Symmetry, Integrability and Geometry: Methods and Applications SIGMA **5** (2009), 111, 11 pages

Second-Order Conformally Equivariant Quantization in Dimension 1|2

Najla MELLOULI

Institut Camille Jordan, UMR 5208 du CNRS, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France
E-mail: *mellouli@math.univ-lyon1.fr*

Received September 22, 2009, in final form December 13, 2009; Published online December 28, 2009
doi:10.3842/SIGMA.2009.111

Abstract. This paper is the next step of an ambitious program to develop conformally equivariant quantization on supermanifolds. This problem was considered so far in (super)dimensions 1 and 1|1. We will show that the case of several odd variables is much more difficult. We consider the supercircle $S^{1|2}$ equipped with the standard contact structure. The conformal Lie superalgebra $\mathcal{K}(2)$ of contact vector fields on $S^{1|2}$ contains the Lie superalgebra osp(2|2). We study the spaces of linear differential operators on the spaces of weighted densities as modules over osp(2|2). We prove that, in the non-resonant case, the spaces of second order differential operators are isomorphic to the corresponding spaces of symbols as osp(2|2)-modules. We also prove that the conformal equivariant quantization map is unique and calculate its explicit formula.

Key words: equivariant quantization; conformal superalgebra

2000 Mathematics Subject Classification: 17B10; 17B68; 53D55

1 Introduction and the main results

The concept of equivariant quantization first appeared in [8] and [3]. The general idea is to identify, in a canonical way, the space of linear differential operators on a manifold acting on weighted densities with the corresponding space of symbols. Such an identification is called a quantization (or symbol) map. It turns out that for an arbitrary projectively/conformally flat manifold, there exists a unique quantization map commuting with the action of the group of projective/conformal transformations.

Equivariant quantization on supermanifolds was initiated by [1] and further investigated in [5]. In these works, the authors considered supermanifolds of dimension 1|1. This is in part due to the fact that, in the super cases considered, one has to take into account a non-integrable distribution, namely the contact structure, see [9, 6], dubbed "SUSY" structure in [1].

In this paper, we consider the space of linear differential operators on the supercircle $S^{1|2}$ acting from the space of λ-densities to the space of μ-densities, where λ and μ are (real or complex) numbers. This space of operators is naturally a module over the Lie superalgebra of contact vector fields (see Section 3 below) also known as the stringy superalgebra $\mathcal{K}(2)$, see [6]. We denote these modules by $\mathcal{D}_{\lambda,\mu}\left(S^{1|2}\right)$.

Our main result concerns the spaces containing second-order differential operators, $\mathcal{D}^{\frac{3}{2}}_{\lambda\mu}\left(S^{1|2}\right)$ and $\mathcal{D}^{2}_{\lambda\mu}\left(S^{1|2}\right)$. The space $\mathcal{D}^{\frac{3}{2}}_{\lambda\mu}\left(S^{1|2}\right)$ is contained in $\mathcal{D}^{2}_{\lambda\mu}\left(S^{1|2}\right)$, so we will be interested to the space $\mathcal{D}^{2}_{\lambda\mu}\left(S^{1|2}\right)$. This space is not isomorphic to the corresponding space of symbols, as a $\mathcal{K}(2)$-module. The obstructions to the existence of such an isomorphism are given by (the infinitesimal version of) the Schwarzian derivative, see [11] and references therein. We thus restrict the module structure on $\mathcal{D}^2_{\lambda\mu}\left(S^{1|2}\right)$ to the orthosymplectic Lie superalgebra $\mathrm{osp}\,(2|2)$ naturally embedded to $\mathcal{K}(2)$.

The main result of this paper is as follows.

Theorem 1. (i) *The space* $\mathcal{D}^2_{\lambda\mu}\left(S^{1|2}\right)$ *and the corresponding space of symbols are isomorphic as* $\mathrm{osp}\,(2|2)$*-modules, provided* $\mu-\lambda\neq 0,\frac{1}{2},1,\frac{3}{2},2$.

(ii) *The above isomorphism is unique.*

The particular values of λ and μ such that $\mu-\lambda\in\{0,\frac{1}{2},1,\frac{3}{2},2\}$ are called *resonant*. We do not study here the corresponding "resonant modules" of differential operators. Note that these modules are of particular interest and deserve further study.

We think that a similar result holds for the space of differential operators of arbitrary order, but such a result is out of reach so far. We would like to mention however, that most of the known interesting examples of differential operators in geometry and mathematical physics are of order 2. This allows one to expect concrete applications of the above theorem.

2 Geometry of the supercircle $S^{1|2}$

The supercircle $S^{1|2}$ is a supermanifold of dimension $1|2$ which generalizes the circle S^1. To fix the notation, let us give the basic definitions of geometrical objects on $S^{1|2}$, see [9, 6] for more details.

We define the supercircle $S^{1|2}$ by describing its graded commutative algebra of (complex valued) functions that we note by $C^\infty\left(S^{1|2}\right)$, consisting of the elements

$$f\left(x,\xi_1,\xi_2\right)=f_0(x)+\xi_1 f_1(x)+\xi_2 f_2(x)+\xi_1\xi_2 f_{12}(x),$$

where x is the Fourier image of the angle parameter on S^1 and ξ_1, ξ_2 are odd Grassmann coordinates, i.e., $\xi_i^2=0$, $\xi_1\xi_2=-\xi_2\xi_1$ and where $f_0,f_{12},f_1,f_2\in C^\infty(S)$ are functions with complex values. We define the parity function p by setting $p\left(x\right)=0$ and $p\left(\xi_i\right)=1$.

2.1 Vector fields and differential forms

Any *vector field on* $S^{1|2}$ is a derivation of the algebra $C^\infty\left(S^{1|2}\right)$, it can be expressed as

$$X=f\partial_x+g_1\partial_{\xi_1}+g_2\partial_{\xi_2}$$

with $f,g_i\in C^\infty\left(S^{1|2}\right)$, $\partial_x=\frac{\partial}{\partial x}$ and $\partial_{\xi_i}=\frac{\partial}{\partial\xi_i}$, for $i=1,2$. The space of vector fields on $S^{1|2}$ is a Lie superalgebra which we note by $\mathrm{Vect}\left(S^{1|2}\right)$.

Any *differential form* is a skew-symmetric multi-linear map (over $C^\infty\left(S^{1|2}\right)$) from $\mathrm{Vect}\left(S^{1|2}\right)$ to $C^\infty\left(S^{1|2}\right)$. To fix the notation, we set $\langle\partial_{u_i},du_j\rangle=\delta_{ij}$, for $u=(x,\xi_1,\xi_2)$. The space of differential forms $\Omega^1\left(S^{1|2}\right)$ is a right $C^\infty\left(S^{1|2}\right)$-module and a left $\mathrm{Vect}\left(S^{1|2}\right)$-module, the action being given by the Lie derivative, i.e., $\langle X,L_Y\alpha\rangle:=\langle\left[X,Y\right],\alpha\rangle$ for any $Y,X\in\mathrm{Vect}\left(S^{1|2}\right)$, $\alpha\in\Omega^1\left(S^{1|2}\right)$.

2.2 The Lie superalgebra of contact vector fields

The *standard contact* structure on $S^{1|2}$ is defined by the data of a linear distribution $\langle\overline{D}_1,\overline{D}_2\rangle$ on $S^{1|2}$ generated by the odd vector fields

$$\overline{D}_1=\partial_{\xi_1}-\xi_1\partial_x,\qquad \overline{D}_2=\partial_{\xi_2}-\xi_2\partial_x.$$

This contact structure can also be defined as the kernel of the differential 1-form:

$$\alpha = dx + \xi_1 d\xi_1 + \xi_2 d\xi_2.$$

We refer to [14] for more details.

A vector field X on $S^{1|2}$ is called a *contact vector field* if it preserves the contact distribution, that is, satisfies the condition:

$$\left[X, \overline{D}_1\right] = \psi_{1_X} \overline{D}_1 + \psi_{2_X} \overline{D}_2, \qquad \left[X, \overline{D}_2\right] = \phi_{1_X} \overline{D}_1 + \phi_{2_X} \overline{D}_2,$$

where $\psi_{1_X}, \psi_{2_X}, \phi_{1_X}, \phi_{2_X} \in C^\infty\left(S^{1|2}\right)$ are functions depending on X. The space of the contact vector fields is a Lie superalgebra which we note by $\mathcal{K}(2)$.

The following fact is well-known.

Lemma 1. *Every contact vector field* (*see* [9]) *can be expressed, for some function* $f \in C^\infty\left(S^{1|2}\right)$, *by*

$$X_f = f\partial_x - (-1)^{p(f)} \tfrac{1}{2} \left(\overline{D}_1(f)\, \overline{D}_1 + \overline{D}_2(f)\, \overline{D}_2\right). \tag{1}$$

The function f is said to be a *contact Hamiltonian* of the field X_f. The space $C^\infty\left(S^{1|2}\right)$ is therefore identified with the Lie superalgebra $\mathcal{K}(2)$ and equipped with the structure of Lie superalgebra with respect to the contact bracket:

$$\{f, g\} = fg' - f'g - (-1)^{p(f)} \tfrac{1}{2} \left(\overline{D}_1(f)\, \overline{D}_1(g) + \overline{D}_2(f)\, \overline{D}_2(g)\right), \tag{2}$$

where $f' = \partial_x(f)$.

2.3 Conformal symmetry: the orthosymplectic superalgebra

The *conformal* (or projective) structure on the supercircle $S^{1|2}$ (see [13]) is defined by the action of the 4|4-dimensional Lie superalgebra $\mathrm{osp}(2|2)$. This action is spanned by the contact vector fields with the contact Hamiltonians:

$$\left\{1, x, x^2, \xi_1\xi_2;\ \xi_1, \xi_2, x\xi_1, x\xi_2\right\}.$$

The embedding of $\mathrm{osp}(2|2)$ into $\mathcal{K}(2)$ is then given by (1).

The subalgebra $\mathrm{Aff}(2|2)$ of $\mathrm{osp}(2|2)$ spanned by the contact vector fields with the contact Hamiltonians $\{1, x, \xi_1\xi_2;\ \xi_1, \xi_2\}$ will be called the *affine* Lie superalgebra.

2.4 Modules of weighted densities

We introduce a family of $\mathcal{K}(2)$-modules with a parameter. For any contact vector field, we define a family of differential operators of order one on $C^\infty\left(S^{1|2}\right)$

$$L^\lambda_{X_f} := X_f + \lambda f', \tag{3}$$

where the parameter λ is an arbitrary (complex) number and the function is considered as a 0-order differential operator of left multiplication by this function. The map $X_f \mapsto L^\lambda_{X_f}$ is a homomorphism of Lie superalgebras. We thus obtain a family of $\mathcal{K}(2)$-modules on $C^\infty\left(S^{1|2}\right)$ that we note by $\mathcal{F}_\lambda\left(S^{1|2}\right)$ and that we call spaces of *weighted densities* of weight λ.

Viewed as vector spaces, but not as $\mathcal{K}(2)$-modules, the spaces $\mathcal{F}_\lambda\left(S^{1|2}\right)$ are isomorphic to $C^\infty\left(S^{1|2}\right)$.

The space of weighted densities possesses a Poisson superalgebra structure with respect to the contact bracket $\{\cdot,\cdot\}:\mathcal{F}_\lambda\left(S^{1|2}\right)\otimes\mathcal{F}_\mu\left(S^{1|2}\right)\to\mathcal{F}_{\lambda+\mu+1}\left(S^{1|2}\right)$ given explicitly by

$$\{f,g\}=\mu f'g-\lambda fg'-(-1)^{p(f)}\,\tfrac{1}{2}\left(\overline{D}_1(f)\,\overline{D}_1(g)+\overline{D}_2(f)\,\overline{D}_2(g)\right)$$

for all $f\in\mathcal{F}_\lambda\left(S^{1|2}\right)$, $g\in\mathcal{F}_\mu\left(S^{1|2}\right)$.

Note that if $f\in\mathcal{F}_{-1}\left(S^{1|2}\right)$, then $\{f,g\}=L^\mu_{X_f}(g)$. The space $\mathcal{F}_{-1}\left(S^{1|2}\right)$ is a subalgebra isomorphic to $\mathcal{K}(2)$, see formula (2).

3 Differential operators on the spaces of weighted densities

In this section we introduce the space of differential operators acting on the spaces of weighted densities and the corresponding space of symbols on $S^{1|2}$. We refer to [10, 7, 5, 2] for further details. This space is naturally a module over the Lie superalgebra $\mathcal{K}(2)$.

We also define a $\mathcal{K}(2)$-invariant "finer filtration" on the modules of differential operators that plays the key role in this paper. The graded $\mathcal{K}(2)$-module associated to the finer filtration is called the *module of symbols*.

3.1 Definition of the modules $\mathcal{D}^{(k)}_{\lambda\mu}\left(S^{1|2}\right)$

Let $\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$ be the space of linear differential operators $A:\mathcal{F}_\lambda\left(S^{1|2}\right)\to\mathcal{F}_\mu\left(S^{1|2}\right)$.

This space is naturally filtered:

$$\mathcal{D}^{(0)}_{\lambda\mu}(S^{1|2})\subset\mathcal{D}^{(1)}_{\lambda\mu}(S^{1|2})\subset\cdots\subset\mathcal{D}^{(k-1)}_{\lambda\mu}(S^{1|2})\subset\mathcal{D}^{(k)}_{\lambda\mu}(S^{1|2})\subset\cdots,$$

where $\mathcal{D}^{(k)}_{\lambda\mu}\left(S^{1|2}\right)$ is the space of linear differential operators of order k.

The space $\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$ and every subspace $\mathcal{D}^{(k)}_{\lambda\mu}\left(S^{1|2}\right)$ is naturally a module over the Lie superalgebra of contact vector fields $\mathcal{K}(2)$. The above filtration is of course $\mathcal{K}(2)$-invariant. Note that, in the case $\lambda=\mu=0$, the space of differential operators is a module over the full Lie superalgebra $\mathrm{Vect}\left(S^{1|2}\right)$ and the above filtration is $\mathrm{Vect}\left(S^{1|2}\right)$-invariant.

3.2 The finer filtration: modules $\mathcal{D}^{k}_{\lambda\mu}\left(S^{1|2}\right)$

It turns out that there is another, finer filtration:

$$\mathcal{D}^{0}_{\lambda\mu}(S^{1|2})\subset\mathcal{D}^{\frac12}_{\lambda\mu}(S^{1|2})\subset\mathcal{D}^{1}_{\lambda\mu}(S^{1|2})\subset\mathcal{D}^{\frac32}_{\lambda\mu}(S^{1|2})\subset\mathcal{D}^{2}_{\lambda\mu}(S^{1|2})\subset\cdots\tag{4}$$

on the space of differential operators on $S^{1|2}$. This finer filtration is invariant with respect to the action of $\mathcal{K}(2)$ (but it cannot be invariant with respect to the action of the full algebra of vector fields).

Proposition 1. *Every differential operator can be expressed in the form*

$$A=\sum_{\ell,m,n}a_{\ell,m,n}\left(\partial_x\right)^\ell\,\overline{D}_1^m\overline{D}_2^n,\tag{5}$$

where $a_{\ell,m,n}\in C^\infty\left(S^{1|2}\right)$, the index ℓ is arbitrary while $m,n\leq1$, and where only finitely many terms are non-zero.

Proof. If $A\in\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$, then $A=\sum a_{\ell,m,n}\left(\partial_x\right)^\ell\partial^m_{\xi_1}\partial^n_{\xi_2}$ and since one has:

$$\partial_{\xi_1}=\overline{D}_1+\xi_1\partial_x,\qquad\partial_{\xi_2}=\overline{D}_2+\xi_2\partial_x,\qquad\overline{D}_i^2=-\partial_x,$$

we have the form desired. ■

For every (half)-integer k, we denote by $\mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$ the space of differential operators of the form

$$A = \sum_{\ell+\frac{m}{2}+\frac{n}{2}\leq k} a_{\ell,m,n}\left(\partial_x\right)^\ell \overline{D}_1^m \overline{D}_2^n, \tag{6}$$

where $a_{\ell,m,n} \in C^\infty\left(S^{1|2}\right)$. Furthermore, since $\partial_x = -\overline{D}_1^2 = -\overline{D}_2^2$, we can assume $m, n \leq 1$.

Proposition 2. *The form* (6) *is stable with respect to the action of* $\mathcal{K}(2)$.

Proof. Let X_f be a contact vector field, see formula (1). The action of X_f on the space $\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$ is given by

$$\mathcal{L}_{X_f}(A) = L^\mu_{X_f} \circ A - (-1)^{p(f)p(A)} A \circ L^\lambda_{X_f}, \tag{7}$$

where $L^\lambda_{X_f}$ is the Lie derivative (3). The invariance of the form (6) is subject to a straightforward calculation. ■

Remark 1. 1) It is worth noticing that for k integer, one has

$$\mathcal{D}^{(k)}_{\lambda\mu}\left(S^{1|2}\right) \subset \mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$$

but these modules do not coincide. Indeed, the module $\mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$ contains operators proportional to $\partial_x^{k-1}\overline{D}_1\overline{D}_2$ which are, of course, of order $k+1$. An element of $\mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$ will be called k-differential operator, it does not have to be of order k, it can be of order $k+1$.

2) A similar finer filtration exists for an arbitrary contact manifold, cf. [12, 4]. In the 1|1-dimensional case this finer filtration was used in [5].

Example 1. The module $\mathcal{D}^2_{\lambda\mu}\left(S^{1|2}\right)$ will be particularly interesting for us. Every differential operator $A \in \mathcal{D}^2_{\lambda\mu}\left(S^{1|2}\right)$ can be expressed in the form

$$\begin{aligned} A &= a_{0,0,0} + a_{0,1,0}\overline{D}_1 + a_{0,0,1}\overline{D}_2 + a_{1,0,0}\partial_x + a_{0,1,1}\overline{D}_1\overline{D}_2 + a_{1,1,0}\partial_x\overline{D}_1 \\ &\quad + a_{1,0,1}\partial_x\overline{D}_2 + a_{2,0,0}\partial_x^2 + a_{1,1,1}\partial_x\overline{D}_1\overline{D}_2. \end{aligned}$$

3.3 Space of symbols of differential operators

We consider the graded $\mathcal{K}(2)$-module associated to the fine filtration (4):

$$\operatorname{gr}\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right) = \bigoplus_{i=0}^{\infty} \operatorname{gr}^{\frac{i}{2}}\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right),$$

where $\operatorname{gr}^k\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right) = \mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)/\mathcal{D}^{k-\frac{1}{2}}_{\lambda\mu}\left(S^{1|2}\right)$ for every (half)integer k. This module is called the *space of symbols* of differential operators.

The image of a differential operator A under the natural projection

$$\sigma_{\rm pr}: \ \mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right) \to \operatorname{gr}^k\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$$

defined by the filtration (4) is called the *principal symbol*.

We need to know the action of the Lie superalgebra $\mathcal{K}(2)$ on the space of symbols.

Proposition 3. *If k is an integer, then*

$$\operatorname{gr}^k\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right) = \mathcal{F}_{\mu-\lambda-k} \oplus \mathcal{F}_{\mu-\lambda-k}$$

Proof. By definition (see formula (6)), a given operator $A \in \mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$ with integer k is of the form

$$A = F_1 \partial_x^k + F_2 \partial_x^{k-1} \overline{D}_1 \overline{D}_2 + \cdots ,$$

where $\cdots$ stand for lower order terms. The principal symbol of A is then encoded by the pair (F_1, F_2). From (7), one can easily calculate the $\mathcal{K}(2)$-action on the principal symbol:

$$L_{X_f}(F_1, F_2) = \left(L^{\mu-\lambda-k}_{X_f}(F_1), L^{\mu-\lambda-k}_{X_f}(F_2)\right).$$

In other words, both F_1 and F_2 transform as $(\mu - \lambda - k)$-densities. ■

Surprisingly enough, the situation is more complicated in the case of half-integer k.

Proposition 4. *If k is a half-integer, then the $\mathcal{K}(2)$-action is as follows:*

$$L_{X_f}(F_1, F_2) = \left(L^{\mu-\lambda-k}_{X_f}(F_1) - \tfrac{1}{2}\overline{D}_1\overline{D}_2(f)\, F_2, L^{\mu-\lambda-k}_{X_f}(F_2) + \tfrac{1}{2}\overline{D}_1\overline{D}_2(f)\, F_1\right). \tag{8}$$

Proof. Directly from (7). ■

This means that the space of symbols of half-integer contact order is not isomorphic to the space of weighted densities. It would be nice to understand the geometric nature of the action (8).

Corollary 1. *The module* $\mathrm{gr}\, \mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$ *depends only on* $\mu - \lambda$.

Following [8, 3, 5] and to simplify the notation, we will denote by $\mathcal{S}_{\mu-\lambda}\left(S^{1|2}\right)$ the full space of symbols $\mathrm{gr}\mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$ and $\mathcal{S}^k_{\mu-\lambda}\left(S^{1|2}\right)$ the space of symbols of contact order k.

Corollary 2. *The space* $\mathcal{D}^k_{\lambda\mu}\left(S^{1|2}\right)$ *is isomorphic to* $\mathcal{S}^k_{\mu-\lambda}\left(S^{1|2}\right)$ *as a module over the affine Lie superalgebra* $\mathrm{Aff}\,(2|2)$.

Proof. To define an $\mathrm{Aff}\,(2|2)$-equivariant quantization map, it suffice to consider the inverse of the principal symbol: $Q = \sigma_{\mathrm{pr}}^{-1}$. ■

A linear map $Q : \mathcal{S}_{\mu-\lambda}\left(S^{1|2}\right) \to \mathcal{D}_{\lambda\mu}\left(S^{1|2}\right)$ is called a *quantization map* if it is bijective and preserves the principal symbol of every differential operator, i.e., $\sigma_{\mathrm{pr}} \circ Q = \mathrm{Id}$. The inverse map $\sigma = Q^{-1}$ is called a *symbol map*.

4 Conformally equivariant quantization on $S^{1|2}$

In this section we prove the main results of this paper – Theorem 1 – on the existence and uniqueness of the conformally equivariant quantization map on the space $\mathcal{D}^2_{\lambda\mu}\left(S^{1|2}\right)$. We calculate this quantization map explicitly.

We already proved that the space $\mathcal{D}^2_{\lambda\mu}\left(S^{1|2}\right)$ is isomorphic to the corresponding space of symbols as a module over the affine Lie superalgebra $\mathrm{Aff}\,(2|2)$. We will now show how to extend this isomorphism to that of the $\mathrm{osp}\,(2|2)$-modules.

4.1 Equivariant quantization map in the case of $\frac{1}{2}$-differential operators

Let us first consider the quantization of symbols of $\frac{1}{2}$-differential operators. By linearity, we can assume that the symbols of differential operators are homogeneous (purely even or purely odd). Since for any symbol $(F_1, F_2) \in \mathcal{S}^{\frac{1}{2}}_{\mu-\lambda}\left(S^{1|2}\right)$, we have $p(F_1) = p(F_2)$, we can define *parity of the symbol* (F_1, F_2) as $p(F) := p(F_1) = p(F_2)$.

Proposition 5. *The unique* osp(2|2)*-equivariant quantization map associates the following $\frac{1}{2}$-differential operator to a symbol $(F_1, F_2) \in \mathcal{S}^{\frac{1}{2}}_{\mu-\lambda}\left(S^{1|2}\right)$ provided $\mu \neq \lambda$:*

$$Q\left(F_1, F_2\right) = F_1\overline{D}_1 + F_2\overline{D}_2 + (-1)^{p(F)}\frac{\lambda}{\lambda-\mu}\left(\overline{D}_1\left(F_1\right) + \overline{D}_2\left(F_2\right)\right).$$

Proof. First, one easily checks that this quantization map is, indeed, osp(2|2)-equivariant. Let us prove the uniqueness.

Consider first an arbitrary differentiable linear map $Q : \mathcal{S}^{\frac{1}{2}}_{\mu-\lambda}\left(S^{1|2}\right) \to \mathcal{D}^{\frac{1}{2}}_{\lambda\mu}\left(S^{1|2}\right)$ preserving the principal symbol. Such a map is of the form:

$$Q\left(F_1, F_2\right) = F_1\overline{D}_1 + F_2\overline{D}_2 + \widetilde{Q}^{(1)}_1\left(F_1\right) + \widetilde{Q}^{(1)}_2\left(F_2\right),$$

where $\widetilde{Q}^{(1)}_1$ and $\widetilde{Q}^{(1)}_2$ are differential operators with coefficients in $\mathcal{F}_{\mu-\lambda}$, cf. formula (5).

One then easily checks the following:

a) This map commutes with the action of the vector fields $D_1, D_2 \in$ osp(2|2), where $D_i = \partial_{\xi_i} + \xi_i\partial_x$, if and only if the differential operators $\widetilde{Q}^{(1)}_1$ and $\widetilde{Q}^{(1)}_2$ are with constant coefficients.

b) This map commutes with the linear vector fields X_x, X_{ξ_1}, X_{ξ_2} if and only if the differential operators $\widetilde{Q}^{(1)}_1$ and $\widetilde{Q}^{(1)}_2$ are of order 1 and moreover have the form:

$$\widetilde{Q}^{(1)}_1\left(F_1\right) = C_{11}\overline{D}_1\left(F_1\right), \qquad \widetilde{Q}^{(1)}_2\left(F_2\right) = C_{12}\overline{D}_2\left(F_2\right),$$

where C_{11}, C_{12} are arbitrary constants.

We thus determined the general form of a quantization map commuting with the action of the affine subalgebra Aff(2|2).

c) This map commutes with $X_{\xi_1\xi_2}$ if and only if $C_{11} = C_{12}$.

In order to satisfy the full condition of osp(2|2)-equivariance, it remains to impose the equivariance with respect to the vector field X_{x^2}.

d) The above quantization map commutes with the action of X_{x^2} if and only if C_{11}, C_{12} satisfy the following condition:

$$2\left(\mu-\lambda-\tfrac{1}{2}\right)C_{11} + C_{12} = (-1)^{p(F)+1}2\lambda,$$
$$C_{11} + 2\left(\mu-\lambda-\tfrac{1}{2}\right)C_{12} = (-1)^{p(F)+1}2\lambda$$

If $\mu - \lambda \neq 0$, this system can be easily solved and the solution is $C_{11} = C_{12} = (-1)^{p(F)}\frac{\lambda}{\lambda-\mu}$. ■

4.2 Equivariant quantization map in the case of 1-differential operators

Let us consider the next case. All the calculations are similar (yet more involved) to the above calculations.

Proposition 6. *The unique* osp(2|2)*-equivariant quantization map associates the following differential operator to a symbol $(F_1, F_2) \in \mathcal{S}^{1}_{\mu-\lambda}\left(S^{1|2}\right)$:*

$$Q\left(F_1, F_2\right) = F_1\partial_x + F_2\overline{D}_1\overline{D}_2 + (-1)^{p(F)}\frac{1}{2\left(\mu-\lambda-1\right)}\left(\overline{D}_1\left(F_1\right)\overline{D}_1 + \overline{D}_2\left(F_1\right)\overline{D}_2\right)$$

$$+ (-1)^{p(F)} \frac{\lambda + \frac{1}{2}}{\mu - \lambda - 1} \left(\overline{D}_2 (F_2) \overline{D}_1 - \overline{D}_1 (F_2) \overline{D}_2\right)$$
$$- \frac{\lambda}{\mu - \lambda - 1} \left(\partial_x (F_1) - \frac{1 + 2\lambda}{2(\mu - \lambda) - 1} \overline{D}_1 \overline{D}_2 (F_2)\right) \tag{9}$$

provided $\mu - \lambda \neq \frac{1}{2}, 1$.

Proof. First, we check by a straightforward calculation that an arbitrary Aff $(2|2)$-equivariant quantization map is given by

$$Q(F_1, F_2) = F_1 \partial_x + F_2 \overline{D}_1 \overline{D}_2 + \widetilde{Q}_1^{(1)}(F_1) + \widetilde{Q}_2^{(1)}(F_2) + \widetilde{Q}_1^{(2)}(F_1) + \widetilde{Q}_2^{(2)}(F_2),$$

where the differential operators $\widetilde{Q}_1^{(n)}$ and $\widetilde{Q}_2^{(n)}$ are of order n and have the form:

$$\widetilde{Q}_1^{(1)}(F_1) = C_{11} \overline{D}_1 (F_1) \overline{D}_1 + C_{13} \overline{D}_2 (F_1) \overline{D}_2,$$
$$\widetilde{Q}_1^{(2)}(F_1) = C_{21} \partial_x (F_1),$$
$$\widetilde{Q}_2^{(1)}(F_2) = C_{12} \overline{D}_2 (F_2) \overline{D}_1 + C_{14} \overline{D}_1 (F_2) \overline{D}_2,$$
$$\widetilde{Q}_2^{(2)}(F_2) = C_{22} \overline{D}_1 \overline{D}_2 (F_2).$$

The above quantization map commutes with the action of X_{x^2} if and only if the coefficients C_{ij} satisfy the following system of linear equations:

$$2(\mu - \lambda - 1) C_{11} = (-1)^{p(F)},$$
$$2(\mu - \lambda - 1) C_{12} = (-1)^{p(F)} (1 + 2\lambda),$$
$$2(\mu - \lambda - 1) C_{13} = (-1)^{p(F)},$$
$$2(\mu - \lambda - 1) C_{14} = (-1)^{p(F)+1} (1 + 2\lambda),$$
$$(\mu - \lambda - 1) C_{21} = -\lambda,$$
$$(2(\mu - \lambda) - 1) C_{22} = (-1)^{p(F)} \lambda C_{12} - \lambda C_{14}.$$

Solving this system, one obtains the formula (9). ■

4.3 Equivariant quantization map in the case of $\frac{3}{2}$-order differential operators

Consider now the space of differential operators $\mathcal{D}^{\frac{3}{2}}_{\lambda\mu}\left(S^{1|2}\right)$.

Proposition 7. *The unique* osp $(2|2)$*-equivariant quantization map associates the following differential operator to a symbol* $(F_1, F_2) \in \mathcal{S}^{\frac{3}{2}}_{\mu-\lambda}\left(S^{1|2}\right)$*:*

$$Q(F_1, F_2) = F_1 \partial_x \overline{D}_1 + F_2 \partial_x \overline{D}_2 + (-1)^{p(F)} \frac{\lambda + \frac{1}{2}}{\lambda - \mu + 1} \left(\overline{D}_1 (F_1) + \overline{D}_2 (F_2)\right) \partial_x$$
$$+ (-1)^{p(F)} \frac{1}{2(\lambda - \mu + 1)} \left(\overline{D}_2 (F_1) - \overline{D}_1 (F_2)\right) \overline{D}_1 \overline{D}_2$$
$$+ \frac{(\lambda + \frac{1}{2})(\lambda - \mu + \frac{1}{2})}{(\lambda - \mu + 1)^2} \left(\partial_x (F_1) \overline{D}_1 + \partial_x (F_2) \overline{D}_2\right)$$
$$- \frac{\lambda + \frac{1}{2}}{2(\lambda - \mu + 1)^2} \left(\overline{D}_1 \overline{D}_2 (F_1) \overline{D}_2 - \overline{D}_1 \overline{D}_2 (F_2) \overline{D}_1\right)$$
$$+ (-1)^{p(F)} \frac{\lambda(\lambda + \frac{1}{2})}{(\lambda - \mu + 1)^2} \left(\partial_x \overline{D}_1 (F_1) + \partial_x \overline{D}_2 (F_2)\right). \tag{10}$$

Proof. An arbitrary Aff (2|2)-equivariant quantization map is given by

$$Q(F_1,F_2) = F_1\partial_x\overline{D}_1 + F_2\partial_x\overline{D}_2 + \widetilde{Q}_1^{(1)}(F_1) + \widetilde{Q}_2^{(1)}(F_2) + \widetilde{Q}_1^{(2)}(F_1) + \widetilde{Q}_2^{(2)}(F_2) + \widetilde{Q}_1^{(3)}(F_1) + \widetilde{Q}_2^{(3)}(F_2),$$

where

$$\widetilde{Q}_1^{(1)}(F_1) = C_{11}\overline{D}_1(F_1)\partial_x + C_{13}\overline{D}_2(F_1)\overline{D}_1\overline{D}_2,$$
$$\widetilde{Q}_1^{(2)}(F_1) = C_{21}\partial_x(F_1)\overline{D}_1 + C_{23}\overline{D}_1\overline{D}_2(F_1)\overline{D}_2,$$
$$\widetilde{Q}_1^{(3)}(F_1) = C_{31}\partial_x\overline{D}_1(F_1),$$
$$\widetilde{Q}_2^{(1)}(F_2) = C_{14}\overline{D}_1(F_2)\overline{D}_1\overline{D}_2 + C_{12}\overline{D}_2(F_2)\partial_x,$$
$$\widetilde{Q}_2^{(2)}(F_2) = C_{24}\partial_x(F_2)\overline{D}_2 + C_{22}\overline{D}_1\overline{D}_2(F_2)\overline{D}_1,$$
$$\widetilde{Q}_2^{(3)}(F_2) = C_{32}\partial_x\overline{D}_2(F_2).$$

The above quantization map commutes with the action of X_{x^2} if and only if the coefficients C_{ij} satisfy the system of linear equations:

$$-2\left(\mu-\lambda-\tfrac{3}{2}\right)C_{11} - C_{12} = (-1)^{p(F)}(1+2\lambda),$$
$$-C_{11} - 2\left(\mu-\lambda-\tfrac{3}{2}\right)C_{12} = (-1)^{p(F)}(1+2\lambda),$$
$$-2\left(\mu-\lambda-\tfrac{3}{2}\right)C_{13} + C_{14} = (-1)^{p(F)},$$
$$C_{13} - 2\left(\mu-\lambda-\tfrac{3}{2}\right)C_{14} = (-1)^{p(F)+1},$$
$$2\left(\mu-\lambda-\tfrac{3}{2}\right)C_{21} - C_{22} = -(1+2\lambda),$$
$$-C_{21} + 4(\mu-\lambda-1)C_{22} = -C_{12} + (1+2\lambda)C_{14} - C_{24},$$
$$C_{23} + 2\left(\mu-\lambda-\tfrac{3}{2}\right)C_{24} = -(1+2\lambda),$$
$$4(\mu-\lambda-1)C_{23} + C_{24} = C_{11} + (1+2\lambda)C_{13} + C_{21},$$
$$4(\lambda-\mu+1)C_{31} - C_{32} = (-1)^{p(F)}2\lambda(C_{11}+C_{21}),$$
$$-C_{31} + 4(\lambda-\mu+1)C_{32} = (-1)^{p(F)}2\lambda(C_{12}+C_{24}).$$

Solving this system, one obtains the formula (10). ■

4.4 Case of 2-contact order differential operators

The last case we consider is the space of differential operators $\mathcal{D}^2_{\lambda\mu}\left(S^{1|2}\right)$. The proof of the following statement is again similar to that of Proposition 6; we will omit some details of calculations.

Proposition 8. *The unique* osp (2|2)*-equivariant quantization map associates the following differential operator to a symbol* $(F_1,F_2) \in \mathcal{S}^2_{\mu-\lambda}\left(S^{1|2}\right)$*:*

$$Q(F_1,F_2) = F_1\partial_x^2 + F_2\partial_x\overline{D}_1\overline{D}_2 + (-1)^{p(F)}\frac{1}{\mu-\lambda-2}\left(\overline{D}_1(F_1)\partial_x\overline{D}_1 + \overline{D}_2(F_1)\partial_x\overline{D}_2\right)$$
$$+ (-1)^{p(F)}\frac{\lambda+1}{\mu-\lambda-2}\left(\overline{D}_2(F_2)\partial_x\overline{D}_1 - \overline{D}_1(F_2)\partial_x\overline{D}_2\right)$$
$$+ \frac{2\lambda+1}{\lambda-\mu+2}\partial_x(F_1)\partial_x + \frac{\lambda+1}{\lambda-\mu+2}\partial_x(F_2)\overline{D}_1\overline{D}_2$$
$$- \frac{1}{(\lambda-\mu+2)(2\lambda-2\mu+3)}\overline{D}_1\overline{D}_2(F_1)\overline{D}_1\overline{D}_2$$

$$+\frac{(2\lambda+1)(\lambda+1)}{(\lambda-\mu+2)(2\lambda-2\mu+3)}\overline{D}_1\overline{D}_2(F_2)\,\partial_x$$
$$-(-1)^{p(F)}\frac{2\lambda+1}{(\lambda-\mu+2)(2\lambda-2\mu+3)}\left(\partial_x\overline{D}_1(F_1)\,\overline{D}_1+\partial_x\overline{D}_2(F_1)\,\overline{D}_2\right)$$
$$-(-1)^{p(F)}\frac{(2\lambda+1)(\lambda+1)}{(\lambda-\mu+2)(2\lambda-2\mu+3)}\left(\partial_x\overline{D}_2(F_2)\,\overline{D}_1-\partial_x\overline{D}_1(F_2)\,\overline{D}_2\right)$$
$$+\frac{\lambda(2\lambda+1)}{(\lambda-\mu+2)(2\lambda-2\mu+3)}\partial_x^2(F_1)$$
$$+\frac{\lambda(2\lambda+1)(\lambda+1)}{(\lambda-\mu+2)(2\lambda-2\mu+3)(\lambda-\mu+1)}\partial_x\overline{D}_1\overline{D}_2(F_2), \tag{11}$$

provided $\mu-\lambda\neq 1,\frac{3}{2},2$.

Proof. An arbitrary Aff $(2|2)$-equivariant quantization map is given by

$$Q(F_1,F_2)=F_1\partial_x^2+F_2\partial_x\overline{D}_1\overline{D}_2+\widetilde{Q}_1^{(1)}(F_1)+\widetilde{Q}_2^{(1)}(F_2)$$
$$+\widetilde{Q}_1^{(2)}(F_1)+\widetilde{Q}_2^{(2)}(F_2)+\widetilde{Q}_1^{(3)}(F_1)+\widetilde{Q}_2^{(3)}(F_2)+\widetilde{Q}_1^{(4)}(F_1)+\widetilde{Q}_2^{(4)}(F_2),$$

where

$$\widetilde{Q}_1^{(1)}(F_1)=C_{11}\overline{D}_1(F_1)\,\partial_x\overline{D}_1+C_{13}\overline{D}_2(F_1)\,\partial_x\overline{D}_2,$$
$$\widetilde{Q}_1^{(2)}(F_1)=C_{21}\partial_x(F_1)\,\partial_x+C_{23}\overline{D}_1\overline{D}_2(F_1)\,\overline{D}_1\overline{D}_2,$$
$$\widetilde{Q}_1^{(3)}(F_1)=C_{31}\partial_x\overline{D}_1(F_1)\,\overline{D}_1+C_{33}\partial_x\overline{D}_2(F_1)\,\overline{D}_2,$$
$$\widetilde{Q}_1^{(4)}(F_1)=C_{41}\partial_x^2(F_1),$$
$$\widetilde{Q}_2^{(1)}(F_2)=C_{14}\overline{D}_1(F_2)\,\partial_x\overline{D}_2+C_{12}\overline{D}_2(F_2)\,\partial_x\overline{D}_1,$$
$$\widetilde{Q}_2^{(2)}(F_2)=C_{24}\partial_x(F_2)\,\overline{D}_1\overline{D}_2+C_{22}\overline{D}_1\overline{D}_2(F_2)\,\partial_x,$$
$$\widetilde{Q}_2^{(3)}(F_2)=C_{34}\partial_x\overline{D}_1(F_2)\,\overline{D}_2+C_{32}\partial_x\overline{D}_2(F_2)\,\overline{D}_1,$$
$$\widetilde{Q}_2^{(4)}(F_2)=C_{42}\partial_x\overline{D}_1\overline{D}_2(F_2).$$

The above quantization map commutes with the action of X_{x^2} and therefore is osp $(2|2)$-equivariant if and only if the coefficients C_{ij} are as in (11). ■

Acknowledgements

I am grateful to H. Gargoubi and V. Ovsienko for the statement of the problem and constant help. I am also pleased to thank D. Leites for critical reading of this paper and a number of helpful suggestions.

References

[1] Cohen P., Manin Yu., Zagier D., Automorphic pseudodifferential operators, in Algebraic Aspects of Integrable Systems, *Progr. Nonlinear Differential Equations Appl.*, Vol. 26, Birkhäuser Boston, Boston, MA, 1997, 17–47.

[2] Conley C., Conformal symbols and the action of contact vector fields over the superline, *J. Reine Angew. Math.* **633** (2009), 115–163, arXiv:0712.1780.

[3] Duval C., Lecomte P., Ovsienko V., Conformally equivariant quantization: existence and uniqueness, *Ann. Inst. Fourier (Grenoble)* **49** (1999), 1999–2029, math.DG/9902032.

[4] Fregier Y., Mathonet P., Poncin N., Decomposition of symmetric tensor fields in the presence of a flat contact projective structure, *J. Nonlinear Math. Phys.* **15** (2008), 252–269, math.DG/0703922.

[5] Gargoubi H., Mellouli N., Ovsienko V., Differential operators on supercircle: conformally equivariant quantization and symbol calculus, *Lett. Math. Phys.* **79** (2007), 51–65, math-ph/0610059.

[6] Grozman P., Leites D., Shchepochkina I., Lie superalgebras of string theories, *Acta Math. Vietnam.* **26** (2001), 27–63, hep-th/9702120.

[7] Grozman P., Leites D., Shchepochkina I., Invariant operators on supermanifolds and standard models, in Multiple Facets of Quantization and Supersymmetry, Editors M. Olshanetski and A. Vainstein, World Sci. Publ., River Edge, NJ, 2002, 508–555, math.RT/0202193.

[8] Lecomte P.B.A., Ovsienko V.Yu., Projectively invariant symbol calculus, *Lett. Math. Phys.* **49** (1999), 173–196, math.DG/9809061.

[9] Leites D., Supermanifold theory, Petrozavodsk, 1983 (in Russian).

[10] Leites D., Kochetkov Yu., Weintrob A., New invariant differential operators on supermanifolds and pseudo-(co)homology, in General Topology and Applications (Staten Island, NY, 1989), *Lecture Notes in Pure and Appl. Math.*, Vol. 134, Dekker, New York, 1991, 217–238.

[11] Michel J.-P., Duval C., On the projective geometry of the supercircle: a unified construction of the super cross-ratio and Schwarzian derivative, *Int. Math. Res. Not. IMRN* **2008** (2008), no. 14, Art. ID rnn054, 47 pages, arXiv:0710.1544.

[12] Ovsienko V., Vector fields in the presence of a contact structure, *Enseign. Math. (2)* **52** (2006), 215–229, math.DG/0511499.

[13] Ovsienko V.Yu., Ovsienko O.D., Chekanov Yu.V., Classification of contact-projective structures on the supercircle, *Russian Math. Surveys* **44** (1989), no. 3, 212–213.

[14] Shchepochkina I.M., How to realize Lie algebras by vector fields, *Theoret. and Math. Phys.* **147** (2006), 821–838, math.RT/0509472.