Harrison Cocycles and the Group of Galois Coobjects

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Abstract

Let H be a commutative faithfully flat Hopf algebra over a commutative ring R. We give an exact sequence describing the group of H-Galois coobjects. The other terms in the sequence are Harrison cohomology groups. This generalizes an exact sequence due to Early and Kreimer and Yokogawa.

Résumé

Soit H une algèbre de Hopf commutative fidèlement plate sur un anneau commutatif R. Nous étudions une suite exacte qui décrit le groupe des co-objets H-Galois. Les autres termes de la suite sont des groupes de cohomologie de Harrison. Cela généralise une suite exacte due à Early, Kreimer et Yukogawa.

Introduction

Let H be a finite (i.e. a finitely generated projective) cocommutative Hopf algebra over a commutative ring R. Chase and Sweedler [4] introduced the notion of H-Galois object, generalizing classical Galois theory. Isomorphism classes of H-Galois objects form a group $\mathrm{Gal}(R,H)$. The multiplication on $\mathrm{Gal}(R,H)$ is induced by the cotensor product $\square_H$. Early and Kreimer [5] and, independently, Yokogawa [13] showed that $\mathrm{Gal}(R,H)$ fits into an exact sequence

$$(1) \qquad 1 \longrightarrow H^2(H,R,\mathbb{G}_m) \xrightarrow{\alpha} \mathrm{Gal}(R,H) \xrightarrow{\beta} H^1(H,R,\mathrm{Pic}) \xrightarrow{\gamma} H^3(H,R,\mathbb{G}_m)$$

Here the cohomology groups are Sweedler cohomology groups, cf. [11]. The definition of a Galois object can be generalized to the situation where H is not necessarily finitely generated or projective ([9]). The idea is the following: consider an H-comodule algebra A. Then we have a pair of adjoint functors between the category R-mod and the category of relative (A,H)-modules. This category consists of

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R-modules equipped with an A-action and an H-coaction satisfying a certain compatibility relation. If H is finite, then relative Hopf modules correspond to (right) $A^{\mathrm{opp}}\#H^*$-modules, and this explains the relation with the theory of Chase and Sweedler. If the adjunction is a category equivalence, then we say that A is an H-Galois object.

The question that we are interested in is the following: can we generalize the exact sequence (1) to the situation where the Hopf algebra H is not necessarily finitely generated and projective? The proofs exhibited in [5] and [13] make intensive use of the fact that the Hopf algebra H (and the H-Galois objects) are finitely generated and projective. This allows to switch back and forth between H-comodule algebras and H^*-module coalgebras. For example, the map β is given by forgetting the algebra structure, followed by taking the dual. We then obtain an H^*-module, representing a Sweedler cocycle. Of course these duality arguments no longer hold in the case where H is infinite. Another problem is the fact that the cotensor product is not naturally associative (unless we work over a field instead of a commutative ring). Moreover, we cannot prove that the cotensor product of two H-Galois objects is again an H-Galois object.

In this note, we propose to work with H-module coalgebras instead of H-comodule algebras. In [9], Schneider introduces a Galois theory for H-module coalgebras, leading to the notion of H-Galois coobject. If H is finite, then the dual of an H-Galois coobject is an H^*-Galois object. We will show that, for H commutative, the set of isomorphism classes of H-Galois coobjects forms a group $\mathrm{Gal}^{\mathrm{co}}(R,H)$. The operation is now induced by the tensor product $\otimes_H$. $\mathrm{Gal}^{\mathrm{co}}(R,H)$ fits into an exact sequence, and, in the case where H is finite, a duality argument shows that the exact sequence (1) follows from this new sequence.

When we try to add the H^3-term to the sequence, we face a phenomenon that is typical for the infinite case. We have to restrict attention to a subgroup of the group of Galois coobjects. This subgroup is defined as follows: consider Galois coobjects that have normal basis after we take a faithfully flat base extension. We will say that such a Galois coobject has a *geometric normal basis.* Thus a Galois coobject C has a geometric normal basis if $C \otimes S \cong H \otimes S$ as $H \otimes S$-modules for some faithfully flat commutative R-algebra S. If H is finite then all Galois coobjects have a geometric normal basis, we can take a Zariski covering for S. We have to apply a similar construction for the Picard group, and then we can state the generalized exact sequence, see Theorem 3.4.

Along the way, we obtain two results that seem to be new even in the finite case: we have an explicit construction for the inverse of an H-Galois coobject (Theorem 2.2), and, conversely, if an H-module coalgebra is a twisted form of H as an H-module and is invertible as an H-module coalgebra, then it is an H-Galois coobject

(Corollary 3.5).
Some additional difficulties arise if we try to construct a similar theory for Galois objects; moreover, the formalism turns out to be much more complicated in this situation, and this is why the author has the opnion that the coalgebra formalism is the natural formalism for this type of problem.
For standard results and terminology about Hopf algebras, we refer to the literature, e.g. [1], [7] or [11]. The reader should keep in mind that we work here over a commutative ring, while the monographs cited above restrict attention to Hopf algebras over a field.

1 Notations and preliminary results

Throughout this paper, H will be a commutative Hopf algebra over a commutative ring R, and assume that H is faithfully flat as an R-module. For the comultiplication on H we will use Sweedler's sigma notation ([10]):

$$\Delta(h) = \sum h_1 \otimes h_2$$

A left H-*module coalgebra* is an R-module C such that C is a left H-module and an R-coalgebra satisfying the compatibility relations

$$\Delta_C(h \rightharpoonup c) = \sum (h_1 \rightharpoonup c_1) \otimes (h_2 \rightharpoonup c_2) \tag{2}$$

$$\varepsilon_C(h \rightharpoonup c) = \varepsilon_H(h)\varepsilon_C(c) \tag{3}$$

for all $h \in H$ and $c \in C$. The left action of H on C is denoted by $\rightharpoonup$. If H is commutative, it makes no sense to distinguish between left and right H-module coalgebras.
Let C be a left H-module coalgebra. Then a left (H, C)-Hopf module M is an R-module that is a left H-module and a left C-comodule such that

$$\rho_M(h \cdot m) = \sum h_1 \rightharpoonup m_{(-1)} \otimes h_2 m_{(0)} \tag{4}$$

for all $m \in M$ and $h \in H$. In the sequel, ${}^C_H\mathcal{M}(H)$ will denote the category of left (H, C)-Hopf modules and H-linear C-colinear maps.

Proposition 1.1 — *With notations as above, consider the functors*

$$F: \ {}^C_H\mathcal{M}(H) \longrightarrow R\text{-mod} : M \mapsto R \otimes_H M = \overline{M}$$
$$G: \ R\text{-mod} \longrightarrow {}^C_H\mathcal{M}(H) : N \mapsto C \otimes N$$

Then G is a right adjoint to F.

Proof. This result is a special case of [3, Theorem 1.3]. We restrict to giving a brief sketch of the proof. R is an H-module via the map ε. In fact, $\overline{M} = M/\mathrm{Ker}\varepsilon M$, and in $\overline{M}$ we have the following identity:

$$\overline{hm} = \varepsilon(h)\overline{m}$$

for all $h \in H$ and $m \in M$. For any $M \in {}^C_H\mathcal{M}(H)$ and $N \in R$-mod we consider the maps

$$\alpha : \mathrm{Hom}^C_H(M, C \otimes N) \longrightarrow \mathrm{Hom}_R(\overline{M}, N)$$
$$\beta : \mathrm{Hom}_R(\overline{M}, N) \longrightarrow \mathrm{Hom}^C_H(M, C \otimes N)$$

given by

$$\alpha(f)(\overline{m}) = (\varepsilon_C \otimes I_N)(f(m))$$
$$\beta(g)(m) = \sum m_{(-1)} \otimes g(\overline{m}_{(0)})$$

for all $f \in \mathrm{Hom}_{{}^C_H\mathcal{M}(H)}(M, C \otimes N)$, $g \in \mathrm{Hom}_R(\overline{M}, N)$ and $m \in M$. A straightforward verification shows that f and g are well-defined and each others inverses. This finishes the proof. □

From the adjointness of the functors F and G in Proposition 1.1, it follows that for all $M \in {}^C_H\mathcal{M}(H)$ and $N \in R$-mod we have natural maps

$$\psi_M : M \longrightarrow G(F(M)) = C \otimes \overline{M}$$
$$\phi_N : F(G(N)) = \overline{C} \otimes N \longrightarrow N$$

given by

$$\psi_M(m) = \sum m_{(-1)} \otimes \overline{m}_{(0)}$$
$$\phi_N(\sum_i \overline{c}_i \otimes n_i) = \sum_i \varepsilon(c_i) n_i$$

Definition 1.2 — *With notations as above, an H-module coalgebra C is called an H-Galois coobject if the functors F and G from Proposition 1.1 are inverse equivalences, or, equivalently, if ψ_M and ϕ_N are isomorphisms for all $M \in {}^C_H\mathcal{M}(H)$ and $N \in R$-mod.*

We will now establish some necessary and sufficient conditions for an H-module coalgebra to be an H-Galois coobject. It is clear that ϕ_N is an isomorphism for all $N \in R$-mod if and only if the canonical map

$$\phi_C : \overline{C} \longrightarrow R : \overline{c} \mapsto \varepsilon(c)$$

is an isomorphism.
Observe that $H\otimes C$ can be given the structure of left (H,C)-Hopf module as follows:

$$k(h\otimes c) = kh\otimes c$$
$$\rho_{H\otimes C}(h\otimes c) = \sum h_1 \rightharpoonup c_1\otimes h_2\otimes c_2$$

for all $h,k\in H$ and $c\in C$. It is readily verified that condition 4 is satisfied:

$$\begin{aligned}\rho_{H\otimes C}(kh\otimes c) &= \sum k_1h_1\rightharpoonup c_1\otimes k_2h_2\otimes c_2\\ &= \sum k_1(h\otimes c)_{(-1)}\otimes k_2(h\otimes c)_{(0)}\end{aligned}$$

A necessary condition for M to be an H-Galois coobject is therefore that $\delta=\psi_{H\otimes C}$ is an isomorphism. Let us describe δ. First we remark that $F(H\otimes C)=\overline{H}\otimes C=C$, since $\overline{H}\cong R$. Indeed, the maps

$$I\otimes\varepsilon_H:\ \overline{H}=R\otimes_H H\longrightarrow R\qquad \eta\otimes 1:\ R\longrightarrow\overline{H}=R\otimes_H H$$

are well-defined and each others inverses.
Now $G(F(H\otimes C))=C\otimes C$, where H acts and C coacts on the first factor:

$$h(c\otimes d) = h\rightharpoonup c\otimes d$$
$$\rho_{C\otimes C}(c\otimes d) = \sum c_1\otimes c_2\otimes d$$

$\delta=\psi_{H\otimes C}$ is given by the formula

$$\delta(h\otimes c) = \sum(h_1\rightharpoonup c_1)\otimes\varepsilon(h_2)c_2 = \sum(h\rightharpoonup c_1)\otimes c_2$$

Theorem 1.3 — *Let H be a commutative, faithfully flat Hopf algebra. For a left H-module coalgebra C, the following conditions are equivalent:*

1. *C is an H-Galois coobject;*
2. – *$\overline{C}=R$;*
   – *$\delta=\psi_{H\otimes C}:\ H\otimes C\longrightarrow C\otimes C:\ h\otimes c\mapsto\sum(h\rightharpoonup c_1)\otimes c_2$ is an isomorphism;*
   – *C is flat as an R-module.*
3. – *$\delta=\psi_{H\otimes C}:\ H\otimes C\longrightarrow C\otimes C:\ h\otimes c\mapsto\sum(h\rightharpoonup c_1)\otimes c_2$ is an isomorphism;*
   – *C is faithfully flat as an R-module.*

Proof. For full detail, we refer to [9] or to [3], where more general results are given. The reader might object that the results in [3] are valid only if we work over a field k, but it can be verified that the above Theorem is true over a commutative ring. □

Corollary 1.4 — *Let H be a commutative, faithfully flat Hopf algebra. Then H viewed as a left H-module coalgebra is an H-Galois coobject.*

Proof. We only have to show that the map

$$\delta : H \otimes H \longrightarrow H \otimes H : \ h \otimes k \mapsto \sum hk_1 \otimes k_2$$

has an inverse. This inverse is given by the formula

$$\delta^{-1}(h \otimes k) = \sum hS(k_1) \otimes k_2$$

□

Harrison cohomology

For $i = 0, 1, \cdots, n+1$, consider the maps $\varepsilon_i : \ H^{\otimes n} \longrightarrow H^{\otimes n+1}$ defined as follows:

$$\varepsilon_i(h_1 \otimes \cdots \otimes h_n) = \begin{cases} 1 \otimes h_1 \otimes \cdots \otimes h_n & \text{if } i = 0 \\ h_1 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_n & \text{if } i = 1, \cdots, n \\ h_1 \otimes \cdots \otimes h_n \otimes 1 & \text{if } i = n+1 \end{cases}$$

Let P be a covariant functor from flat commutative R-algebras to abelian groups, and consider

$$\Delta_n = \sum_{i=0}^{n+1} (-1)^i P(\varepsilon_i) : \ P(H^{\otimes n}) \longrightarrow P(H^{\otimes n+1})$$

We obtain a complex $0 \longrightarrow P(H) \xrightarrow{\Delta_1} P(H^{\otimes 2}) \xrightarrow{\Delta_2} P(H^{\otimes 3}) \xrightarrow{\Delta_3} \cdots$. The corresponding cohomology groups$H^n_{\text{Harr}}(H, R, P)$ are called the *Harrison cohomology* groups with values in P.

2 The group of Galois coobjects

Consider the set of isomorphism classes of H-Galois coobjects. In this Section, we will show that this set forms a group under the operation induced by the tensor product over H.

If C and D are two H-module coalgebras, then $C \otimes_H D$ is again an H-module coalgebra. The action and comultiplication are given by the formulas

$$h \rightharpoonup (c \otimes d) = h \rightharpoonup c \otimes d = c \otimes h \rightharpoonup d$$

and

$$\Delta(c \otimes d) = \sum (c_1 \otimes d_1) \otimes (c_2 \otimes d_2)$$

We leave it to the reader to verify that Δ is well-defined. Obviously, H is itself an H-module coalgebra, and we have an H-module coalgebra isomorphism

$$H \otimes_H C \longrightarrow C : \ h \otimes c \mapsto h \rightharpoonup c$$

for every H-module coalgebra C.

Proposition 2.1 — *Suppose that C and D are two H-Galois coobjects. Then $C\otimes_H D$ is also an H-Galois coobject.*

Proof. As we have seen above, $C\otimes_H D$ is an H-module coalgebra. Let us first show that $C\otimes_H D$ is flat as an R-module. Suppose that

$$0\longrightarrow M'\longrightarrow M\longrightarrow M''\longrightarrow 0$$

is an exact sequence of R-modules, and consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C\otimes H\otimes D\otimes M' & \longrightarrow & C\otimes H\otimes D\otimes M & \longrightarrow & C\otimes H\otimes D\otimes M'' & \longrightarrow & 0\\
 & & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow & & \\
0 & \longrightarrow & C\otimes D\otimes M' & \longrightarrow & C\otimes D\otimes M & \longrightarrow & C\otimes D\otimes M'' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C\otimes_H D\otimes M' & \longrightarrow & C\otimes_H D\otimes M & \longrightarrow & C\otimes_H D\otimes M'' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The two top rows are exact since C, H and D are flat R-modules. The three columns are exact because the tensor product is right exact. The Lemma of 5 now implies that the bottom row is exact, and therefore $C\otimes_H D$ is exact.
Observe next that

$$\overline{C\otimes_H D}=C\otimes_H D\otimes_H R\cong C\otimes_H R\cong R$$

Finally, the map

$$\delta:\ H\otimes C\otimes_H D\longrightarrow C\otimes_H D\otimes C\otimes_H D:\ h\otimes c\otimes d\mapsto\sum h\rightharpoonup c_1\otimes d_1\otimes c_2\otimes d_2$$

is an isomorphism. Indeed, observe that we have natural isomorphisms

$$H\otimes C\otimes_H D\cong H\otimes_H H\otimes C\otimes_H D\cong (H\otimes C)\otimes_{H\otimes H}(H\otimes D)$$

We may therefore view δ as the map

$$\delta:\ (H\otimes C)\otimes_{H\otimes H}(H\otimes D)\longrightarrow C\otimes_H D\otimes C\otimes_H D$$

given by

$$\delta=\tau_{23}\circ(\delta_C\otimes\delta_D)$$

and this map is an isomorphism. $C\otimes_H D$ now satisfies all the conditions of Proposition 1.3, and is therefore an H-Galois coobject. $\square$

Theorem 2.2 — *Suppose that H is a Hopf algebra that is faithfully flat as an R-module, and let C be an H-Galois coobject. Then there exists an H-Galois coobject D such that $C \otimes_H D \cong H$ as H-module coalgebras. As a coalgebra, $D = C^{\mathrm{cop}}$. The action of H on D is given by the formula*

$$h \overset{D}{\rightharpoonup} d = S(h) \overset{C}{\rightharpoonup} d$$

for all $h \in H$ and $d \in D$.

Proof. Let $K = H^{\mathrm{cop}}$ as a coalgebra, with H-action given by $h \overset{K}{\rightharpoonup} d = S(h)k$ for all $h, k \in H$. Then the antipode $S: H \to K$ is an isomorphism of H-comodule algebras. It therefore suffices to show that $C \otimes_H D \cong K$ as H-module coalgebras. Since $\overline{C} = R$, R is the coequalizer of the maps

$$\begin{cases} \varepsilon_H \otimes I_C \\ \quad \psi_C \end{cases} : H \otimes C \rightrightarrows C \xrightarrow{\varepsilon_C} R \longrightarrow 0$$

Now H is flat as an R-module, and this implies that H is the coequalizer of the maps

$$\begin{cases} I_H \otimes \varepsilon_H \otimes I_C \\ \quad I_H \otimes \psi_C \end{cases} : H \otimes H \otimes C \rightrightarrows H \otimes C \xrightarrow{I_H \otimes \varepsilon_C} H \longrightarrow 0$$

Recall from Corollary 1.4 that

$$\delta_H^{-1} \otimes I_C : H \otimes H \otimes C \longrightarrow H \otimes H \otimes C : h \otimes k \otimes c \mapsto \sum hS(k_1) \otimes k_2 \otimes c$$

is an isomorphism. Therefore H is also the coequalizer of the maps

$$\begin{cases} \beta = (I_H \otimes \varepsilon_C \otimes I_C) \circ (\delta_H^{-1} \otimes I_C) \\ \alpha = (I_H \otimes \psi_C) \circ (\delta_H^{-1} \otimes I_C) \end{cases} : H \otimes H \otimes C \rightrightarrows H \otimes C \xrightarrow{I_H \otimes \varepsilon_C} H \longrightarrow 0$$

One easily verifies that

$$\alpha(h \otimes k \otimes c) = \sum hS(k_1) \otimes (k_2 \rightharpoonup c)$$
$$\beta(h \otimes k \otimes c) = hS(k) \otimes c$$

Now consider the map

$$\delta' : H \otimes H \otimes C \longrightarrow D \otimes H \otimes C : h \otimes k \otimes c \mapsto \sum (h \rightharpoonup c_1) \otimes k \otimes c_2$$

and observe that the diagram

$$\begin{array}{ccc}
H\otimes H\otimes C & \xrightarrow{\delta'} & D\otimes H\otimes C \\
\beta\downarrow\downarrow\alpha & & I\otimes\psi_C\downarrow\downarrow\psi_D\otimes I_C \\
H\otimes C & \xrightarrow{\delta} & D\otimes C \\
\downarrow I_H\otimes\varepsilon_C & & \downarrow \\
H & & D\otimes_H C \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

commutes. Indeed,

$$\begin{aligned}((\psi_D\circ I_C)\circ\delta)(h\otimes k\otimes c) &= \sum S(k)h\rightharpoonup c_1\otimes c_2\\ &= (\delta\circ\beta)(h\otimes k\otimes c)\end{aligned}$$

and

$$\begin{aligned}(\delta\circ\alpha)(h\otimes k\otimes c) &= \sum((hS(k_1)k_2)\rightharpoonup c_1)\otimes(k_3\rightharpoonup c_2)\\ &= \sum(h\rightharpoonup c_1)\otimes(k\rightharpoonup c_2)\\ &= ((I\otimes\psi_C)\circ\delta')(h\otimes k\otimes c)\end{aligned}$$

Now δ and δ' are isomorphisms, and the two columns in the above diagram are exact. It therefore follows that δ descends to an isomorphism $H\cong D\otimes_H C$.

We still have to show that D is an H-Galois coobject. It is clear that D is faithfully flat as an R-module, since $D=C$ as an R-module, and it therefore suffices to show that

$$\delta_D:\ H\otimes D\longrightarrow D\otimes D:\ h\otimes d\mapsto\sum(S(h)\rightharpoonup d_2)\otimes d_1$$

is an isomorphism.

We first show that δ is surjective. Take $d\otimes e\in D\otimes D$, and let $\delta_C^{-1}(e\otimes d)=\sum_i h_i\otimes c_i\in H\otimes C$. Then

$$\begin{aligned}d\otimes e &= \sum_i c_{i_2}\otimes h_i\rightharpoonup c_{i_1}\\ &= \sum_i(S(h_{i_3})h_{i_2})\rightharpoonup c_{i_2}\otimes h_{i_1}\rightharpoonup c_{i_1}\\ &= \delta_D\Big(\sum_i h_{i_2}\otimes h_{i_1}\rightharpoonup c_i\Big)\end{aligned}$$

Let us finally show that δ_D is injective. Suppose that

$$\delta_D(\sum_i \ell_i \otimes d_i) = \sum_i S(\ell_i) \rightharpoonup d_{i_2} \otimes d_{i_1} = 0$$

Then

$$\begin{aligned} 0 &= \sum_i d_{i_1} \otimes S(\ell_i) \rightharpoonup d_{i_2} \\ &= \sum_i (\ell_{i_3} S(\ell_{i_2})) \rightharpoonup d_{i_1} \otimes S(\ell_{i_1}) \rightharpoonup d_{i_2} \\ &= \delta_C\Big(\sum_i \ell_{i_2} \otimes S(\ell_{i_1}) \rightharpoonup d_i\Big) \end{aligned}$$

Now δ_C is injective, and therefore

$$\sum_i \ell_{i_2} \otimes S(\ell_{i_1}) \rightharpoonup d_i = 0$$

Applying $\tau \circ \Delta_H$ to the first factor, we obtain

$$\sum_i \ell_{i_3} \otimes \ell_{i_2} \otimes S(\ell_{i_1}) \rightharpoonup d_i = 0$$

Now let the second factor act on the third one. This yields that

$$\sum_i \ell_i \otimes d_i = 0$$

and it follows that δ_D is injective. □

From Corollary 1.4, Proposition 2.1 and Proposition 2.2, we may conclude the following result.

Theorem 2.3 — *Suppose that H is a commutative, faithfully flat Hopf algebra. Then* $\mathrm{Gal}^{\mathrm{co}}(R,H)$*, the set of isomorphism classes of H-Galois coobjects, forms a group under the operation induced by the tensor product over H. We call this group the group of H-Galois coobjects.*

We will now show that $\mathrm{Gal}^{\mathrm{co}}(R,H)$ fits into an exact sequence.

Theorem 2.4 — *Let H be a commutative faithfully flat Hopf algebra. Then we have an exact sequence*

$$1 \longrightarrow H^2_{\mathrm{Harr}}(H,R,\mathbb{G}_m) \xrightarrow{\alpha} \mathrm{Gal}^{\mathrm{co}}(R,H) \xrightarrow{\beta} H^1_{\mathrm{Harr}}(H,R,\mathrm{Pic}) \tag{5}$$

Proof. Definition of the map α.
Take a Harrison cocycle $u = \sum u_i \otimes v_i \in \mathbb{G}_m(H^{\otimes 2})$. Let C be equal to H as an H-module, and define a comultiplication Δ_C on C by the rule

$$\Delta_C(c) = u\Delta(c)$$

for all $c \in C$. From the cocycle relation, it follows easily that Δ_C is coassociative. Let us show that C has a counit. Since $u = \sum_i u_i \otimes v_i$ is a cocycle, we have that

$$\sum_{i,j} u_j u_{i_1} \otimes v_j u_{i_2} \otimes v_i = \sum_{i,j} u_i \otimes u_j v_{i_1} \otimes v_j v_{i_2}$$

Apply $I_H \otimes \varepsilon_H \otimes I_H$ to both sides to obtain

$$\sum_{i,j} u_j u_i \otimes \varepsilon(v_j) v_i = \sum_{i,j} u_i \otimes \varepsilon(u_j) v_j v_i$$

or

$$(1 \otimes \sum_j \varepsilon(v_j) u_j) u = (1 \otimes \sum_j \varepsilon(u_j) v_j) u$$

or, since u is invertible

$$\sum_j \varepsilon(v_j) u_j = \sum_j \varepsilon(u_j) v_j$$

Observe that $\sum_j \varepsilon(v_j) u_j$ is invertible in H, since $\varepsilon_H \otimes I_H$ is a multiplicative map. We now define

$$\varepsilon_C(c) = (\sum_j \varepsilon(v_j) u_j)^{-1} \varepsilon_H(c) = (\sum_j \varepsilon(u_j) v_j)^{-1} \varepsilon_H(c)$$

It is straightforward to show that ε_C is a counit, and it follows that C is a coalgebra. Left multiplication by elements of H makes C into an H-module coalgebra. To prove that C is an H-Galois coobject, it suffices to show that

$$\delta_C : H \otimes C \longrightarrow C \otimes C : h \otimes c \mapsto \sum h u_i c_1 \otimes v_i c_2 = u \delta_H(h \otimes c)$$

is an isomorphism. This is obvious, since u is invertible.
We define $\alpha([u]) = [C]$. It is straightforward to show that α is a well-defined monomorphism.
Definition of the map β.
Let C be an H-Galois coobject. We claim that C considered as an invertible H-module (forget the comultiplication) is a Harrison cocycle with values in Pic. Observe that

$$\mathrm{Pic}(\varepsilon_0)(C) = (H \otimes H) \otimes_{\varepsilon_0} C = H \otimes C$$
$$\mathrm{Pic}(\varepsilon_2)(C) = (H \otimes H) \otimes_{\varepsilon_2} C = C \otimes H$$

and therefore

$$\mathrm{Pic}(\varepsilon_0)(C) \otimes_{H \otimes H} \mathrm{Pic}(\varepsilon_2)(C) \cong C \otimes C$$

On the other hand

$$\mathrm{Pic}(\varepsilon_1)(C) = (H \otimes H) \otimes_{\varepsilon_1} C$$

is generated by monomials of the form

$$(h \otimes k) \otimes_{\varepsilon_1} c$$

subject to the relation

$$(h \otimes k) \otimes_{\varepsilon_1} l \cdot c = \sum (hl_1 \otimes kl_2) \otimes_{\varepsilon_1} c$$

for all $h, k, l \in H$ and $c \in C$. Consider the map

$$\theta : \ (H \otimes H) \otimes_{\varepsilon_1} C \longrightarrow H \otimes C$$

given by

$$\theta((h \otimes k) \otimes_{\varepsilon_1} c) = \sum hS(k_1) \otimes k_2 c$$

θ is well-defined, since

$$\begin{aligned} \theta\left(\sum (hl_1 \otimes kl_2) \otimes_{\varepsilon_1} c\right) &= \sum hl_1 S(l_2) S(k_1) \otimes k_2 l_3 c \\ &= \sum hS(k_1) \otimes k_2 l_3 c \\ &= \theta((h \otimes k) \otimes_{\varepsilon} lc) \end{aligned}$$

θ is an isomorphism of R-modules. Its inverse is given by

$$\theta^{-1}(h \otimes c) = (h \otimes 1) \otimes_{\varepsilon_1} c$$

Indeed,

$$\begin{aligned} \theta^{-1}(\theta((h \otimes k) \otimes_{\varepsilon} c)) &= \sum (hS(k_1) \otimes 1) \otimes_{\varepsilon_1} k_2 c \\ &= \sum (hS(k_1) k_2 \otimes k_3) \otimes_{\varepsilon_1} c \\ &= (h \otimes k) \otimes_{\varepsilon} c \end{aligned}$$

and

$$\theta(\theta^{-1}(h \otimes c)) = \theta((h \otimes 1) \otimes_{\varepsilon_1} c) = h \otimes c$$

It follows that

$$\delta_C \circ \theta : \ (H \otimes H) \otimes_{\varepsilon_1} C \longrightarrow H \otimes C$$

is an isomorphism of R-modules. $\delta_C \circ \theta$ can be described explicitly as follows:

$$\begin{aligned} (\delta_C \circ \theta)((h \otimes k) \otimes_{\varepsilon_1} c) &= \delta_C(\sum hS(k_1) \otimes k_2 c) \\ &= \sum hS(k_1) k_2 c_1 \otimes k_3 c_2 \\ &= (h \otimes k) \Delta_C(c) \end{aligned} \tag{6}$$

From (6), it follows easily that $\delta_C \circ \theta$ is $H \otimes H$-linear.
We now define $\beta([C]) = [C]$.
Exactness at $\mathrm{Gal}^{co}(R, H)$.
Suppose that $C = H$ as an H-module. From the coassociativity of the comultiplication on C, it follows that $u = \Delta_C(1)$ is a Harrison 2-cocycle. It is straightforward to show that $[C] = \alpha([u])$. □

$\mathrm{Gal}^{co}(R, H)$ is now described completely if we can add one more term to the long exact sequence (5). The obvious candidate for this next term is the third Harrison cohomology group $H^3_{\mathrm{Harr}}(H, R, \mathbb{G}_m)$. If H is finite, then this works: the exact sequence (5) (or at least a dual version of it), with the H^3-term added to it is then exact. This was shown independently by Early and Kreimer [5] and Yokogawa [13]. In the general case, we are only able to describe a subgroup of $\mathrm{Gal}^{co}(R, H)$, that coincides with the full $\mathrm{Gal}^{co}(R, H)$ if H is finite. This is what we will be doing in the sequel.

3 Galois coobjects with geometric normal basis

A Galois coobject C has normal basis if $C \cong H$ as an H-module, or, equivalently, if $[C] \in \mathrm{Ker}(\gamma)$ in Theorem 2.4. It follows from Theorem 2.4 that $\mathrm{Gal}^{co}_{\mathrm{nb}}(R, H)$, the subgroup of $\mathrm{Gal}^{co}(R, H)$ consisting of Galois objects with a normal basis, is isomorphic to the second Harrison cohomology group $H^2_{\mathrm{Harr}}(H, R, \mathbb{G}_m)$. This statement, which is much older then the exact sequence (5) is known as the *normal basis Theorem* , and goes back to several authors, cf. for example [6] and [8]. We now introduce the following geometric version of Galois object with a normal basis.

Definition 3.1 — *Let A be a commutative faithfully flat R-algebra, and H a commutative faithfully flat Hopf algebra. An invertible A-module I has a geometric normal basis if there exists a faithfully flat commutative R-algebra S such that $I \otimes S \cong A \otimes S$ as S-modules. An H-Galois coobject C has a geometric normal basis if it has a geometric normal basis as an invertible H-module.*

Obviously, the subsets of $\mathrm{Gal}^{co}(R, H)$ and $\mathrm{Pic}(A)$ consisting of isomorphism classes of objects with geometric normal basis are subgroups. These subgroups will be denoted by $\mathrm{Gal}^{co}_{\mathrm{gnb}}(R, H)$ and $\mathrm{Pic}_{\mathrm{gnb}}(R, A)$. We have the following inclusions:

$$\mathrm{Gal}^{co}_{\mathrm{nb}}(R, H) \subset \mathrm{Gal}^{co}_{\mathrm{gnb}}(R, H) \subset \mathrm{Gal}^{co}(R, H)$$

$$\mathrm{Pic}_{\mathrm{gnb}}(R, A) \subset \mathrm{Pic}(A)$$

Lemma 3.2 — *If A (resp. H) is faithfully projective as an R-module, then*

$$\mathrm{Gal}^{co}_{\mathrm{gnb}}(R, H) = \mathrm{Gal}^{co}(R, H)$$

and

$$\mathrm{Pic}_{\mathrm{gnb}}(R,A)=\mathrm{Pic}(A)$$

Proof. Let I be an invertible H-module, and take $p\in\mathrm{Spec}(R)$. Then $H_p=H\otimes R_p$ is a finitely generated projective R_p-algebra and is therefore semilocal. Thus $I\otimes R_p$ is free of rank one as an H_p-module. A standard argument now shows that there is a Zariski covering $S=R_{f_1}\times\cdots\times R_{f_n}$ of R such that $I\otimes S$ is free of rank one as an $H\otimes S$-module. □

With notations as in Theorem 2.4, we have that

$$\mathrm{Gal}^{\mathrm{co}}_{\mathrm{gnb}}(R,H)=\gamma^{-1}(\mathrm{Pic}^{\mathrm{co}}_{\mathrm{gnb}}(R,H))$$

and

$$\mathrm{Im}(\beta)=\mathrm{Gal}^{\mathrm{co}}_{\mathrm{nb}}(R,H)\subset\mathrm{Gal}^{\mathrm{co}}_{\mathrm{gnb}}(R,H)$$

The exact sequence (5) therefore restricts to an exact sequence

$$1\longrightarrow H^2_{\mathrm{Harr}}(H,R,\mathbb{G}_m)\xrightarrow{\beta}\mathrm{Gal}^{\mathrm{co}}_{\mathrm{gnb}}(R,H)\longrightarrow H^1_{\mathrm{Harr}}(H,R,\mathrm{Pic}_{\mathrm{gnb}}(R,\bullet))$$

Before extending this sequence, let us state the following technical Lemma.

Lemma 3.3 — *Suppose that S is a faithfully flat R-algebra, and that C is an H-module coalgebra. If $S\otimes C$ is an $S\otimes H$-Galois coobject, then A is an H-Galois coobject.*

Proof. C is a faithfully flat R-module, because $S\otimes C$ is a faithfully flat S-module, and S is a faithfully flat R-algebra. Furthermore

$$\delta_S:\ (S\otimes H)\otimes_S(S\otimes C)=S\otimes(H\otimes C)\longrightarrow(S\otimes C)\otimes_S(S\otimes C)=S\otimes(C\otimes C)$$

defined by

$$\delta_S(s\otimes(h\otimes c))=\sum s\otimes(h\rightharpoonup c_1)\otimes c_2$$

is an isomorphism of S-modules. The fact that S is a faithfully flat R-algebra implies that

$$\delta:\ H\otimes C\longrightarrow C\otimes S$$

is an isomorphism of R-modules. □

Theorem 3.4 — *Let H be a commutative faithfully flat Hopf algebra. Then we have the following exact sequence*

(7)

$$1\longrightarrow H^2_{\mathrm{Harr}}(H,R,\mathbb{G}_m)\xrightarrow{\alpha}\mathrm{Gal}^{\mathrm{co}}_{\mathrm{gnb}}(R,H)\xrightarrow{\beta}H^1_{\mathrm{Harr}}(H,R,\mathrm{Pic}_{\mathrm{gnb}}(R,\bullet))\xrightarrow{\gamma}H^3_{\mathrm{Harr}}(H,R,\mathbb{G}_m)$$

Proof. Definition of the map γ.
Take a cocycle $C \in Z^1_{\mathrm{Harr}}(H, R, \mathrm{Pic}_{\mathrm{gnb}}(R, \bullet))$. We have an isomorphism

$$f: \ H^{\otimes 2} \otimes_{\varepsilon_1} C \longrightarrow C \otimes C$$

of $H^{\otimes 2}$-modules. Consider the following maps.

$$\widehat{u}: \ C \longrightarrow C \otimes C$$

given by

$$\widehat{u}(c) = f((1 \otimes 1) \otimes_{\varepsilon_2} c)$$

and

$$\zeta_1, \ \zeta_2: \ H^{\otimes 2} \otimes C \longrightarrow C^{\otimes 3}$$

given by

$$\zeta_1(h \otimes k \otimes c) = (h \otimes k \otimes 1) \rightharpoonup \big(((\widehat{u} \otimes I_C) \circ \widehat{u})(c)\big)$$
$$\zeta_2(h \otimes k \otimes c) = (h \otimes k \otimes 1) \rightharpoonup \big(((I_C \otimes \widehat{u}) \circ \widehat{u})(c)\big)$$

It is clear that $\widehat{u}$ makes C into a coassociative coalgebra if and only if $\zeta_1 = \zeta_2$. Suppose for a moment that $C \cong H$ as an H-module. Then for all $h \in H$, we have

$$\begin{aligned}\widehat{u}(h) &= f((1 \otimes 1) \otimes_{\varepsilon_2} h)\\ &= f((\sum h_1 \otimes h_2) \otimes_{\varepsilon_2} 1)\\ &= (\sum h_1 \otimes h_2) f((1 \otimes 1) \otimes_{\varepsilon_2} 1)\\ &= \widehat{u}(1)\Delta_H(h)\end{aligned}$$

Now write

$$u = \widehat{u}(1) = \sum u^1 \otimes u^2 = \sum U^1 \otimes U^2$$

and let

$$f^{-1}(1 \otimes 1) = v \otimes_{\varepsilon_2} 1$$

with

$$v = \sum v^1 \otimes v^2 = \sum V^1 \otimes V^2 \in H \otimes H$$

Then $1 \otimes 1 = f(f^{-1}(1 \otimes 1)) = uv$, and $v = u^{-1}$. Observe next that the map

$$\alpha: \ H^{\otimes 3} \longrightarrow H^{\otimes 3}: \ h \otimes k \otimes l \mapsto \sum hl_1 \otimes kl_2 \otimes l_3$$

is bijective; the inverse of α is given by the formula

$$\alpha^{-1}(h \otimes k \otimes l) = \sum hS(l_2) \otimes kS(l_1) \otimes l_3$$

We now have that

$$\begin{aligned}\zeta_1(h\otimes k\otimes l) &= \sum(h\otimes k\otimes 1)(\widehat{u}\otimes I_H)(h^1l_1\otimes u^2l_2)\\ &= \sum(h\otimes k\otimes 1)(U^1u_1^1l_1\otimes U^2u_2^1l_2\otimes u^2l_3)\\ &= \sum(U^1\otimes U^2\otimes 1)(u_1^1\otimes u_2^1\otimes u_2)(hl_1\otimes kl_2\otimes l_3)\\ &= \varepsilon_3(u)\varepsilon_1(u)\alpha(h\otimes k\otimes l)\end{aligned}$$

and, in a similar way,

$$\zeta_2(h\otimes k\otimes l) = \varepsilon_0(u)\varepsilon_2(u)\alpha(h\otimes k\otimes l)$$

Write $m(\varepsilon_i(u))$ for the map given by multiplication by $\varepsilon_i(u)$. Then

$$\begin{aligned}\zeta_2 &= m(\varepsilon_0(u))\circ m(\varepsilon_2(u))\circ\alpha\\ \zeta_1^{-1} &= \alpha^{-1}\circ m(\varepsilon_1(v))\circ m(\varepsilon_3(v))\end{aligned}$$

and therefore

$$\zeta_2\circ\zeta_1^{-1} = m(\varepsilon_0(u))\circ m(\varepsilon_2(u))\circ m(\varepsilon_1(v))\circ m(\varepsilon_3(v)) = m(\Delta_2(u))$$

is given by multiplication by the coboundary $\Delta_2(u)$.
We now return to the general case. Let S be a faithfully flat extension of R such that $S\otimes C\cong S\otimes H$ as $S\otimes H$-modules. The map

$$\zeta_1\otimes I_S:\ C^{\otimes 3}\otimes S\longrightarrow H^{\otimes 2}\otimes C\otimes S$$

is bijective (see above), and this implies that ζ_1 is also bijective (S is faithfully flat). Consider the map

$$\zeta_2\circ\zeta_1^{-1}:\ C^{\otimes 3}\xrightarrow{\zeta_1^{-1}}H^{\otimes 2}\otimes C\xrightarrow{\zeta_2}C^{\otimes 3}$$

Then the map $I_S\otimes(\zeta_2\circ\zeta_1^{-1})$ is given by multiplication by a coboundary in $B^3_{\mathrm{Harr}}(S\otimes H,S,\mathbb{G}_m)$, and is an isomorphism of $S\otimes H^{\otimes 3}$-modules. $\zeta_2\circ\zeta_1^{-1}$ is therefore an isomorphism of (rank one) $H^{\otimes 3}$-modules, and is given by multiplication by a unit $x\in\mathbb{G}_m(H^{\otimes 3})$. Moreover $1_S\otimes x\in B^3_{\mathrm{Harr}}(S\otimes H,S,\mathbb{G}_m)\subset Z^3_{\mathrm{Harr}}(S\otimes H,S,\mathbb{G}_m)$ is a cocycle, and thus x is a cocycle in $Z^3_{\mathrm{Harr}}(H,R,\mathbb{G}_m)$. We define $\delta([C])=[x]$. We leave it to the reader to show that δ is well-defined: if we repeat the above arguments with a different isomorphism $f':\ H^{\otimes 2}\otimes_{\varepsilon_2}C\longrightarrow C^{\otimes 2}$ then we obtain a cocycle x' that is cohomologous to x.
Exactness at $H^1_{\mathrm{Harr}}(H,R,\mathrm{Pic}_{\mathrm{gnb}}(R,\bullet))$.
It is clear that $\delta\circ\beta=1$. If C is an H-Galois coobject, then we can choose the isomorphism

$$f:\ H^{\otimes 2}\otimes_{\varepsilon_2}C\longrightarrow C^{\otimes 2}$$

as follows:

$$f((h \otimes k) \otimes_{\varepsilon_2} c) = \sum h_1 c_1 \otimes k_2 c_2$$

(see the end of the proof of Theorem 2.4). Now the map $\widehat{u}$ defined above is nothing else then the comultiplication on Δ_C, and therefore $\zeta_1 = \zeta_2$ (C is coassociative), and $x = 1$.

Conversely, if $\delta([C]) = [x]$, with $x = \Delta_2(y^{-1})$ a coboundary, then we replace the isomorphism f by f' given by

$$f'((h \otimes k) \otimes_{\varepsilon_2} c) = yf((h \otimes k) \otimes_{\varepsilon_2} c)$$

Then it follows immediately that

$$\begin{aligned} \widehat{u}'(c) &= y\widehat{u}(c) \\ \zeta_1' &= m(\varepsilon_3(y)) \circ m(\varepsilon_1(y)) \circ \zeta_1 \\ \zeta_2' &= m(\varepsilon_0(y)) \circ m(\varepsilon_2(y)) \circ \zeta_1 \end{aligned}$$

and consequently

$$\zeta_2' \circ \zeta_1' = m(\Delta_2(y)) \circ \zeta_2 \circ \zeta_1 = 1$$

such that $\widehat{u}'$ makes C into a coassociative coalgebra. Finally, observe that $S \otimes C$ is nothing else then $S \otimes H$ with comultiplication twisted by the Harrison cocycle $1_S \otimes \widehat{u}'$. Therefore $S \otimes C$ is an $S \otimes H$-Galois coobject, and, by the previous proposition, C is an H-Galois coobject. □

We have allready seen that an H-Galois coobject C is invertible as an H-module coalgebra, that is, there exists an H-module coalgebra D such that $C \otimes_H D \cong H$ as H-module coalgebras. For an H-module coalgebra with geometric normal basis, the converse also holds.

Corollary 3.5 — *Let C and D be H-module coalgebras such that $C \otimes_H D \cong H$ as H-module coalgebras, and suppose that C (and therefore D) has a geometric normal basis. Then C and D are H-Galois coobjects.*

Proof. From Proposition 3.3, it follows that we can assume that C and D have normal basis, that is, $C \cong D \cong H$ as H-modules. Write $\Delta_C(1) = u$ and $\Delta_D(1) = v$, and consider the canonical isomorphism $f : C \otimes_H D \to H : c \otimes d \mapsto cd$. The H-module coalgebra structure on $C \otimes_H D$ induces an H-module coalgebra structure on H. The new comultiplication is given by

$$\widetilde{\Delta}(1) = uv$$

Let $\widetilde{H}$ be equal to H as an H-module with the new comultiplication $\widetilde{\Delta}$. By assumption, $\widetilde{H}$ is isomorphic to H as an H-module coalgebra. From the exactness of the sequence (5), it follows that $uv \in B^2(H, R, \mathbb{G}_m)$ is a coboundary. It follows in particular that u and v are invertible. From the fact that C and D are coassociative, it follows that u and v are Harrison cocycles, and we allready know that in this case C and D are H-Galois coobjects. □

Remark 3.6. Yokogawa [13] has shown that the exact sequence (1) can be extended to an infinite sequence of infinite length. To this end, he introduces Sweedler cohomology with values in the category of invertible modules, in the spirit of the cohomology introduced by Villamayor and Zelinsky in [12]. Yokogawa's observation can be generalized to our situation. This leads to some new cohomology groups $H^n_{\text{Harr}}(H, R, \underline{\underline{\text{Pic}}}_{\text{gnb}}(R, \bullet))$. It can be shown that

$$H^1_{\text{Harr}}(H, R, \underline{\underline{\text{Pic}}}_{\text{gnb}}(R, \bullet)) \cong \text{Gal}^{\text{co}}_{\text{gnb}}(R, H)$$

Moreover, we have a long exact sequence

(8)

$$\begin{aligned} 1 \to H^2_{\text{Harr}}(H, R, \mathbb{G}_m) &\xrightarrow{\alpha_1} H^1_{\text{Harr}}(H, R, \underline{\underline{\text{Pic}}}_{\text{gnb}}(R, \bullet)) \xrightarrow{\beta_1} H^1_{\text{Harr}}(H, R, \text{Pic}_{\text{gnb}}(R, \bullet)) \\ \xrightarrow{\gamma_1} H^3_{\text{Harr}}(H, R, \mathbb{G}_m) &\xrightarrow{\alpha_2} H^2_{\text{Harr}}(H, R, \underline{\underline{\text{Pic}}}_{\text{gnb}}(R, \bullet)) \xrightarrow{\beta_2} H^2_{\text{Harr}}(H, R, \text{Pic}_{\text{gnb}}(R, \bullet)) \\ \xrightarrow{\gamma_2} & \quad \cdots \end{aligned}$$

We omit the details, since they are not that much different from the ones in Yokogawa's paper.

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