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Hypergeometric τ Functions of the q-Painlevé Systems of Type $(A_2 + A_1)^{(1)}$

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Abstract. We consider a q-Painlevé III equation and a q-Painlevé II equation arising from a birational representation of the affine Weyl group of type $(A_2 + A_1)^{(1)}$. We study their hypergeometric solutions on the level of τ functions.

Key words: q-Painlevé system; hypergeometric function; affine Weyl group; τ function

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1 Introduction

We consider a q-analog of the Painlevé III equation (q-P$_{\rm III}$) [8, 12, 13, 32]

$$g_{n+1} = \frac{q^{2N+1}c^2}{f_n g_n} \frac{1 + a_0 q^n f_n}{a_0 q^n + f_n}, \qquad f_{n+1} = \frac{q^{2N+1}c^2}{f_n g_{n+1}} \frac{1 + a_2 a_0 q^{n-m} g_{n+1}}{a_2 a_0 q^{n-m} + g_{n+1}}, \tag{1.1}$$

and that of the Painlevé II equation (q-P$_{\rm II}$) [12, 30, 20]

$$X_{k+1} = \frac{q^{2N+1}c^2}{X_k X_{k-1}} \frac{1 + a_0 q^{k/2} X_k}{a_0 q^{k/2} + X_k}, \tag{1.2}$$

for the unknown functions $f_n = f_n(m, N)$, $g_n = g_n(m, N)$, and $X_k = X_k(N)$ and the independent variables $n, k \in \mathbb{Z}$. Here $m, N \in \mathbb{Z}$ and $a_0, a_2, c, q \in \mathbb{C}^\times$ are parameters. These equations arise from a birational representation of the (extended) affine Weyl group of type $(A_2 + A_1)^{(1)}$.

Note that substituting

$$m = 0, \qquad a_2 = q^{1/2},$$

and putting

$$f_k(0, N) = X_{2k}(N), \qquad g_k(0, N) = X_{2k-1}(N),$$

in (1.1) yield (1.2). This procedure is called a symmetrization of (1.1), which comes from the terminology used for Quispel–Roberts–Thompson (QRT) mappings [28, 29].

It is well known that the τ functions play a crucial role in the theory of integrable systems [19], and it is also possible to introduce them in the theory of Painlevé systems [5, 6, 7, 13, 21, 22, 24, 25, 26, 27]. A representation of the affine Weyl groups can be lifted on the level of the τ functions [10, 11, 33], which gives rise to various bilinear equations of Hirota type satisfied the τ functions.

The hypergeometric solutions of various Painlevé and discrete Painlevé systems are expressible in the form of ratio of determinants whose entries are given by hypergeometric type functions.

Usually, they are derived by reducing the bilinear equations to the Plücker relations by using the contiguity relations satisfied by the entries of determinants [2, 3, 4, 8, 9, 13, 14, 15, 16, 20, 23, 31]. This method is elementary, but it encounters technical difficulties for Painlevé systems with large symmetries. In order to overcome this difficulty, Masuda has proposed a method of constructing hypergeometric solutions under a certain boundary condition on the lattice where the τ functions live (*hypergeometric* τ *functions*), so that they are consistent with the action of the affine Weyl groups. Although this requires somewhat complex calculations, the merit is that it is systematic and that it can be applied to the systems with large symmetries. Masuda has carried out the calculations for the q-Painlevé systems with $E_7^{(1)}$ and $E_8^{(1)}$ symmetries [17, 18] and presented explicit determinant formulae for their hypergeometric solutions.

The purpose of this paper is to apply the above method to the q-Painlevé systems with the affine Weyl group symmetry of type $(A_2+A_1)^{(1)}$ and present the explicit formulae of the hypergeometric τ functions. The hypergeometric τ functions provide not only determinant formulae but also important information originating from the geometry of lattice of the τ functions. The result has been already announced in [12] and played an essential role in clarifying the mechanism of reduction from hypergeometric solutions of (1.1) to those of (1.2).

This paper is organized as follows: in Section 2, we first review hypergeometric solutions of q-$\mathrm{P_{III}}$ and then those of q-$\mathrm{P_{II}}$. We next introduce a representation of the affine Weyl group of type $(A_2+A_1)^{(1)}$. In Section 3, we construct the hypergeometric τ functions of q-$\mathrm{P_{III}}$ and those of q-$\mathrm{P_{II}}$. We find that the symmetry of the hypergeometric τ functions of q-$\mathrm{P_{III}}$ are connected with Heine's transform of the basic hypergeometric series ${}_2\varphi_1$.

We use the following conventions of q-analysis throughout this paper [1].

q-Shifted factorials:

$$(a;q)_k = \prod_{i=1}^{k}\left(1-aq^{i-1}\right).$$

Basic hypergeometric series:

$${}_s\varphi_r\left(\begin{matrix}a_1,\dots,a_s\\ b_1,\dots,b_r\end{matrix};q,z\right) = \sum_{n=0}^{\infty}\frac{(a_1,\dots,a_s;q)_n}{(b_1,\dots,b_r;q)_n(q;q)_n}\left[(-1)^n q^{n(n-1)/2}\right]^{1+r-s} z^n,$$

where

$$(a_1,\dots,a_s;q)_n = \prod_{i=1}^{s}(a_i;q)_n.$$

Jacobi theta function:

$$\Theta(a;q) = (a;q)_\infty\left(qa^{-1};q\right)_\infty.$$

Elliptic gamma function:

$$\Gamma(a;p,q) = \frac{(pqa^{-1};p,q)_\infty}{(a;p,q)_\infty},$$

where

$$(a;p,q)_k = \prod_{i,j=0}^{k-1}\left(1-p^iq^ja\right).$$

It holds that

$$\Theta(qa;q) = -a^{-1}\Theta(a;q), \qquad \Gamma(qa;q,q) = \Theta(a;q)\Gamma(a;q,q).$$

2 q-P$_{\rm III}$ and q-P$_{\rm II}$

2.1 Hypergeometric solutions of q-P$_{\rm III}$ and q-P$_{\rm II}$

First, we review the hypergeometric solutions of q-P$_{\rm III}$ and q-P$_{\rm II}$. The hypergeometric solutions of q-P$_{\rm III}$ have been constructed as follows:

Proposition 2.1 ([8]). *The hypergeometric solutions of* q-P$_{\rm III}$, (1.1), *with* $c = 1$ *are given by*

$$f_n = -a_0 q^n \frac{\psi_{N+1}^{n,m-1}\psi_N^{n,m}}{\psi_{N+1}^{n,m}\psi_N^{n,m-1}}, \qquad g_n = {a_0}^{-1} a_2 q^{-n-m+1} \frac{\psi_{N+1}^{n,m}\psi_N^{n-1,m-1}}{\psi_{N+1}^{n-1,m-1}\psi_N^{n,m}},$$

where $\psi_N^{n,m}$ $(N \in \mathbb{Z}_{\geq 0})$ *is an* $N \times N$ *determinant defined by*

$$\psi_N^{n,m} = \begin{vmatrix} F_{n,m} & F_{n+1,m} & \cdots & F_{n+N-1,m} \\ F_{n-1,m} & F_{n,m} & \cdots & F_{n+N-2,m} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n-N+1,m} & F_{n-N+2,m} & \cdots & F_{n,m} \end{vmatrix}, \qquad \psi_0^{n,m} = 1,$$

and $F_{n,m}$ *is an arbitrary solution of the systems*

$$F_{n+1,m} - F_{n,m} = -{a_0}^2 q^{2n} F_{n,m-1}, \tag{2.1}$$

$$F_{n,m+1} - F_{n,m} = -{a_2}^{-2} q^{2m+2} F_{n-1,m}. \tag{2.2}$$

The general solution of (2.1) and (2.2) is given by

$$\begin{aligned} F_{n,m} = {} & \frac{A_{n,m}}{({a_2}^{-2}q^{2m+2};q^2)_\infty}\, {}_1\varphi_1\left(\begin{matrix}0\\ {a_2}^2 q^{-2m}\end{matrix}; q^2, {a_2}^2{a_0}^2 q^{2n-2m}\right) \\ & + B_{n,m}\frac{\Theta({a_0}^2{a_2}^2 q^{2n-2m-2};q^2)}{({a_2}^2 q^{-2m-2};q^2)_\infty \Theta({a_0}^2 q^{2n};q^2)}\, {}_1\varphi_1\left(\begin{matrix}0\\ {a_2}^{-2}q^{2m+4}\end{matrix}; q^2, {a_0}^2 q^{2n+2}\right), \end{aligned} \tag{2.3}$$

where $A_{n,m}$ and $B_{n,m}$ are periodic functions of period one with respect to n and m, i.e.,

$$A_{n,m} = A_{n+1,m} = A_{n,m+1}, \qquad B_{n,m} = B_{n+1,m} = B_{n,m+1}.$$

The explicit form of the hypergeometric solutions of q-P$_{\rm II}$ are given as follows:

Proposition 2.2 ([20]). *The hypergeometric solutions of* q-P$_{\rm II}$, (1.2), *with* $c = 1$ *are given by*

$$X_k = -a_0 q^{k/2+N} \frac{\phi_{N+1}^k \phi_N^{k-1}}{\phi_{N+1}^{k-1}\phi_N^k}, \tag{2.4}$$

where ϕ_N^k $(N \in \mathbb{Z}_{\geq 0})$ *is an* $N \times N$ *determinant defined by*

$$\phi_N^k = \begin{vmatrix} G_k & G_{k-1} & \cdots & G_{k-N+1} \\ G_{k+2} & G_{k+1} & \cdots & G_{k-N+3} \\ \vdots & \vdots & \ddots & \vdots \\ G_{k+2N-2} & G_{k+2N-3} & \cdots & G_{k+N-1} \end{vmatrix}, \qquad \phi_0^k = 1, \tag{2.5}$$

and G_k *is an arbitrary solution of the system*

$$G_{k+1} - G_k + {a_0}^{-2} q^{-k} G_{k-1} = 0. \tag{2.6}$$

The general solution of (2.6) is given by

$$\begin{aligned} G_k &= A_k\Theta(ia_0q^{(2k+1)/4};q^{1/2})\ {}_1\varphi_1\left(\begin{matrix}0\\ -q^{1/2}\end{matrix};q^{1/2},-ia_0q^{(3+2k)/4}\right)\\ &\quad + B_k\Theta(-ia_0q^{(2k+1)/4};q^{1/2})\ {}_1\varphi_1\left(\begin{matrix}0\\ -q^{1/2}\end{matrix};q^{1/2},ia_0q^{(3+2k)/4}\right), \end{aligned} \tag{2.7}$$

where A_k and B_k are periodic functions of period one, i.e.,

$$A_k = A_{k+1}, \qquad B_k = B_{k+1}.$$

2.2 Projective reduction from q-P$_{\rm III}$ and q-P$_{\rm II}$

We formulate the family of Bäcklund transformations of q-P$_{\rm III}$ and q-P$_{\rm IV}$ as a birational representation of the affine Weyl group of type $(A_2+A_1)^{(1)}$. Here, q-P$_{\rm IV}$ is a q-analog of the Painlevé IV equation discussed in [13]. We refer to [21] for basic ideas of this formulation.

We define the transformations s_i $(i=0,1,2)$ and π on the variables f_j $(j=0,1,2)$ and parameters a_k $(k=0,1,2)$ by

$$\begin{aligned} &s_i(a_j) = a_j{a_i}^{-a_{ij}}, \qquad && s_i(f_j) = f_j\left(\frac{a_i+f_i}{1+a_if_i}\right)^{u_{ij}},\\ &\pi(a_i) = a_{i+1}, && \pi(f_i) = f_{i+1}, \end{aligned}$$

for $i,j\in\mathbb{Z}/3\mathbb{Z}$. Here the symmetric 3×3 matrix

$$A = (a_{ij})_{i,j=0}^2 = \begin{pmatrix} 2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix},$$

is the Cartan matrix of type $A_2^{(1)}$, and the skew-symmetric one

$$U = (u_{ij})_{i,j=0}^2 = \begin{pmatrix} 0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix},$$

represents an orientation of the corresponding Dynkin diagram. We also define the transformations w_j $(j=0,1)$ and r by

$$\begin{aligned} &w_0(f_i) = \frac{a_ia_{i+1}(a_{i-1}a_i+a_{i-1}f_i+f_{i-1}f_i)}{f_{i-1}(a_ia_{i+1}+a_if_{i+1}+f_if_{i+1})}, && w_0(a_i)=a_i,\\ &w_1(f_i) = \frac{1+a_if_i+a_ia_{i+1}f_if_{i+1}}{a_ia_{i+1}f_{i+1}(1+a_{i-1}f_{i-1}+a_{i-1}a_if_{i-1}f_i)}, && w_1(a_i)=a_i,\\ &r(f_i) = \frac{1}{f_i}, && r(a_i)=a_i, \end{aligned}$$

for $i\in\mathbb{Z}/3\mathbb{Z}$.

Proposition 2.3 ([13])**.** *The group of birational transformations $\langle s_0,s_1,s_2,\pi,w_0,w_1,r\rangle$ forms the affine Weyl group of type $(A_2+A_1)^{(1)}$, denoted by $\widetilde{W}((A_2+A_1)^{(1)})$. Namely, the transformations satisfy the fundamental relations*

$$\begin{aligned} &{s_i}^2 = (s_is_{i+1})^3 = \pi^3 = 1, \qquad \pi s_i = s_{i+1}\pi \qquad (i\in\mathbb{Z}/3\mathbb{Z}),\\ &{w_0}^2 = {w_1}^2 = r^2 = 1, \qquad rw_0 = w_1r, \end{aligned}$$

and the action of $\widetilde{W}(A_2^{(1)}) = \langle s_0,s_1,s_2,\pi\rangle$ and that of $\widetilde{W}(A_1^{(1)}) = \langle w_0,w_1,r\rangle$ commute with each other.

In general, for a function $F=F(a_i,f_j)$, we let an element $w\in\widetilde{W}((A_2+A_1)^{(1)})$ act as $w.F(a_i,f_j)=F(a_i.w,f_j.w)$, that is, w acts on the arguments from the right. Note that $a_0a_1a_2=q$ and $f_0f_1f_2=qc^2$ are invariant under the action of $\widetilde{W}((A_2+A_1)^{(1)})$ and $\widetilde{W}(A_2^{(1)})$, respectively. We define the translations T_i $(i=1,2,3,4)$ by

$$T_1=\pi s_2s_1,\qquad T_2=s_1\pi s_2,\qquad T_3=s_2s_1\pi,\qquad T_4=rw_0,\tag{2.8}$$

whose action on parameters a_i $(i=0,1,2)$ and c is given by

$$\begin{aligned}
&T_1:\ (a_0,a_1,a_2,c)\mapsto\big(qa_0,q^{-1}a_1,a_2,c\big),\\
&T_2:\ (a_0,a_1,a_2,c)\mapsto(a_0,qa_1,q^{-1}a_2,c),\\
&T_3:\ (a_0,a_1,a_2,c)\mapsto\big(q^{-1}a_0,a_1,qa_2,c\big),\\
&T_4:\ (a_0,a_1,a_2,c)\mapsto(a_0,a_1,a_2,qc).
\end{aligned}$$

Note that T_i $(i=1,2,3,4)$ commute with each other and $T_1T_2T_3=1$. The action of T_1 on the f-variables can be expressed as

$$T_1(f_1)=\frac{qc^2}{f_1f_0}\frac{1+a_0f_0}{a_0+f_0},\qquad T_1(f_0)=\frac{qc^2}{f_0T_1(f_1)}\frac{1+a_2a_0T_1(f_1)}{a_2a_0+T_1(f_1)}.\tag{2.9}$$

Or, applying ${T_1}^n{T_2}^m{T_4}^N$ $(n,m,N\in\mathbb{Z})$ on (2.9) and putting

$$f_{i,N}^{n,m}={T_1}^n{T_2}^m{T_4}^N(f_i)\qquad(i=0,1,2),$$

we obtain

$$f_{1,N}^{n+1,m}=\frac{q^{2N+1}c^2}{f_{1,N}^{n,m}f_{0,N}^{n,m}}\frac{1+a_0q^nf_{0,N}^{n,m}}{a_0q^n+f_{0,N}^{n,m}},\qquad f_{0,N}^{n+1,m}=\frac{q^{2N+1}c^2}{f_{0,N}^{n,m}f_{1,N}^{n+1,m}}\frac{1+a_2a_0q^{n-m}f_{1,N}^{n+1,m}}{a_2a_0q^{n-m}+f_{1,N}^{n+1,m}},$$

which is equivalent to q-$\mathrm{P_{III}}$. Then T_1 and T_i $(i=2,4)$ are regarded as the time evolution and Bäcklund transformations of q-$\mathrm{P_{III}}$, respectively. We here note that we also obtain q-$\mathrm{P_{IV}}$ by identifying T_4 as a time evolution [13].

In order to formulate the symmetrization to q-$\mathrm{P_{II}}$, it is crucial to introduce the transformation R_1 defined by

$$R_1=\pi^2s_1,\tag{2.10}$$

which satisfies

$${R_1}^2=T_1.$$

Considering the projection of the action of R_1 on the line $a_2=q^{1/2}$, we have

$$\begin{aligned}
&R_1:\ (a_0,a_1,c)\mapsto(q^{1/2}a_0,q^{-1/2}a_1,c),\\
&R_1(f_0)=\frac{qc^2}{f_0f_1}\frac{1+a_0f_0}{a_0+f_0},\qquad R_1(f_1)=f_0.
\end{aligned}\tag{2.11}$$

Applying ${R_1}^k{T_4}^N$ on (2.11) and putting

$$f_{i,N}^{k}={R_1}^k{T_4}^N(f_i)\qquad(i=0,1,2),$$

we have

$$f_{0,N}^{k+1}=\frac{q^{2N+1}c^2}{f_{0,N}^kf_{0,N}^{k-1}}\frac{1+a_0q^{k/2}f_{0,N}^k}{a_0q^{k/2}+f_{0,N}^k},$$

which is equivalent to q-P$_{\rm II}$. Then R_1 and T_4 are regarded as the time evolution and a Bäcklund transformation of q-P$_{\rm II}$, respectively.

In general, we can derive various discrete Painlevé systems from elements of infinite order of affine Weyl groups that are not necessarily translations by taking a projection on a certain subspace of the parameter space. We call such a procedure a projective reduction [12]. The symmetrization is a kind of the projective reduction.

2.3 Birational representation of $\widetilde{W}((A_2+A_1)^{(1)})$ on the τ function

We introduce the new variables τ_i and $\overline{\tau}_i$ ($i\in\mathbb{Z}/3\mathbb{Z}$) by letting

$$f_i=q^{1/3}c^{2/3}\,\frac{\overline{\tau}_{i+1}\tau_{i-1}}{\tau_{i+1}\overline{\tau}_{i-1}},$$

and lift a representation to the affine Weyl group on their level:

Proposition 2.4 ([33]). *We define the action of s_i $(i=0,1,2)$, π, w_j $(j=0,1)$, and r on τ_k and $\overline{\tau}_k$ $(k=0,1,2)$ by the following formulae:*

$$\begin{aligned}
&s_i(\tau_i)=\frac{u_i\tau_{i+1}\overline{\tau}_{i-1}+\overline{\tau}_{i+1}\tau_{i-1}}{{u_i}^{1/2}\overline{\tau}_i}, && s_i(\tau_j)=\tau_j\quad(i\neq j),\\
&s_i(\overline{\tau}_i)=\frac{v_i\overline{\tau}_{i+1}\tau_{i-1}+\tau_{i+1}\overline{\tau}_{i-1}}{{v_i}^{1/2}\tau_i}, && s_i(\overline{\tau}_j)=\overline{\tau}_j\quad(i\neq j),\\
&\pi(\tau_i)=\tau_{i+1}, && \pi(\overline{\tau}_i)=\overline{\tau}_{i+1},\\
&w_0(\overline{\tau}_i)=\frac{{a_{i+1}}^{1/3}(\overline{\tau}_i\tau_{i+1}\tau_{i+2}+u_{i-1}\tau_i\overline{\tau}_{i+1}\tau_{i+2}+{u_{i+1}}^{-1}\tau_i\tau_{i+1}\overline{\tau}_{i+2})}{{a_{i+2}}^{1/3}\overline{\tau}_{i+1}\overline{\tau}_{i+2}}, && w_0(\tau_i)=\tau_i,\\
&w_1(\tau_i)=\frac{{a_{i+1}}^{1/3}(\tau_i\overline{\tau}_{i+1}\overline{\tau}_{i+2}+v_{i-1}\overline{\tau}_i\tau_{i+1}\overline{\tau}_{i+2}+{v_{i+1}}^{-1}\overline{\tau}_i\overline{\tau}_{i+1}\tau_{i+2})}{{a_{i+2}}^{1/3}\tau_{i+1}\tau_{i+2}}, && w_1(\overline{\tau}_i)=\overline{\tau}_i,\\
&r(\tau_i)=\overline{\tau}_i, && r(\overline{\tau}_i)=\tau_i,
\end{aligned}$$

with

$$u_i=q^{-1/3}c^{-2/3}a_i,\qquad v_i=q^{1/3}c^{2/3}a_i,$$

where $i,j\in\mathbb{Z}/3\mathbb{Z}$. Then, $\langle s_0,s_1,s_2,\pi,w_0,w_1,r\rangle$ forms the affine Weyl group $\widetilde{W}((A_2+A_1)^{(1)})$.

We define the τ function $\tau_N^{n,m}$ $(n,m,N\in\mathbb{Z})$ by

$$\tau_N^{n,m}={T_1}^n{T_2}^m{T_4}^N(\tau_1).$$

We note that

$$\tau_0=\tau_0^{-1,0},\qquad \tau_1=\tau_0^{0,0},\qquad \tau_2=\tau_0^{0,1},\qquad \overline{\tau}_0=\tau_1^{-1,0},\qquad \overline{\tau}_1=\tau_1^{0,0},\qquad \overline{\tau}_2=\tau_1^{0,1},\quad (2.12)$$

and

$$f_{0,N}^{n,m}=q^{(2N+1)/3}c^{2/3}\,\frac{\tau_{N+1}^{n,m}\tau_N^{n,m+1}}{\tau_N^{n,m}\tau_{N+1}^{n,m+1}},\qquad f_{1,N}^{n,m}=q^{(2N+1)/3}c^{2/3}\,\frac{\tau_{N+1}^{n,m+1}\tau_N^{n-1,m}}{\tau_N^{n,m+1}\tau_{N+1}^{n-1,m}},$$

$$f_{2,N}^{n,m}=q^{(2N+1)/3}c^{2/3}\,\frac{\tau_{N+1}^{n-1,m}\tau_N^{n,m}}{\tau_N^{n-1,m}\tau_{N+1}^{n,m}}.$$

Let us consider the τ functions for q-$\mathrm{P_{II}}$. We set

$$\tau_N^k = {R_1}^k {T_4}^N(\tau_1).$$

Note that

$$\tau_0 = \tau_0^{-2}, \qquad \tau_1 = \tau_0^0, \qquad \tau_2 = \tau_0^{-1}, \qquad \overline{\tau}_0 = \tau_1^{-2}, \qquad \overline{\tau}_1 = \tau_1^0, \qquad \overline{\tau}_2 = \tau_1^{-1}, \tag{2.13}$$

and

$$f_{0,N}^k = q^{(2N+1)/3} c^{2/3} \frac{\tau_{N+1}^k \tau_N^{k-1}}{\tau_N^k \tau_{N+1}^{k-1}}.$$

In general, it follows that

$$\tau_N^{n,0} = \tau_N^{2n}, \qquad \tau_N^{n,1} = \tau_N^{2n-1}.$$

For convenience, we introduce α_i, γ, and Q by

$${\alpha_i}^6 = a_i, \qquad \gamma^6 = c, \qquad Q^6 = q.$$

3 Hypergeometric τ functions of the q-Painlevé systems of type $(A_2 + A_1)^{(1)}$

In this section, we construct the hypergeometric τ functions of q-$\mathrm{P_{III}}$ and q-$\mathrm{P_{II}}$. We define the hypergeometric τ functions of q-$\mathrm{P_{III}}$ by $\tau_N^{n,m}$ consistent with the action of $\langle T_1, T_2, T_3, T_4\rangle$. We also define the hypergeometric τ functions of q-$\mathrm{P_{II}}$ by τ_N^k consistent with the action of $\langle R_1, T_4\rangle$. Here, we mean $\tau(\alpha)$ consistent with a action of transformation r as

$$r.\tau(\alpha) = \tau(\alpha.r).$$

We then regard $\tau_N^{n,m}$ as function in α_0 and α_2, i.e.,

$$\tau_N^{n,m} = \tau_N^{0,0}(Q^n\alpha_0, Q^{-m}\alpha_2).$$

We also regard τ_N^k as function in α_0, i.e.,

$$\tau_N^k = \tau_N^0(Q^{k/2}\alpha_0).$$

3.1 Hypergeometric τ functions of q-$\mathrm{P_{III}}$

We construct the hypergeometric τ functions of q-$\mathrm{P_{III}}$. By the action of the affine Weyl group, $\tau_N^{n,m}$ is determined as a rational function in $\tau_0^{n,m}$ and $\tau_1^{n,m}$ (or τ_i and $\overline{\tau}_i$). Thus, our purpose is determining $\tau_0^{n,m}$ and $\tau_1^{n,m}$ consistent with the action of $\langle T_1, T_2, T_3, T_4\rangle$ and constructing $\tau_N^{n,m}$ under the condition

$$\gamma = 1, \tag{3.1}$$

and the boundary condition

$$\tau_N^{n,m} = 0 \quad (N < 0). \tag{3.2}$$

First we consider the condition for $\tau_0^{n,m}$ which follows from the boundary condition (3.2). We use the bilinear equations obtained in [12]:

Proposition 3.1. *The following bilinear equations hold:*

$$\tau_{N+1}^{n,m}\tau_{N-1}^{n,m}+Q^{4n-8m+4}{\alpha_1}^{-4}{\alpha_2}^4\left(\tau_N^{n,m}\right)^2-Q^{n-2m+1}{\alpha_1}^{-1}\alpha_2\tau_N^{n,m+1}\tau_N^{n,m-1}=0, \tag{3.3}$$

$$\tau_{N+1}^{n,m}\tau_{N-1}^{n,m}+Q^{4n+4m}{\alpha_0}^4{\alpha_2}^{-4}\left(\tau_N^{n,m}\right)^2-Q^{n+m}\alpha_0{\alpha_2}^{-1}\tau_N^{n+1,m+1}\tau_N^{n-1,m-1}=0, \tag{3.4}$$

$$\tau_{N+1}^{n,m}\tau_{N-1}^{n,m}+Q^{-8n+4m-4}{\alpha_0}^{-4}{\alpha_1}^4\left(\tau_N^{n,m}\right)^2-Q^{-2n+m-1}{\alpha_0}^{-1}\alpha_1\tau_N^{n+1,m}\tau_N^{n-1,m}=0. \tag{3.5}$$

By putting $N=0$ in (3.3)–(3.5), we get

$$Q^{4n-8m+4}{\alpha_1}^{-4}{\alpha_2}^4\left(\tau_0^{n,m}\right)^2-Q^{n-2m+1}{\alpha_1}^{-1}\alpha_2\tau_0^{n,m+1}\tau_0^{n,m-1}=0, \tag{3.6}$$

$$Q^{4n+4m}{\alpha_0}^4{\alpha_2}^{-4}\left(\tau_0^{n,m}\right)^2-Q^{n+m}\alpha_0{\alpha_2}^{-1}\tau_0^{n+1,m+1}\tau_0^{n-1,m-1}=0, \tag{3.7}$$

$$Q^{-8n+4m-4}{\alpha_0}^{-4}{\alpha_1}^4\left(\tau_0^{n,m}\right)^2-Q^{-2n+m-1}{\alpha_0}^{-1}\alpha_1\tau_0^{n+1,m}\tau_0^{n-1,m}=0. \tag{3.8}$$

We set

$$\tau_0^{n,m}=\Gamma(Q^{2n-m+1}{\alpha_0}^2\alpha_2;Q,Q)\Gamma(Q^{-n+2m-1}{\alpha_1}^2\alpha_0;Q,Q)\Gamma(Q^{-n-m}{\alpha_2}^2\alpha_1;Q,Q)A_0^{n,m}. \tag{3.9}$$

From (3.6)–(3.8), the following equations hold:

$$(A_0^{n,m})^2=A_0^{n,m+1}A_0^{n,m-1}, \tag{3.10}$$

$$(A_0^{n,m})^2=A_0^{n+1,m+1}A_0^{n-1,m-1}, \tag{3.11}$$

$$(A_0^{n,m})^2=A_0^{n+1,m}A_0^{n-1,m}. \tag{3.12}$$

We next determine $\tau_0^{n,m}$ and $\tau_1^{n,m}$. From (2.8) and Proposition 2.4, we see that the action of T_1, T_2, and T_3 are given by

$$T_i(\tau_{i-1})=\tau_i, \tag{3.13}$$

$$T_i(\overline{\tau}_{i-1})=\overline{\tau}_i, \tag{3.14}$$

$$T_i(\tau_{i+1})=\frac{{\alpha_{i-1}}^6\tau_i\overline{\tau}_{i+1}+Q^2\overline{\tau}_i\tau_{i+1}}{Q{\alpha_{i-1}}^3\overline{\tau}_{i-1}}, \tag{3.15}$$

$$T_i(\overline{\tau}_{i+1})=\frac{Q^2{\alpha_{i-1}}^6\tau_{i+1}\overline{\tau}_i+\overline{\tau}_{i+1}\tau_i}{Q{\alpha_{i-1}}^3\tau_{i-1}}, \tag{3.16}$$

$$T_i(\tau_i)=\frac{1}{{\alpha_{i+1}}^3{\alpha_{i-1}}^6}\frac{{\tau_i}^2}{\tau_{i-1}}+\frac{\alpha_{i+1}{\alpha_{i-1}}^4}{{\alpha_i}^2}\frac{\tau_i\overline{\tau}_i}{\overline{\tau}_{i-1}}+\frac{{\alpha_i}^2{\alpha_{i-1}}^2}{\alpha_{i+1}}\frac{\overline{\tau}_i\tau_i\tau_{i+1}}{\overline{\tau}_{i+1}\tau_{i-1}}+{\alpha_{i+1}}^3\frac{{\overline{\tau}_i}^2\tau_{i+1}}{\overline{\tau}_{i+1}\overline{\tau}_{i-1}}, \tag{3.17}$$

$$\begin{aligned}T_i(\overline{\tau}_i)=&\frac{1}{{\alpha_{i+1}}^3{\alpha_{i-1}}^6}\frac{{\overline{\tau}_i}^2}{\overline{\tau}_{i-1}}+{\alpha_i}^2{\alpha_{i+1}}^5{\alpha_{i-1}}^8\frac{\tau_i\overline{\tau}_i}{\tau_{i-1}}\\&+\frac{1}{{\alpha_i}^2{\alpha_{i+1}}^5{\alpha_{i-1}}^2}\frac{\tau_i\overline{\tau}_i\overline{\tau}_{i+1}}{\tau_{i+1}\overline{\tau}_{i-1}}+{\alpha_{i+1}}^3\frac{{\tau_i}^2\overline{\tau}_{i+1}}{\tau_{i+1}\tau_{i-1}},\end{aligned} \tag{3.18}$$

where $i=1,2,3$.

Lemma 3.1. *If τ_i and $\overline{\tau}_i$ are consistent with (3.13)–(3.16), then they are also consistent with (3.17) and (3.18).*

Proof. Applying T_{i-1} on (3.16) and using (3.13) and (3.14), we have

$$T_i(\tau_i)=\frac{Q}{{\alpha_{i+1}}^3{\alpha_{i-1}}^3}\frac{\tau_i}{\overline{\tau}_{i+1}}T_i(\overline{\tau}_{i+1})+\frac{{\alpha_{i+1}}^2{\alpha_{i-1}}^2}{\alpha_i}\frac{\overline{\tau}_i}{\overline{\tau}_{i+1}}T_i(\tau_{i+1}). \tag{3.19}$$

By using (3.15) and (3.16) for (3.19), we get (3.17). Similarly, applying T_{i-1} on (3.15) and using (3.13) and (3.14), we have

$$T_i(\overline{\tau}_i)=\frac{1}{Q{\alpha_{i+1}}^3{\alpha_{i-1}}^3}\frac{\overline{\tau}_i}{\tau_{i+1}}T_i(\tau_{i+1})+Q{\alpha_{i-1}}^3{\alpha_{i+1}}^3\frac{\tau_i}{\tau_{i+1}}T_i(\overline{\tau}_{i+1}). \tag{3.20}$$

By using (3.15) and (3.16) for (3.20), we get (3.18). ■

From (2.12), we rewrite (3.15) and (3.16) as follows:

$$\tau_0^{-1,0}\tau_1^{0,1} - Q^{-1}{\alpha_1}^3\tau_0^{-1,1}\tau_1^{0,0} + Q^{-2}{\alpha_1}^6\tau_0^{0,1}\tau_1^{-1,0} = 0, \tag{3.21}$$
$$\tau_0^{0,0}\tau_1^{-1,0} - Q^{-1}{\alpha_2}^3\tau_0^{-1,-1}\tau_1^{0,1} + Q^{-2}{\alpha_2}^6\tau_0^{-1,0}\tau_1^{0,0} = 0, \tag{3.22}$$
$$\tau_0^{0,1}\tau_1^{0,0} - Q^{-1}{\alpha_0}^3\tau_0^{1,1}\tau_1^{-1,0} + Q^{-2}{\alpha_0}^6\tau_0^{0,0}\tau_1^{0,1} = 0, \tag{3.23}$$
$$\tau_0^{0,1}\tau_1^{-1,0} - Q{\alpha_1}^3\tau_0^{0,0}\tau_1^{-1,1} + Q^{2}{\alpha_1}^6\tau_0^{-1,0}\tau_1^{0,1} = 0, \tag{3.24}$$
$$\tau_0^{-1,0}\tau_1^{0,0} - Q{\alpha_2}^3\tau_0^{0,1}\tau_1^{-1,-1} + Q^{2}{\alpha_2}^6\tau_0^{0,0}\tau_1^{-1,0} = 0, \tag{3.25}$$
$$\tau_0^{0,0}\tau_1^{0,1} - Q{\alpha_0}^3\tau_0^{-1,0}\tau_1^{1,1} + Q^{2}{\alpha_0}^6\tau_0^{0,1}\tau_1^{0,0} = 0. \tag{3.26}$$

We set

$$\tau_1^{n,m} = -Q^{2n+2m}{\alpha_0}^2{\alpha_2}^{-2}\frac{\Theta(-Q^{-6n}{\alpha_0}^{-6};Q^6)\Theta(-Q^{6m}{\alpha_2}^{-6};Q^6)}{\Theta(Q^{-6(n-m)}{\alpha_0}^{-6}{\alpha_2}^{-6};Q^6)}\tau_0^{n,m}F_{n,m-1}. \tag{3.27}$$

Here, $F_{n,m}$ is equivalent to (2.3) because we obtain (2.1) and (2.2) from (3.24)–(3.26) and (3.21)–(3.23), respectively. If we assume $A_0^{n,m}$ is an arbitrary constant, it does not contradict (3.10)–(3.18). Therefore, we may set $A_0^{n,m} = 1$.

Finally we construct $\tau_N^{n,m}$.

Theorem 3.1. *Under the assumption* (3.1) *and* (3.2)*, the hypergeometric τ functions of q-$\mathrm{P_{III}}$ are given as the follows:*

$$\begin{aligned}\tau_N^{n,m} &= (-1)^{N(N+1)/2}Q^{-2(2n-m)N^2+6nN}{\alpha_0}^{-4N^2+6N}{\alpha_2}^{-2N^2}\\ &\times\left(\frac{\Theta(-Q^{-6n}{\alpha_0}^{-6};Q^6)\Theta(-Q^{6m}{\alpha_2}^{-6};Q^6)}{\Theta(Q^{-6(n-m)}{\alpha_0}^{-6}{\alpha_2}^{-6};Q^6)}\right)^N\\ &\times\Gamma(Q^{2n-m+1}{\alpha_0}^2\alpha_2;Q,Q)\Gamma(Q^{-n+2m-1}{\alpha_1}^2\alpha_0;Q,Q)\\ &\times\Gamma(Q^{-n-m}{\alpha_2}^2\alpha_1;Q,Q)\psi_N^{n,m-1},\end{aligned} \tag{3.28}$$

where

$$\psi_N^{n,m} = \begin{vmatrix} F_{n,m} & F_{n+1,m} & \cdots & F_{n+N-1,m}\\ F_{n-1,m} & F_{n,m} & \cdots & F_{n+N-2,m}\\ \vdots & \vdots & \ddots & \vdots\\ F_{n-N+1,m} & F_{n-N+2,m} & \cdots & F_{n,m}\end{vmatrix}, \qquad \psi_0^{n,m} = 1, \qquad \psi_{-N}^{n,m} = 0 \quad (N > 0),$$

and

$$\begin{aligned}F_{n,m} &= \frac{A_{n,m}}{({a_2}^{-2}q^{2m+2};q^2)_\infty}\,{}_1\varphi_1\left(\begin{matrix}0\\ {a_2}^2q^{-2m}\end{matrix};q^2,{a_2}^2{a_0}^2q^{2n-2m}\right)\\ &+ B_{n,m}\frac{\Theta({a_0}^2{a_2}^2q^{2n-2m-2};q^2)}{({a_2}^2q^{-2m-2};q^2)_\infty\Theta({a_0}^2q^{2n};q^2)}\,{}_1\varphi_1\left(\begin{matrix}0\\ {a_2}^{-2}q^{2m+4}\end{matrix};q^2,{a_0}^2q^{2n+2}\right).\end{aligned} \tag{3.29}$$

Here, $A_{n,m}$ and $B_{n,m}$ are periodic functions of period one with respect to n and m.

Proof. We set

$$\begin{aligned}\tau_N^{n,m} &= (-1)^{N(N+1)/2}Q^{-2(2n-m)N^2+6nN}{\alpha_0}^{-4N^2+6N}{\alpha_2}^{-2N^2}\\ &\times\left(\frac{\Theta(-Q^{-6n}{\alpha_0}^{-6};Q^6)\Theta(-Q^{6m}{\alpha_2}^{-6};Q^6)}{\Theta(Q^{-6(n-m)}{\alpha_0}^{-6}{\alpha_2}^{-6};Q^6)}\right)^N\end{aligned}$$

$$\times\, \Gamma(Q^{2n-m+1}{\alpha_0}^2\alpha_2;Q,Q)\Gamma(Q^{-n+2m-1}{\alpha_1}^2\alpha_0;Q,Q)\Gamma(Q^{-n-m}{\alpha_2}^2\alpha_1;Q,Q)\psi_N^{n,m-1}.$$

From (3.2), (3.9), and (3.27), we find

$$\psi_N^{n,m}=0 \quad (N<0), \qquad \psi_0^{n,m}=1, \qquad \psi_1^{n,m}=F_{n,m}.$$

Furthermore, it is easily verified that $\psi_N^{n,m}$ satisfy

$$\psi_{N+1}^{n,m}\psi_{N-1}^{n,m}-\left(\psi_N^{n,m}\right)^2+\psi_N^{n+1,m}\psi_N^{n-1,m}=0, \tag{3.30}$$

from (3.5). In general, (3.30) admits a solution expressed in terms of the Toeplitz type determinant

$$\psi_N^{n,m}=\det\left(c_{n-i+j,m}\right)_{i,j=1,\dots,N} \quad (N>0),$$

under the boundary conditions

$$\psi_N^{n,m}=0 \quad (N<0), \qquad \psi_0^{n,m}=1, \qquad \psi_1^{n,m}=c_{n,m},$$

where $c_{n,m}$ is an arbitrary function. Therefore we have completed the proof. ■

3.2 Hypergeometric τ functions of q-$\mathrm{P_{II}}$

In this section, we construct the hypergeometric τ functions of q-$\mathrm{P_{II}}$ by two methods.

3.2.1 Hypergeometric τ functions of q-$\mathrm{P_{II}}$ (I)

We construct the hypergeometric τ functions of q-$\mathrm{P_{II}}$ by using those of q-$\mathrm{P_{III}}$. We here note that $\tau_N^{n,m}$ consistent with the action of $\langle s_2,T_1,T_2,T_3,T_4\rangle$ is also consistent with the action of R_1 because

$$R_1=s_2{T_2}^{-1}.$$

Therefore, we construct $\tau_N^{n,m}$ consistent with the action of $\langle s_2,T_1,T_2,T_3,T_4\rangle$. The action of s_2 on $\tau_N^{n,m}$ is

$$s_2(\tau_N^{n,m})=\tau_N^{n-m,-m}. \tag{3.31}$$

We consider only $\tau_0^{n,m}$ and $\tau_1^{n,m}$ because $\tau_N^{n,m}$ is determined as a rational function in $\tau_0^{n,m}$ and $\tau_1^{n,m}$. It easily verified that $\tau_0^{n,m}$, (3.28) (or (3.9)), is consistent with the action of s_2. When $N=1$, we rewrite (3.31) as

$$s_2(F_{n,m-1})={\alpha_2}^{-12}Q^{-12m}\frac{\Theta({\alpha_0}^{-12}Q^{12m-12n};Q^{12})}{\Theta({\alpha_0}^{-12}{\alpha_2}^{-12}Q^{-12n};Q^{12})}F_{n-m,-m-1}, \tag{3.32}$$

from (3.28). Moreover, by using (3.29), (3.32) can be rewritten as

$$\frac{s_2(A_{n,m})-B_{n,m}}{({\alpha_2}^{12}Q^{12m};Q^{12})_\infty}\,{}_1\varphi_1\left(\begin{matrix}0\\ {\alpha_2}^{-12}Q^{-12m+12}\end{matrix};Q^{12},{\alpha_0}^{12}Q^{12n-12m+12}\right)$$
$$=\frac{(A_{n,m}-s_2(B_{n,m}))\Theta({\alpha_0}^{12}Q^{12n-12m};Q^{12})}{({\alpha_2}^{-12}Q^{-12m};Q^{12})_\infty\Theta({\alpha_0}^{12}{\alpha_2}^{12}Q^{12n};Q^{12})}\,{}_1\varphi_1\left(\begin{matrix}0\\ {\alpha_2}^{12}Q^{12m+12}\end{matrix};Q^{12},{\alpha_0}^{12}{\alpha_2}^{12}Q^{12n+12}\right),$$

which implies that $\tau_1^{n,m}$ is also consistent with the action of s_2 when

$$s_2(A_{n,m})=B_{n,m}. \tag{3.33}$$

Lemma 3.2. *Under the assumption* (3.33)*, the hypergeometric τ functions* (3.28) *are consistent with the action of $\langle s_2, T_1, T_2, T_3, T_4\rangle$.*

Therefore we easily obtain the following theorem:

Theorem 3.2. *Setting*

$$R_1(A_{n,m}) = B_{n,m}, \qquad (3.34)$$
$$\alpha_2 = Q^{1/2},$$

and putting

$$\tau_N^{2n} = \tau_N^{n,0}, \qquad \tau_N^{2n-1} = \tau_N^{n,1},$$

we obtain the hypergeometric τ functions of q-$\mathrm{P_{II}}$. Here $\tau_N^{n,m}$ is given by (3.28)*.*

In general, the entries of determinants of the hypergeometric τ functions of Painlevé systems are expressed by two-parameter family of the functions satisfying the contiguity relations. However the hypergeometric τ functions of q-$\mathrm{P_{II}}$ in Theorem 3.2 have only one parameter because of the condition (3.34). In the next section, we construct the hypergeometric τ functions of q-$\mathrm{P_{II}}$ which admits two parameters.

3.2.2 Hypergeometric τ functions of q-$\mathrm{P_{II}}$ (II)

We construct the hypergeometric τ functions of q-$\mathrm{P_{II}}$ whose ratios correspond to the hypergeometric solutions of q-$\mathrm{P_{II}}$ in Proposition 2.2. By the action of the affine Weyl group, τ_N^k is determined as a rational function of τ_0^k and τ_1^k (or τ_i and $\overline{\tau}_i$). Thus, our purpose is determining τ_0^k and τ_1^k consistent with the action of $\langle R_1, T_4\rangle$ and constructing τ_N^k under the conditions

$$\alpha_2 = Q^{1/2}, \qquad \gamma = 1, \qquad (3.35)$$

and the boundary condition

$$\tau_N^k = 0 \qquad (N < 0). \qquad (3.36)$$

First we consider the condition for τ_0^k which follows from the boundary condition (3.36). We use the bilinear equation obtained in [12]:

Proposition 3.2. *The following bilinear equation holds:*

$$\tau_{N+1}^k\tau_{N-1}^{k+1} - Q^{(k-4N+1)/2}\gamma^{-2}\alpha_0\tau_N^{k+2}\tau_N^{k-1} - Q^{-k+4N-1}\gamma^4{\alpha_0}^{-2}\tau_N^{k+1}\tau_N^k = 0. \qquad (3.37)$$

By putting $N = 0$ in (3.37), we get

$$Q^{3(k+1)/2}{\alpha_0}^3\tau_0^{k+2}\tau_0^{k-1} + \tau_0^{k+1}\tau_0^k = 0. \qquad (3.38)$$

We set

$$\tau_0^k = \Gamma\big(Q^{(2k+3)/2}{\alpha_0}^2; Q, Q\big)\Gamma\big(Q^{-k/2}{\alpha_0}^{-1}; Q, Q\big)\Gamma\big(Q^{(-k+3)/2}{\alpha_0}^{-1}; Q, Q\big)A_1^k. \qquad (3.39)$$

From (3.38), A_1^k satisfies

$$A_1^{k+2}A_1^{k-1} = A_1^{k+1}A_1^k. \qquad (3.40)$$

We next determine τ_0^k and τ_1^k. From (2.10) and Proposition 2.4, we see that the action of R_1 on τ_0^k and τ_1^k is given by

$$R_1(\tau_0) = \tau_2, \tag{3.41}$$

$$R_1(\tau_1) = \frac{Q^{-2}{\alpha_0}^6\tau_1\overline{\tau}_2 + \overline{\tau}_1\tau_2}{Q^{-1}{\alpha_0}^3\overline{\tau}_0}, \tag{3.42}$$

$$R_1(\tau_2) = \tau_1, \tag{3.43}$$

$$R_1(\overline{\tau}_0) = \overline{\tau}_2, \tag{3.44}$$

$$R_1(\overline{\tau}_1) = \frac{Q^2{\alpha_0}^6\overline{\tau}_1\tau_2 + \tau_1\overline{\tau}_2}{Q{\alpha_0}^3\tau_0}, \tag{3.45}$$

$$R_1(\overline{\tau}_2) = \overline{\tau}_1. \tag{3.46}$$

From (2.13), we rewrite (3.42) and (3.45) as

$$Q{\alpha_0}^{-3}\tau_1^{-2}\tau_0^1 - Q^2{\alpha_0}^{-6}\tau_1^0\tau_0^{-1} - \tau_0^0\tau_1^{-1} = 0, \tag{3.47}$$

$$Q^{-1}{\alpha_0}^{-3}\tau_0^{-2}\tau_1^1 - Q^{-2}{\alpha_0}^{-6}\tau_0^0\tau_1^{-1} - \tau_1^0\tau_0^{-1} = 0, \tag{3.48}$$

respectively. Setting

$$\tau_1^k = \frac{\tau_0^k}{\Theta(Q^{3k+1}{\alpha_0}^6; Q^3)} G_k, \tag{3.49}$$

then the systems (3.47) and (3.48) reduce to (2.6). Therefore G_k is equivalent to (2.7). If we assume A_1^k is an arbitrary constant, it does not contradict (3.40)–(3.46). Therefore, we may put $A_1^k = 1$.

Finally we present an explicit formula for τ_N^k.

Theorem 3.3. *Under the assumption* (3.35) *and* (3.36)*, the hypergeometric τ functions of q-$\mathrm{P_{II}}$ are given as the follows:*

$$\tau_N^k = (-1)^{N(N-1)/2} Q^{N(N-1)(k+N)} {\alpha_0}^{2N(N-1)} \times \frac{\Gamma(Q^{(2k+3)/2}{\alpha_0}^2; Q, Q)\Gamma(Q^{-k/2}{\alpha_0}^{-1}; Q, Q)\Gamma(Q^{(-k+3)/2}{\alpha_0}^{-1}; Q, Q)}{\Theta(Q^{3k+1}{\alpha_0}^6; Q^3)^N} \phi_N^k,$$

where

$$\phi_N^k = \begin{vmatrix} G_k & G_{k-1} & \cdots & G_{k-N+1} \\ G_{k+2} & G_{k+1} & \cdots & G_{k-N+3} \\ \vdots & \vdots & \ddots & \vdots \\ G_{k+2N-2} & G_{k+2N-3} & \cdots & G_{k+N-1} \end{vmatrix}, \qquad \phi_0^k = 1, \qquad \phi_{-N}^k = 0 \quad (N > 0),$$

and

$$G_k = A_k \Theta\big(ia_0 q^{(2k+1)/4}; q^{1/2}\big)\ {}_1\varphi_1\left(\begin{matrix} 0 \\ -q^{1/2} \end{matrix}; q^{1/2}, -ia_0 q^{(3+2k)/4}\right) + B_k \Theta\big(-ia_0 q^{(2k+1)/4}; q^{1/2}\big)\ {}_1\varphi_1\left(\begin{matrix} 0 \\ -q^{1/2} \end{matrix}; q^{1/2}, ia_0 q^{(3+2k)/4}\right).$$

Here, A_k and B_k are periodic functions of period one.

Proof. We set

$$\tau_N^k = (-1)^{N(N-1)/2} Q^{N(N-1)(k+N)} {\alpha_0}^{2N(N-1)} \times \frac{\Gamma(Q^{(2k+3)/2}{\alpha_0}^2;Q,Q)\Gamma(Q^{-k/2}{\alpha_0}^{-1};Q,Q)\Gamma(Q^{(-k+3)/2}{\alpha_0}^{-1};Q,Q)}{\Theta(Q^{3k+1}{\alpha_0}^6;Q^3)^N}\phi_N^k.$$

From (3.36), (3.39), and (3.49), we find that

$$\phi_N^k = 0 \quad (N<0), \qquad \phi_0^k = 1, \qquad \phi_1^k = G_k.$$

From (3.37), ϕ_N^k satisfies

$$\phi_{N+1}^k\phi_{N-1}^{k+1} - \phi_N^k\phi_N^{k+1} + \phi_N^{k+2}\phi_N^{k-1} = 0, \tag{3.50}$$

which is a variant of the discrete Toda equation. Under the conditions

$$\phi_N^k = 0 \quad (N<0), \qquad \phi_0^k = 1, \qquad \phi_1^k = c_k,$$

where c_k is an arbitrary function. Equation (3.50) admits a solution expressed by

$$\phi_N^k = \det\left(c_{k+2i-j-1}\right)_{i,j=1,\ldots,N} \qquad (N>0).$$

This complete the proof. ■

3.3 Relation between the hypergeometric τ functions of q-P$_{\mathrm{III}}$ and Heine's transform

Masuda showed that the consistency of a certain reflection transformation to the hypergeometric τ functions of type $E_8^{(1)}$ correspond to Bailey's four term transformation formula [18]. It is also shown that the consistency of a certain reflection transformation to the hypergeometric τ functions of type $E_7^{(1)}$ correspond to limiting case of Bailey's ${}_{10}\varphi_9$ transformation formula [17]. We here show that the consistency of s_0 to the hypergeometric τ functions of q-P$_{\mathrm{III}}$ give rise to a transformation of ${}_1\varphi_1$ which is obtained by Heine's transform for ${}_2\varphi_1$.

The action of s_0 on $\tau_N^{n,m}$ is

$$s_0(\tau_N^{n,m}) = \tau_N^{-n,m-n}. \tag{3.51}$$

We consider only $\tau_0^{n,m}$ and $\tau_1^{n,m}$ because $\tau_N^{n,m}$ is determined as a rational function in $\tau_0^{n,m}$ and $\tau_1^{n,m}$. It easily verified that $\tau_0^{n,m}$, (3.28) (or (3.9)), is consistent with the action of s_0. When $N=1$, (3.51) implies

$$s_0(F_{n,m-1}) = \frac{\Theta({\alpha_2}^{-12}Q^{-12n+12m};Q^{12})}{\Theta({\alpha_0}^{-12}{\alpha_2}^{-12}Q^{12m};Q^{12})}F_{-n,m-n-1}, \tag{3.52}$$

from (3.28). Moreover, by using (3.29), (3.52) can be rewritten as

$$s_0(A_{n,m})\,{}_1\varphi_1\left(\begin{matrix}0\\ {\alpha_0}^{12}{\alpha_2}^{12}Q^{-12m+12}\end{matrix};Q^{12},{\alpha_2}^{12}Q^{12n-12m+12}\right) - A_{n,m}\frac{({\alpha_2}^{12}Q^{12n-12m+12};Q^{12})_\infty}{({\alpha_0}^{12}{\alpha_2}^{12}Q^{-12m+12};Q^{12})_\infty}\,{}_1\varphi_1\left(\begin{matrix}0\\ {\alpha_2}^{12}Q^{12n-12m+12}\end{matrix};Q^{12},{\alpha_2}^{12}{\alpha_0}^{12}Q^{-12m+12}\right) - B_{n,m}{\alpha_0}^{12}Q^{-12n}\,\frac{({\alpha_0}^{-12}{\alpha_2}^{-12}Q^{12m},{\alpha_2}^{-12}Q^{-12n+12m+12};Q^{12})_\infty}{\Theta({\alpha_0}^{12}Q^{-12n};Q^{12})}$$

$$\times\ {}_1\varphi_1\left(\begin{matrix}0\\ {\alpha_2}^{-12}Q^{-12n+12m+12}\end{matrix};Q^{12},{\alpha_0}^{12}Q^{-12n+12}\right)$$
$$+\ s_0(B_{n,m})\,\frac{({\alpha_0}^{-12}{\alpha_2}^{-12}Q^{12m};Q^{12})_\infty\Theta({\alpha_2}^{12}Q^{12n-12m};Q^{12})}{({\alpha_0}^{12}{\alpha_2}^{12}Q^{-12m};Q^{12})_\infty\Theta({\alpha_0}^{-12}Q^{12n};Q^{12})}$$
$$\times\ {}_1\varphi_1\left(\begin{matrix}0\\ {\alpha_0}^{-12}{\alpha_2}^{-12}Q^{12m+12}\end{matrix};Q^{12},{\alpha_0}^{-12}Q^{12n+12}\right)=0. \tag{3.53}$$

In particular, setting

$$s_0(A_{n,m})=A_{n,m},\qquad B_{n,m}=0,$$

in (3.53), we obtain

$$ {}_1\varphi_1\left(\begin{matrix}0\\ {\alpha_0}^{12}{\alpha_2}^{12}Q^{-12m+12}\end{matrix};Q^{12},{\alpha_2}^{12}Q^{12n-12m+12}\right) \tag{3.54}$$
$$=\frac{({\alpha_2}^{12}Q^{12n-12m+12};Q^{12})_\infty}{({\alpha_0}^{12}{\alpha_2}^{12}Q^{-12m+12};Q^{12})_\infty}\,{}_1\varphi_1\left(\begin{matrix}0\\ {\alpha_2}^{12}Q^{12n-12m+12}\end{matrix};Q^{12},{\alpha_2}^{12}{\alpha_0}^{12}Q^{-12m+12}\right).$$

Equation (3.54) corresponds to a specialization of Heine's transform. Actually, by putting

$$a=b^{-1}c,\qquad d=b^{-1}z,$$

in Heine's transform [34]

$$ {}_2\varphi_1\left(\begin{matrix}a,b\\ c\end{matrix};q,d\right)=\frac{(a,bd;q)_\infty}{(c,d;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}a^{-1}c,d\\ bd\end{matrix};q,a\right),$$

we obtain

$$ {}_2\varphi_1\left(\begin{matrix}b^{-1}c,b\\ c\end{matrix};q,b^{-1}z\right)=\frac{(b^{-1}c,z;q)_\infty}{(c,b^{-1}z;q)_\infty}\,{}_2\varphi_1\left(\begin{matrix}b,b^{-1}z\\ z\end{matrix};q,b^{-1}c\right). \tag{3.55}$$

Taking the limit $b\to\infty$ in (3.55) leads to

$$ {}_1\varphi_1\left(\begin{matrix}0\\ c\end{matrix};q,z\right)=\frac{(z;q)_\infty}{(c;q)_\infty}\,{}_1\varphi_1\left(\begin{matrix}0\\ z\end{matrix};q,c\right),$$

which is equivalent to (3.54).

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