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# HURWITZ SPACES

*by*

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***Abstract.*** — This paper is intended to serve as a general introduction to the theory of Hurwitz spaces and as an overview over the different methods for their construction.

***Résumé* (Espaces de Hurwitz).** — Cet article a pour but de donner une introduction à la théorie des espaces de Hurwitz et un aperçu des différentes méthodes pour leur construction.

## 1. Introduction

**1.1. The classical Hurwitz space and the moduli of curves.** — The classical Hurwitz space first appeared in the work of Clebsch [**Cle72**] and Hurwitz [**Hur91**] as an auxiliary object to study the moduli space of curves. Let $X$ be a smooth projective curve of genus $g$ over $\mathbb{C}$. A rational function $f : X \to \mathbb{P}^1$ of degree $n$ is called *simple* if there are at least $n-1$ points on $X$ over every point of $\mathbb{P}^1$. Such a cover has exactly $r := 2g + 2n - 2$ branch points. Let $\mathcal{H}_{n,r}$ denote the set of isomorphism classes of simple branched covers of $\mathbb{P}^1$ of degree $n$ with $r$ branch points. Hurwitz [**Hur91**] showed that the set $\mathcal{H}_{n,r}$ has a natural structure of a complex manifold. In fact, one can realize $\mathcal{H}_{n,r}$ as a finite unramified covering

$$\Psi_{n,r} : \mathcal{H}_{n,r} \longrightarrow \mathcal{U}_r := \mathbb{P}^r - \Delta_r,$$

where $\Delta_r$ is the discriminant hypersurface. (Note that the space $\mathcal{U}_r$ has a natural interpretation as the set of all subsets of $\mathbb{P}^1$ of cardinality $r$. The map $\Psi_{n,r}$ sends the class of a simple cover $f : X \to \mathbb{P}^1$ to the branch locus of $f$.) Using a combinatorial calculation of Clebsch [**Cle72**] which describes the action of the fundamental group of $\mathcal{U}_r$ on the fibers of $\Psi_{n,r}$, Hurwitz showed that $\mathcal{H}_{n,r}$ is connected.

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Later Severi [**Sev21**] proved that for $n \geq g+1$ every curve $X$ of genus $g$ admits a simple cover $f : X \to \mathbb{P}^1$ of degree $n$. In other words, the natural map

$$\mathcal{H}_{n,r} \longrightarrow \mathcal{M}_g$$

which maps the class of the cover $f : X \to \mathbb{P}^1$ to the class of the curve $X$ is surjective. Using the connectedness of $\mathcal{H}_{n,r}$, Severi concluded that $\mathcal{M}_g$ is connected.

Although $\mathcal{M}_g$ is an algebraic variety and can be defined over $\mathbb{Z}$, the proof of its connectedness sketched above is essentially topological. It therefore does not immediately yield the connectedness of $\mathcal{M}_g \otimes \mathbb{F}_p$ for a prime $p$. In order to fill this gap, Fulton [**Ful69**] gave a purely algebraic construction of the Hurwitz space $\mathcal{H}_{n,r}$. In his theory, $\mathcal{H}_{n,r}$ is a scheme, of finite type over $\mathbb{Z}$, which represents a certain moduli functor. It is equipped with a natural étale morphism $\Psi_{n,r} : \mathcal{H}_{n,r} \to \mathcal{U}_r$ which becomes finite when restricted to $\operatorname{Spec} \mathbb{Z}[1/n!]$. In this setup, Fulton was able to prove the irreducibility of $\mathcal{H}_{n,r} \otimes \mathbb{F}_p$ for every prime $p > n$, using the irreducibility of $\mathcal{H}_{n,r} \otimes \mathbb{C}$. With the same reasoning as above, one can deduce the irreducibility of $\mathcal{M}_g \otimes \mathbb{F}_p$ for $p > g+1$. (At about the same time, Deligne and Mumford proved the irreducibility of $\mathcal{M}_g \otimes \mathbb{F}_p$ for all $p$, using much more sophisticated methods.)

Further applications of Hurwitz spaces to the moduli of curves were given by Harris and Mumford [**HM82**]. They construct a compactification $\bar{\mathcal{H}}_{n,r}$ of $\mathcal{H}_{n,r}$. Points on the boundary $\partial\bar{\mathcal{H}}_{n,r} := \bar{\mathcal{H}}_{n,r} - \mathcal{H}_{n,r}$ correspond to a certain type of degenerate covers between singular curves called *admissible covers*. The map $\mathcal{H}_{n,r} \to \mathcal{M}_g$ extends to a map $\bar{\mathcal{H}}_{n,r} \to \bar{\mathcal{M}}_g$, where $\bar{\mathcal{M}}_g$ is the Deligne–Mumford compactification of $\mathcal{M}_g$. The geometry of this map near the boundary yields interesting results on the geometry of $\bar{\mathcal{M}}_g$.

**1.2. Hurwitz spaces in Galois theory.** — Branched covers of the projective line have more applications besides the moduli of curves. For instance, in the context of the regular inverse Galois problem one is naturally led to study Galois covers $f : X \to \mathbb{P}^1$ with a fixed Galois group $G$. Here arithmetic problems play a prominent role, e.g. the determination of the minimal field of definition of a Galois cover.

Fried [**Fri77**] first pointed out that the geometry of the moduli spaces of branched covers of $\mathbb{P}^1$ with a fixed Galois group $G$ and a fixed number of branch points carries important arithmetic information on the individual covers that are parameterized. Matzat [**Mat91**] reformulated these ideas in a field theoretic language and gave some concrete applications to the regular inverse Galois problem. Fried and Völklein [**FV91**] gave the following precise formulation of the connection between geometry and arithmetic. For a field $k$ of characteristic 0, let $\mathcal{H}_{r,G}(k)$ denote the set of isomorphism classes of pairs $(f, \tau)$, where $f : X \to \mathbb{P}^1_k$ is a *regular* Galois cover with $r$ branch points, defined over $k$, and $\tau : G \xrightarrow{\sim} \operatorname{Gal}(X/\mathbb{P}^1)$ is an isomorphism of $G$ with the Galois group of $f$. Suppose for simplicity that $G$ is center-free. Then it is proved in [**FV91**] that the set $\mathcal{H}_{r,G}(k)$ is naturally the set of $k$-rational points of a smooth variety $\mathcal{H}_{r,G}$, defined over $\mathbb{Q}$. Moreover, we have a finite étale cover of $\mathbb{Q}$-varieties

$$\Psi_{r,G} : \mathcal{H}_{r,G} \longrightarrow \mathcal{U}_r$$

whose associated topological covering map is determined by an explicit action of the fundamental group of $\mathcal{U}_r$ (called the *Hurwitz braid group*) on the fibres of $\Psi_{r,G}$. Using this braid action, it is shown in [**FV91**] that $\mathcal{H}_{r,G}$ has at least one absolutely irreducible component defined over $\mathbb{Q}$ if $r$ is sufficiently large. This has interesting consequences for the structure of the absolute Galois group of $\mathbb{Q}$, see [**FV91**].

In some very special cases one can show, using the braid action on the fibres of $\Psi_{r,G}$, that $\mathcal{H}_{r,G}$ has a connected component which is a rational variety over $\mathbb{Q}$ and hence has many rational points. Then these rational points correspond to regular Galois extensions of $\mathbb{Q}(t)$ with Galois group $G$. For instance, [**Mat91**], §9.4, gives an example with $r = 4$ and $G = M_{24}$. This example yields the only known regular realizations of the Mathieu group $M_{24}$.

**1.3. The general construction.** — In [**FV91**] the Hurwitz space $\mathcal{H}_{r,G}$ is first constructed as a complex manifold. It is then shown to have a natural structure of a $\mathbb{Q}$-variety with the property that $k$-rational points on $\mathcal{H}_{r,G}$ correspond to $G$-Galois covers defined over $k$, but only for fields $k$ of characteristic 0 (and assuming that $G$ is center-free). From the work of Fulton one can expect that there exists a scheme $\mathcal{H}_{r,G,\mathbb{Z}}$ of finite type over $\mathbb{Z}$ such that $k$-rational points correspond to tamely ramified $G$-Galois covers over $k$ for *all* fields $k$. Moreover, $\mathcal{H}_{r,G,\mathbb{Z}}$ should have good reduction at all primes $p$ which do not divide the order of $G$. One can also expect that the construction of Harris and Mumford extends to the Galois situation and yields a nice compactification $\bar{\mathcal{H}}_{r,G,\mathbb{Z}}$ of $\mathcal{H}_{r,G,\mathbb{Z}}$, at least over $\mathbb{Z}[1/|G|]$. These expectations are proved in [**Wew98**], in a more general context.

If the group $G$ has a nontrivial center, then the Hurwitz space $\mathcal{H}_{r,G,\mathbb{Z}}$ is only a *coarse* and not a *fine* moduli space. For instance, a $k$-rational point on $\mathcal{H}_{r,G,\mathbb{Z}}$ corresponds to a tame $G$-cover $f : X \to \mathbb{P}^1_{\bar{k}}$ defined over the algebraic closure of $k$. The field $k$ is the *field of moduli*, but not necessarily a field of definition of $f$. To deal with this difficulty it is very natural to work with algebraic stacks.

The point of view of algebraic stacks has further advantages. For instance, even if $G$ is center-free, the construction of the Harris–Mumford compactification $\bar{\mathcal{H}}_{r,G}$ of $\mathcal{H}_{r,G}$ becomes awkward without the systematic use of stacks. It also provides a much clearer understanding of the connection of Hurwitz spaces with the moduli space of curves with level structure, see [**Rom02**]. Finally, Hurwitz spaces as algebraic stacks are useful for the computation of geometric properties of the moduli of curves, e.g. Picard groups.

The present paper is intended to serve as a general introduction to the theory of Hurwitz spaces and as an overview over the different methods for their construction. For applications to arithmetic problems and Galois theory, we refer to the other contributions of this volume.

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## 2. Hurwitz spaces as coarse moduli spaces

In this section we define the Hurwitz space $\mathcal{H}_{r,G}$ as a coarse moduli space, using the language of schemes.

**2.1. Basic definitions.** — Let $S$ be a scheme. By a *curve* over $S$ we mean a smooth and proper morphism $X \to S$ whose (geometric) fibres are connected and 1-dimensional. If $X$ is a curve over $S$, a *cover* of $X$ is a finite, flat and surjective $S$-morphism $f : Y \to X$, where $Y$ is another curve over $S$. We denote by $\mathrm{Aut}(f)$ the group of automorphisms of $Y$ which leave $f$ fixed.

A cover $f : Y \to X$ is called *Galois* if it is separable and the group $\mathrm{Aut}(f)$ acts transitively on every (geometric) fibre of $f$. It is called *tame* if there exists a smooth relative divisor $D \subset X$ such that the following holds: (a) the natural map $D \to S$ is finite and étale, (b) the restriction of $f : Y \to X$ to the open subset $U := X - D$ is étale, and (c) for every geometric point $s : \mathrm{Spec}\, k \to D$, the ramification index of $f$ along $D$ at $s$ is $> 1$ and prime to the residue characteristic of $s$. If this is the case, the divisor $D$ is called the *branch locus* of $f$. If the degree of $D \to S$ is constant and equal to $r$, we say that the cover $f$ has $r$ *branch points*.

Let $G$ be a finite group and $X$ a curve over $S$. A *$G$-cover* of $X$ is a Galois cover $f : Y \to X$ together with an isomorphism $\tau : G \xrightarrow{\sim} \mathrm{Aut}(f)$. Usually we will identify the group $\mathrm{Aut}(f)$ with $G$.

Two $G$-covers $f_1 : Y_1 \to X$ and $f_2 : Y_2 \to X$ of the same curve $X$ over $S$ are called *isomorphic* if there exists an isomorphism $h : Y_1 \xrightarrow{\sim} Y_2$ such that $f_2 \circ h = f_1$ and $g \circ h = h \circ g$ for all $g \in G$.

**2.2.** Suppose that $S = \mathrm{Spec}\, k$, where $k$ is a field. Then a curve $X$ over $S$ is uniquely determined by its function field $K := k(X)$. A cover $f : Y \to X$ corresponds one-to-one to a finite, separable and regular field extension $L/K$ (here 'regular' means that $k$ is algebraically closed in $L$). The cover $f$ is Galois (resp. tame) if and only if the extension $L/K$ is Galois (resp. tamely ramified at all places of $K$ which are trivial on $k$).

**2.3.** Let us fix a finite group $G$ and an integer $r \geq 3$. For a scheme $S$, we denote by $\mathbb{P}^1_S$ the relative projective line over $S$. Define

$$\mathcal{H}_{r,G}(S) := \{\, f : X \xrightarrow{G} \mathbb{P}^1_S \;\mid\; \deg(D/S) = r \,\}/_{\cong}$$

as the set of isomorphism classes of tame $G$-covers of $\mathbb{P}^1_S$ with $r$ branch points. If $S = \operatorname{Spec} k$ then $\mathcal{H}_{r,G}$ is the set of $G$-Galois extensions of the rational function field $k(t)$, up to isomorphism.

**2.4.** The functor $S \mapsto \mathcal{H}_{r,G}(S)$ is a typical example of a *moduli problem.* One would like to show that there is a *fine moduli space* representing this functor, i.e. a scheme $\mathcal{H}$ together with an isomorphism of functors (from schemes to sets)

$$\mathcal{H}_{r,G}(S) \cong \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}).$$

Unfortunately, this is true only under an additional assumption (if and only if the group $G$ is center free). Fortunately, one can prove a slightly weaker result without this extra assumption (see e.g. [**Wew98**]).

***Theorem 2.1***. — *There exists a scheme $\mathcal{H} = \mathcal{H}_{r,G,\mathbb{Z}}$, smooth and of finite type over $\mathbb{Z}$, together with a morphism of functors (from schemes to sets)*

$$\mathcal{H}_{r,G}(S) \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}), \tag{1}$$

*such that the following holds.*

(i) *Suppose there is another scheme $\mathcal{H}'$ and a morphism of functors $\mathcal{H}_{r,G}(S) \to \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}')$. Then there exists a unique morphism of schemes $\mathcal{H} \to \mathcal{H}'$ which makes the following diagram commute:*

$$\begin{array}{ccc} \mathcal{H}_{r,G}(S) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}) \\ & \searrow & \downarrow \\ & & \operatorname{Hom}_{\mathbb{Z}}(S, \mathcal{H}'). \end{array}$$

(ii) *If $S$ is the spectrum of an algebraically closed field then* (1) *is a bijection.*

We say that the scheme $\mathcal{H} = \mathcal{H}_{r,G,\mathbb{Z}}$ is the *coarse moduli space* associated to the functor $S \mapsto \mathcal{H}_{r,G}(S)$, and call it the *Hurwitz space* for tame $G$-Galois covers of $\mathbb{P}^1$ with $r$ branch points.

In particular the theorem says that for any algebraically closed field $k$ the set $\mathcal{H}_{r,G}(k)$ (i.e. the set of isomorphism classes of regular and tamely ramified $G$-Galois extensions of $k(t)$) has a natural structure of a smooth $k$-variety $\mathcal{H}_{r,G,k}$. For $k$ of characteristic zero this was first proved by Fried and Völklein, see [**FV91**]. In §4 we will prove it for an arbitrary field $k$.

Let $(f, \tau)$ be a $G$-cover over a scheme $S$. It follows immediately from the definition that the group of automorphisms of the pair $(f, \sigma)$ is the center of $G$. It is a general fact that a coarse moduli space representing objects with no nontrivial automorphisms is actually a fine moduli space. Hence we deduce from Theorem 2.1:

***Corollary 2.2***. — *Suppose that the center of $G$ is trivial. Then* (1) *is a bijection for all schemes $S$. In other words, the scheme $\mathcal{H}_{r,G,\mathbb{Z}}$ is a fine moduli space.*

In particular, if $G$ is center-free then for any field $k$ (not necessarily algebraically closed) the set $\mathcal{H}_{r,G}(k)$ (i.e. the set of isomorphism classes of regular and tamely ramified $G$-Galois extensions of $k(t)$) can be identified with the set of $k$-rational points of a smooth $k$-variety $\mathcal{H}_{r,G,k}$.

**2.5.** Theorem 2.1 has many variants and generalizations. One variant which is important in arithmetic applications comes up when we weaken the notion of 'isomorphism' between two $G$-covers. For instance, two $G$-covers $f_1 : Y_1 \to \mathbb{P}^1_S$ and $f_2 : Y_2 \to \mathbb{P}^1_S$ over the same scheme $S$ are called *weakly isomorphic* if there exist isomorphisms $\phi : Y_1 \xrightarrow{\sim} Y_2$ and $\psi : \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ such that $f_2 \circ \phi = \psi \circ f_1$ and $g \circ \phi = \phi \circ g$ for all $g \in G$. (Note that $\psi$ may be regarded as an element of $\mathrm{PGL}_2(S)$.) Replacing 'isomorphism class' by 'weak isomorphism class' in the definition of the moduli problem $\mathcal{H}_{r,G}$, we get a new moduli problem $S \mapsto \mathcal{H}^{\mathrm{red}}_{r,G}(S) := \mathcal{H}_{r,G}(S)/\mathrm{PGL}_2(S)$. Theorem 2.1 carries over to this new situation and shows the existence of a coarse moduli space $\mathcal{H}^{\mathrm{red}}_{r,G}$, called a *reduced Hurwitz space*. It is easy to see that the natural map $\mathcal{H}_{r,G} \to \mathcal{H}^{\mathrm{red}}_{r,G}$ identifies $\mathcal{H}^{\mathrm{red}}_{r,G}$ with the quotient of $\mathcal{H}_{r,G}$ under the natural action of $\mathrm{PGL}_2$. Hence $\mathcal{H}^{\mathrm{red}}_{r,G}$ is normal. In general, it is not smooth over $\mathbb{Z}$.

There are many more variants of the moduli problem $\mathcal{H}_{r,G}$. For instance, one can regard $G$-Galois covers $f : Y \to \mathbb{P}^1$ as *mere covers* (i.e. one forgets the isomorphism $\tau : G \xrightarrow{\sim} \mathrm{Aut}(f)$), look at non Galois covers, or one can order the branch points. All these variants are important in applications, and arithmetic questions dealing with their differences can be quite subtle. See e.g. the other contributions for this volume. However, as far as the construction of the corresponding Hurwitz spaces (i.e. the proof of the relevant version of Theorem 2.1) is concerned, it makes no essential difference which variant one is looking at. In fact, in [**Wew98**] a much more general version of Theorem 2.1 is proved from which all the special cases discussed above can be deduced. This general approach will be discussed in §4.3. In the first three sections of the present article we restrict ourselves to the moduli problem $\mathcal{H}_{r,G}$.

## 3. Analytic construction

In this section we describe the Hurwitz space $\mathcal{H}_{r,G}$ as an analytic space, using Riemann's Existence Theorem. We also prove the analog of Theorem 2.1 in the context of analytic spaces.

**3.1. Riemann's Existence Theorem.** — Let us fix, for the moment, a finite subset $D = \{t_1, \ldots, t_r\} \subset \mathbb{P}^1_{\mathbb{C}}$, of cardinality $r \geq 0$. We denote by $\mathcal{H}_{D,G,\mathbb{C}}$ the set of isomorphism classes of $G$-covers of $\mathbb{P}^1_{\mathbb{C}}$ with branch locus $D$.

Set $U := \mathbb{P}^1_{\mathbb{C}} - D$ and choose a basepoint $x_0 \in U$. By elementary topology, there exists a presentation

$$\pi_1(U, x_0) = \langle \gamma_1, \ldots, \gamma_r \mid \prod_i \gamma_i = 1 \rangle,$$

where $\gamma_i$ is represented by a simple closed loop winding around the missing point $t_i$, counterclockwise.

Let $f : X \to \mathbb{P}^1_{\mathbb{C}}$ be a $G$-cover with branch locus $D$. We consider $f$ as a finite, holomorphic map between compact Riemann surfaces. The restriction of $f$ to $V := f^{-1}(U)$ is a regular covering projection with group of deck transformations $G$, see [**Spa66**], §2. It corresponds to a surjective homomorphism $\rho_f : \pi_1(U, x_0) \to G$ (well defined up to composition by an inner automorphism of $G$). Set $g_i := \rho_f(\gamma_i)$. Then $\boldsymbol{g} = (g_1, \ldots, g_r)$ is an element of the set

$$\mathcal{E}_r(G) := \{ \ \boldsymbol{g} = (g_1, \ldots, g_r) \ \mid \ g_i \in G - \{1\}, \ G = \langle g_i \rangle, \ \prod_i g_i \ = \ 1 \ \}.$$

We let $G$ act on $\mathcal{E}_r(G)$ by simultaneous conjugation and denote by

$$\mathrm{ni}_r(G) := \mathcal{E}_r(G)/G$$

the set of (inner) *Nielsen classes*. The most fundamental fact about $G$-covers over $\mathbb{C}$ is the following theorem, which is sometimes called Riemann's Existence Theorem.

***Theorem 3.1***. — *The correspondence $f \mapsto \boldsymbol{g}$ induces a bijection*

$$\mathcal{H}_{D,G,\mathbb{C}} \xrightarrow{\sim} \mathrm{ni}_r(G).$$

For a proof, see e.g. [**Völ96**]. It is not hard to see that an element $\boldsymbol{g} \in \mathrm{ni}_r(G)$ gives rise to a ramified Galois cover $f^{\mathrm{an}} : Y^{\mathrm{an}} \to \mathbb{P}^1_{\mathbb{C}}$ of compact Riemann surfaces. The most substantial part of the proof of Theorem 3.1 consists in showing that $f^{\mathrm{an}}$ is actually algebraic.

**3.2. Deformation of covers.** — Fix an integer $r \geq 0$ and set

$$\mathcal{U}_{r,\mathbb{C}} := \{ \ D \subset \mathbb{P}^1_{\mathbb{C}} \ \mid \ |D| = r \ \}.$$

The set $\mathcal{U}_{r,\mathbb{C}}$ has a natural structure of a complex manifold. Let $\mathcal{H}_{r,G,\mathbb{C}}$ be the set of isomorphism classes of $G$-covers of $\mathbb{P}^1_{\mathbb{C}}$ with $r$ branch points. Let

$$\Psi_r : \mathcal{H}_{r,G,\mathbb{C}} \longrightarrow \mathcal{U}_{r,\mathbb{C}}$$

denote the map which associates to a $G$-cover $f : X \to \mathbb{P}^1_{\mathbb{C}}$ the branch locus of $f$.

We endow the set $\mathcal{H}_{r,G,\mathbb{C}}$ with a topology, as follows. Fix a $G$-cover $f : X \to \mathbb{P}^1_{\mathbb{C}}$ with branch locus $D = \{t_1, \ldots, t_r\}$, and let $\rho : \pi_1(\mathbb{P}^1_{\mathbb{C}} - D) \to G$ denote the corresponding group homomorphism. Let $C_1, \ldots, C_r \subset \mathbb{P}^1_{\mathbb{C}}$ be disjoint disk-like neighborhoods of the points $t_1, \ldots, t_r$. Let $\mathcal{U}(C_i)$ denote the subset of $\mathcal{U}_{r,\mathbb{C}}$ consisting of divisors $D' = \{t'_1, \ldots, t'_r\} \in \mathcal{U}_{r,\mathbb{C}}$ with $t'_i \in C_i$. Note that the subsets $\mathcal{U}(C_i)$ form a basis of open neighborhoods of the point $D \in \mathcal{U}_{r,\mathbb{C}}$. For any $D' \in \mathcal{U}(C_i)$ we have natural isomorphisms

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} - D) \cong \pi_1(\mathbb{P}^1_{\mathbb{C}} - (\cup_i C_i)) \cong \pi_1(\mathbb{P}^1_{\mathbb{C}} - D'). \tag{2}$$

We define the subset $\mathcal{H}(f, C_i) \subset \mathcal{H}_{r,G,\mathbb{C}}$ as the set of isomorphism classes of $G$-covers $f' : X' \to \mathbb{P}^1_{\mathbb{C}}$ with the following properties. First, the branch locus $D'$ of $f'$ is contained in $\mathcal{U}(C_i)$; second, the composition of the homomorphism $\rho' : \pi_1(\mathbb{P}^1_{\mathbb{C}} - D') \to G$ corresponding to $f'$ with the isomorphism (2) agrees with $\rho$, up to an inner automorphism of $G$. The topology we put on $\mathcal{H}_{r,G,\mathbb{C}}$ is the unique topology in which the sets $\mathcal{H}(f, C_i)$ form a basis of open neighborhoods of the point corresponding to $f$.

***Proposition 3.2***. — (i) *The map $\Psi_r$ is a covering projection.*

(ii) *The topological space $\mathcal{H}_{r,G,\mathbb{C}}$ has a unique structure of a complex manifold such that $\Psi_r$ is biholomorphic.*

*Proof.* — By construction, we have the following decomposition into open and closed subsets:

$$\Psi_r^{-1}(\mathcal{U}(C_i)) = \coprod_f \mathcal{H}(f, C_i),$$

where $f$ runs over all $G$-covers with branch locus $D$, up to isomorphism. Moreover, for each $f$ the induced map $\mathcal{H}(f, C_i) \to \mathcal{U}(C_i)$ is a bijection. This proves (i). Statement (ii) is a direct consequence of (i). □

***Remark 3.3***. — By elementary topology, the space $\mathcal{H}_{r,G,\mathbb{C}}$ is determined, as a covering of $\mathcal{U}_{r,\mathbb{C}}$, by a natural action of the fundamental group of $\mathcal{U}_{r,\mathbb{C}}$ on the fiber $\mathcal{H}_{D,G,\mathbb{C}} \cong \mathrm{ni}_r(G)$. The fundamental group of $\mathcal{U}_{r,\mathbb{C}}$, which is called the *Hurwitz braid group* on $r$ strands, has a well known presentation by generators and relations. Moreover, the action of the generators of this presentation on $\mathrm{ni}_r(G)$ are given by simple and explicit formulas, see e.g. [**FV91**].

So from a topological point of view, we have a rather explicit description of the Hurwitz space $\mathcal{H}_{r,G,\mathbb{C}}$. For instance, the connected components of $\mathcal{H}_{r,G,\mathbb{C}}$ correspond to the orbits of the braid action on the set $\mathrm{ni}_r(G)$.

### 3.3. $\mathcal{H}_{r,G,\mathbb{C}}$ is a coarse moduli space.

— We consider the set $\mathcal{H}_{r,G,\mathbb{C}}$ as a complex manifold. Then we have the following result.

***Proposition 3.4***. — *Let $S$ be a complex analytic space and $f : X \to \mathbb{P}^1_S$ a tame $G$-cover over $S$ with $r$ branch points. Let*

$$\varphi_f : S \to \mathcal{H}_{r,G,\mathbb{C}}$$

*be the map which assigns to a point $s \in S$ the isomorphism class of the fiber of $f : X \to \mathbb{P}^1_S$ over $s$. Then $\varphi_f$ is an analytic morphism.*

*Proof.* — Let $\psi_f : S \to \mathcal{U}_{r,\mathbb{C}}$ denote the composition of the map $\varphi_f$ with $\Psi_r$. By definition, for $s \in S$ the point $\psi_f(s) \in \mathcal{U}_{r,\mathbb{C}}$ corresponds to the branch divisor of the fiber of $f$ over the point $s$. Since the branch divisor of the fibres of $f$ vary analytically with $s$, $\psi_f$ is an analytic map. It also follows immediately from the definition of the topology on $\mathcal{H}_{r,G,\mathbb{C}}$ that $\varphi_f$ is continuous. Since $\Psi_r$ is a local isomorphism of complex manifolds, we deduce that $\varphi_f$ is analytic. □

For an analytic space $S$, let $\mathcal{H}_{r,G}(S)$ denote the set of isomorphism classes of $G$-covers of $\mathbb{P}^1_S$ with $r$ branch points (as in §2.3). The proposition shows that we have a natural morphism of functors (from analytic spaces to sets)

$$\mathcal{H}_{r,G}(S) \longrightarrow \mathrm{Hom}_{(\mathrm{An})}(S, \mathcal{H}_{r,G,\mathbb{C}}). \tag{3}$$

It yields an analog of Theorem 2.1 in the context of analytic spaces. (We leave to the reader the task of formulating the correct notion of 'coarse moduli space' in the context of analytic spaces.)

**Theorem 3.5**. — *The morphism of functors* (3) *identifies $\mathcal{H}_{r,G,\mathbb{C}}$ with the coarse moduli space of the functor $S \mapsto \mathcal{H}_{r,G}(S)$.*

*Proof.* — Let $\mathcal{H}'$ be an analytic space and

$$G(S) : \mathcal{H}_{r,G}(S) \longrightarrow \mathrm{Hom}_{(\mathrm{An})}(S, \mathcal{H}')$$

a morphism of functors in $S$ (from analytic spaces to sets). If we evaluate this morphism on the analytic space $S = \{s\}$ consisting of a single point, then we get a map $g : \mathcal{H}_{r,G,\mathbb{C}} \to \mathcal{H}'$. We claim that $g$ is analytic. Once this claim is proved, it is clear that the composition of the morphism (3) with the morphism induced by $g$ is equal to $G$ (compare with the diagram of Theorem 2.1 (i)). Moreover, $g$ is the unique map with this property. Therefore, it remains to show that $g$ is analytic. Let $f : X \to \mathbb{P}^1$ be a $G$-cover and $s \in \mathcal{H}_{r,G,\mathbb{C}}$ the corresponding point. Let $S := \mathcal{H}(f, C_i) \subset \mathcal{H}_{r,G,\mathbb{C}}$ be one of the basic neighborhoods of $s$ constructed in §3.2 above. Let $\tilde{D} \subset \mathbb{P}^1_S$ be the relative divisor corresponding to the isomorphism $S \xrightarrow{\sim} \mathcal{U}(C_i)$. Using the fact that the projection $\mathbb{P}^1_S - \tilde{D} \to S$ is a topological fibration, it is easy to show that there exists a unique family of $G$-covers $\tilde{f} : \tilde{X} \to \mathbb{P}^1_S$ over $S$ with branch locus $\tilde{D}$ and such that $\tilde{f}_s = f$. The morphism of functors $G$ applied to the isomorphism class of $\tilde{f}$ yields an analytic map $\tilde{g} : S \to \mathcal{H}'$. By functoriality we have $\tilde{g} = g|_S$. This shows that $g$ is analytic and finishes the proof of the theorem. □

## 4. Algebraic construction

In this section we prove a certain weak version of Theorem 2.1. Essentially we show that there is a scheme $\mathcal{H}_{r,G,\mathbb{Z}}$, smooth and of finite type over $\mathbb{Z}$, such that for any algebraically closed field $k$ there is a functorial bijection between the set of $k$-rational points of $\mathcal{H}_{r,G,\mathbb{Z}}$ and the set of isomorphism classes of tame $G$-covers $f : X \to \mathbb{P}^1_k$

with $r$ branch points, defined over $k$. This suffices for many applications. The proof we give is somewhat similar to the proof of Theorem 3.5, in the sense that it relies heavily on the nice deformation theory of tamely ramified covers. The topological arguments used in the last section are replaced by the use of étale morphism. For basic facts about étale morphisms, see the first chapter of [**G$^+$71**] or of [**Mil80**].

Throughout this section, we fix an integer $r \geq 3$ and a finite group $G$.

**4.1. The configuration space $\mathcal{U}_{r,\mathbb{Z}}$.** — For $r \geq 1$, let $\mathcal{U}_{r,\mathbb{Z}}$ denote the open subset of $\mathbb{P}^r_{\mathbb{Z}}$ defined by the condition

$$\Delta(c_0 : \ldots : c_r) := \mathrm{discr}(c_0 T^r + \cdots + c_n) \neq 0.$$

Let $D_{\mathrm{univ}} \subset \mathbb{P}^1 \times_{\mathbb{Z}} \mathcal{U}_{r,\mathbb{Z}}$ be the 'universal' smooth divisor of degree $r$, given by the equation $c_0 T^r + \cdots + c_n = 0$. To any morphism of schemes $\psi : S \to \mathcal{U}_{r,\mathbb{Z}}$ we associate the divisor $D_\psi := D_{\mathrm{univ}} \times_\psi S \subset \mathbb{P}^1_S$. This gives a one-to-one correspondence between morphisms $S \to \mathcal{U}_{r,\mathbb{Z}}$ and divisors $D \subset \mathbb{P}^1_S$ such that the projection $D \to S$ is finite and étale, of degree $r$. (This makes $\mathcal{U}_{r,\mathbb{Z}}$ a fine moduli space.) The morphism $\psi : S \to \mathcal{U}_{r,\mathbb{Z}}$ corresponding to a divisor $D \subset \mathbb{P}^1_S$ is called the *classifying map* of $D$.

**4.2. The field of moduli of a $G$-cover.** — Let $k$ be an algebraically closed field and $f : X \to \mathbb{P}^1_k$ a $G$-cover, defined over $k$. In this subsection we associate to $f$ two subfields of $k$, denoted by $k_0$ and $k_m$. They depend only on the isomorphism class of $f$ as a $G$-cover.

***Definition 4.1***. — Let $D \subset \mathbb{P}^1_k$ be the branch locus of $f$. The *branch locus field* of $f$ is the residue field $k_0$ of the image of the classifying map $\mathrm{Spec}\, k \to \mathcal{U}_{r,\mathbb{Z}}$ of $D$. In other words, $k_0$ is the smallest field of definition of the divisor $D \subset \mathbb{P}^1$.

***Definition 4.2***. — Let $\sigma : k \xrightarrow{\sim} k$ be a field automorphism. We extend $\sigma$ to an automorphism of the rational function field $k(t)$ by prescribing that $\sigma(t) = t$. We denote by $\tilde{\sigma}$ the corresponding automorphism of $\mathbb{P}^1_k$ (where we regard $k(t)$ as the function field of $\mathbb{P}^1_k$, in the standard way). The *twist of $f$ by $\sigma$* is the $G$-cover $f^\sigma : X^\sigma \to \mathbb{P}^1_k$ occurring in the following commutative diagram with Cartesian squares:

$$\begin{array}{ccc} X^\sigma & \longrightarrow & X \\ {\scriptstyle f^\sigma}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{P}^1_k & \xrightarrow{\tilde{\sigma}} & \mathbb{P}^1_k \end{array}$$

The *field of moduli* of $f$ is the subfield $k_m \subset k$ of elements of $k$ fixed by the group of automorphisms

$$A_f = \{\, \sigma : k \xrightarrow{\sim} k \;\mid\; f^\sigma \cong f \,\}.$$

Note that the condition $f^\sigma \cong f$ is equivalent to the condition that $\sigma : k(t) \xrightarrow{\sim} k(t)$ extends to an automorphism of the function field $L := k(X)$ of $X$ which commutes with the action of $G$.

By construction, we have $k_0 \subset k_m \subset k$. Moreover, if $k' \subset k$ is a field of definition of $f$ then $k_m \subset k'$. We have the following important finiteness result.

***Theorem 4.3***. — *Let $k$ be an algebraically closed field and $f : X \to \mathbb{P}^1_k$ a tame $G$-cover defined over $k$. Then $f$ has a field of definition $k' \subset k$ which is a finite separable extension of the branch locus field $k_0$. In particular, the field of moduli of $f$ is a finite separable extension of $k_0$.*

A well known application of this theorem is the 'obvious direction' of Belyi's theorem: if $X$ is a smooth projective curve over $\mathbb{C}$ and $f : X \to \mathbb{P}^1$ is a rational function with only three branch points, then $X$ can be defined over a number field.

Theorem 4.3 is equivalent to the assertion that the tame fundamental group $\pi_1^t(\mathbb{P}^1_{k_0^{\rm sep}} - D)$ does not change under base change with the extension $k/k_0^{\rm sep}$. The corresponding fact for smooth projective schemes is proved in [**G$^+$71**], see Corollaire X.1.8 and Théorème IX.6.1. In [**G$^+$71**], Exposé XIII, the tame fundamental group of an affine curve over an algebraically closed field is studied. However, as the referee pointed out, there seems to be no statement in *loc.cit.* of which Theorem 4.3 is a direct consequence.

Note that the tameness assumption in Theorem 4.3 is necessary, see [**G$^+$71**], Remarque X.1.10. We will prove Theorem 4.3 in §4.4 below. The proof is part of our algebraic construction of the Hurwitz space $\mathcal{H}_{r,G}$. For a different proof of Theorem 4.3 in the case where $k$ has characteristic 0, see e.g. [**Völ96**], §7, or [**KÖ4**]. Both these proofs use the Riemann Existence Theorem, see §3.

The following fact is very useful, in particular for applications to the regular inverse Galois problem.

***Proposition 4.4***. — *Let $f : X \to \mathbb{P}^1_k$ be a $G$-cover, with field of moduli $k_m \subset k$. Suppose that the group $G$ has trivial center. Then $f$ has a unique model over $k_m$. In particular, $k_m$ is a field of definition.*

In general, the obstruction for $k_m$ to be a field of definition is represented by an element of the Galois cohomology group $H^2(k_m, Z_G)$, where $Z_G$ denotes the center of $G$. See [**DD97**].

**4.3. Algebraic deformation theory.** — Let $k$ be an algebraically closed field. We denote by $\mathcal{C}_k$ the category of Noetherian complete local rings with residue field $k$. Homomorphisms between objects of $\mathcal{C}_k$ are local ring homomorphisms which induce the identity on $k$.

Let $f : X \to \mathbb{P}^1_k$ be a $G$-cover defined over $k$, and let $R$ be an object of $\mathcal{C}_k$. A *deformation* of $f$ over $R$ is a $G$-cover $f_R : X_R \to \mathbb{P}^1_R$ defined over the ring $R$ together with an isomorphism of $G$-covers between $f$ and the special fiber $f_R \otimes k$. Usually we will identify $f$ and $f_R \otimes k$. An isomorphism between two deformations $f_R$ and $f'_R$ is an isomorphism of $G$-covers over $R$ which induces the identity on $f$.

Let $t_1, \ldots, t_r$ denote the branch points of $f$. It is no restriction to assume that $f$ is unramified over infinity. We may hence consider the $t_i$ as elements of $k$. The next proposition is the algebraic version of the fact that one can deform a cover in a unique way by moving its branch points. It is our main tool for the proof of the Theorems 4.3 and 4.11 (just as the corresponding topological fact was the main tool for the proof of Theorem 3.5).

***Proposition 4.5***. — *Assume that the cover $f$ is tamely ramified. Let $R$ be an object of $\mathcal{C}_k$, and let $\tilde{t}_1, \ldots, \tilde{t}_r \in R$ be elements which lift $t_1, \ldots, t_r \in k$. Then there exists a deformation $f_R$ of $f$ with branch points $\tilde{t}_1, \ldots, \tilde{t}_r$. It is unique up to unique isomorphism.*

Proposition 4.5 is a special case of the deformation theory of tame covers, see [**G$^+$71**] and §5 below. Here is a useful lemma which follows from the uniqueness statement in Proposition 4.5.

***Lemma 4.6***. — *Let $R$ be a Noetherian, normal and integral domain, $S = \operatorname{Spec} R$, and $f_1 : X_1 \to \mathbb{P}^1_S$, $f_2 : X_2 \to \mathbb{P}^1_S$ two tame $G$-covers over $S$.*

(i) *Let $\lambda, \lambda' : f_1 \xrightarrow{\sim} f_2$ be two isomorphisms of $G$-covers. If there exists a geometric point $s : \operatorname{Spec} k \to S$ such that $\lambda_s = \lambda'_s$, then $\lambda = \lambda'$.*

(ii) *Suppose that $f_1$ and $f_2$ have the same branch locus $D \subset \mathbb{P}^1_S$. Let $s : \operatorname{Spec} k \to S$ be a geometric point and $\lambda_s : f_{1,s} \xrightarrow{\sim} f_{2,s}$ an isomorphism between the fibres of $f_1$ and $f_2$ over $s$. Then there exists an étale neighborhood $S' \to S$ of the point $s$ and an isomorphism $\lambda : f_1 \times_S S' \xrightarrow{\sim} f_2 \times_S S'$ which extends $\lambda_s$.*

(iii) *Let $K$ denote the fraction field of $R$. Any isomorphism $\lambda_K : f_1 \times_S \operatorname{Spec} K \xrightarrow{\sim} f_2 \times_S \operatorname{Spec} K$ extends to a unique isomorphism $\lambda : f_1 \xrightarrow{\sim} f_2$.*

*Proof*. — Let $\lambda, \lambda'$ be as in (i), and let $W \subset S$ denote the locus of points $s : \operatorname{Spec} k \to S$ such that $\lambda_s = \lambda'_s$. It is easy to see that $W$ is a closed subset of $V$. Therefore, $W = \operatorname{Spec} R/I$, where $I \lhd R$ is an ideal. Suppose that $s : \operatorname{Spec} k \to S$ has image in $W$. Let $\hat{R}$ denote the strict completion of $R$ at $s$. Since $\lambda_s = \lambda'_s$, we can think of $f_i \times_S \operatorname{Spec} \hat{R}$ as two deformations of the same $G$-cover over $k$; then $\lambda \times_S \operatorname{Spec} \hat{R}$ and $\lambda' \times_S \operatorname{Spec} \hat{R}$ are isomorphisms of these two deformations. The uniqueness statement in Proposition 4.5 implies that $\lambda \times_S \operatorname{Spec} \hat{R} = \lambda' \times_S \operatorname{Spec} \hat{R}$. But since $R$ is contained in $\hat{R}$, this shows that $I = 0$ and $W = S$ and proves (i).

Suppose now that $f_1$ and $f_2$ have the same branch locus and that there exists a point $s : \operatorname{Spec} k \to S$ and an isomorphism $\lambda_s : f_{1,s} \xrightarrow{\sim} f_{2,s}$. Let $\hat{R}$ denote the strict completion of $R$ at $s$. By Proposition 4.5 and the assumption on the branch locus, the isomorphism $\lambda_s$ extends to an isomorphism $\hat{\lambda} : f_1 \times_S \operatorname{Spec} \hat{R} \xrightarrow{\sim} f_2 \times_S \operatorname{Spec} \hat{R}$. Since $X_1$ and $X_2$ are of finite presentation over $R$, there exists a subring $R' \subset \hat{R}$, which is a finitely generated $R$-algebra, such that $\hat{\lambda}$ descends to an isomorphism $\lambda : f_1 \times_S S' \xrightarrow{\sim} f_2 \times_S S'$ over $S' = \operatorname{Spec} R'$. Let $s' : \operatorname{Spec} k \to S'$ be the point corresponding to the composition of the inclusion $R' \subset \hat{R}$ and the natural map $\hat{R} \to k$.

Since $S'$ is of finite type over $S$ and $R' \subset \hat{R}$, the map $S' \to S$ is étale in a neighborhood of $s'$, see e.g. [**Har77**], Ex. III.10.4. Replacing $S'$ by an open subset containing the image of $s'$, we may assume that $S' \to S$ is étale. This proves (ii).

The schemes $X_1$ and $X_2$ are smooth over the normal scheme $S$ and hence normal. Therefore, $X_i$ is the normalization of $\mathbb{P}^1_S$ in the function field of $X_i \times_S \operatorname{Spec} K$. It follows immediately that any isomorphism $\lambda_K : f_1 \times_S \operatorname{Spec} K \xrightarrow{\sim} f_2 \times_S \operatorname{Spec} K$ extends to a unique isomorphism $\lambda : f_1 \xrightarrow{\sim} f_2$. The lemma is proved. □

***Remark 4.7***. — With a little extra work, the same reasoning as in the proof of Lemma 4.6 yields the following statement. Let $S$ be an arbitrary scheme and $f_1, f_2$ two tame $G$-covers over $S$. Then the functor which associates to an $S$-scheme $S'$ the set of isomorphisms from $f_1 \times_S S'$ to $f_2 \times_S S'$ is representable by a scheme $\operatorname{Iso}_S(f_1, f_2)$. This scheme is *finite and étale* over the closed subscheme of $S$ defined by the condition that $f_1$ and $f_2$ have the same branch locus. Note that this statement immediately implies Lemma 4.6.

***Definition 4.8***. — Let $S$ be a connected scheme and $f : X \to \mathbb{P}^1_S$ a tame $G$-cover with $r$ branch points over $S$. Let $\psi : S \to \mathcal{U}_{r,\mathbb{Z}}$ be the classifying map of the branch locus of $f$, see §4.1. If $\psi$ is étale then $f$ is called a *versal family* of $G$-covers over $S$.

***Definition 4.9***. — Let $k$ be an algebraically closed field and $f : X \to \mathbb{P}^1_k$ a tame $G$-cover over $k$ with $r$ branch point. A *versal algebraic deformation* of $f$ is a triple $(\tilde{f}, s, \lambda)$, where

- $\tilde{f} : \tilde{X} \to \mathbb{P}^1_S$ is a versal family of $G$-covers over a scheme $S$,
- $s : \operatorname{Spec} k \to S$ is a $k$-rational point, and
- $\lambda : f \xrightarrow{\sim} \tilde{f}_s$ is an isomorphism of $G$-covers between $f$ and the pullback of $\tilde{f}$ via $s$.

A morphism between two versal deformations $(\tilde{f}_i, s_i, \lambda_i)$ defined over schemes $S_i$, $i = 1, 2$, is a morphism $h : S_1 \to S_2$ such that $h(s_1) = s_2$, together with an isomorphism $\tilde{h} : \tilde{f}_1 \cong \tilde{f}_2 \times_{S_2} S_1$ of $G$-covers over $S_1$ which identifies $\lambda_1$ with the pullback of $\lambda_2$. We write $\tilde{h} : \tilde{f}_1 \to \tilde{f}_2$ to denote this morphism.

We would like to point out that the image of the $k$-rational point $s$ is in general *not* a closed point of the scheme $S$, and that the residue field of this point is *never* equal to $k$. So a 'versal algebraic deformation' is something quite different from a 'deformation', as defined at the beginning of this subsection.

***Proposition 4.10***. — *Let $f : X \to \mathbb{P}^1_k$ be a tame $G$-cover with $r$ branch points, defined over an algebraically closed field $k$.*

(i) *There exists a versal algebraic deformation $(\tilde{f}, s, \lambda)$ of $f$.*
(ii) *Given two versal algebraic deformations of $f$, $\tilde{f}_1$ and $\tilde{f}_2$, there exists a third one, $\tilde{f}_3$, together with morphisms $\tilde{f}_3 \to \tilde{f}_1$ and $\tilde{f}_3 \to \tilde{f}_2$.*

*Proof.* — We start with Part (ii) of the proposition. Let $(\tilde{f}_i, s_i, \lambda_i)$, $i = 1, 2$, be two versal algebraic deformations of $f$. Let $S_3$ be the connected component of $S_1 \times_{\mathcal{U}_{r,\mathbb{Z}}} S_2$ which contains the image of $s_3 := (s_1, s_2)$. The projections $h_1 : S_3 \to S_1$ and $h_2 : S_3 \to S_2$ are étale and we have $h_1(s_3) = s_1$ and $h_2(s_3) = s_2$. After replacing $S_3$ by a sufficiently small étale neighborhood of $s_3$, the isomorphism $\lambda_2 \circ \lambda_1^{-1} : \tilde{f}_1 \otimes k \xrightarrow{\sim} \tilde{f}_2 \otimes k$ extends to an isomorphism $\tilde{f}_1 \times S_3 \cong \tilde{f}_2 \times S_3$, by Lemma 4.6 (ii). Part (ii) of the proposition follows immediately.

To prove Part (i), let us first assume that $k = k_0^{\mathrm{sep}}$, where $k_0 \subset k$ is the branch locus field of $f$ (see §4.2). This is the case, for instance, when $k$ is the algebraic closure of its prime field (which is either $\mathbb{Q}$ or $\mathbb{F}_p$ for a prime $p$). Without loss of generality, we may assume that $f$ is unramified over infinity. Hence we may regard the branch points $t_1, \ldots, t_r$ of $f$ as elements of $k$. If $k$ has characteristic $p > 0$, then we write $W(k)$ to denote the ring of Witt vectors over $k$; if $k$ is of characteristic 0, then we set $W(k) := k$. In any case, we set $\hat{R} := W(k)[[s_1, \ldots, s_r]]$, the ring of formal power series over $W(k)$ in $r$ variables. Lift the elements $t_i \in k$ to elements of $W(k)$ with the same name. Set $\tilde{t}_i := t_i + s_i \in \hat{R}$. By the existence part of Proposition 4.5, there exists a unique deformation of $f$ to a $G$-cover $\hat{f} : \hat{X} \to \mathbb{P}^1_{\hat{R}}$ over $\hat{R}$ with branch points $\tilde{t}_1, \ldots, \tilde{t}_r$. (The uniqueness part of Proposition 4.5 shows that $\hat{f}$ is the *universal deformation* of $f$.)

Since projective curves are of finite presentation, there exists a subring $R \subset \hat{R}$ which is of finite type over $\mathbb{Z}$ and such that $\hat{f}$ descends to a finite map $\tilde{f} : \tilde{X} \to \mathbb{P}^1_S$ between flat projective curves over $S = \operatorname{Spec} R$. Let $s : \operatorname{Spec} k \to S$ be the point obtained by composing the inclusion $R \hookrightarrow \hat{R}$ with the canonical map $\hat{R} \to k$. The fiber $f_s$ of $\tilde{f}$ over $s$ is canonically isomorphic to $f$. We denote this canonical isomorphism by $\lambda$. By standard arguments, the locus of points on $S$ where the fiber of $\tilde{f}$ is a tame $G$-cover is open in $S$. Therefore, after restricting $S$ to an open subset containing the point $s$, we may assume that $\tilde{f}$ is a tame $G$-cover over $S$. Let $\psi : S \to \mathcal{U}_{r,\mathbb{Z}}$ be the classifying map of the branch locus of $\tilde{f}$. The map $\psi$ induces an isomorphism between the strict completions of the local rings at $s$ and $\psi(s)$, because both these rings are equal to $\hat{R}$ (here we use the assumption $k = k_0^{\mathrm{sep}}$!). Since $\psi$ is of finite type, it follows that $\psi$ is étale in a neighborhood of $s$ (see e.g. [**Wew99**], Proposition 5.2.3.(v) and [**Mil80**], Proposition I.3.8). Therefore, after shrinking $S$ we may assume that $\psi : S \to \mathcal{U}_{r,\mathbb{Z}}$ is étale everywhere. Then $(\tilde{f}, s, \lambda)$ is a versal algebraic deformation of $f$. This proves (i), under the assumption $k = k_0^{\mathrm{sep}}$.

We now give the proof in the general case. As above, we note that there exists a subring $R \subset k$ which is of finite type over $\mathbb{Z}$ such that $f$ descends to a tame $G$-cover $f_R$ over $\operatorname{Spec} R$. Since $R$ is of finite type over $\mathbb{Z}$, there exists a geometric point $t : \operatorname{Spec} \ell \to \operatorname{Spec} R$, where $\ell$ is the algebraic closure of its prime field. Write $f_t$ for the pullback of $f_R$ via $t$, and let $\ell_0 \subset \ell$ denote the branch locus field of $f_t$. Then $\ell = \ell_0^{\mathrm{sep}}$, and hence we may apply to $f_t$ the case of the proposition which we have already proved. Let $(\tilde{f}, s, \lambda)$ over $S$ be the resulting versal algebraic deformation of $f_t$. Let $(S', s')$ be an étale neighborhood of the pointed scheme $(S \times_{\mathcal{U}_{r,\mathbb{Z}}} \operatorname{Spec} R, (s,t))$ (the map $S \to \mathcal{U}_{r,\mathbb{Z}}$ used in the definition of the fiber product is the classifying map of the branch locus of $\tilde{f}$). We obtain a commutative diagram of pointed schemes

$$\begin{array}{ccc} (S', s') & \longrightarrow & (\operatorname{Spec} R, t) \\ \downarrow & & \downarrow \\ (S, s) & \longrightarrow & (\mathcal{U}_{r,\mathbb{Z}}, t_0) \end{array}$$

in which the horizontal arrows are étale. Consider the two tame $G$-covers $\tilde{f} \times_S S'$ and $f_R \times_{SpecR} S'$. By the commutativity of the above diagram, they have the same branch locus. Moreover, their fibers at the point $s'$ are both isomorphic to $f_t$. Therefore, by Lemma 4.6 (ii) we may assume, after shrinking the neighborhood $(S', s')$, that the isomorphism of the fibers extends to an isomorphism

$$\tilde{f} \times_S S' \cong f_R \times_{SpecR} S' \tag{4}$$

over $S'$. The map $S' \to \operatorname{Spec} R$ being étale, we can lift the tautological geometric point $\operatorname{Spec} k \to \operatorname{Spec} R$ to a geometric point $s'' : \operatorname{Spec} k \to S'$. Specializing the isomorphism (4) at $s''$, we obtain an isomorphism

$$f = (f_R \times S')_{s''} \cong (\tilde{f} \times S')_{s''} = \tilde{f}_{s'''},$$

where $s''' : \operatorname{Spec} k \to S$ is the composition of $s''$ with $S' \to S$. We may therefore regard $\tilde{f}$ as a versal algebraic deformation of $f$. This completes the proof of Proposition 4.10. □

**4.4. Proof of Theorem 4.3.** — We will now derive Theorem 4.3 from Proposition 4.10. Let $(\tilde{f}, s, \lambda)$ be a versal algebraic deformation of $f$. Let $k'$ be the residue field of the image of $s : \operatorname{Spec} k \to S$. By definition, $k'$ is a field of definition for the $G$-cover $f$. Let $\psi : S \to \mathcal{U}_{r,\mathbb{Z}}$ be the (étale) classifying map of the branch locus of $f_S$. The image of the composition of $s : \operatorname{Spec} k \to S$ with $\psi$ is a point on $\mathcal{U}_{r,\mathbb{Z}}$ whose residue field is, by definition, the branch locus field $k_0$ of $f$. We obtain the following commutative diagram:

$$\begin{array}{ccc} \operatorname{Spec} k' & \longrightarrow & S \\ \downarrow & & \downarrow \psi \\ \operatorname{Spec} k_0 & \longrightarrow & \mathcal{U}_{r,\mathbb{Z}}. \end{array}$$

The horizontal arrows and the right vertical arrow are unramified morphisms. It follows that the left vertical arrow is also unramified. This means that $k'/k_0$ is a finite separable extension (see [**Har77**], Exercise III.10.3). Theorem 4.3 follows. □

**4.5. The algebraic Hurwitz space $\mathcal{H}_{r,G,\mathbb{Z}}$.** — Here is the main result of this section.

***Theorem 4.11***. — *There exists a scheme $\mathcal{H} = \mathcal{H}_{r,G,\mathbb{Z}}$ of finite type over $\mathbb{Z}$, an étale morphism $\pi : \mathcal{H} \to \mathcal{U}_{r,\mathbb{Z}}$ and, for each algebraically closed field $k$, a bijection*

$$f \longmapsto [f]_k$$

*between (a) the set of isomorphism classes of tame $G$-covers $f : X \to \mathbb{P}^1_k$ defined over $k$ with $r$ branch points and (b) the set of $k$-rational points $\operatorname{Spec} k \to \mathcal{H}$. Furthermore, the following holds.*

(i) *Let $f$ be a tame $G$-cover $f$ with $r$ branch points, defined over an algebraically closed field $k$. Let $\sigma : k \xrightarrow{\sim} k$ be a field isomorphism. Then we have*

$$[f^\sigma]_k = [f]_k^\sigma.$$

*Therefore, the field of moduli of $f$ is equal to the residue field of the point $[f]_k$.*

(ii) *The point $\pi([f]_k) : \operatorname{Spec} k \to \mathcal{U}_{r,\mathbb{Z}}$ is the classifying map of the branch locus $D \subset \mathbb{P}^1_k$ of $f$.*

(iii) *For $k = \mathbb{C}$, the bijection $f \mapsto [f]_k$ gives rise to an isomorphism of complex manifolds $\mathcal{H}_{r,G,\mathbb{C}} \cong \mathcal{H} \otimes_{\mathbb{Z}} \mathbb{C}$.*

This theorem is a special case of Theorem 2.1. Combining Theorem 4.11 with Proposition 4.4, we obtain:

***Corollary 4.12***. — *If the group $G$ has trivial center, then for every field $k$ the set of $k$-rational points on $\mathcal{H}_{r,G,\mathbb{Z}}$ is in natural bijection with the set of isomorphism classes of $G$-covers with $r$ branch points defined over $k$.*

*Proof*. — (of Theorem 4.11) In the first step of the proof we define a scheme $\mathcal{H}'$ which is, in some sense, a good candidate for the Hurwitz scheme $\mathcal{H}$. Let $K_0$ denote the function field of $\mathcal{U}_{r,\mathbb{Z}}$. We identify $K_0$ with the rational function field $\mathbb{Q}(\tilde{c}_1, \dots, \tilde{c}_r)$ in such a way that the generic point $\operatorname{Spec} K_0 \to \mathcal{U}_{r,\mathbb{Z}}$ corresponds to the 'generic' divisor $D_{\mathrm{gen}} \subset \mathbb{P}^1_{K_0}$ with equation $t^r + \tilde{c}_1 t^{r-1} + \cdots + \tilde{c}_r = 0$. Choose an embedding $K_0 \hookrightarrow \mathbb{C}$ and let $(f_\mu : X_\mu \to \mathbb{P}^1_{\mathbb{C}})_{\mu \in M}$ be a system of representatives of the orbits of the action of $\operatorname{Aut}(\mathbb{C}/K_0)$ on the set of isomorphism classes of $G$-covers over $\mathbb{C}$ with branch locus $D_{\mathrm{gen}} \subset \mathbb{P}^1_{\mathbb{C}}$. It follows from the Riemann's Existence Theorem (Theorem §3.1) that $M$ is a finite set. For each $\mu \in M$, we let $K_\mu$ denote the field of moduli of the $G$-cover $f_\mu$. By Theorem 4.3, $K_\mu$ is a finite extension of $K_0$. We let $\mathcal{H}'_\mu$ denote the normalization of $\mathcal{U}_{r,\mathbb{Z}}$ in $K_\mu$. Recall that, by definition, $\mathcal{H}'_\mu$ is a normal connected scheme with function field $K_\mu$ and that there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K_\mu & \longrightarrow & \mathcal{H}'_\mu \\ \downarrow & & \downarrow{\scriptstyle \pi'_\mu} \\ \operatorname{Spec} K_0 & \longrightarrow & \mathcal{U}_{r,\mathbb{Z}}, \end{array}$$

where $\pi'_\mu$ is a finite morphism of schemes. We define the scheme

$$\mathcal{H}' := \coprod_{\mu \in M} \mathcal{H}'_\mu$$

as the disjoint union of the $\mathcal{H}'_\mu$ and denote by $\pi' : \mathcal{H}' \to \mathcal{U}_{r,\mathbb{Z}}$ the map whose restriction to $\mathcal{H}'_\mu$ is equal to $\pi'_\mu$.

Let $\tilde{f} : \tilde{X} \to \mathbb{P}^1_S$ be a versal family of $G$-covers, defined over a scheme $S$. By definition, the classifying map $\psi : S \to \mathcal{U}_{r,\mathbb{Z}}$ of the branch locus of $\tilde{f}$ is étale. Since $\mathcal{U}_{r,\mathbb{Z}}$ is regular and in particular normal, the same holds for $S$ (see [**G**$^+$**71**], Corollaire I.9.2). Let $K$ be the function field of $S$. The map $\psi : S \to \mathcal{U}_{r,\mathbb{Z}}$ induces an embedding $K_0 \hookrightarrow K$. Choose an embedding $K \hookrightarrow \mathbb{C}$ which extends the embedding $K_0 \hookrightarrow \mathbb{C}$ fixed at the beginning of this subsection. The pullback $\tilde{f} \times_S \operatorname{Spec} \mathbb{C}$ is a $G$-cover over $\mathbb{C}$ with branch locus $D_{\mathrm{gen}} \subset \mathbb{P}^1_{\mathbb{C}}$. We may therefore choose the embedding $K \subset \mathbb{C}$ in such a way that $\tilde{f} \times_S \operatorname{Spec} \mathbb{C}$ is isomorphic to the $G$-cover $f_\mu$, for some $\mu \in M$. It follows that $K \subset \mathbb{C}$ is a field of definition for $f_\mu$. Therefore, $K$ contains the field $K_\mu$, the field of moduli of $f_\mu$. Recall that the scheme $\mathcal{H}'_\mu \subset \mathcal{H}'$ was defined as the normalization of $\mathcal{U}_{r,\mathbb{Z}}$ in the field extension $K_\mu/K_0$. Using the normality of $S$, the universal property of normalization (see e.g. [**Har77**], Exercise II.3.8) implies that $\psi : S \to \mathcal{U}_{r,\mathbb{Z}}$ factors through the map $\pi'_\mu : \mathcal{H}'_\mu \to \mathcal{U}_{r,\mathbb{Z}}$, resulting in a dominant map $\varphi : S \to \mathcal{H}'_\mu$ which induces the inclusion $K_\mu \subset K$. We consider $\varphi$ as a map $S \to \mathcal{H}'$ and call it the *chart* associated to the versal family $\tilde{f}$. (The next proposition shows that $\varphi$ does not depend on the choice of the embedding $K \hookrightarrow \mathbb{C}$.)

Let $k$ be an algebraically closed field and $f : X \to \mathbb{P}^1_k$ a tame $G$-cover with $r$ branch points over $k$. Let $(\tilde{f}, s, \lambda)$ be a versal algebraic deformation of $f$ (which exists by Proposition 4.10 (i)) and $\varphi : S \to \mathcal{H}'$ the associated chart. We define the $k$-rational point $[f]_k : \operatorname{Spec} k \to \mathcal{H}'$ as the composition of the point $s$ with $\varphi$.

***Proposition 4.13***. — (i) *The chart $\varphi : S \to \mathcal{H}'$ is étale.*

(ii) *Let $(\tilde{f}_i, s_i, \lambda_i)$, $i = 1, 2$, be two versal algebraic deformations of $f$ and $\varphi_i : S_i \to \mathcal{H}'$ the associated charts. Let $\tilde{h} : \tilde{f}_1 \to \tilde{f}_2$ be a morphism of algebraic deformations and $h : S_1 \to S_2$ the underlying morphism of the base schemes. Then $\varphi_2 \circ h = \varphi_1$. In particular, the map $\varphi_i$ is independent of the choice of the embedding of the function field of $S_i$ into $\mathbb{C}$.*

(iii) *The point $[f]_k$ defined above depends only on the isomorphism class of the $G$-cover $f$. Moreover, for all $\sigma \in \operatorname{Aut}(k)$ we have $[f]_k^\sigma = [f^\sigma]_k$.*

(iv) *If $f'$ is another tame $G$-cover over $k$ with $r$ branch points and $[f] = [f']$, then $f \cong f'$.*

(v) *The image of the map $f \mapsto [f]_k$ is an open subscheme of $\mathcal{H}'$.*

*Proof*. — Statement (i) follows from the fact that $\psi = \pi' \circ \varphi$ is étale and from Lemma 4.14 below. Let $\tilde{h} : \tilde{f}_1 \to \tilde{f}_2$, $h : S_1 \to S_2$ and $\varphi_i$ be as in (ii). Let $\psi_i : S_i \to \mathcal{U}_{r,\mathbb{Z}}$

be the classifying map of the branch locus of $\tilde{f}_i$. The existence of the isomorphism $\tilde{h}$ shows that $\psi_2 \circ h = \psi_1$. Let $K_i$ denote the function field of $S_i$. Recall that the definition of the chart $\varphi_i$ depended on the choice of an embedding $K_i \hookrightarrow \mathbb{C}$. This embedding extends the fixed embedding $K_0 \hookrightarrow \mathbb{C}$ and was chosen such that $\tilde{f}_i \otimes_{S_i} \mathbb{C}$ is isomorphic to the cover $f_{\mu_i}$, for a unique $\mu_i \in M$. In particular, $K_{\mu_i}$, the field of moduli of $f_{\mu_i}$, is contained in $K_i$. The map $\varphi_i : S_i \to \mathcal{H}'_{\mu_i}$ is characterized by the two properties that $\pi'_{\mu_i} \circ \varphi_i = \psi_i$ and that it induces the inclusion $K_{\mu_i} \subset K_i$ on the function fields. Let $\sigma : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ be an automorphism of $\mathbb{C}$ whose restriction to $K_2$ is equal to the inclusion $K_2 \hookrightarrow K_1$ induced by the map $h$. Then the morphism $\tilde{h} : \tilde{f}_1 \to \tilde{f}_2$ induces an isomorphism

$$\tilde{f}_1 \otimes_{S_1} \mathbb{C} \xrightarrow{\sim} (\tilde{f}_2 \otimes_{S_2} \mathbb{C})^{\sigma}.$$

It follows now from the definition of the field of moduli that the restriction of $\sigma$ to $K_{\mu_2}$ is the identity $K_{\mu_2} = K_{\mu_1}$. Together with the equality $\psi_2 \circ h = \psi_1$, this implies the desired equality $\varphi_2 \circ h = \varphi_1$ and proves (ii). Statement (iii) follows easily from (ii) and Proposition 4.10 (ii).

To prove (iv), let $f' : X' \to \mathbb{P}^1_k$ be another tame $G$-cover defined over $k$. Choose a versal algebraic deformation $(\tilde{f}', s', \lambda')$ of $f'$ and let $\varphi' : S' \to \mathcal{H}'$ be the associated chart. We assume that $\varphi(s) = \varphi'(s')$, and we have to show that $f \cong f'$. Let $S''$ be a connected normal scheme, $s'' : \operatorname{Spec} k \to S''$ a $k$-rational point and $h : S'' \to S$ and $h : S'' \to S'$ quasi-finite and dominant maps such that $\varphi \circ h = \varphi' \circ h'$, $h(s'') = s$ and $h'(s'') = s'$ (e.g. as in the proof of Proposition 4.10 (ii)). By construction, the generic fibres of the $G$-covers $\tilde{f} \times_S S''$ and $\tilde{f}' \times_{S'} S''$ have the same field of moduli. Therefore, after replacing $S''$ by a finite cover, we may assume that the generic fibres are isomorphic. It follows now from Lemma 4.6 (iii) that $\tilde{f} \times_S S''$ and $\tilde{f}' \times_{S'} S''$ are isomorphic. Specializing this isomorphism at the point $s''$, we get an isomorphism $f \xrightarrow{\sim} f'$. This proves (iv). (The preceeding argument can be summarized by saying that, if two versal families over the same scheme $S$ have the same chart, then there exists a finite étale cover $S' \to S$ over which they become isomorphic. See also Remark 4.7.)

Let $s' : \operatorname{Spec} k \to S$ be any point; then $(\tilde{f}, s', \mathrm{Id})$ is a versal deformation of $f_{s'}$ with chart $\varphi$. Therefore, $[f_{s'}]_k = \varphi(s')$. We conclude that every point in the image of $\varphi$ is in the image of the map $f \mapsto [f]_k$. But $\varphi$ is an open morphism because it is étale by (i). Statement (v) follows. This finishes the proof of the proposition. □

We define $\mathcal{H} := \mathcal{H}_{r,G,\mathbb{Z}}$ as the union of the images of all charts $\varphi : S \to \mathcal{H}'$. By Proposition 4.13, this is an open subscheme of $\mathcal{H}'$, and the map $f \mapsto [f]_k$ is a bijection between $\mathcal{H}_{r,G}(k)$ and the set of $k$-rational points of $\mathcal{H}$. So Part (i) of Theorem 4.11 is proved. Part (ii) is obvious. Set $\pi := \pi'|_{\mathcal{H}} : \mathcal{H} \to \mathcal{U}_{r,\mathbb{Z}}$. The following lemma, applied to the maps $\varphi : S \to \mathcal{H}$ and $\pi : \mathcal{H} \to \mathcal{U}_{r,\mathbb{Z}}$, shows that $\pi$ and all charts $\varphi$ are étale.

***Lemma 4.14***. — *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Assume that the following holds.*

(i) *$Y$ and $Z$ are normal,*
(ii) *$f$ and $g$ are dominant, and*
(iii) *$g \circ f$ is étale.*

*Then $f$ and $g|_{f(X)}$ are étale.*

*Proof.* — The map $g \circ f$ is unramified, by (iii). Hence $f$ is unramified as well. Using (i), (ii) and [**Mil80**], Theorem I.3.20, we conclude that $f$ is étale.

Let $x \in X$ and set $y := f(x)$, $z := g(y)$. Since $g \circ f$ is étale, $X_z := X \times_Z k(z)$ is of the form $\coprod_{x'} \operatorname{Spec} k(x')$, where $x'$ runs over $(g \circ f)^{-1}(z)$ and $k(x')/k(z)$ are all finite separable field extensions. Moreover, since $f$ is étale the induced morphism $f_z : X_z \to Y_z := Y \times_Z k(z)$ is étale as well. The last two assertions together show that $Y_z \cap f(X)$ is of the form $\coprod_{y'} \operatorname{Spec} k(y')$, where $y'$ runs over $g^{-1}(z) \cap f(X)$ and $k(y')/k(z)$ are all finite separable field extensions. Hence $g|_{f(X)}$ is unramified. Using again (i), (ii) and [**Mil80**], Theorem I.3.20, we conclude that $g|_{f(X)}$ is étale. □

It remains to prove Part (iii) of Theorem 4.11. Part (i) of Theorem 4.11 in the case $k = \mathbb{C}$ and the Riemann Existence Theorem show that we have a bijection

$$t : \mathcal{H}(\mathbb{C}) \xrightarrow{\sim} \mathcal{H}_{r,G,\mathbb{C}}.$$

We only have to show that $t$ is an analytic map. Let $\tilde{f} : \tilde{X} \to \mathbb{P}^1_S$ be a versal family of $G$-covers and $\varphi : S \to \mathcal{H}$ the associated chart. Since $\varphi$ is étale, the induced map $\varphi^{\mathrm{an}} : S(\mathbb{C}) \to \mathcal{H}(\mathbb{C})$ of analytic spaces is a local isomorphism (see [**G$^+$71**], Proposition XII.3.1 and Remarque XII.3.3). Therefore, it suffices to show that for all charts $\varphi$ the composition $t \circ \varphi^{\mathrm{an}}$ is analytic. But $t \circ \varphi^{\mathrm{an}}$ is obviously the classifying map of the analytification of the $G$-cover $\tilde{f}$ and is analytic by Proposition 3.4. This concludes the proof of Theorem 4.11. □

***Remark 4.15***. — (i) Part (iii) of Theorem 4.11 shows that the morphism $\pi \otimes \mathbb{Q} : \mathcal{H}_{r,G,\mathbb{Z}} \otimes \mathbb{Q} \to \mathcal{U}_{r,\mathbb{Z}} \otimes \mathbb{Q}$ is finite and hence $\mathcal{H}_{r,G,\mathbb{Z}} \otimes \mathbb{Q} = \mathcal{H}' \otimes \mathbb{Q}$.

(ii) Let $n$ be the order of $G$. Using more or less standard arguments (good reduction and the valuation criterion of properness) one can conclude from (i) that $\pi \otimes \mathbb{Z}[1/n] : \mathcal{H}_{r,G,\mathbb{Z}} \otimes \mathbb{Z}[1/n] \to \mathcal{U}_{r,\mathbb{Z}} \otimes \mathbb{Z}[1/n]$ is still finite. Furthermore, $\mathcal{H}_{r,G,\mathbb{Z}}$ has good reduction at all primes $p$ not dividing $n$. In particular, the connected components of $\mathcal{H}_{r,G,\mathbb{Z}} \otimes \bar{\mathbb{F}}_p$ are in natural bijection with the connected components of $\mathcal{H}_{r,G,\mathbb{C}}$. See [**Ful69**] and [**Wew98**].

(iii) For primes $p$ dividing $n$, the reduction $\mathcal{H}_{r,G,\mathbb{Z}} \otimes \mathbb{F}_p$ is not very well understood. However, a few cases have been successfully studied, see e.g. [**BW04**] and [**Rom**].

## 5. Admissible covers

In the tame case, it is possible to compactify the Hurwitz schemes by adjoining coverings of stable curves of a particularly nice type. The resulting projective schemes are helpful in a number of questions : for example, it is often easier to find rational points on the boundary. More applications will be given in the other lectures of the conference.

We construct here a compactification denoted $\overline{\mathcal{H}}^{\mathrm{in}}_{r,G}$, whose objects are called *admissible G-covers*. The branch locus map $\Psi\colon \mathcal{H}^{\mathrm{in}}_{r,G} \to \mathcal{U}_r$ extends to a map on this moduli space, with values in the moduli space of genus 0 stable marked curves of Knudsen-Mumford. We denote the latter by $\overline{\mathcal{U}}_r$ (but be careful : this is not exactly the same one as in [**Wew98**]).

The moduli space $\overline{\mathcal{H}}^{\mathrm{in}}_{r,G}$ being *a posteriori* normal, we could define it like in Section 4, using a normalization of $\overline{\mathcal{U}}_r$ in some field extension. However, the map $\Psi$ is not étale any more on the compactification, so we get into trouble when we want to prove that the variety we obtain is the desired moduli space. Instead we will present two other constructions of the moduli space : either by Geometric Invariant Theory, or by the use of theorems of representability for tamely ramified stacks.

### 5.1. Definitions.

— For a double point on a curve, the tangent space at the point splits canonically into two one-dimensional subspaces which we call the *branches*.

***Definition 5.1***. — Let $G$ be a finite group. Let $k$ be an algebraically closed field of characteristic zero. An *admissible G-cover* of a curve of genus 0 is a pair $(f, \alpha)$ where

(i) $f : Y \to X$ is a cover of stable curves with $X$ of genus 0.

(ii) $\alpha : G \hookrightarrow \mathrm{Aut}_X(Y)$ is an injective group homomorphism such that for any node $y \in Y$, for any $g \in G$ with $g(y) = y$, $\alpha(g)$ respects the branches at $y$ and acts on them by characters that are inverse to each other.

(iii) $f$ factors through an isomorphism $Y/G \simeq X$.

The branch divisor is defined as the image by $f$ of the (ramification) divisor defined by the sheaf of ideals $\mathcal{I} := f^*\omega_X \otimes \omega_Y^{-1}$, where $\omega_?$ denotes dualizing sheaves. By condition (ii) the support of the branch divisor of an admissible $G$-cover is included in the smooth locus of $X$. In the case of smooth curves, this is the same divisor as in section 1. Also as before, if the degree of the branch divisor is constant and equal to $r$ we say that *$f$ has $r$ branch points*.

The definition of an isomorphism of admissible $G$-covers is the same as in the smooth case.

The moduli space for admissible $G$-covers with $r$ branch points will be called $\overline{\mathcal{H}}^{\mathrm{in}}_{r,G}$.

### 5.2. The Geometric Invariant Theory construction.

— It is possible to adapt the construction of $\overline{\mathcal{M}}_g$ as a GIT quotient by Gieseker and Mumford [**Gie82**], to the case of Hurwitz moduli spaces. This was worked out by Bertin, and we now sketch the steps of his proof, in the case of $\overline{\mathcal{H}}^{\mathrm{in}}_{r,G}$. We keep the data $g, r, G$ as above. For clarity, for each of the three main steps below we indicate the result which is the analog in the construction of $\overline{\mathcal{M}}_g$ in [**Gie82**].

We recall the notion of the *Hilbert points* of a curve $X \subset \mathbb{P}^N$ in projective space over a field $k$. Let $L$ denote the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to $X$. From the theorems of Serre it follows that there is an $\nu_0$ such that : for all $\nu \geq \nu_0$ the Euler characteristic $\chi(\nu) = \sum_i (-1)^i h^i(X, L^\nu)$ is just equal to $h^0(X, L^\nu)$, and we have a surjection

$$H^0(\mathbb{P}^N, \mathcal{O}(\nu)) \to H^0(X, L^\nu) \to 0$$

Taking exterior powers yields another surjection

$$\overset{\chi(\nu)}{\wedge} H^0(\mathbb{P}^N, \mathcal{O}(\nu)) \to \overset{\chi(\nu)}{\wedge} H^0(X, L^\nu) \simeq k$$

This defines the *$\nu$-th Hilbert point of $X$* as a point in the projective space $\mathbb{P}(\wedge H^0(\mathbb{P}^N, \mathcal{O}(\nu)))$. Observe that $\mathrm{PGL}_{N+1}$ acts naturally on $\mathbb{P}(\wedge H^0(\mathbb{P}^N, \mathcal{O}(\nu)))$, so there is a notion of GIT-stability for this action.

Now fix $m \geq 10$, $N = (2m-1)(g-1) - 1$ and the Hilbert polynomial $P(x) = (2mx-1)(g-1)$. Let $H$ be the Hilbert scheme parameterizing closed subschemes of $\mathbb{P}^N$ with Hilbert polynomial $P$. Let $Z \to H$ be the universal curve. Finally, fix an injection $G \subset \mathrm{PGL}_{N+1}$ (this is not a restriction, since we are interested in $G$-curves and their canonical embeddings).

***Theorem 5.2***. — *There exists $\nu$ such that for any fiber of $Z \to H$ which is smooth and nondegenerate (i.e. included in no hyperplane), the $\nu$-th Hilbert point of $Z_h$ is GIT-stable for the action of* $\mathrm{PGL}_{N+1}$.

This result is essentially [**Gie82**], th. 1.0.0, p. 26. In the sequel $\nu$ is fixed like in the theorem. Let $H^{\mathrm{ss}}$ be the open subset of $H$ of points $h$ such that the $\nu$-th Hilbert point of $Z_h$ is GIT-semi-stable. Now define $U \subset H^{\mathrm{ss}}$ by the two conditions "$Z_h$ is connected" and "$Z_h \subset \mathbb{P}^N$ is the $m$-canonical embedding".

***Theorem 5.3***. — *$U$ is closed in $H^{\mathrm{ss}}$. Furthermore, the fibres of $Z|_U \to U$ are stable curves in the sense of Deligne-Mumford.*

This is the conjunction of [**Gie82**], th. 1.0.1, p. 35 and prop. 2.0.0, p. 88. The scheme $U$ is smooth (by deformation theory), hence so is the fixed point subscheme $T := U^G$. (This is a simple calculation ; it follows from the fact that the characteristics do not divide the order of $G$.) By construction, the fibres of $Z|_T \to T$ are endowed with an action of $G$. Among the irreducible components of $T$ for which the fiber $Z_t$ over the generic point is a smooth curve, let $T_0 \subset T$ be the component for which the genus of $Z_t/G$ is zero, and the number of branch points of $Z_t \to Z_t/G$ equal to $r$.

***Theorem 5.4***. — *$T_0//\mathrm{PGL}_{N+1}$ is a coarse moduli space for $\overline{\mathcal{H}}^{\mathrm{in}}_{r,G}$.*

This is the analog of [**Gie82**], th. 2.0.2, p. 93 in the equivariant case.

We can notice that some facts from deformation theory are needed in the course of the proof, e.g. in order to show that $U$ is smooth. This is also necessary at the end, if we want to prove that the points of $T_0//\mathrm{PGL}_{N+1}$ are exactly the $G$-covers as we defined them in 5.1. The necessary results from deformation theory are presented in the next subsection.

**5.3. The construction via algebraic stacks.** — This construction uses general arguments based on extensions of the theory of Grothendieck of representation for unramified functors. It involves algebraic stacks which are somehow more sophisticated objects. However, the reader who does not know about stacks should feel comfortable enough, in the sequel of this section, if he keeps in mind the following ideas. Roughly, a stack is a category, whose isomorphism classes we want to classify.

In our main case of interest the category is denoted by $\overline{\mathsf{H}}^{\mathrm{in}}_{r,G}$. It is determined by the categories of families over $S$ denoted by $\overline{\mathsf{H}}^{\mathrm{in}}_{r,G}(S)$, for varying $S$ :

- the objects in the category $\overline{\mathsf{H}}^{\mathrm{in}}_{r,G}(S)$ are the families of admissible covers with $r$ branch points over $S$,
- the morphisms between two given families are the $G$-equivariant isomorphisms of covers over $S$.

A particular case is when the stack is representable by a scheme (i.e. it is equal to its moduli space), which means essentially that the category considered is equivalent to a category with unique isomorphisms. Thus the presence of morphisms is the main difference between moduli spaces and moduli stacks. The utility of stacks is that very often, they are "close enough" to a scheme so that we can really do geometry in the same way as we usually do with schemes. For more details, the interested reader is advised to look at e.g. [**DM69**], §4 or the Appendix of [**Vis89**].

Let $\mathsf{S}$ be a smooth Deligne-Mumford stack with a normal crossings divisor $\mathsf{D}$. Let $\mathsf{M}$ be a stack over $\mathsf{S}$. Assume that $\mathsf{M}$ is a stack with respect to the fpqc topology, locally of finite presentation, and has finite diagonal. In the case where $\mathsf{S} = \overline{\mathsf{U}}_r$ with $\mathsf{D} = \overline{\mathsf{U}}_r - \mathsf{U}_r$, and $\mathsf{M} = \overline{\mathsf{H}}^{\mathrm{in}}_{r,G}$, these assumptions are verified by standard arguments. For example, the fact that the diagonal is finite is equivalent to the finiteness of the schemes of automorphisms of coverings, well-known by [**DM69**].

***Theorem 5.5***. — *Assume* $\mathsf{M} \to \mathsf{S}$ *is as above. If* $\mathsf{M}$ *is formally tamely ramified along* $\mathsf{D}$ *(and formally étale over* $\mathsf{S} - \mathsf{D}$*), then it is algebraic and tamely ramified along* $\mathsf{D}$.

The proof of this is theorem 1.3.3 and remark 2.1.3 of [**Wew98**]. Thus, for the application to $\mathsf{M} = \overline{\mathsf{H}}^{\mathrm{in}}_{r,G}$, we only have to check formal tame ramification. Showing this property is essentially a question of deformation theory.

Let $R$ be a Noetherian complete local ring with separably closed residue field $k$. Let $X$ be a stable curve over $R$, marked with a divisor $D \subset X$ of degree $r$ (we will be interested only in the case where $X$ has geometric genus 0). Let $X_0$ be the special fiber of $X$. For each double point $x_i \in X$ in the special fiber, we can choose local étale coordinates $u_i, v_i$ lying in the Henselian ring $\widetilde{\mathcal{O}}_{X,x_i}$ such that $t_i := u_i v_i \in R$.

Now let $f_0 \colon Y_0 \to X_0$ be an admissible $G$-cover, with branch locus $D$. Choose one point $y_i \in f_0^{-1}(x_i)$ above each $x_i$. By the definition (5.1(ii)), we can choose local étale coordinates $\bar{p}_i, \bar{q}_i$ at $y_i$ such that $\bar{\tau}_i := \bar{p}_i \bar{q}_i \in R$ and the map $\widetilde{\mathcal{O}}_{X_0,x_i} \to \widetilde{\mathcal{O}}_{Y_0,y_i}$ is given by $\bar{u}_i \mapsto \bar{p}_i^{n_i}$, $\bar{v}_i \mapsto \bar{q}_i^{n_i}$ with the stabilizer of $G$ at $y_i$ acting by inverse characters. In particular, we have $\bar{\tau}_i^{n_i} = \bar{t}_i$.

If a cover $f \colon Y \to X$ extends $f_0$ there is a unique choice of coordinates $p_i, q_i$ at each point $y_i$, satisfying $u_i \mapsto p_i^{n_i}$, $v_i \mapsto q_i^{n_i}$ and lifting $\bar{p}_i$, $\bar{q}_i$. Thus for $\tau_i := p_i q_i$ we have $\tau_i^{n_i} = t_i$.

Define $\mathrm{Def}(f_0)$ to be the set of isomorphism classes of lifts $f$ of $f_0$, and $T(\{t_i\})$ to be the set of families $\{\tau_i\}$ such that $\tau_i \in R$ and $\tau_i^{n_i} = t_i$. As we said above there is a map $\mathrm{Def}(f_0) \to T(\{t_i\})$.

***Theorem 5.6***. — *The natural map* $\mathrm{Def}(f_0) \to T(\{t_i\})$ *is a bijection. Moreover, if any two lifts of* $f_0$ *are isomorphic, then the isomorphism between them is unique.*

This result appears in several places, e.g. [**Wew99**] or [**Moc95**]. This shows that $\overline{\mathsf{H}}^{\mathrm{in}}_{r,G}$ is formally tamely ramified along the divisor of singular curves in $\overline{\mathsf{U}}_r$. Hence, by theorem 5.5, $\overline{\mathsf{H}}^{\mathrm{in}}_{r,G}$ is an algebraic stack. Its coarse moduli space ([**KM97**]) is $\overline{\mathcal{H}}^{\mathrm{in}}_{r,G}$.

## 6. Picard groups of Hurwitz stacks

We end these notes with a paragraph about the information on the covers which is *not* captured by the coarse moduli spaces. We chose to present Picard groups because some recent results give nice examples. The computations are similar in spirit to the computation of the Picard group of the moduli of elliptic curves (see Mumford [**Mum65**]). One can expect that the invariants obtained in this way reflect the difference between the cases of tame and wild ramification. In this respect the case $p = 2$ is however quite exceptional, since it is known to be the only case where the moduli space of the corresponding wild covers is smooth [**BM00**].

We recall that given a Deligne-Mumford algebraic stack $\mathcal{M}$, its Picard group is by definition the group of isomorphism classes of invertible sheaves on $\mathcal{M}$ (an invertible sheaf is given by invertible sheaves on every atlas of $\mathcal{M}$, together with compatible isomorphisms between the sheaves for different choices of atlases). The operation is induced by the tensor product of invertible sheaves. In the case where $\mathcal{M}$ is a quotient stack $[X/G]$, the Picard group is just the group of isomorphism classes of $G$-linearized invertible sheaves on $X$, denoted $\mathrm{Pic}_G(X)$.

**6.1. Hyperelliptic curves in characteristic $\neq 2, g+1$ after Arsie-Vistoli.** — Arsie and Vistoli describe stacks of cyclic covers of projective space $\mathbb{P}^n$ as quotient stacks. A particular case is the computation of the Picard group of the stack of genus $g$ hyperelliptic curves (as double covers of $\mathbb{P}^1$), in characteristics $\neq 2, g+1$.

To simplify the exposition we restrict the choice of the base scheme to a field with characteristic different from some "bad" primes. Except for this point, we keep the same generality as in [**AV04**]. So let $n, r, d$ be integers and $k$ a field of characteristic $p \geq 0$ such that $p \not| 2rd$. In this subsection all schemes are schemes over $k$. For our main concern, which is hyperelliptic curves, the first two definitions with their discussion can be better understood in example 6.3 below.

***Definition 6.1***. — Let $S$ be a scheme and $Y$ an $S$-scheme. A *(relative) uniform cyclic cover* of $Y$ is a morphism of $S$-schemes $f : X \to Y$ together with an action of $\mu_r$ on $X/S$, with the following local property. Any point $q \in Y$ has an open affine neighborhood $V = \mathrm{Spec}\,(R)$ such that $f^{-1}(V)$ is isomorphic to $\mathrm{Spec}\,(R[x]/x^r - h)$ with the obvious action of $\mu_r$, and the divisor of $Y$ defined by $h$ is a relative Cartier divisor (the branch divisor of $f$).

The last condition means that $h \in R$ is a nonzerodivisor and $R/h$ is flat over $S$. Any such covering is affine, hence is determined by the algebra structure on $f_*\mathcal{O}_X$. Due to the $\mu_r$ action this structure is particularly simple : there is an invertible sheaf $\mathcal{L} \in \mathrm{Pic}(Y)$ such that $f_*\mathcal{O}_X$ decomposes into eigenspaces

$$f_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{L} \oplus \mathcal{L}^2 \oplus \cdots \oplus \mathcal{L}^{r-1}$$

Furthermore there is an injective morphism $\phi : \mathcal{L}^r \to \mathcal{O}_Y$ and multiplication in $f_*\mathcal{O}_X$ is given by

$$\text{the product of } s \in \mathcal{L}^i \text{ with } t \in \mathcal{L}^j \text{ is } \begin{cases} s \otimes t \in \mathcal{L}^{i+j} & \text{if } i+j < r \\ (\phi \otimes \mathrm{id})(s \otimes t) \in \mathcal{L}^{i+j-r} & \text{if } i+j \geq r \end{cases}$$

It follows that, up to isomorphism, it is equivalent to consider a uniform cyclic cover or a triple $(Y, \mathcal{L}, \phi)$.

***Definition 6.2***. — Let $f : X \to Y$ be a uniform cyclic cover as above.

If $Y \to S$ is a Brauer-Severi scheme, i.e. it is isomorphic to $\mathbb{P}^n_S$ locally for the étale topology, we say that $f$ has *branch degree* $d$ if the sheaf $\mathcal{L}$ has degree $d$ on any fiber over $S$.

We denote by $\mathsf{H}(n, r, d)$ the stack of uniform cyclic covers of degree $r$ and branch degree $d$ of Brauer-Severi schemes of relative dimension $n$. The morphisms are $\mu_r$-equivariant isomorphisms. We denote by $\mathsf{H}_{\mathrm{sm}}(n, r, d)$ the substack of uniform cyclic covers such that $X$ is a smooth $S$-scheme.

***Example 6.3***. — Let $X$ be a smooth hyperelliptic curve over $S$, in the usual sense. There is given an involution $\tau : X \to X$, and the quotient $Y = X/\tau$ is a curve over $S$ whose geometric fibres are projective lines. The branch divisor of $\pi\colon X \to Y$ has degree equal to $g+1$ where $g$ is the genus of $X$. Thus $(X, \tau)$ belongs to $\mathsf{H}_{\mathrm{sm}}(1, 2, g+1)$.

When $Y = \mathbb{P}^n_S$, thanks to the $\mu_r$ action we can write down an equation for a uniform cyclic cover. The coefficients of the equation lie in an affine space. This is the key point of the result.

More precisely, let

$$\mathbb{A}(n, rd) := \quad \text{set of homogeneous forms of degree } rd \text{ in } n+1 \text{ variables}$$

$$\mathbb{A}_{\mathrm{sm}}(n, rd) := \quad \text{open set of smooth forms, i.e. with nonzero discriminant}$$

Thus $\mathbb{A}(n, rd)$ is just affine space of dimension $\binom{rd+n}{n}$. The natural action of the linear group $\mathrm{GL}_{n+1}$ on $\mathbb{A}(n, rd)$ by changes of variables, factors through $\mathrm{GL}_{n+1}/\mu_d$. (The $d$-th roots of unity are viewed as scalar diagonal matrices.) This action stabilizes $\mathbb{A}_{\mathrm{sm}}(n, rd)$.

***Lemma 6.4***. — *The discriminant of the generic form in $\mathbb{A}(n, rd)$ defines an irreducible hypersurface $\Delta$ of degree $(n+1)(rd-1)^n$. Moreover, $\mathbb{A}_{\mathrm{sm}}(n, rd)$ is the complement in $\mathbb{A}(n, rd)$ of $\Delta$.*

This is classical; see e.g. [**GKZ94**], chap. 1, 1.3 and 4.15.

***Theorem 6.5***. — *$\mathsf{H}(n, r, d)$ is isomorphic to the quotient stack*

$$[\mathbb{A}(n, rd)/(\mathrm{GL}_{n+1}/\mu_d)]$$

*The same holds with "sm" subscripts.*

*Proof*. — The proof is similar to the classical construction of $\mathsf{M}_g$ as the quotient $[\mathrm{Hilb}^P_{\mathbb{P}^{5g-6}}/\mathrm{PGL}_{5g-6}]$. Precisely, one introduces a stack whose objects are the objects of $\mathsf{H}(n, r, d)$ together with a rigidification $\psi : (Y, \mathcal{L}) \simeq (\mathbb{P}^n_S, \mathcal{O}(-d))$. This stack is isomorphic to $\mathbb{A}(n, rd)$ : intuitively the uniform cyclic covers in here are given by an equation $y^r = F(x_0, \ldots, x_n)$ where $F$ is a homogeneous form. Then one identifies the automorphism group of $(\mathbb{P}^n_S, \mathcal{O}(-d))$ (the model of the rigidification) with $\mathrm{GL}_{n+1}/\mu_d$. The result follows. □

***Theorem 6.6***. — *The Picard group of $\mathsf{H}_{\mathrm{sm}}(n, r, d)$ is finite cyclic of order*

$$r(rd-1)^n \gcd(n+1, d)$$

*Proof*. — From now on we set $G = \mathrm{GL}_{n+1}/\mu_d$ and we denote by $A$ the polynomial ring of functions of $\mathbb{A}(n, rd)$. By theorem 6.5 we have $\mathrm{Pic}(\mathsf{H}_{\mathrm{sm}}(n, r, d)) \simeq \mathrm{Pic}_G(\mathbb{A}_{\mathrm{sm}}(n, rd))$. We use the surjective restriction map $r : \mathrm{Pic}_G(\mathbb{A}(n, rd)) \to \mathrm{Pic}_G(\mathbb{A}_{\mathrm{sm}}(n, rd))$. When the non-equivariant Picard group is trivial, $\mathrm{Pic}_G$ is just the set of isomorphism classes of $G$-linearizations of the structure sheaf. Thus

$$\mathrm{Pic}_G(\mathbb{A}(n, rd)) \simeq H^1(G, A^\times) = H^1(G, k^\times) = \mathrm{Hom}(G, k^\times) = \widehat{G}$$

The epimorphism $\mathrm{GL}_{n+1} \to G$ induces an injection on character groups $\widehat{G} \hookrightarrow \widehat{\mathrm{GL}}_{n+1}$. Since $\widehat{\mathrm{GL}}_{n+1}$ is generated by the determinant, a suitable power generates $\widehat{G}$. One finds immediately that the generator is $\mathfrak{det}$ such that

$$\mathfrak{det}(\bar{g}) = (\det(g))^{d/\gcd(n+1,d)} \qquad \text{where } g \in \mathrm{GL}_{n+1} \text{ is a lift of } \bar{g} \in G$$

It remains to identify the kernel of $r$. Let $f = 0$ be an equation for $\Delta$. By definition, if a linearization of $\mathcal{O}_{\mathbb{A}(n,rd)} = \widetilde{A}$ induces the trivial linearization of $\mathcal{O}_{\mathbb{A}_{\mathrm{sm}}(n,rd)} = \widetilde{A[1/f]}$ then it is conjugated to the trivial linearization by an element $\lambda \in A[1/f]^{\times}$. As $f$ is irreducible (lemma 6.4) we have $A[1/f]^{\times} = \{af^n, a \in k^{\times}, n \in \mathbb{Z}\} \simeq k^{\times} \times \mathbb{Z}$ so it only remains to compute the "weight" of $f$. One checks that

$$\bar{g}.f = \left(\mathfrak{det}(\bar{g})\right)^{-r(rd-1)^n \gcd(n+1,d)} f$$

(it is enough to check this for $g$ a homothety; then use lemma 6.4). The result follows. □

The proof above is slightly different from the one in [**AV04**], although it essentially amounts to the same thing. Arsie and Vistoli use the equivariant cycle groups of [**EG98**]. In any case we recognize an equivariant analog of the classical exact sequence $\mathbb{Z} \to \mathrm{Pic}(X) \to \mathrm{Pic}(U) \to 0$ ([**Har77**], II, 6.5 and 6.16).

***Corollary 6.7***. — *The Picard group of the stack* $\mathsf{H}_{\mathrm{sm}}(1, 2, g+1)$ *(the stack of smooth hyperelliptic curves) over a field of characteristic prime to* $2(g+1)$*, is cyclic of order* $4g+2$ *if* $g$ *is even, and* $8g+4$ *if* $g$ *is odd.*

**6.2. Hyperelliptic curves in char 2 after Bertin.** — In the case where the field $k$ has characteristic 2, Bertin [**Ber06**] adapts the arguments in order to compute the Picard group of the stack of hyperelliptic curves. The main change is that $\mathbb{Z}/2\mathbb{Z}$ is not anymore isomorphic to the diagonalizable $\mu_2$. Therefore the description of $\mathbb{Z}/2\mathbb{Z}$-covers of arbitrary schemes, in terms of invertible sheaves, is a little more complicated. However, for covers of Brauer-Severi schemes, the situation is better locally on the base.

In this subsection we put $\mathsf{H}_g := \mathsf{H}_{\mathrm{sm}}(1, 2, g+1) \otimes \mathbb{F}_2$ (definition 6.2).

***Lemma 6.8***. — *Let* $g \geq 1$ *and* $m = g+1$. *Let* $X \to S$ *be a smooth hyperelliptic curve of genus* $g$ *with involution* $\tau$. *Let* $f\colon X \to Y$ *be the quotient by* $\tau$. *Then, étale locally on* $S$, $f$ *can be described by a triple* $(Y, \mathcal{L}, \phi)$ *where* $f_*\mathcal{O}_X \simeq \mathcal{O}_Y \oplus \mathcal{L}$ *and* $\phi\colon \mathcal{L}^2 \to \mathcal{O}_Y \oplus \mathcal{L}$.

*Proof*. — Let $\mathcal{L}$ be the cokernel of the natural injection $\mathcal{O}_Y \to f_*\mathcal{O}_X$. Using $\chi(\mathcal{L}) = \chi(f_*\mathcal{O}_X) - \chi(\mathcal{O}_Y)$ and the Riemann-Roch theorem, we have

$$\deg(\mathcal{L}) = \chi(\mathcal{L}) - 1 = -g - 1 = -m$$

Hence after an étale extension $S' \to S$ we can assume that $Y \simeq \mathbb{P}^1_S$ and $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1_S}(-m)$. Using Serre duality one shows $\mathrm{Ext}^1(\mathcal{O}(-m), \mathcal{O}) = 0$ so the exact sequence $0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathcal{L} \to 0$ splits. Thus $f_*\mathcal{O}_X \simeq \mathcal{O}_Y \oplus \mathcal{L}$. The multiplication in the sheaf of algebras is known once we know how to multiply sections of $\mathcal{L}$. This is given by a morphism $\phi\colon \mathcal{L}^2 \to \mathcal{O}_Y \oplus \mathcal{L}$. □

It is the same thing to consider $\phi$ as above, or two morphisms $\mathcal{L}^2 \to \mathcal{L}$ and $\mathcal{L}^2 \to \mathcal{O}_Y$. These correspond to global sections $A \in H^0(\mathcal{O}(m))$ and $B \in H^0(\mathcal{O}(2m))$. Therefore, the analog of the affine space $\mathbb{A}(n, rd)$ in 6.1 is affine space $\mathbb{A}^{3m+2} = \mathbb{A}^{m+1} \times \mathbb{A}^{2m+1}$. Intuitively, over the complement of $\infty$ in $\mathbb{P}^1$ the curve described by $A, B$ has equation

$$y^2 = A(x,1)y + B(x,1)$$

***Lemma 6.9***. — *The locus in $\mathbb{A}^{3m+2}$ of the pairs of forms $(A, B)$ such that the corresponding curve is smooth and separable over $\mathbb{P}^1$, is the complement of an irreducible hypersurface $\Delta$.*

We refer to [**Ber06**] for the proof. Note that it is more subtle in this case than in the odd characteristic case. Now, in order to prove the analog of theorem 6.5 one uses rigidifications $\psi : (Y, \mathcal{O}_Y \oplus \mathcal{L}) \simeq (\mathbb{P}^1_S, \mathcal{O} \oplus \mathcal{O}(-m))$. It can be computed that the automorphism group of the model is $(\mathbb{G}_a)^{m+1} \rtimes (\mathrm{GL}_2/\mu_m)$. Here, $(\mathbb{G}_a)^{m+1}$ is the group of global sections of $\mathcal{O}(m)$ and the action of $\mathrm{GL}_2/\mu_m$ is the natural action on homogeneous coordinates. The result of this is the following :

***Theorem 6.10***. — *We have an isomorphism of stacks $\mathsf{H}_g \simeq [\mathbb{A}^{3m+2} - \Delta/G]$ where $G = (\mathbb{G}_a)^{m+1} \rtimes (\mathrm{GL}_2/\mu_m)$.*

***Corollary 6.11***. — *The Picard group of $\mathsf{H}_g$ is finite cyclic of order $4g+2$ if $g$ is even, and $8g+4$ is $g$ is odd.*

To prove this, one proceeds like in the proof of 6.6, using an exact sequence $\mathbb{Z} \to \mathrm{Pic}_G(\mathbb{A}^{3m+2}) \to \mathrm{Pic}_G(\mathbb{A}^{3m+2} - \Delta) \to 0$. The character group of $G$ is again generated by a power of the determinant because the unipotent part $(\mathbb{G}_a)^{m+1}$ has no characters.

It is remarkable that the result is the same, whether we are in odd characteristic or in characteristic two.

**6.3. An example with $p > 2$.** — In this subsection, we content ourselves with giving the Picard group of the stack of a family of covers of degree $p$ of $\mathbb{P}^1$, the so-called *Potts curves.* This shows that the coincidence of the Picard groups for hyperelliptic curves in all characteristics is an exceptional phenomenon, not to be expected in general.

Fix a prime $p > 2$ and a field $k$ of characteristic different from 2. By definition, a Potts curve is a hyperelliptic curve of genus $p-1$ which is a Galois covering of degree $p$ of $\mathbb{P}^1$. The stack of Potts curves is a 1-dimensional algebraic stack over $k$, denoted $\mathsf{P}$. Its Picard group is as follows.

***Theorem 6.12***. — (i) *Assume* $\mathrm{char}(k) \neq p$. *Then $\mathsf{P}$ has $p-1$ smooth connected components which are all isomorphic. The Picard group of any of them is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2p\mathbb{Z}$.*

(ii) *Assume* char$(k) = p$. *Then* P *is irreducible and nonreduced. Let* $U$ *be the subset of* $k[\,z \mid z^{(p-1)/2}\,][X, 1/X]$ *consisting of elements that map to* 1 *under the specialization* $z = 0$. *This is a multiplicative subgroup of the group of units of* $k[\,z \mid z^{(p-1)/2}\,][X, 1/X]$. *Then the Picard group of* P *is isomorphic to* $\mathbb{Z}/2\mathbb{Z} \times U$, *in particular it is infinite.*

Here, the strategy of the proof is to relate invertible sheaves on the stack with invertible sheaves on the coarse moduli space. This leads to the (nontrivial) result even when the moduli space has trivial Picard group. This method and the one used for hyperelliptic curves are somehow different, since in [**AV04**], [**Ber06**] one compares sheaves on the stack with sheaves on an atlas, which "covers" the stack, and in [**Rom**] one compares sheaves on the stack with sheaves on the moduli space, which "is covered" by the stack.

The details of the proof of theorem 6.12 can be found in [**Rom**].

## References


[AV04] A. Arsie & A. Vistoli – Stacks of cyclic covers of projective spaces, *Compos. Math.* **140** (2004), no. 3, p. 647–666.

[Ber06] J. Bertin – Le champ des courbes hyperelliptiques en caractéristique deux, *Bull. Sci. Math.* **130** (2006), no. 5, p. 403–427.

[BM00] J. Bertin & A. Mézard – Déformations formelles des revêtements sauvagement ramifiés de courbes algébriques, *Invent. Math.* **141** (2000), p. 195–238.

[BW04] I. Bouw & S. Wewers – Reduction of covers and Hurwitz spaces, *J. Reine Angew. Math.* **574** (2004), p. 1–49.

[Cle72] A. Clebsch – Zur Theorie der Riemann'schen Flächen, *Math. Ann.* **6** (1872), p. 216–230.

[DD97] P. Dèbes & J.-C. Douai – Algebraic covers: Field of moduli versus field of definition, *Ann. Sci. École Norm. Sup.* **30** (1997), p. 303–338.

[DM69] P. Deligne & D. Mumford – The irreducibility of the space of curves of given genus, *Publ. Math. IHES* **36** (1969), p. 75–109.

[EG98] D. Edidin & W. Graham – Equivariant intersection theory (With an appendix by Angelo Vistoli: The Chow ring of $\mathcal{M}_2$), *Invent. Math.* **131, No.3** (1998), p. 595–644.

[Fri77] M. Fried – Fields of definition of function fields and Hurwitz families — groups as Galois groups, *Comm. Alg.* **5** (1977), p. 17–82.

[Ful69] W. Fulton – Hurwitz schemes and the irreducibility of the moduli of algebraic curves, *Ann. Math.* **90** (1969), p. 542–575.

[FV91] M. Fried & H. Völklein – The inverse Galois problem and rational points on moduli spaces, *Math. Ann.* **290** (1991), p. 771–800.

[G$^+$71] A. Grothendieck et al. – *Revêtement étale et groupe fondemental*, Lecture Notes in Math., no. 224, Springer-Verlag, 1971.

[Gie82] D. Gieseker – *Lectures on moduli of curves*, Tata Institute of Fundamental Research, Bombay. Springer-Verlag, 1982.

[GKZ94] I. Gel'fand, M. Kapranov & A. Zelevinsky – *Discriminants, resultants, and multidimensional determinants*, Birkhäuser, 1994.

[Har77] R. Hartshorne – *Algebraic geometry*, GTM, no. 52, Springer-Verlag, 1977.

[HM82] J. Harris & D. Mumford – On the Kodaira dimension of the moduli space of curves, *Invent. Math.* **67** (1982), p. 23–86.

[Hur91] A. Hurwitz – Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.* **39** (1891), p. 1–61.

[KÖ4] B. Köck – Belyi's theorem revisited, *Beiträge Algebra Geom.* **45** (2004), no. 1, p. 253–265.

[KM97] S. Keel & S. Mori – Quotients by groupoids, *Ann. Math. (2)* **145, No.1** (1997), p. 193–213.

[Mat91] B. Matzat – Zöpfe und Galoissche Gruppen, *J. Reine Angew. Math.* **420** (1991), p. 99–159.

[Mil80] J. Milne – *Etale cohomology*, Princeton University Press, 1980.

[MM99] G. Malle & B. H. Matzat – *Inverse Galois theory*, Monographs in Mathematics, Springer, 1999.

[Moc95] S. Mochizuki – The geometry of the compactification of the Hurwitz scheme, *Publ. RIMS* **31, no. 3** (1995), p. 355–441.

[Mum65] D. Mumford – Picard groups of moduli problems, in *Arithmetical Algebraic Geom., Proc. Conf. Purdue Univ. 1963*, Harper & Row, New York, 1965, p. 33–81.

[Rom] M. Romagny – The stack of Potts curves and its fibre at a prime of wild ramification, To appear in J. Algebra.

[Rom02] ———, Sur quelques aspects des champs de revêtements de courbes algébriques, Thèse de l'Université Joseph Fourier, 2002.

[Rom05] ———, Group actions on stacks and applications, *Michigan Math. J.* **53** (2005), no. 1, p. 209–236.

[Sev21] F. Severi – *Vorlesungen über algebraische Geometrie*, Teubner-Verlag, 1921.

[Spa66] E. Spanier – *Algebraic topology*, Springer-Verlag, 1966.

[Vis89] A. Vistoli – Intersection theory on algebraic stacks and on their moduli spaces, *Invent. Math.* **97** (1989), p. 613–670.

[Völ96] H. Völklein – *Groups as Galois groups*, Cambridge Studies in Adv. Math., no. 53, Cambridge Univ. Press, 1996.

[Wew98] S. Wewers – Construction of Hurwitz spaces, Ph.D. Thesis, Essen, 1998, available at: http://www.math.uni-bonn.de/people/wewers.

[Wew99] ———, Deformation of tame admissible covers of curves, in *Aspects of Galois Theory* (H. Völklein, ed.), London Math. Soc. Lecture Note Series, no. 256, Cambridge Univ. Press, 1999, p. 239–282.

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