijection [Ja2, 6.20; see also J7, Sect. 8.4] from ideals of U to submodules of $M(\lambda)$ an equivalent assertion is that the $M(w\lambda) : w \in W$ generate a distributive lattice of subspaces of $M(\lambda)$. Unfortunately we do not know this to hold except in the case when the simple components of π_λ are at most of rank 2 and then it is a trivial consequence of Verma modules being multiplicity-free. Since $o(\alpha)$ itself has only components of type A_1 or type A_2 we can reduce it to this situation by varying λ so that it is fixed on $o(\alpha)$. Had λ been integral the modules which concern us, namely the $M(w\lambda) : w \in W_{o(\alpha)}$, viewed as subspaces of $U(\mathfrak{n}^-)$ can be assumed to be *independent* of λ and so the result follows in this case. In the general case choose a weight δ such that $(\delta,\beta) \neq 0$ if $\beta \in \pi_\lambda \backslash o(\alpha)$ and $(\delta,\beta) = 0$ if $\beta \in o(\alpha)$ and consider $\lambda_c := \lambda + c\delta : c \in \mathbb{R}$. Now $M(s_\beta w_{o(\alpha)}\lambda_c) \supset M(w_{o(\alpha)}\lambda_c), \forall \beta \in o(\alpha), c \in \mathbb{R}$ and the quotient is simple for c in general position. Thus the inequality of formal characters

$$ch \sum_{\beta\in o(\alpha)} M(s_\beta w_{o(\alpha)}\lambda_c) \geq \sum_{\beta\in o(\alpha)} chM(s_\beta w_{o(\alpha)}\lambda_c) - (|o(\alpha)| - 1)chM(w_{o(\alpha)}\lambda_c)$$

which by the above is an equality for c in general position, is an equality for arbitrary c (because the right hand side is independent of c and the left hand side can only become smaller at special values of c). Finally let M_c^r (resp. M_c^ℓ) denote the sum of the Verma submodules of $M(\lambda)_c$ corresponding to the ideal in the right (resp. left) hand side of the lemma. Given the inclusion $\subset$, then $M_c^\ell \subset M_c^r$ with equality if and only if the inequality $chM_c^\ell \leq chM_c^r$ is an equality. Since the left hand side has been shown to be independent of c, the inclusion $\supset$ for c in general position gives the required assertion.

For the asserted inclusion $\subset$ recall 4.17 and suppose that case 1) holds. Recall (4.6 (i)) that $P_\beta^2 = P_\beta, \forall \beta \in \pi$. Since the members of $o(\alpha)$ are pairwise orthogonal one obtains

$$\bigcap_{\substack{\beta,\gamma\in o(\alpha)\\ \text{distinct}}} (P_\beta + P_\gamma) \supset \prod_{\substack{\beta,\gamma\in o(\alpha)\\ \text{distinct}}} (P_\beta + P_\gamma) \supset \sum_{\beta\in o(\alpha)} \left(\prod_{\gamma\in o(\alpha)\backslash\{\beta\}} P_\gamma \right) = \sum_{\beta\in o(\alpha)} P_{s_\beta w_{o(\alpha)}}.$$

Distributivity gives the reverse inclusion.

Case 2) of 4.17 is similar. First write $o(\alpha) = \{\beta_i, \gamma_i\}_{i=1}^k$ so that $\{\beta_i, \gamma_i\}$ forms a system of type A_2 for each i. Then

$$\prod_{i=1}(P_{\beta_i}P_{\gamma_i} + P_{\gamma_i}P_{\beta_i}) \subset \bigcap_{i=1}^k (P_{\beta_i} \cap P_{\gamma_i})$$

whilst

$$\sum_{i=1}^{k} \prod_{j(\neq i)=1}^{k} (P_{\beta_j} P_{\gamma_j} + P_{\gamma_j} P_{\beta_j}) \subset \sum_{i=1}^{k} \bigcap_{j(\neq i)=1}^{k} (P_{\beta_j} \cap P_{\gamma_j})$$
$$\subset \bigcap_{j=i+1}^{k} \bigcap_{i=1}^{k} (P_{\beta_i} \cap P_{\gamma_i} + P_{\beta_j} \cap P_{\gamma_j})$$
$$\subset \bigcap_{\substack{\beta,\gamma \in o(\alpha) \\ \text{of type } A_1^2}} (P_\beta + P_\gamma)$$

and

$$\sum_{\beta \in o(\alpha)} P_{s_\beta w_{o(\alpha)}} = \sum_{i=1}^{k} (P_{\beta_i} P_{\gamma_i} + P_{\gamma_i} P_{\beta_i}) \left(\prod_{j(\neq i)=1}^{k} P_{\beta_j} P_{\gamma_j} P_{\beta_j} \right).$$

Combined these give the inclusion $\subset$. Distributivity gives the reverse inclusion. □

Corollary (5.11) — *For each G orbit $o(\alpha)$ in π one has*

$$U \left(\sum_{\beta \in c(\alpha)} p_{s_\beta w_{o(\alpha)}} \right) U \subset \sum_{\beta \in o(\alpha)} P_{s_\beta w_{o(\alpha)}}.$$

Proof. It is enough to show that $\left(\sum_{\beta \in o(\alpha)} P_{s_\beta w_{0(\alpha)}} \right)^G \supset \sum_{\beta \in o(\alpha)} p_{s_\beta w_{c(\alpha)}}$. Clearly $\{P_\beta + P_\gamma : \beta, \gamma \in o(\alpha)$ of type $A_1^2\}$ is a union of G orbits. Combining 4.12 (ii), 5.7, 5.8 it follows that the corresponding union of classes is $\{p_\beta + p_\gamma : \beta, \gamma \in c(\alpha)$ of type $A_1^2\}$. Combined with 4.5 (i) and 2.6 the required assertion follows from 5.10. □

5.12. We may generalize the notion of reduced decomposition by saying that $w = w_1 w_2$ is a reduced decomposition whenever lengths add, that is $\ell(w) = \ell(w_1) + \ell(w_2)$.

Recall that G induces Dynkin diagram automorphisms of π. We may view G as a subgroup of Γ (notation 5.3) and then consider its induced action on W. Set $W^G = \{w \in W | g(w) = w, \forall g \in G\}$. Obviously $w_{\hat{\pi}} \in W^G$ for any G stable subset $\hat{\pi}$ of π.

Lemma — (i) *Take $w \in W^G$ and $\alpha \in \pi$ such that $\ell(s_\alpha w) < \ell(w)$. Then there exists $w' \in W^G$ such that $w = w_{o(\alpha)} w'$ is a reduced decomposition.*

(ii) *There exists a reduced decomposition $w_\pi = w_{o(\alpha_1)} w_{o(\alpha_2)} \cdots w_{o(\alpha_n)}$ of the unique longest element.*

(iii) *Suppose $\alpha, \beta \in \pi$ lie in distinct G orbits. Then one may take $\alpha_1 = \alpha, \alpha_2 = \beta$ in (ii).*

Proof. The hypothesis of (i) is equivalent to $w\alpha < 0$ which further implies $w\beta < 0$ for all $\beta \in o(\alpha)$ and hence all $\beta \in \mathbb{N}o(\alpha) \cap \Delta^+$. By say [J7, A.1.1] this gives the desired conclusion. Obviously (ii) results from (i). For (iii) it is enough to observe that $w_\pi w_{o(\alpha)}\beta < 0$ if $\beta \notin o(\alpha)$. ☐

5.13. Through 5.12 (ii), 5.11 with $\alpha = \alpha_1$ and 5.9 (i) with $\hat{\pi} = o(\alpha_i) : i > 1$ we obtain the

Lemma — *For all $\alpha \in \pi$ one has*

$$\sum_{\beta \in c(\alpha)} Up_{s_\beta w_\pi o} \subset \sum_{\beta \in o(\alpha)} P_{s_\beta w_\pi}.$$

5.14. Let π_1 be a connected component of π. Given $\alpha, \beta \in \pi_1$ in distinct G orbits which are neighbours in D_π, let $o[\alpha, \beta]$ denote the set of all ordered pairs $\gamma \in o(\alpha), \delta \in o(\beta)$ which are neighbours in D_π. Give $c[\alpha, \beta]$ a similar meaning.

Let $\mathcal{M}$ denote the set of all $U - V$ bimodules M such that ${}_UM,\ M_V$ are finitely generated and (cf. 4.4) that $d_U(M) = d'_V(M) = d(U)$. Given $M, M' \in \mathcal{M}$ we say that $M = M'$ up to codimension 2 if there exists $M'' \in \mathcal{M}$ such that $M'' \subset M, M'' \subset M'$, $M/M'' = M'/M''$ and $d(U) - d(M/M'') > 2, d(U) - d(M'/M'') > 2$. Notice that if N is a left V module then $d_V(N) = d_U(UN)$.

Lemma — *Fix $\alpha, \beta \in \pi_1$ in distinct G orbits which are neighbours in D_π. Then*

$$\sum_{\gamma,\delta \in c[\alpha,\beta]} Up_{s_\gamma s_\delta w_\pi o} \subset \sum_{\gamma,\delta \in o[\alpha,\beta]} P_{s_\gamma s_\delta w_\pi}.$$

up to codimension 2.

Proof. As in 5.13 this follows from 5.9 (i), 5.11, 5.12 (iii) given that certain obvious unwanted terms can be ignored. For example if $\gamma \in o(\alpha), \delta \in o(\beta)$ are not neighbours then $d(U) - d(P_{s_\gamma s_\delta w_\pi}/(P_{s_\gamma w_\pi} + P_{s_\delta w_\pi})) = 4$. There is also an extra little verification in type A^k_{2n} when $D_{o(\alpha)}$ is of type A^k_2. For example when $k = 1, n = 2$ writing $s_{\alpha_i} = s_i$ and $P_{\alpha_i} = P_i$ the term $P_4P_2P_3P_1P_4P_2P_3P_2$ equals $P_{s_3s_4s_1w_\pi}$ which in turn equals $P_{s_3s_4w_\pi} + P_{s_1w_\pi}$ up to codimension 2. ☐

5.15. Recall [Ja2, 6.20] the bijection $I \mapsto IM(\lambda)$ of ideals of U to submodules of $M(\lambda)$. In this $P_{\min} = P_{w_\pi}$ becomes the simple Verma module $M(w_\pi\lambda)$ and $P_{s_\beta w_\pi}/P_{w_\pi}$ the simple quotient $L(s_\beta w_\pi) = M(s_\beta w_\pi\lambda)/M(w_\pi\lambda)$. Moreover $Ann_U L(s_\beta w_\pi) = P_\beta$. Similar considerations apply to V.

Proposition — *For all $\alpha \in \pi$ one has*

$$\sum_{\beta \in c(\alpha)} Up_{s_\beta w_\pi o} = \sum_{\beta \in c(\alpha)} p_{s_\beta w_\pi o} U = \sum_{\beta \in o(\alpha)} P_{s_\beta w_\pi}.$$

Proof. We prove only the equality of the first and third terms since equality of the second and third terms is similar. Set

$$Q = \sum_{\beta \in o(\alpha)} P_{s_\beta w_\pi}, \qquad q = \sum_{\beta \in c(\alpha)} p_{s_\beta w_{\pi^o}}.$$

By 5.13 we have $Uq \subset Q$ and we must show that equality holds. By 5.9 (iii) it is enough to show that $U\bar{q} = \bar{Q}$ where $\bar{q} = q/p_{\min}, \bar{Q} = Q/P_{\min}$. By the remarks preceding the proposition $\bar{Q}$ is a semisimple module for the semisimple noetherian ring $\bar{U} := U/\bigcap_{\beta \in o(\alpha)} P_\beta$. Similarly $\bar{q}$ is a semisimple module for the semisimple noetherian ring $\bar{V} := V/\bigcap_{\beta \in c(\alpha)} p_\beta$. Moreover 4.5 (i) gives $\bar{V} = \bar{U}^G$ and 4.3 implies that $\bar{U}$ is finitely generated as a left and a right $\bar{V}$ module. Let S be the set of regular elements of $\bar{V}$. By Goldie's theorem S is an Ore subset of $\bar{V}$ and then $A := \bar{V}S^{-1}$ coincides with the ring of fractions of $\bar{V}$ and is a semisimple artinian ring [H, Chap.7]. Then $B := \bar{U}S^{-1}$ has the structure of a finitely generated A module and is hence artinian. Left multiplication by a regular element of $\bar{U}$ is hence a bijective map. It follows (as noted in [JS, 3.7]) that B coincides with the ring of fractions of $\bar{U}$ which is also semisimple, artinian. Moreover $B^G = A$. Similarly $U\bar{q}S^{-1}$ and $\bar{Q}S^{-1}$ are $B - A$ bimodules. Admit that the inclusion $U\bar{q}S^{-1} \subset \bar{Q}S^{-1}$ is an equality, equivalently that $\bar{Q}/U\bar{q}$ is an S torsion module. The noetherianity of U implies that $\bar{Q}$ and hence $\bar{Q}/U\bar{q}$ is finitely generated as a left U module. Consequently there exists $s \in S$ such that $s \in Ann_{\bar{V}}(\bar{Q}/U\bar{q}) =: p$. Then by 4.3 $(*)$ and [B, 1.3] one has $d_{\bar{U}}(\bar{Q}/U\bar{q}) = d'_{\bar{V}}(\bar{Q}/U\bar{q}) \leq d(\bar{V}/p) < d(\bar{V}) = d(\bar{U})$. Now $\bar{Q}$ is finitely generated as a right $\bar{U}$ module and so $\bar{Q}$ and hence $\bar{Q}/U\bar{q}$ is finitely generated as a right $\bar{V}$ module. Then the previous inequality implies that $d(\bar{U}/Ann_{\bar{U}}(\bar{Q}/U\bar{q})) < d(\bar{U})$. Yet the annihilators $P_\beta : \beta \in o(\alpha)$ of the simple quotients of $\bar{Q}$ satisfy $d(U/P_\beta) = d(\bar{U})$ and so $(Ann_{\bar{U}}(\bar{Q}/U\bar{q}))\bar{Q} = \bar{Q}$. This forces $\bar{Q}/U\bar{q} = 0$ which is the assertion of the proposition. It remains to show $U\bar{q}S^{-1} = \bar{Q}S^{-1}$.

It is clear from 5.13 that $Q^G \supset q$. Recall that $\left(\bigcap_{\beta \in o(\alpha)} P_\beta\right) Q \subset P_{\min}$ and so 4.5 (i) and 5.9 (iii) give $\left(\bigcap_{\beta \in c(\alpha)} p_\beta\right) Q^G \subset p_{\min}$. Recall the bijection $I \mapsto IM(\mu)$ of ideals of V to submodules of $M(\mu)$ and the fact [Ja1, 5.22] that any submodule, say $Q^G M(\mu)$, of $M(\mu)$ strictly containing some $\sum_{\beta \in \pi'} M(s_\beta w_{\pi^o}\mu)$ must contain a Verma submodule, which is either $M(s_\gamma w_{\pi^o}\mu) : \gamma \in \pi \setminus \pi'$ or $M(s_\beta s_\gamma w_{\pi^o}\mu)$: $\beta, \gamma \in \pi'$. From this and the previous inclusion one checks as in say [J2, Sect.7] that $Q^G M(\mu)/qM(\mu)$ can at most have a simple quotient of the form $L(s_\beta s_\gamma w_{\pi^o}\mu)$ with $\beta, \gamma \in c(\alpha)$ of type $A_1 \times A_1$. Then the corresponding simple subquotient of V has GK dimension equal to $d(V) - 4$. We conclude that $Q^G = q$ and hence $\bar{Q}^G = \bar{q}$, both up to codimension 2. Obviously $\bar{V}$ modules of GK dimension strictly less than $d(\bar{V}) = d(V) - 2$ are S torsion and so $\bar{q}S^{-1} = (\bar{Q}S^{-1})^G$. Set $M = \bar{Q}S^{-1}$. It remains to show that $M = BM^G$. In this we want to apply 3.8 but first it is necessary to replace B by a simple ring.

Since the minimal primes of $\bar{U}$ form a G orbit it follows that G permutes the minimal central idempotents $e_1, e_2, \cdots, e_k : k = |o(\alpha)|$ of B. Set $H = Stab_G e_1$. Then we may write $G = \coprod_{i=1}^k g_i H$ where $g_i \in G$ satisfies $g_i(e_1) = e_i$. Set $B_1 = Be_1 = e_1 B$. Define a linear map $\theta : B_1 \longrightarrow B$ by $\theta(b) = \sum_{i=1}^k g_i(b)$. Then $g_i(b)g_j(b') = 0$ for all $b, b' \in Be_1$ and $i \neq j$. From this it follows that θ is a homomorphism (and obviously injective) of the simple artinian ring B_1 into B. Obviously $\theta(B_1)$ is G stable and $\theta(B_1)^G = B^G$. Set $M_1 = e_1 M$ and define an injection $\theta : M_1 \longrightarrow M$ by $\theta(m) = \sum_{i=1}^k g_i(m)$. Then $\theta(bm) = \theta(b)\theta(m), \forall b \in B_1, m \in M_1$. Again $\theta(M_1)$ is G stable and $\theta(M_1)^G = M^G$. One has $\mathrm{End}_{\theta(B_1)}\theta(M_1) \cong \mathrm{End}_{B_1} M_1 = \mathrm{End}_{Fract(U/P_\alpha)} Fract(P_{s_\alpha w_\pi}/P_{w_\pi})$ which by [J4, I, 3.2] is a simple artinian ring (and even isomorphic to $Fract(U/P_{-w_\pi \alpha})$). Finally $\theta(M_1)^G$ is a direct sum of the $Fract(p_{s_\beta w_{\pi^o}}/p_{w_{\pi^o}}) : \beta \in c(\alpha)$ each of which is a non-zero module for the simple direct summand $Fract(V/p_\beta)$ of A. Thus 3.8 applies to give $\theta(B_1)M^G = \theta(M_1)$. Multiplying on the left by the $e_i : i = 1, 2, \cdots, k$ gives $BM^G = M$ as required. □

5.16. Set $\beta^\vee = 2\beta/(\beta,\beta), \forall \beta \in \pi$. By [CD, 8.6] the Goldie rank $rk(U/P_\beta)$ of U/P_β equals $(\lambda, \beta^\vee)$. Moreover this value is clearly independent of the choice of β in its G orbit $o(\alpha)$. Similarly $rk(V/p_\beta) = (\mu, \beta^\vee)$ but this may depend on the choice of β in its class $c(\alpha)$. Let μ_i denote the restriction of μ to the connected component $c(\pi_i)$ of π^0. Our main result established below is essentially that λ is a linear combination of the μ_i with positive integer coefficients. However since we do not know if the Dynkin diagrams of π_1 and $c(\pi_i)$ coincide we can only prove the following modification of this result. Indeed define $\tilde{\mu}_i$ to be μ_i if π_1 and $c(\pi_i)$ have the same Dynkin diagram. Otherwise long and short roots are interchanged on passing from π_1 to $c(\pi_i)$. Then define $\tilde{\mu}_i$ as a weight of π_1 by $(\tilde{\mu}_i, \alpha^\vee) = (\mu_i, c(\alpha)^\vee)$ if α is long in π_1 ; and $(\tilde{\mu}_i, \alpha) = (\mu_i, c(\alpha))$ if α is short in π_1. In other words if we set $r = (c(\alpha), c(\alpha))/(\alpha, \alpha)$ for α short, then the coefficient of the fundamental weight ω_β occurring in μ_i is multiplied by r to obtain $\tilde{\mu}_i$ exactly when β is a short root. One may remark that any Goldie rank polynomial computed with respect to μ_i defined on the dual Dynkin diagram reversing long and short roots divides the corresponding Goldie rank polynomial computed with respect to $\tilde{\mu}_i$.

Theorem — *There exist strictly positive integers m_i such that*

$$\lambda = \sum_{i=1}^n m_i \tilde{\mu}_i.$$

Proof. As before we can assume without loss of generality that π is a G orbit of a connected component π_1. The assertion then results from 3.7 and 5.14 using an analysis similar to that given in 5.15 so we shall be brief. Let α, β be neighbours in

D_{π_1} and set

$$\bar{Q} = \sum_{\gamma,\delta\in o[\alpha,\beta]} P_{s_\gamma s_\delta w_\pi} \Big/ \sum_{\delta\in o(\alpha,\beta)} P_{s_\delta w_\pi}, \bar{q} = \sum_{\gamma,\delta\in c[\alpha,\beta]} p_{s_\gamma s_\delta w_{\pi^o}} \Big/ \sum_{\delta\in c(\alpha,\beta)} p_{s_\delta w_{\pi^o}}.$$

By 5.14 and 5.15 we have $U\bar{q} \subset \bar{Q}$ up to codimension 2. Moreover $\bar{Q}$ (resp. $\bar{q}$) is a semisimple module for the semisimple noetherian ring $\bar{U} := U/\bigcap_{\gamma\in o(\alpha)} P_\gamma$ (resp. $\bar{V} = V/\bigcap_{\gamma\in c(\alpha)} p_\gamma$). Let S be the set of regular elements of $\bar{V}$. Then as in 5.14 the rings $A := \bar{V}S^{-1} = Fract\bar{V}, B := \bar{U}S^{-1} = Fract\bar{U}$ are semisimple artinian with $B^G = A$. Moreover $U\bar{q}S^{-1}$ is a $B - A$ submodule of $\bar{Q}S^{-1}$ with $(\bar{Q}S^{-1})^G = \bar{q}S^{-1}$.

Set $M = \bar{Q}S^{-1}$. We are almost ready to apply 3.7. As in 5.15 we may replace B, M by $\theta(B_1), \theta(M_1)$ defined similarly and we remark that

$$\mathrm{End}_{\theta(B_1)}\theta(M_1) = \mathrm{End}_{Fract(U/P_\alpha)} Fract(P_{s_\alpha s_\beta w_\pi}/(P_{s_\beta w_\pi} + P_{s_\alpha w_\pi})).$$

This by [J4, I, 3.2] is a simple artinian ring containing $Fract(U/P_{w_\pi \beta})$. Moreover the Goldie rank of the former divided by the Goldie rank of the latter is $z_{s_\alpha s_\beta} = -(\alpha^\vee, \beta)$. (See also [P, 2.6]). Finally $\theta(M_1)^G$ is a direct sum of the $Fract(p_{s_\gamma s_\delta w_{\pi^o}}/p_{s_\gamma w_{\pi^o}} + p_{s_\delta w_{\pi^o}})$ each of whose endomorphism ring as a left $Fract(V/p_\gamma)$ module can be similarly expressed. Application of 3.7 then gives the assertion of the theorem. □

Remark. It is perhaps useful to consider the case when G orbits in π are all trivial and α, β is of type A_2. Then the above endomorphism ring is exactly $Fract(U/P_{-w_\pi \beta})$. Moreover 3.7 (or just 3.4) gives $(\lambda, \alpha^\vee)/(\lambda, \beta^\vee) = (\mu, \alpha^\vee)/(\mu, \beta^\vee)$.

6 Skew Field Extensions

6.1. The aim of this section is to analyze to what extent one may strengthen Theorem 4.20 replacing Coxeter by Dynkin. Unfortunately we find that except for special values of λ this cannot be achieved at the level of rings of fractions. For this we return to the point raised in 3.6 and construct an interesting counterexample. For any ring A and any integer r we denote by $M_r(A)$ the ring of $r \times r$ matrices over A.

6.2. We start with some general and fairly well-known analysis. Let B be a simple, artinian ring and G a finite subgroup of $\mathrm{Aut}B$. Let S be the unique up to isomorphism simple B module and $K := \mathrm{End}_B S$ which is a skew field. For all $g \in G$ let S^g be the B module which is S as an additive group with B action $g(b).s = bs, \forall b \in B, s \in S$; equivalently there is an additive group isomorphism $\varphi_g : S \longrightarrow S^g$ satisfying $g(b)\varphi_g(s) = \varphi_g(bs)$. Yet $S^g \xrightarrow{\sim} S$ as a B module so one can just view φ_g as an element of $\mathrm{Aut}S$. Then the action of g becomes conjugation by

φ_g. Let $\tilde{\varphi}_g$ be a second element of Aut$S$ with the above properties. One easily checks that $\varphi_{g^{-1}}\varphi_g$ and $\varphi_{g^{-1}}\tilde{\varphi}_g$ are non-zero elements of $\text{End}_B S = K$. Thus $\tilde{\varphi}_g \in \varphi_g K$. Similarly $\tilde{\varphi}_g \in K\varphi_g$.

Take $\theta \in \text{End}_B S$. Then $\theta\varphi_g(bs) = \theta(g(b)\varphi_g(s)) = g(b)\theta(\varphi_g(s))$ and so $\theta\varphi_g \in \varphi_g K$ by our previous observation. Again $\varphi_g(\theta(bs)) = \varphi_g(b\theta(s)) = g(b)\varphi_g(\theta(s))$ and so $\varphi_g\theta \in K\varphi_g$. Consequently $K\varphi_g = \varphi_g K$, equivalently K is stable for conjugation by φ_g, hence is G stable. Finally one checks that $\varphi_g\varphi_{g'} \in \varphi_{gg'}K$. These properties exactly mean that

$$\bigoplus_{g\in G} \varphi_g K = \bigoplus_{g\in G} K\varphi_g =: K * G$$

inherits a cross product structure. Moreover its action on S defines an algebra homomorphism $K * G \longrightarrow \text{End}S$ whose image we denote by C. Since $K * G$ is semisimple, artinian so is C. Clearly $\text{End}_C S = (\text{End}_K S)^G = B^G =: A$. Finally $C = \text{End}_A S$ by the second commutant theorem [H, 4.3.2].

6.3. To simplify the subsequent analysis we further assume that A is simple. Let T be the unique up to isomorphism simple A module and set $L = \text{End}_A T$. Let r be the dimension of T over L and s the multiplicity of T in S. Then $C = \text{End}_A S = M_s(L)$, whilst $A = M_r(L)$. The question raised in 3.6 is whether rkB/rkA can be read off from the action of G on S. For the moment observe that $r = rkA,\ s = rkC$ and $rs = \dim_L S$.

Suppose $\hat{B}$ is a simple, artinian ring containing B and assume that the action of G on B extends to $\hat{B}$. Suppose further that $\hat{A} := \hat{B}^G$ is simple. One may pose the above question in a strong form by asking if $rk\hat{B}/rk\hat{A} = rkB/rkA$? Consider the special case when $rk\hat{B} = rkB$. Then a minimal idempotent $f \in B$ is also a minimal idempotent for $\hat{B}$ and so $\hat{K} := f\hat{B}f \supset fBf = K$ is a skew-field and the unique up to isomorphism simple $\hat{B}$ module $\hat{S}$ can be taken to be $S \otimes_K \hat{K}$. Then $\hat{C}, \hat{L}, \hat{r}, \hat{s}$ may be defined with respect to $\hat{B}, G$ as in 6.2 and one obtains $\hat{A} = M_{\hat{r}}(\hat{L}),\ \hat{C} = C \otimes_K \hat{K} = M_{\hat{s}}(\hat{L})$ and $\hat{r}\hat{s} = \dim_{\hat{L}} \hat{S}$. Then $\dim_{\hat{K}} \hat{C} = \dim_K C = s \dim_K U = (s/r) \dim_K S = (s/r) rkB$, whilst for the same reason $\dim_{\hat{K}} \hat{C} = (\hat{s}/\hat{r}) rk\hat{B}$. Thus $m := \hat{r}/r = \hat{s}/s$ and our question becomes is $m = 1$? Consider the special case when $s = 1$. Then U identifies with K which consequently must be a subfield of L. Furthermore $C = L,\ \hat{C} = L \otimes_K \hat{K} = M_{\hat{s}}(\hat{L})$ and we are asking if $\hat{s} = 1$. To specify the algebra structure of $L \otimes_K \hat{K}$ we recall that L is an image of $K * G$ and that G acts by automorphisms on $\hat{K}$. It is clear that there is an additive group isomorphism $L \otimes_K \hat{K} \xrightarrow{\sim} \hat{K} \otimes_K L$ and a skew-field embedding of L (resp. $\hat{K}$) in $\hat{K} \otimes_K L$ defined by $\ell \mapsto 1 \otimes \ell$ (resp. $\hat{k} \mapsto \hat{k} \otimes 1$) such that the multiplication in $\hat{C}$ is given by

$$(*) \qquad (\hat{k}_1 \otimes g_1, \hat{k}_2 \otimes g_2) \mapsto \hat{k}_1 g_1(\hat{k}_2) \otimes g_1 g_2, \forall \hat{k}_1, \hat{k}_2 \in \hat{K}, g_1, g_2 \in G.$$

For our purposes it is relevant to further impose that $M_{\hat{s}}(\hat{L})$ is isomorphic to $\mathrm{End}_{\hat{L}} L$.

6.4. We give a number of examples for which $\hat{s} > 1$ in the above.

Example (1). Let G be the group of quaternions and $K = \mathbb{R}$ the real field. Then G admits a four dimensional simple module over $\mathbb{R}$ which further defines a surjection of $\mathbb{R}G$ onto the skew-field of quaternions $\mathbb{H}$. Take $L = \mathbb{H}$ and $\hat{K} = \mathbb{C}$ with the trivial action of G and set $\hat{L} = \mathbb{C}$. Then $\hat{K} \otimes_K L = \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C}) \cong \mathrm{End}_{\mathbb{C}}\mathbb{H} = \mathrm{End}_{\hat{L}} L$. Thus $\hat{s} = 2$. In this (classical) example the base field (namely $\mathbb{R}$) is not algebraically closed and the rings A, B are finite dimensional over their centres so it is not so relevant to our present interest.

Example (2). Take $\hat{K}$ to be the first Weyl skew-field over $\mathbb{C}$ which we may represent as $\mathbb{C}\left(y, \frac{\partial}{\partial y}\right)$ and define $\theta \in \mathrm{Aut}\hat{K}$ by $\theta(y) = -y, \theta\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial y} + y^{-1}$. Then θ^2 is conjugation by y^2 on $\hat{K}$ whilst on the subfield $K := \mathbb{C}\left(y^2, y\frac{\partial}{\partial y}\right)$, the automorphism θ restricts to conjugation by y. Identify $\hat{K}, L$ by their images in $\hat{K} \otimes_K L$ where L is generated over K by an element z satisfying $z^2 = -y^2$ and $zkz^{-1} = \theta(k), \forall k \in \hat{K}$ viewed as an element of $\hat{K} \otimes_K L$. Set $x = \frac{\partial}{\partial y}$. Then $\hat{K} \otimes_K L$ has generators x, y, z satisfying

$$(*) \qquad xy - yx = 1, yz + zy = 0, zx + xz = y^{-1}z, z^2 = -y^2.$$

In particular L has generators $z^{-1}yx, z$. It is also a first Weyl skew-field over $\mathbb{C}$. Indeed $\hat{K}, \hat{L}, K$ are all isomorphic.

The element $u := y^{-1}z$ satisfies $u^2 - 1, uz + zu = 0, yu + uy = 0, xyu = uxy$. Thus conjugation by u defines an involution σ of $\hat{K}$ leaving K pointwise fixed. Define G to be generated by σ and so isomorphic to $\mathbb{Z}_2$.

One has $\hat{K} = K + yK$. Thus the structure of $\hat{K}$ as a right K module may be represented by taking $1 = \binom{1}{0}, y = \binom{0}{1}$. Then through left multiplication the generators y, yx of $\hat{K}$ become the elements of $\mathrm{End}_K\hat{K}$ represented as the matrices $\begin{pmatrix} 0 & y^2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} yx & 0 \\ 0 & 1+yx \end{pmatrix}$; whilst $\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $z \in \mathrm{End}_K\hat{K}$ is just $y\sigma = \begin{pmatrix} 0 & -y^2 \\ 0 & 1 \end{pmatrix}$. One thus easily checks that y, yx, z generate $\mathrm{End}_K\hat{K}$. This establishes the isomorphism (of algebras) $\hat{K} \otimes_K L \xrightarrow{\sim} \mathrm{End}_K\hat{K}$ which is furthermore isomorphic to $\mathrm{End}_{\hat{L}} L$ as required.

Since $\hat{K}$ and L are isomorphic we can view the above as an example of an algebra isomorphism $L \otimes_K L \xrightarrow{\sim} \mathrm{End}_K L$. However this is quite different to the well-known [H, 4.13] isomorphism $L \otimes_F L^{op} \xrightarrow{\sim} \mathrm{End}_F L$ for any skew-field L finite dimensional over its centre F since the algebra structure in the first case is defined by 6.3$(*)$ rather than just component-wise.

Example (3). Let ω be a primitive n^{th} root of unity. Take $\hat{K}$ as in example 2 and $\sigma \in \mathrm{Aut}\hat{K}$ defined by $\sigma(x) = \omega x, \sigma(y) = \omega^{-1}y$ with $G =< \sigma >\cong \mathbb{Z}_n$. Set $K = \hat{K}^G$. Then $\dim_K\hat{K} = n$. The elements $yx, y\sigma \in \hat{K} * G$ generate a subfield L isomorphic to $\hat{K}$. One checks as in example 2 that

$$\hat{K} \otimes_K L \xrightarrow{\sim} \hat{K} * G \xrightarrow{\sim} \mathrm{End}_K \hat{K} \xrightarrow{\sim} M_n(K).$$

Example (4). Take $\hat{K}, G$ as in example 2; but with $\sigma(x) = x, \sigma(y) = y$. Define $L \subset \hat{K} * G$ as in example 3. Set $K = L \cap \hat{K}$ which is the Weyl skew-field $\mathbb{C}(yx, y^2)$ but no longer $\hat{K}^G$. One checks from 6.3(∗) that $\hat{K} \otimes_K L \xrightarrow{\sim} L \oplus L$.

6.5. Let us show how example 2 relates to the problem discussed in 6.1. Here eight rings are involved and these will all be constructed as subrings of the ring of 2×2 matrices of the first Weyl skew field which we shall represent as $M_2(\mathbb{C}(a, b))$ where $a = \partial/\partial b$.

Define $x, y, z, x', y', z' \in M_2(\mathbb{C}(a, b))$ by the formulae

$$yx = y'x' = \begin{pmatrix} ba & 0 \\ 0 & 1 + ba \end{pmatrix}, y = \begin{pmatrix} 0 & b^2 \\ 1 & 0 \end{pmatrix}, z = z' = \begin{pmatrix} 0 & -b^2 \\ 1 & 0 \end{pmatrix}, y' = \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}.$$

It is clear that we may define $\sigma \in \mathrm{Aut}M_2(\mathbb{C}(a, b))$ satisfying $\sigma(y') = -y'$ and fixing xy, z, y'. Take $G =< \sigma >\cong \mathbb{Z}_2$. From the analysis of example 2 the relations 6.4(∗) hold for x, y, z and one further checks that they also hold for the primed elements.

Set $L = \mathbb{C}(a, b), K = \mathbb{C}(ab, b^2), \hat{B} = M_2(L)$. Then $\hat{B}$ is generated by yx, y, z, y' and by inverting regular elements. We shall designate this briefly as $\hat{B} = \mathbb{C}(yx, y, z, y')$. Then $\hat{A} := \hat{B}^G = \mathbb{C}(yx, y, z)$ which as we have seen in example 2 is isomorphic to $M_2(K)$. Define $B = \mathbb{C}(yx, y', z) = \mathbb{C}(y'x', y', z')$ which is again isomorphic to $M_2(K)$. Then $A := B^G = \mathbb{C}(yx, z)$ and is isomorphic to L.

In 3.4 take $M = B$ considered as a left B module. Then $B' = \mathrm{End}_B M$ identifies with B and $A' := B'^G$ with $\mathrm{End}_A M^G = \mathrm{End}_A A = A$. Set $\tau := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \hat{B}$. Then conjugation by τ restricts to an automorphism of B and $\hat{B}$ identifies with the cross product $B* < \tau >$. Then $M = B$ may be viewed as a $\hat{B}$ module extending the B action through conjugation by τ. One checks that $\hat{B}' := \mathrm{End}_{\hat{B}} M = B'^\tau$ identifies with the subfield $\mathbb{C}(y'x', y')$. Then $\hat{A}' := \hat{B}'^G = \mathbb{C}(y'x', y'^2) = \mathbb{C}(yx, y^2) = \mathbb{C}(yx, z^2)$. One has $M^G = A$ and as a right $\hat{A}'$ module $\mathrm{End}_{\hat{A}'} M^G = \mathbb{C}(yx, y, z) = \hat{A} = \hat{B}^G$. Finally $\hat{B}'^G = \mathrm{End}_{\hat{B}^G} M^G = \hat{A}'$.

The above may be summarized by the following diagram of ring inclusions.

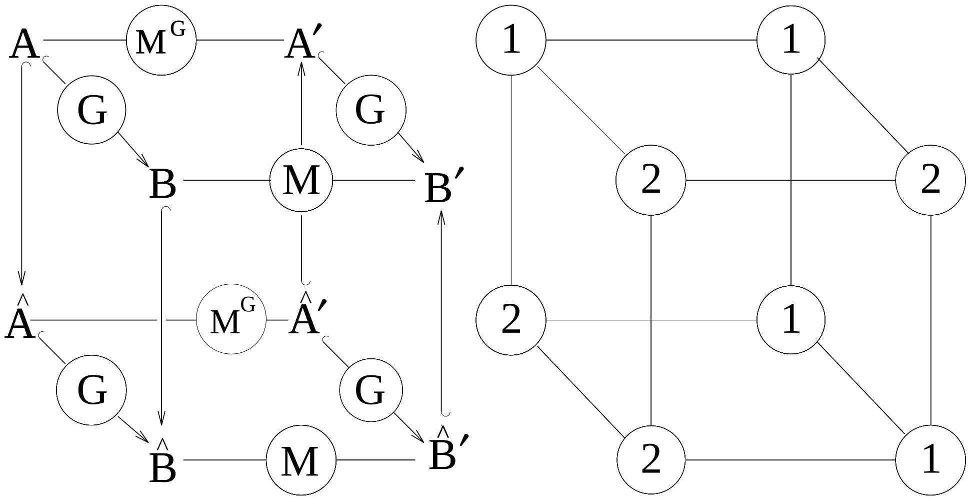

Figure 1.

In the above $A \hookrightarrow \textcircled{G} \rightarrow B$ designates that $B = A^G$ whilst $B - \textcircled{M} - B'$ designates that $B' = \mathrm{End}_B M$. The diagram on the right gives the Goldie ranks of the corresponding rings.

6.6. We are not yet able to show that the above inclusions of rings (of fractions) can be lifted to give an example with $\mathfrak{g}$ of type B_2 and $\mathfrak{g}'$ of type C_2 in the conclusion of Theorem 4.20. The question is the following. Does there exist a finite subgroup G of $\mathrm{Aut}(U(\mathfrak{g})/AnnM(\lambda))$ such that $(U(\mathfrak{g})/AnnM(\lambda))^G \cong U(\mathfrak{g}')/AnnM(\mu)$ with $P_\alpha^G = p_\beta, P_\beta^G = p_\alpha$ and $(\beta,\beta)/(\alpha,\alpha) > 1$? To be specific take $\mathfrak{g}$ of type B_2 and $\mathfrak{g}'$ of type C_2 so then $(\beta,\beta) = 2(\alpha,\alpha)$. Recalling 5.16 set

$$\bar{Q} = P_{s_\beta s_\alpha w_\pi}/(P_{s_\alpha w_\pi} + P_{s_\beta w_\pi}), \bar{q} = p_{s_{\beta'} s_{\alpha'} w_{\pi^o}}/(p_{s_{\alpha'} w_{\pi^o}} + p_{s_{\beta'} w_{\pi^o}}).$$

One has $U\bar{q} = \bar{Q}$ but we cannot say if $\alpha = \alpha', \beta = \beta'$ that is if the root lengths remain in the same relative proportions, or become reversed. Define S as in 5.16 and set $M = \bar{Q}S^{-1}$ which is a right $B' = FractU(\mathfrak{g})/P_\alpha$ module. As noted in 5.16 its endomorphism ring B has $z_{s_\alpha s_\beta} = -(\alpha^\vee,\beta) = 2$ times the Goldie rank of $Fract(U(\mathfrak{g})/P_\beta)$. On the other hand if $\alpha' = \beta, \beta' = \alpha$ then M^G is a right $A' := B'^G = FractU(\mathfrak{g}')/p_\beta$ module and its endomorphism ring $A := B^G$ has $z_{s_\beta s_\alpha} = -(\beta^\vee,\alpha) = 1$ times the Goldie rank of $Fract\ U(\mathfrak{g}')/p_\alpha$. This is perfectly compatible with the Goldie ranks in the top half of Figure 1 if we take $\lambda = \omega_\alpha + \rho, \mu = \rho$. On the other hand if $\alpha' = \alpha, \beta' = \beta$ and taking $\lambda = \mu, \mu = \rho$ the resulting Goldie ranks are compatible with those occurring in the bottom half of Figure 1. *Consequently we cannot strengthen Theorem 4.20 in the desired manner by passage to rings of fractions unless* $(\lambda,\alpha^\vee)$ *is not divisible by* $z_{s_\alpha s_\beta}$. Finally notice in the first example that G has to be quite big. Indeed $P_{\max}$ is the annihilator of a four dimensional module, whilst $P^G_{\max} = p_{\max}$ is the annihilator of the trivial module. Consequently G must admit (3.6) a four dimensional irreducible projective representation.

6.7. As noted above we can strengthen Theorem 4.20 in the following "indivisible Goldie rank" case. In this we can assume that π is the G orbit of a connected component π_1 of π and moreover that π_1 is not simply-laced. Then in π_1 there is a unique pair of neighbouring roots α, β with $(\beta, \beta)/(\alpha, \alpha) =: z > 1$.

Theorem — *Suppose that z does not divide $(\lambda, \alpha^\vee)$. Then one may replace Coxeter by Dynkin in 4.20 and $\tilde{\mu}_i$ by μ_i in 5.16.*

Proof. The required strengthening is obtained by applying the result noted in italics in 6.6. □

Remark. Of course this applies in particular when $\lambda = \rho$ and refines Polo's result in [P, Thm. 7.1].

Index of Notations

Symbols used frequently are given below in the place where they are first defined:

1.1 $\mathfrak{g}, U(\mathfrak{g})$.

2.2 $O(P), I(P), O(P) \geq O(P')$.

2.3 $\varphi, I(p)$.

2.4 $C(p)$.

2.5 $C(p) \geq C(p')$.

2.6 Φ.

4.1 $\mathfrak{n}^+, \mathfrak{h}, \mathfrak{n}^-, \Delta, \Delta^+, \rho, M(\lambda), U_\lambda, \Delta_\lambda, W_\lambda, \Delta^+_\lambda, \pi_\lambda$.

4.2 $V_\mu, \Delta^o_\mu, \pi^o_\mu$.

4.3 d_B, d'_B.

4.4 $C(P), \mathrm{cod}$.

4.5 $P_\alpha, p_\alpha, o(\alpha), c(\alpha)$.

4.6 $\tau, P_{\pi'}, P^{\pi'}, P^\alpha, P_{\max}, P^{\pi''}_{\pi'}, P^\beta_{\pi'}$.

4.9 $\hat{\pi}$.

4.11 $\mathrm{Spec}^{\hat{\pi}}$.

4.12 $\mathrm{Spec}_{\hat{\pi}}$.

4.14 $\Delta_{\pi'}$.

4.17 $D_{\pi'}$.

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