Toeplitz Operators, Kähler Manifolds, and Line Bundles*

Tatyana FOTH

Department of Mathematics, University of Western Ontario, London, Ontario N6A 5B7, Canada
E-mail: *tfoth@uwo.ca*
URL: http://www.math.uwo.ca/~tfoth/

Received August 23, 2007, in final form October 23, 2007; Published online October 26, 2007
Original article is available at http://www.emis.de/journals/SIGMA/2007/101/

Abstract. This is a survey paper. We discuss Toeplitz operators in Kähler geometry, with applications to geometric quantization, and review some recent developments.

Key words: Kähler manifolds; holomorphic line bundles; geometric quantization; Toeplitz operators

2000 Mathematics Subject Classification: 32Q15; 53D50

1 Historical remarks

Toeplitz operators have been studied by analysts for many years. In [17] a *Toeplitz operator* on $S^1 = \{z \in \mathbb{C}|\ |z| = 1\}$ is defined as follows. Denote by μ the normalized Lebesgue measure, and by $e_n = e_n(z) = z^n$, $z \in S^1$, $n \in \mathbb{Z}$. Functions e_n are bounded, measurable, and they form an orthonormal basis in $L^2 = L^2(S^1, \mu)$. A function $f \in L^2$ is called *analytic* if $\int_{S^1} f\bar{e}_n d\mu = 0$ for all $n < 0$, and the *Hardy space* H^2 is defined as the space of all functions in L^2 which are analytic. Denote by $P : L^2 \to H^2$ the orthogonal projector. Let φ be a bounded measurable function on S^1. The corresponding *Toeplitz operator* $T_\varphi : H^2 \to H^2$ is defined by $T_\varphi = P \circ M_\varphi$, where M_φ is the operator of multiplication by φ. The function φ is called *the symbol of* T_φ. We immediately observe: for $\varphi(z) = 1$ T_φ is the identity operator, and for $\alpha, \beta \in \mathbb{C}$ and bounded measurable functions f, g on S^1 we have: $T_{\alpha f+\beta g} = \alpha T_f + \beta T_g$. These operators have various remarkable properties. For example, Theorem 4 [17] states: A necessary and sufficient condition that an operator on H^2 be a Toeplitz operator is that its matrix (with respect to the orthonormal basis $\{e_n :\ n = 0, 1, 2, \dots\}$) be a Toeplitz matrix.

Remark 1. Otto Toeplitz (1881–1940) was a German born Jewish mathematician, professor in Bonn from 1928 until 1935. A *Toeplitz matrix* is a one-way infinite matrix (a_{ij}) (i.e. $i, j = 0, 1, 2, \dots$) such that $a_{i+1,j+1} = a_{ij}$. For example, the matrix of T_{e_1} in the basis $\{e_n :\ n = 0, 1, 2, \dots\}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Toeplitz operators on bounded domains in $\mathbb{C}^n$, $n \geq 1$, (on various function spaces) have been studied extensively, and it would be a very difficult task to give a comprehensive description of

*This paper is a contribution to the Proceedings of the 2007 Midwest Geometry Conference in honor of Thomas P. Branson. The full collection is available at http://www.emis.de/journals/SIGMA/MGC2007.html

all the work done in this area. See, in particular, [11, 24, 25, 30, 31, 33, 34, 38, 40, 45, 46, 47, 48, 49, 50, 51, 52], and references in [11].

The purpose of this paper is different: we outline how Toeplitz operators appear in complex and symplectic geometry (often in problems coming from mathematical physics) and we overview some recent results in this area.

2 Kähler manifolds, geometric quantization and Toeplitz operators

2.1 Preliminaries

The use of Toeplitz operators in geometric quantization has its origins in work of F. Berezin, L. Boutet de Monvel, and J. Sjöstrand, see, in particular, [1, 2, 3, 12, 15, 16], and see also [37, 44].

Many of the key ideas are contained in [12]. The article [16] laid down the foundations for the analysis. Other important, more recent, papers include [4] and [14].

In the following "smooth" will always mean C^∞. Let W be a strictly pseudoconvex domain in a complex n-dimensional manifold, $n \geq 1$. Assume that the boundary ∂W is smooth and $\overline{W} = W \cup \partial W$ is compact. Let $r \in C^\infty(M)$ be a defining function for W: $r|_W < 0$, $r|_{\partial W} = 0$, $dr \neq 0$ near ∂W. Let $j : \partial W \hookrightarrow \overline{W}$ be the inclusion map. The 1-form $\alpha = j^*\mathrm{Im}(\overline{\partial r})$ is a contact form on ∂W. Denote by ν the measure on ∂W associated to the $(2n-1)$-form $\alpha \wedge (d\alpha)^{n-1}$, and denote $L^2 = L^2(\partial W, \nu)$. Denote by $A(W)$ the space of functions on $\overline{W}$ which are continuous on $\overline{W}$, smooth on ∂W, and holomorphic on W. Define *the Hardy space* $H^2 = H^2(\partial W)$ to be the closure in L^2 of $\{f|_{\partial W} \mid f \in A(W)\}$. Denote by $\Pi : L^2 \to H^2$ the orthogonal projector.

By definition an operator $T : C^\infty(\partial W) \to C^\infty(\partial W)$ is called a *Toeplitz operator of order* k if it is of the form $\Pi Q \Pi$, where Q is a pseudodifferential operator of order k. The *symbol* of T is $\sigma(T) := \sigma(Q)|_\Sigma$ (a function on Σ), where $\sigma(Q)$ is the symbol of Q and

$$\Sigma = \{(x, \xi) \,|\, x \in \partial W,\ \xi = r\alpha_x,\ r > 0\}$$

is a symplectic submanifold of $T^*\partial W$. Note that the symbol is well-defined: if Q_1, Q_2 are pseudodifferential operators and $T = \Pi Q_1 \Pi = \Pi Q_2 \Pi$, then $\sigma(Q_1)|_\Sigma = \sigma(Q_2)|_\Sigma$. Boutet de Monvel and Guillemin also show that for Toeplitz operators T_1, T_2 $\sigma(T_1 T_2) = \sigma(T_1)\sigma(T_2)$, $\sigma([T_1, T_2]) = \{\sigma(T_1), \sigma(T_2)\}$, where $\{\cdot, \cdot\}$ is the intrinsic Poisson bracket on the symplectic manifold Σ, and Toeplitz operators form a ring under composition.

2.2 Berezin–Toeplitz quantization

Let X be a connected compact n-dimensional Kähler manifold, $n \geq 1$. Denote the Kähler form by ω. Assume that ω is integral. There is an (ample) holomorphic Hermitian line bundle $L \to X$, with Hermitian connection ∇, such that $curv(\nabla) = -2\pi i\omega$ (thus $c_1(L) = [\omega]$). Let N be a positive integer. We shall denote by $L^2(X, L^{\otimes N})$ the space of square-integrable sections of $L^{\otimes N}$, and by $H^0(X, L^{\otimes N})$ the space of holomorphic sections of $L^{\otimes N}$. Also denote by $C^\infty(X)$ the space of real-valued smooth functions on X and by $C^\infty_{\mathbb{C}}(X)$ the space of complex-valued smooth functions on X.

The unit disc bundle W in L^* is a strictly pseudoconvex domain. Denote $P = \partial W$ (the unit circle bundle in L^*). In this particular setting, with $k = 0$, the definition of a Toeplitz operator in Section 2.1 leads to the following (revised) definition. Let $f \in C^\infty_{\mathbb{C}}(X)$. The corresponding *Toeplitz operator* (also called *Berezin–Toeplitz operator* in this setting) is

$$T_f = \oplus_{N=0}^{\infty} T_f^{(N)},$$

where $T_f^{(N)} = \Pi^{(N)} \circ M_f^{(N)} \in \mathrm{End}(H^0(X, L^{\otimes N}))$,

$$M_f^{(N)}: \; H^0(X, L^{\otimes N}) \to L^2(X, L^{\otimes N}), \qquad s \mapsto fs,$$

is the operator of multiplication by f and

$$\Pi^{(N)}: \; L^2(X, L^{\otimes N}) \to H^0(X, L^{\otimes N})$$

is the orthogonal projector.

Much has been written on this subject, see, in particular, review papers [7, 41, 43].

We shall list some properties of Berezin–Toeplitz operators. For every N the map

$$C^\infty_{\mathbb{C}}(X) \to \mathrm{End}\big(H^0(X, L^{\otimes N})\big), \qquad f \mapsto T_f^{(N)}$$

is surjective [6, Proposition 4.2].

It is known that for a positive integer m, $f_1, \dots, f_m \in C^\infty(X)$

$$\mathrm{tr}\big(T_{f_1}^{(N)} \cdots T_{f_m}^{(N)}\big) = N^n \left(\int_X f_1 \cdots f_m \frac{\omega^n}{n!} + O\left(\frac{1}{N}\right) \right)$$

as $N \to +\infty$ [6, Section 5].

Theorem 4.2 [6] states that for $f, g \in C^\infty(X)$

$$\big\| N[T_f^{(N)}, T_g^{(N)}] - i T_{\{f,g\}}^{(N)} \big\| = O\left(\frac{1}{N}\right)$$

as $N \to +\infty$, where $\|\cdot\|$ is the operator norm, i.e. $||A^{(N)}||^2 = \sup_{s \in (H^0(X, L^{\otimes N}) - 0)} \frac{\langle As, As \rangle_N}{\langle s, s \rangle_N}$ for $A^{(N)} \in \mathrm{End}(H^0(X, L^{\otimes N}))$, and $\langle \cdot, \cdot \rangle_N$ is the Hermitian inner product on $H^0(X, L^{\otimes N})$. Similar statements, for certain deformations of Lie algebra structure on $\mathrm{End}(H^0(X, L^{\otimes N}))$, were obtained in [26].

Also

$$\big\| T_{fg}^{(N)} - T_f^{(N)} T_g^{(N)} \big\| = O\left(\frac{1}{N}\right)$$

as $N \to +\infty$ [6, p. 291, (2)].

2.3 Other aspects

2.3.1. There is a strong connection between Berezin–Toeplitz quantization and deformation quantization. See, in particular, [8, 18, 29, 32, 35, 36, 39, 42].

2.3.2. Everything discussed in Section 2.2 has a traditional "translation" into the language of physics: it is customary to say that X is the classical phase space, $1/N$ is the Planck's constant, $H^0(X, L^{\otimes N})$ is the space of wave functions, f is a classical Hamiltonian, $T_f^{(N)}$ is the quantum Hamiltonian, and $N \to +\infty$ is the semiclassical limit. There are interesting and difficult results related to semiclassical behaviour of eigenvalues and eigenfunctions of Toeplitz operators, to quantization of maps, and, generally, to the relation between classical dynamics and quantum dynamics. See, in particular, [9, 13, 53, 54, 55].

2.3.3. Symbol calculus of Toeplitz operators has been used to study integrable systems [5].

2.3.4. Lagrangian submanifolds and symplectic reduction are two very important concepts in symplectic geometry. Toeplitz operators in this context have been studied and exploited in [10, 19, 20, 21, 22].

2.3.5. Let (X, ω, J_0) be a compact connected Kähler manifold (here X denotes the underlying smooth manifold, ω is the symplectic structure, J_0 is the complex structure). Denote by $\mathcal{J}$ the space of all complex structures on X compatible with ω, it is an infinite-dimensional Kähler manifold. The group $\mathrm{Symp}(X, \omega)$ acts on $\mathcal{J}$ by pull-back. It was observed by Fujiki [28] and by Donaldson [23] that $J \to s_J$, where $J \in \mathcal{J}$ and s_J is the scalar curvature of the Riemannian metric given by ω and J, is a moment map for this action. In [27] we obtain another proof of this statement, using Toeplitz operators and semiclassical asymptotics.

Acknowledgements

This work supported in part by NSERC.

References

[1] Berezin F., Covariant and contravariant symbols of operators, *Izv. Akad. Nauk SSSR Ser. Mat.* **36** (1972), 1134–1167 (in Russian).

[2] Berezin F., Spectral properties of generalized Toeplitz matrices, *Mat. Sb. (N.S.)* **95 (137)** (1974), 305–325, 328 (in Russian).

[3] Berezin F., General concept of quantization, *Comm. Math. Phys.* **40** (1975), 153–174.

[4] Berman R., Berndtsson B., Sjoestrand J., Asymptotics of Bergman kernels, math.CV/0506367.

[5] Bloch A., Golse F., Uribe A., Dispersionless Toda and Toeplitz operators, *Duke Math. J.* **117** (2003), 157–196.

[6] Bordemann M., Meinrenken E., Schlichenmaier M., Toeplitz quantization of Kähler manifolds and $gl(N)$, $N \to \infty$ limits, *Comm. Math. Phys.* **165** (1994), 281–296.

[7] Borthwick D., Introduction to Kähler quantization, in Quantization, the Segal–Bargmann transform and semiclassical analysis, 1st Summer School in Analysis and Mathematical Physics (Mexico, 1998), *Contemp. Math.* **260** (2000), 91–132.

[8] Borthwick D., Lesniewski A., Upmeier H., Nonperturbative deformation quantization of Cartan domains, *J. Funct. Anal.* **113** (1993), 153–176.

[9] Borthwick D., Paul T., Uribe A., Semiclassical spectral estimates for Toeplitz operators, *Ann. Inst. Fourier (Grenoble)* **48** (1998), 1189–1229.

[10] Borthwick D., Paul T., Uribe A., Legendrian distributions with applications to relative Poincaré series, *Invent. Math.* **122** (1995), 359–402, hep-th/9406036.

[11] Böttcher A., Silbermann B., Analysis of Toeplitz operators, Springer-Verlag, Berlin, 1990.

[12] Boutet de Monvel L., On the index of Toeplitz operators of several complex variables, *Invent. Math.* **50** (1978/79), 249–272.

[13] Boutet de Monvel L., Toeplitz operators – an asymptotic quantization of symplectic cones, in Stochastic Processes and Their Applications in Mathematics and Physics (Bielefeld, 1985), *Math. Appl.*, Vol. 61, Kluwer Acad. Publ., Dordrecht, 1990, 95–106.

[14] Boutet de Monvel L., Logarithmic trace of Toeplitz projectors, *Math. Res. Lett.* **12** (2005), 401–412, math.CV/0412252.

[15] Boutet de Monvel L., Guillemin V., The spectral theory of Toeplitz operators, *Annals of Math. Studies*, Vol. 99, Princeton University Press, Princeton, New Jersey, 1981.

[16] Boutet de Monvel L., Sjöstrand J., Sur la singularite des noyaux de Bergman et de Szego, in Journées: Équations aux Dérivees Partielles de Rennes (1975), *Asterisque*, no. 34–35 (1976), 123–164.

[17] Brown A., Halmos P., Algebraic properties of Toeplitz operators, *J. Reine Angew. Math.* **213** (1963/1964), 89–102.

[18] Charles L., Berezin–Toeplitz operators, a semi-classical approach, *Comm. Math. Phys.* **239** (2003), 1–28.

[19] Charles L., Quasimodes and Bohr–Sommerfeld conditions for the Toeplitz operators, *Comm. Partial Differential Equations* **28** (2003), 1527–1566.

[20] Charles L., Toeplitz operators and Hamiltonian torus actions, *J. Funct. Anal.* **236** (2006), 299–350, math.SG/0405128.

[21] Charles L., Symbolic calculus for Toeplitz operators with half-form, *J. Symplectic Geom.* **4** (2006), 171–198, math.SG/0602167.

[22] Charles L., Semi-classical properties of geometric quantization with metaplectic correction, *Comm. Math. Phys.* **270** (2007), 445–480, math.SG/0602168.

[23] Donaldson S., Remarks on gauge theory, complex geometry and 4-manifold topology, in Fields Medallists' Lectures, *World Sci. Ser. 20th Century Math.*, Vol. 5, World Sci. Publ., River Edge, NJ, 1997, 384–403.

[24] Douglas R., Ideals in Toeplitz algebras, *Houston J. Math.* **31** (2005), 529–539, math.OA/0309438.

[25] Douglas R., Widom H., Toeplitz operators with locally sectorial symbols, *Indiana Univ. Math. J.* **20** (1970/1971), 385–388.

[26] Foth T., Toeplitz operators, deformations, and asymptotics, *J. Geom. Phys.* **57** (2007), 855–861.

[27] Foth T., Uribe A., The manifold of compatible almost complex structures and geometric quantization, *Comm. Math. Phys.*, to appear.

[28] Fujiki A., Moduli space of polarized algebraic manifolds and Kähler metrics, *Sugaku Expositions* **5** (1992), 173–191.

[29] Guillemin V., Star products on compact pre-quantizable symplectic manifolds, *Lett. Math. Phys.* **35** (1995), 85–89.

[30] Jewell N., Toeplitz operators on the Bergman spaces and in several complex variables, *Proc. London Math. Soc. (3)* **41** (1980), 193–216.

[31] Jewell N., Krantz S., Toeplitz operators and related function algebras on certain pseudoconvex domains, *Trans. Amer. Math. Soc.* **252** (1979), 297–312.

[32] Karabegov A., Schlichenmaier M., Identification of Berezin–Toeplitz deformation quantization, *J. Reine Angew. Math.* **540** (2001), 49–76, math.QA/0006063.

[33] Klein E., The numerical range of a Toeplitz operator, *Proc. Amer. Math. Soc.* **35** (1972), 101–103.

[34] Klein E., More algebraic properties of Toeplitz operators, *Math. Ann.* **202** (1973), 203–207.

[35] Klimek S., Lesniewski A., Quantum Riemann surfaces. I. The unit disc, *Comm. Math. Phys.* **146** (1992), 103–122.
Klimek S., Lesniewski A., Quantum Riemann surfaces. II. The discrete series, *Lett. Math. Phys.* **24** (1992), 125–139.
Klimek S., Lesniewski A., Quantum Riemann surfaces. III. The exceptional cases, *Lett. Math. Phys.* **32** (1994), 45–61.

[36] Klimek S., Lesniewski A., Quantum Riemann surfaces for arbitrary Planck's constant, *J. Math. Phys.* **37** (1996), 2157–2165.

[37] Kostant B., Quantization and unitary representations. I. Prequantization, in Lectures in Modern Analysis and Applications. III, *Lecture Notes in Math.*, Vol. 170, Springer, Berlin, 1970, 87–208.

[38] Peller V., Invariant subspaces of Toeplitz operators with piecewise continuous symbols, *Proc. Amer. Math. Soc.* **119** (1993), 171–178.

[39] Reshetikhin N., Takhtajan L., Deformation quantization of Kähler manifolds, in L.D. Faddeev's Seminar on Mathematical Physics,, *Amer. Math. Soc. Transl. Ser. 2*, Vol. 201, Amer. Math. Soc., Providence, RI, 2000, 257–276, math.QA/9907171.

[40] Salinas N., Sheu A., Upmeier H., Toeplitz operators on pseudoconvex domains and foliation C^*-algebras, *Ann. Math. (2)* **130** (1989), 531–565.

[41] Schlichenmaier M., Berezin–Toeplitz quantization of compact Kähler manifolds, in Quantization, Coherent States, and Poisson Structures (Bialowieza, 1995), PWN, Warsaw, 1998, 101–115.

[42] Schlichenmaier M., Deformation quantization of compact Kähler manifolds by Berezin–Toeplitz quantization, in Conférence Moshé Flato 1999, Vol. II (Dijon), *Math. Phys. Stud.*, Vol. 22, Kluwer Acad. Publ., Dordrecht, 2000, 289–306, math.QA/9910137.

[43] Schlichenmaier M., Berezin–Toeplitz quantization and Berezin transform, in Long Time Behaviour of Classical and Quantum Systems (Bologna, 1999), *Ser. Concr. Appl. Math.*, Vol. 1, World Sci. Publ., River Edge, NJ, 2001, 271–287.

[44] Souriau J.-M., Structure of dynamical systems, A symplectic view of physics, *Progress in Mathematics*, Vol. 149, Birkhäuser Boston, Inc., Boston, MA, 1997 (translation of Structure des systèmes dynamiques, Maîtrises de mathématiques Dunod, Paris, 1970).

[45] Stroethoff K., Zheng D., Bounded Toeplitz products on the Bergman space of the polydisk, *J. Math. Anal. Appl.* **278** (2003), 125–135.

[46] Stroethoff K., Zheng D., Bounded Toeplitz products on Bergman spaces of the unit ball, *J. Math. Anal. Appl.* **325** (2007), 114–129.

[47] Upmeier H., Toeplitz operators on bounded symmetric domains, *Trans. Amer. Math. Soc.* **280** (1983), 221–237.

[48] Upmeier H., Toeplitz C^*-algebras on bounded symmetric domains, *Ann. Math. (2)* **119** (1984), 549–576.

[49] Upmeier H., Toeplitz operators on symmetric Siegel domains, *Math. Ann.* **271** (1985), 401–414.

[50] Upmeier H., Toeplitz operators and index theory in several complex variables, *Operator Theory: Advances and Applications*, Vol. 81, Birkhäuser Verlag, Basel, 1996.

[51] Widom H., On the spectrum of a Toeplitz operator, *Pacific J. Math.* **14** (1964), 365–375.

[52] Widom H., Toeplitz operators on H_p, *Pacific J. Math.* **19** (1966), 573–582.

[53] Zelditch S., Index and dynamics of quantized contact transformations, *Ann. Inst. Fourier (Grenoble)* **47** (1997), 305–363, math-ph/0002007.

[54] Zelditch S., Quantum maps and automorphisms, in The Breadth of Symplectic and Poisson Geometry, *Progr. Math.*, Vol. 232, Birkhäuser Boston, Boston, MA, 2005, 623–654, math.QA/0307175.

[55] Zelditch S., Quantum dynamics from the semi-classical point of view, unpublished notes, available at http://mathnt.mat.jhu.edu/zelditch/.