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OPTIMAL RESULTS FOR THE TWO DIMENSIONAL NAVIER-STOKES EQUATIONS WITH LOWER REGULARITY ON THE DATA

by

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Abstract. — We establish existence and uniqueness of solutions in the anisotropic Sobolev space $H^{1,1/2}$ to the two dimensional Navier-Stokes equations with data in $H^{-1,-1/2}$. Our results give a new elementary proof for and extend some of recent results of G. Grubb.

***Résumé* (Résultats optimaux pour les équations de Navier-Stokes en dimension 2 avec des données initiales peu régulières)**

On établit l'existence et l'unicité des solutions dans l'espace de Sobolev anisotrope $H^{1,1/2}$ pour les équations de Navier-Stokes en dimension 2 avec des données dans $H^{-1,-1/2}$. Nos résultats donnent une preuve élémentaire nouvelle de résultats récents de G. Grubb, tout en les complétant.

1. Introduction

Working with divergence free vectorfields, the Navier-Stokes equations take the form

$$u_t - \Delta_x u + (u \cdot \nabla)u = f. \tag{1}$$

In two space dimensions it is known, since the pioneering works by J. Leray [**LE**], O.A. Ladyzhenskaya [**L1**], [**L2**],[**L3**] and J.L. Lions and G. Prodi [**LP**], that under suitable boundary conditions, (1) has a unique solution $u \in L^2(\mathbb{R}, H^1)$ for $f \in L^2(\mathbb{R}, H^{-1})$.

Later on these results have been complemented in various ways. In a recent interesting paper [**G**] G. Grubb gives general existence and uniqueness theorems for the Navier-Stokes equations in scales of L^p-Sobolev, Bessel potential and Besov spaces,

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using fairly complicated pseudo differential operator techniques. In two space dimensions with zero initial data her results include that under suitable conditions, given a source term $f \in H^{-1,-1/2}$ there exists a unique solution $u \in H^{1,1/2}$. However, in order to obtain existence it is assumed in [**G**] that the data is small enough in norm.

In the present note we give an elementary proof of the existence and uniqueness in the case of $f \in H^{-1,-1/2}$ and $u \in H^{1,1/2}$, without assuming smallness on the data. Our approach is in the spirit of the seminal work by J. Leray which appeared in *Acta Mathematica* in 1934, and it is based on the method of [**F1**], which turns out to be adaptable also to this situation. We refer to Theorem 1 below for the precise statement of our result. The main improvement in our result, compared to previously known results, is the regularity gain of the extra half derivative in time for the solution, and at the same time the corresponding wider range of possible irregularities for the source term.

Additional motivation for reconsidering the case $f \in H^{-1,-1/2}$ is provided by the fact that the corresponding result is optimal in a certain sense. Namely, the solution and the source spaces are in complete duality and, moreover, the difference in the smoothness corresponds exactly to the order of the non-linear operator in the respective variables.

An advantage of our approach is that it is completely elementary and self contained. Moreover, it appears to be possible to generalize it to certain situations, where other methods probably fail. For example our argument goes through unchanged if we replace the Laplacian in (1) with a uniformly elliptic linear operator having measurable coefficients (see the remark at the end of the paper).

We briefly mention some recent related results on the two-dimensional case. The papers of H. Amann [**A**] and [**A1**] apply interpolation arguments and semigroup methods to consider data with strong irregularity in space, but nonnegative smoothness in time. The paper [**MS1**] of J. Mattingly and Y. Sinai uses direct estimates on Fourier series to reprove and extend previous results in the case of very high regularity in space. As we are concerned with low regularity for the data in time, it is of interest here to observe that Brickmont, Kupiainen and Lefevre [**BKL1**] use the method of [**MS1**] to treat a very specific situation, where the smoothness of the source term corresponds to that of the white noise process, which barely fails to be locally $H^{-1/2}$ in time (see also [**BKL2**], [**MS2**] and [**KS**] in this connection). We refer to [**G**], [**MS1**], [**FT**], [**A**], [**A1**], [**L3**] and [**T**] and their references for further results.

The structure of the proof (and the note) is as follows: In the first section we consider the linearized operator and prove that it yields an isomorphism between the right spaces. To this end we apply simple Fourier analysis in connection with the Hilbert transform and the half-derivative operator; the conclusion is obtained by an application of the Lax-Milgram lemma. In the second section we first verify a suitable a priori estimate for the solution, which is based on a simple non-homogeneous Sobolev imbedding theorem (see Lemma 3). The existence of a solution in the non-linear case is now deduced from a simple finite dimensional approximation combined with an application of the theory of the Brouwer mapping degree. In turn, the proof of the uniqueness follows the classical lines.

2. The linear case

In the linear case there is no restriction on the space dimension. Let Ω be an open, bounded and connected set in $\mathbb{R}^n$, let $Q = \Omega \times \mathbb{R}$ and let

$$H^{1,1/2}(Q,\mathbb{R}^n) := \Big\{u \in L^2(Q,\mathbb{R}^n);\, \frac{\partial_+^{1/2} u}{\partial t^{1/2}}, \frac{\partial u}{\partial x_i} \in L^2(Q,\mathbb{R}^n) \text{ for } 1 \leqslant i \leqslant n\Big\}. \tag{2}$$

Here the half-derivative $\partial_+^{1/2} u/\partial t^{1/2}$ corresponds to the Fourier-multiplier $(i\tau)^{1/2}$, where τ is the Fourier frequency of t and we use the principal branch of the square root. In a similar manner, the half-derivative $\partial_-^{1/2} u/\partial t^{1/2}$ corresponds to the multiplier $(-i\tau)^{1/2}$. We obtain a Hilbert space with the norm

$$\Big(\iint_Q |u|^2 + \Big|\frac{\partial_+^{1/2} u}{\partial t^{1/2}}\Big|^2 + \sum_{i=1}^n \Big|\frac{\partial u}{\partial x_i}\Big|^2 dxdt\Big)^{1/2}.$$

By the Poincaré inequality, for elements in the closure of compactly supported functions, this is equivalent to the norm

$$\|u\|_{H^{1,1/2}} = \Big(\iint_Q \Big|\frac{\partial_+^{1/2} u}{\partial t^{1/2}}\Big|^2 + \sum_{i=1}^n \Big|\frac{\partial u}{\partial x_i}\Big|^2 dxdt\Big)^{1/2}, \tag{3}$$

which we will use henceforth. Let $\mathcal{V}(Q,\mathbb{R}^n)$ denote the space of divergence free (in the space variables) $\mathcal{D}(Q,\mathbb{R}^n)$-testfunctions. Here $\mathcal{D}$ stands for infinitely differentiable and compactly supported test functions.

We denote the closure in the $H^{1,1/2}(Q,\mathbb{R}^n)$ norm of $\mathcal{V}(Q,\mathbb{R}^n)$ by

$$V_0^{1,1/2}(Q,\mathbb{R}^n) := \overline{\mathcal{V}(Q,\mathbb{R}^n)}. \tag{4}$$

The restriction of an element ξ in the dual space $V_0^{1,1/2}(Q,\mathbb{R}^n)^*$ to the space of divergence free testfunctions $\mathcal{V}(Q,\mathbb{R}^n)$ can be extended to a (non-unique) distribution in $H_0^{1,1/2}(Q,\mathbb{R}^n)^*$.

Lemma 1. — *Given $\xi \in V_0^{1,1/2}(Q,\mathbb{R}^n)^*$, there exist functions $f_0, f_1, \ldots, f_n \in L^2(Q,\mathbb{R}^n)$ such that*

$$\langle \xi, \Phi\rangle = \Big\langle \frac{\partial_+^{1/2} f_0}{\partial t^{1/2}} + \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}, \Phi\Big\rangle; \quad \Phi \in \mathcal{V}(Q,\mathbb{R}^n). \tag{5}$$

Furthermore, given $\varepsilon > 0$ we may always arrange so that $\|f_0\|_{L^2(Q,\mathbb{R}^n)} \leqslant \varepsilon$ (the extension might then of course have a bigger norm).

Proof. — The statement of the Lemma is a direct consequence of the Hahn-Banach theorem, which yields the stated expression apart from the control on the L^2-norm of f_0. For that end take a smooth test function g so that $\|f_0-g\|_{L^2(Q,\mathbb{R}^n)} \leqslant \varepsilon$. Replace f_0 by f_0-g and f_1 by $f_1+\displaystyle\int_{-\infty}^{x_1} \frac{\partial_+^{1/2} g(s,x_2,\ldots,x_n,t)}{\partial t^{1/2}} ds$ in the stated expression (where g is continued as zero outside of Q). This proves the Lemma since one easily verifies that the last term belongs to $L^2(Q,\mathbb{R}^n)$. □

Let $T_0 : V_0^{1,1/2}(Q,\mathbb{R}^n) \to V_0^{1,1/2}(Q,\mathbb{R}^n)^*$ be the operator

$$T_0(u) = \frac{\partial u}{\partial t} - \Delta u, \tag{6}$$

defined by

$$\langle T_0 u, \Phi\rangle = \iint_Q \Big[\Big(\frac{\partial_+^{1/2} u}{\partial t^{1/2}}, \frac{\partial_-^{1/2}\Phi}{\partial t^{1/2}}\Big) + \sum_{i=1}^n \Big(\frac{\partial u}{\partial x_i}, \frac{\partial \Phi}{\partial x_i}\Big)\Big] dxdt\,; \quad \Phi \in V_0^{1,1/2}(Q,\mathbb{R}^n). \tag{7}$$

One should here observe that $(\partial_-^{1/2})^* = \partial_+^{1/2}$ and $(\partial_+^{1/2})^2 = \partial$, and thus (7) is obtained from (6) by a formal integration by parts. The operator T_0 is a well-defined continuous linear operator since the above expression defines a continuous bilinear form on $V_0^{1,1/2}(Q,\mathbb{R}^n)$, as is seen by using the observation that $\partial_-^{1/2} = h\partial_+^{1/2}$, where h is the Hilbert transform. Recall that the Hilbert transform corresponds to the unimodular Fourier multiplier $-i\,\mathrm{sgn}(\tau)$, and hence h is an isometry on L^2.

Definition 1. — We say that a subspace of $L^2(Q,\mathbb{R}^n)$ is *invariant* if it is invariant under the Hilbert transform h in the time direction.

An invariant subspace will then be invariant also under the action of the operator H, defined by $H(u) = 1/\sqrt{2}(u - h(u))$. Observe that the paper [**F1**] introduced the operator H^α, which for the choice $\alpha = 1/4$ corresponds to our H.

The following simple result in the linear case forms the cornerstone of our later treatment of the fully nonlinear equation. It corresponds to the simplest linear case considered in [**F1**, Section 4.1], whence we leave for the reader some easy computational elements of the argument. Recall that an operator $T : V \to V^*$ is coercive if there exists a constant $C > 0$ such that $\langle Tu, u\rangle \geqslant C\|u\|_V^2$ for all $u \in V$.

Proposition 1. — *Let V be a closed invariant subspace of $V_0^{1,1/2}(Q,\mathbb{R}^n)$ and let $f \in V_0^{1,1/2}(Q,\mathbb{R}^n)^*$. Then there exists a unique $u_V(f) \in V$ such that*

$$\langle T_0(u_V), \Phi\rangle = \langle f, \Phi\rangle \quad \textit{for all } \Phi \in V.$$

Proof. — The operator $H : V \to V$ is obviously an isometry as it corresponds to a unimodular Fourier multiplier in the time direction. In particular, it maps divergence free distributions to divergence free distributions.

Recall next certain additional basic properties of the Hilbert transform h and the half derivatives. First of all, h is an isometry on L^2 with the property $h \circ h = -\mathrm{Id}$.

Moreover, one has that $\int_{-\infty}^{\infty}(u,h(u))dt = 0$ assuming that $u \in L^2$. One also has that $\int_{-\infty}^{\infty}(\partial_+^{1/2}u, \partial_-^{1/2}u)dt = 0$ and $\int_{-\infty}^{\infty}(\partial_+^{1/2}u, \partial_-^{1/2}h(u))dt = -\int_{-\infty}^{\infty}|\partial_+^{1/2}u|^2dt$ assuming that the integrals are well-defined. The latter equality is a consequence of the identity $\partial_-^{1/2}h = -\partial_+^{1/2}$. Using the above facts, a straightforward computation shows that the operator $H^* \circ T_0$ (defined by the natural bilinear form on $V \times V$) is a coercive linear operator from V to V^*. By the Lax-Milgram lemma it is an isomorphism, whence the same is true for T_0.

Finally, for the readers comfort we clarify the role of the restrictions of operators in the above argument. Let us denote by P_V the orthogonal projection on V in $V_0^{1,1/2}(Q,\mathbb{R}^n)$, so that $P_{V^*} = P_V^*$ is the orthogonal projection on V^* (which is thus identified with a closed subspace of $(V_0^{1,1/2}(Q,\mathbb{R}^n))^*$. In precise terms the above proof yields that the operator $(P_{V^*}H^*T_0)_{|V} : V \to V^*$ is an isomorphism. However, since H in an isometry on the whole space and $H : V \to V$ is bijective, the same is true for $H^* : V^* \to V^*$. As we have $P_VH = HP_V$, it also holds that $P_{V^*}H^* = H^*P_{V^*}$. We may now deduce that $(P_{V^*}T_0)_{|V} : V \to V^*$ is an isomorphism, and this is equivalent to the statement of the Theorem. □

We explicitly state the special case

Corollary 1. — *The operator* $T_0 : V_0^{1,1/2}(Q,\mathbb{R}^n) \to V_0^{1,1/2}(Q,\mathbb{R}^n)^*$ *is an isomorphism.*

Concerning best approximations we have

Proposition 2. — *Let* $V \subset W$ *be two closed invariant subspaces in* $V_0^{1,1/2}(Q,\mathbb{R}^n)$, *let* $f \in V_0^{1,1/2}(Q,\mathbb{R}^n)^*$ *and let* $u_V(f) \in V$, $u_W(f) \in W$ *be the corresponding solutions from Proposition 1. Then*

$$\|u_V - u_W\|_{H^{1,1/2}} \leqslant 2\|\Phi - u_W\|_{H^{1,1/2}} \quad \textit{for all } \Phi \in V. \tag{8}$$

Proof. — From

$$\langle T_0(u_V - u_W), H(u_V - u_W + u_W - \Phi)\rangle = 0\,; \quad \Phi \in V, \tag{9}$$

one computes as in the proof of Proposition 1 to obtain the inequality

$$a^2 + b^2 \leqslant 2(ac + bd)$$

with

$$a^2 = \iint_Q \left|\frac{\partial_+^{1/2}(u_V - u_W)}{\partial t^{1/2}}\right|^2 dxdt, \qquad b^2 = \iint_Q \left|\frac{\partial(u_V - u_W)}{\partial x_i}\right|^2 dxdt,$$

$$c^2 = \iint_Q \left|\frac{\partial_+^{1/2}(u_W - \Phi)}{\partial t^{1/2}}\right|^2 dxdt, \quad \text{and} \quad d^2 = \iint_Q \left|\frac{\partial(u_W - \Phi)}{\partial x_i}\right|^2 dxdt,$$

The result is now a consequence of the Cauchy-Schwarz inequality. □

Lemma 2. — *There exists a sequence* $V_1 \subset V_2 \subset V_3 \subset \dots$ *of finite dimensional* (*closed*) *invariant subspaces of* $V_0^{1,1/2}(Q,\mathbb{R}^n)$ *such that* $\overline{(\cup_{i=1}^{\infty}V_i) \cap \mathcal{D}(Q,\mathbb{R}^n)} = V_0^{1,1/2}(Q,\mathbb{R}^n)$.

Proof. — The space $V_0^{1,1/2}(Q,\mathbb{R}^n)$ is separable. Choose an orthonormal basis $e_1, e_2, \ldots$, with the additional property that $e_j \in \mathcal{D}(Q,\mathbb{R}^n)$ for all j, and define V_k as the linear hull $V_k = lh(e_1, h(e_1), \ldots e_k, h(e_k))$, where h is the Hilbert transform in the time direction. □

3. The nonlinear case

From now on $n = 2$.

Given $\lambda \in \mathbb{R}$, let T_λ be the (formally defined) operator

$$T_\lambda(u) = \frac{\partial u}{\partial t} - \Delta u + \lambda(u \cdot \nabla)u. \tag{10}$$

Our object in this section is to prove the following theorem, yielding existence and uniqueness of generalized weak solutions to two dimensional Navier-Stokes, with data consisting of sums of half time-derivatives and first order space derivatives of L^2-functions.

Theorem 1. — *For any $\lambda \in \mathbb{R}$ the operator $T_\lambda : V_0^{1,1/2}(Q,\mathbb{R}^2) \to V_0^{1,1/2}(Q,\mathbb{R}^2)^*$ is well-defined by*

$$\langle T_\lambda(u), \Phi\rangle = \iint_Q \Big(\frac{\partial_+^{1/2} u}{\partial t^{1/2}}, \frac{\partial_-^{1/2}\Phi}{\partial t^{1/2}}\Big) + (\nabla u, \nabla\Phi) - \lambda(u \otimes u, \nabla\Phi) dxdt, \tag{11}$$

where $\Phi \in V_0^{1,1/2}(Q,\mathbb{R}^2)$ is arbitrary. Moreover, it is a demicontinuous bijection.

Recall here that demicontinuity means continuity from the norm topology to the weak topology.

In order to prove that T_λ is well defined on $V_0^{1,1/2}(Q,\mathbb{R}^2)$ and maps elements in this space into its dual space, we use the following Sobolev imbedding. This is a special case of results in [**L**], but we give below a simple argument for the readers convenience.

Lemma 3. — *The space $H_0^{1,1/2}(Q,\mathbb{R}^2)$ is continuously imbedded in $L^4(Q,\mathbb{R}^2)$, in fact there exists a constant C such that*

$$\iint_Q |u|^4 dxdt \leqslant C \iint_Q \left|\frac{\partial_+^{1/2} u}{\partial t^{1/2}}\right|^2 dxdt \iint_Q |\nabla u|^2 dxdt \qquad \text{for } u \in \mathcal{D}(Q,\mathbb{R}^2).$$

Furthermore, the imbedding $H_0^{1,1/2}(Q,\mathbb{R}^2)_{|Q'} \hookrightarrow L^2(Q',\mathbb{R}^2)$ is compact for any compact subset $Q' \subset Q$.

Proof. — We have the following chain of imbeddings

$$H_0^{1,1/2}(Q,\mathbb{R}^2) \hookrightarrow H^{1/4}(\mathbb{R}, H^{1/2}(\Omega,\mathbb{R}^2)) \hookrightarrow L^4(\mathbb{R}, H^{1/2}(\Omega,\mathbb{R}^2)) \hookrightarrow L^4(\mathbb{R}, L^4(\Omega,\mathbb{R}^2)).$$

The first imbedding above follows by computing on the Fourier side using the inequality $(1+|\xi|^2)^{1/4}(1+|\tau|^2)^{1/8} \leqslant (1+|\xi|^2)^{1/2} + (1+|\tau|^2)^{1/4}$. The second one is a consequence of the Sobolev imbedding for Hilbert space valued functions on the real line, whose proof runs exactly as in the scalar case. The last imbedding is a standard Sobolev imbedding. The inequality (12) is now obtained from a scaling argument. The statement on the compact imbedding is immediate from the imbeddings $H_0^{1,1/2}(Q,\mathbb{R}^2)_{|Q'} \hookrightarrow H^{1/2}(Q,\mathbb{R}^2)_{|Q'} \hookrightarrow L^2(Q',\mathbb{R}^2)$, where the last imbedding is compact. □

From this lemma we immediately see that if V is a closed invariant subspace of $V_0^{1,1/2}(Q,\mathbb{R}^2)$, then $T_\lambda : V \to V^*$, defined in a natural way by the pairing (11), is a demicontinuos operator, and thus in particular if V is finite dimensional then $T_\lambda : V \to V^*$ is continuous.

From Proposition 1, we have Theorem 1 in the case when $\lambda = 0$. We shall prove it in general by finite dimensional approximation (using invariant subspaces) and continuation in the parameter λ, using the theory of mapping degree. The basic a priori estimate needed for these arguments is provided by the following lemma.

Lemma 4. — *Let V be a closed invariant subspace of $V_0^{1,1/2}(Q,\mathbb{R}^2)$, let $\lambda_0 \in (0,\infty)$ and let $f \in H_0^{1,1/2}(Q,\mathbb{R}^2)^*$. If $|\lambda| \leqslant \lambda_0$ and if $u \in V$ satisfies*

$$\langle T_\lambda(u) - f, \Phi\rangle = 0\,; \quad \Phi \in V, \tag{12}$$

then there exists a constant $C(\lambda_0, f)$ such that

$$\|u\|_{H^{1,1/2}} \leqslant C. \tag{13}$$

Proof. — Given $\varepsilon > 0$, to be fixed later, there exist $f_0, f_1, f_2 \in L^2(Q,\mathbb{R}^2)$ such that

$$\langle f, \Phi\rangle = \Big\langle \frac{\partial_+^{1/2} f_0}{\partial t^{1/2}} + \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}, \Phi\Big\rangle \quad \text{for all } \Phi \in V, \tag{14}$$

with $\|f_0\|_{L^2} \leqslant \varepsilon$.

With $\Phi = u$ in (12) we get, using cancellation for the nonlinear term,

$$\|\nabla u\|_{L^2}^2 \leqslant (\|f_1\|_{L^2} + \|f_2\|_{L^2})\|\nabla u\|_{L^2} + \|f_0\|_{L^2}\left\|\frac{\partial_+^{1/2} u}{\partial t^{1/2}}\right\|_{L^2}. \tag{15}$$

Here one also applies the equality $\|\partial_-^{1/2} u/\partial t^{1/2}\|_{L^2} = \|\partial_+^{1/2} u/\partial t^{1/2}\|_{L^2}$.

With $\Phi = -h(u)$ in (12) we get

$$\left\|\frac{\partial_+^{1/2} u}{\partial t^{1/2}}\right\|_{L^2}^2 \leqslant |\lambda|\|u\|_{L^4}^2\|\nabla u\|_{L^2} + \|f_0\|_{L^2}\left\|\frac{\partial_+^{1/2} u}{\partial t^{1/2}}\right\|_{L^2} + (\|f_1\|_{L^2} + \|f_2\|_{L^2})\|\nabla u\|_{L^2}.$$

Using the Sobolev imbedding and inequality (15) we thus get

$$\begin{aligned}\Big\|\frac{\partial_+^{1/2}u}{\partial t^{1/2}}\Big\|_{L^2}^2 &\leqslant C|\lambda|\Big\|\frac{\partial_+^{1/2}u}{\partial t^{1/2}}\Big\|_{L^2}\|\nabla u\|_{L^2}^2+\|f_0\|_{L^2}\Big\|\frac{\partial_+^{1/2}u}{\partial t^{1/2}}\Big\|_{L^2}\\ &\qquad + (\|f_1\|_{L^2}+\|f_2\|_{L^2})\|\nabla u\|_{L^2}\\ &\leqslant C|\lambda_0|\Big((\|f_1\|_{L^2}+\|f_2\|_{L^2})\|\nabla u\|_{L^2}+\|f_0\|_{L^2}\Big\|\frac{\partial_+^{1/2}u}{\partial t^{1/2}}\Big\|_{L^2}\Big)\Big\|\frac{\partial_+^{1/2}u}{\partial t^{1/2}}\Big\|_{L^2}\\ &\qquad + \|f_0\|_{L^2}\Big\|\frac{\partial_+^{1/2}u}{\partial t^{1/2}}\Big\|_{L^2}+(\|f_1\|_{L^2}+\|f_2\|_{L^2})\|\nabla u\|_{L^2}.\end{aligned}$$

Let us denote $a^2=\|\nabla u\|_{L^2}^2$ and $b^2=\|\partial_+^{1/2}u/\partial t^{1/2}\|_{L^2}^2$. The above estimates take the form

$$\begin{aligned}&a^2\leqslant C_1a+C_2b\\ &b^2\leqslant C\lambda_0\|f_0\|_{L^2}b^2+C_3ab+C_4a+C_5b,\end{aligned}\tag{16}$$

where the constants $C_1,\ldots,C_5$ are finite polynomials in λ_0 and the L^2-norms of f_0,f_1,f_2. We now choose ε so small that $C\lambda_0\|f_0\|_{L^2}\leqslant 1/2$. At this stage the latter inequality yields that

$$b^2\leqslant 2C_3ab+2C_4a+2C_5b.\tag{17}$$

From inequality (16) we see that $a\leqslant C_6+C_7\sqrt{b}$. By substituting this into inequality (17) we infer that the quantity a^2+b^2 is bounded by a constant depending only on λ_0 and the L^2-norms of $f_0,\ldots,f_n$. □

Proof of Theorem 1

Existence. — We start by fixing $f\in V_0^{1,1/2}(Q,\mathbb{R}^2)^*$ and a sequence of finite dimensional subspaces $V_k\subset V_0^{1,1/2}(Q,\mathbb{R}^2)$ satisfying the properties of Lemma 2. We then fix $k\geqslant 1$ and consider the operator $T_\lambda:V_k\to V_k^*$ (defined by the natural bilinear form) as a (nonlinear) perturbation of the linear operator T_0. By Proposition 1 it follows that $T_0:V_k\to V_k^*$ is a (continuous) linear isomorphism. In particular, taking $R>0$ large enough we obtain that $f_{|V_k}\in T_0(B_R(V_k))$ and $f_{|V_k}\notin T_\mu(\partial B_R(V_k))$ for $|\mu|\leqslant|\lambda|$. That the last requirement is feasible follows from the a priori estimate of Lemma 4. Since V_k is finite dimensional the mapping $T_\mu:V_k\to V_k^*$ is jointly continuous with respect to the pair $(u,\mu)\in V_k\times\mathbb{R}$. Everything is now set for applying the theory of Brouwer mapping degree. The degree of T_μ, with respect to the ball $B_R(V_k)$ and the righthand side $f_{|V_k}$, is constant for $|\mu|\leqslant|\lambda|$. As the degree is ±1 for $\mu=0$ we conclude that $f_{|V_k}\in T_\lambda(B_R(V_k))$.

Let $u_k\in V_k$ satisfy $T_\lambda u_k=f_{|V_k}$ in V_k^*. Our a priori estimate shows that

$$\|u_k\|_{H^{1,1/2}(Q,\mathbb{R}^2)}\leqslant C$$

with a constant C independent of k. Hence, by extracting a subsequence if needed, we may assume that

$$u_k \longrightarrow u \quad \text{weakly in } H_0^{1,1/2}(Q, \mathbb{R}^2).$$

Let $\phi \in (\cup_{k=1}^{\infty} V_k) \cap \mathcal{D}(Q, \mathbb{R}^2)$ be arbitrary. For large enough k we have

$$\iint_Q \Big(\frac{\partial_+^{1/2} u_k}{\partial t^{1/2}}, \frac{\partial_-^{1/2} \phi}{\partial t^{1/2}}\Big) + (\nabla u_k, \nabla \phi) - \lambda(u_k \otimes u_k, \nabla \phi) dx dt = \langle \phi, f \rangle.$$

As $k \to \infty$ the first two terms converge to $\iint_Q (\frac{\partial_+^{1/2} u}{\partial t^{1/2}}, \frac{\partial_-^{1/2} \phi}{\partial t^{1/2}}) dx dt$ and $\iint_Q (\nabla u, \nabla \phi) dx dt$, respectively. In order to treat the non-linear term, observe that by the compactness result provided by Lemma 3 we have that $u_k \otimes u_k \to u \otimes u$ in $L^1_{\mathrm{loc}}(Q)$, say. Since ϕ is bounded with compact support we conclude that the non-linear term also converges to what it should. The existence part of the Theorem now follows from the density of $(\cup_{i=1}^{\infty} V_i) \cap \mathcal{D}(Q, \mathbb{R}^2)$ in $V_0^{1,1/2}(Q)$.

Uniqueness. — The proof is essentially the standard argument, found for instance in the reference [**T**, Chapter 3]. However, our situation is slightly different, so we include a sketch of the proof. To begin with, we assume that $u \in V_0^{1,1/2}(Q, \mathbb{R}^2)$ and $v \in V_0^{1,1/2}(Q, \mathbb{R}^2)$ are two solutions corresponding to the same right hand side. Immediately from the definition we obtain $u, v \in L^2(\mathbb{R}, V_0^1(\Omega))$ in a natural sense, where $V_0^1(\Omega)$ stands for the completion of smooth divergence free test functions in $H^1(\Omega)$. The difference $w := u - v$ satisfies

$$w_t = \Delta w + (v \cdot \nabla) v - (u \cdot \nabla) u \quad [\text{ in } V_0^{1,1/2}(Q, \mathbb{R}^2)^*]. \tag{18}$$

From integration by parts and the imbedding of Lemma 3 we infer that the right-hand side also can be interpreted as an element of $(L^2(\mathbb{R}, V_0^1(\Omega))^*$. The latter space may of course be identified with $L^2(\mathbb{R}, V_0^1(\Omega)^*)$. Put together, we have

$$\begin{cases} w \in L^2(\mathbb{R}, V_0^1(\Omega)) \\ w_t \in L^2(\mathbb{R}, V_0^1(\Omega)^*) \end{cases}, \tag{19}$$

where it is easily verified that for almost every t, the right hand side of (18) represents $w_t(t) \in V_0^1(\Omega)^*$. Define $g(t) = \int_\Omega |w(t,x)|^2 dx$. It is standard that the knowledge (19) allows us to deduce that g has an absolutely continuous representative satisfying the equation

$$g(t_2) - g(t_1) = 2 \int_{t_1}^{t_2} (w(t), w_t(t)) dt, \tag{20}$$

where the pairing is with respect to the duality of $V_0^1(\Omega)$.

We next substitute (18) in (20) and differentiate to obtain for almost every t

$$\frac{d}{dt} \|w(t)\|^2_{L^2(\Omega)} + 2\|\nabla w(t)\|^2_{L^2(\Omega)} = -2 \sum_{i,j} \int_\Omega w_i(t) \partial_i v_j(t) w_j(t) dx. \tag{21}$$

Above we applied a standard cancellation of terms. A scaling of the standard Sobolev imbedding yields $\|\phi\|^2_{L^4(\Omega)} \leqslant C\|\phi\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}$. This together with the simple inequality $ab \leqslant \varepsilon a^2 + (1/\varepsilon)b^2$ gives the estimate

$$|(w(t)\otimes v(t), w(t))| \leqslant 2\|\nabla w(t)\|^2_{L^2(\Omega)} + C_1\|w(t)\|^2_{L^2(\Omega)}\|\nabla v(t)\|^2_{L^2(\Omega)}.$$

Let us denote $g(t) = \|w(t)\|^2_{L^2(\Omega)}$ and $h(t) = \|\nabla v(t)\|^2_{L^2(\Omega)}$. Note that $h \in L^1(\mathbb{R})$. By combining the above inequality with (21) we see that

$$g'(t) \leqslant C_1 g(t)h(t).$$

Assume that $g(t) \neq 0$. The Grönwall lemma yields for all $t' < t$ the lower bound

$$g(t') \geqslant C_2 g(t),$$

with the constant $C_2 = \exp(-C_1\int_{-\infty}^{\infty} h(s)ds) > 0$. This contradicts the fact that $g \in L^1(\mathbb{R})$, and we deduce that $g(t) = 0$ for all t. □

We finally consider the situation where the data is supported in positive time. Denote $Q_+ = \Omega\times(0,\infty)$. We say that an element $f \in V_0^{1,1/2}(Q,\mathbb{R}^2)^*$ has support on $\overline{Q_+}$ if $\langle f, v\rangle = 0$ for all $v \in V_0^{1,1/2}(Q)$ that vanish on $\Omega\times(-\infty,0)$.

Lemma 5. — *Assume that* $\operatorname{supp}(T_\lambda(u)) \in \overline{Q_+}$. *Then* $\operatorname{supp}(u) \in \overline{Q_+}$.

Proof. — Let $\chi(t)$ be a smooth decreasing function vanishing on $(0,\infty)$ and satisfying $0 \leqslant \chi \leqslant 1$. Then

$$\langle u_t, \chi u\rangle = \iint_Q \Big(\frac{\partial_+^{1/2}u}{\partial t^{1/2}}, \frac{\partial_-^{1/2}\chi u}{\partial t^{1/2}}\Big)dxdt = \iint_Q -\chi'(t)|u(x,t)|^2 dxdt,$$

which is easily verified by approximating u with smooth functions and observing that the map $H^{1/2} \ni g \mapsto \partial_-^{1/2}(\chi g) \in L^2$ is bounded. Hence, pairing the equation with χu we obtain

$$\iint_Q -\chi'(t)|u(x,t)|^2 dxdt + \iint_Q \chi(t)|\nabla u(x,t)|^2 dxdt = 0.$$

We conclude that $u(x,t) = 0$ almost everywhere on $\Omega\times(-\infty,0)$. □

The next results is an immediate corollary to this lemma and Theorem 1. It solves the problem of homogeneous initial data.

Theorem 2. — *Given* $f \in V_0^{1,1/2}(Q,\mathbb{R}^2)^*$ *with support on* $\overline{Q_+}$ *there exists a unique* $u \in V_0^{1,1/2}(Q,\mathbb{R}^2)$ *such that* $T_\lambda(u) = f$. *Furthermore* u *is supported in* $\overline{Q_+}$.

Observe that if $u \in V_0^{1,1/2}(Q,\mathbb{R}^2)$ is supported in $\overline{Q_+}$, it need not have a trace in the usual sense at $t = 0$, but it is well-known that then the integral $\displaystyle\iint_{Q_+} \frac{u^2(x,t)}{t}dxdt$ converges.

Remarks. — All the arguments go through unchanged if the domain Ω is unbounded but lies in between to parallel hyperplanes.

It is of interest to notice that the method works and all results are true if we replace the Laplacian in the nonlinear operator with a linear uniformly elliptic operator of divergence form $\nabla \cdot (A(x)\nabla u))$, where A is a bounded measurable matrix valued function on Ω. One can even get results along the same lines when one replaces the Laplacian in the equation with a nonlinear elliptic operator.

One of the original motivations for this work was to build a theory which enables one to easily construct finite element schemes for the non stationary Navier-Stokes equations with data that is very rough in time. Our Propositions 1 and 2 together with Theorem 1 provide a starting point for stability estimates covering rough data.

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