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On Non-Point Invertible Transformations of Difference and Differential-Difference Equations

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Abstract. Non-point invertible transformations are completely described for difference equations on the quad-graph and for their differential-difference analogues. As an illustration, these transformations are used to construct new examples of integrable equations and autotransformations of the Hietarinta equation.

Key words: non-point transformation; Darboux integrability; discrete Liouville equation; higher symmetry

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1 Introduction

The present paper is devoted to invertible transformations for both discrete equations of the form

$$u_{i+1,j+1} = F(u_{i,j}, u_{i+1,j}, u_{i,j+1}),$$

and "semi-discrete" chains of the differential equations

$$(u_{i+1})_x = F(x, u_i, u_{i+1}, (u_i)_x).$$

Here i and j are integers, x is a continuous variable, u is a function of i, j and i, x for the first and the second equation, respectively. From now on, we shall omit i and j for brevity and, in particular, write the above equations in the form

$$u_{1,1} = F(u, u_{1,0}, u_{0,1}) \tag{1.1}$$

and

$$(u_1)_x = F(x, u, u_1, u_x). \tag{1.2}$$

We assume that $F_u F_{u_{1,0}} F_{u_{1,0}} \neq 0$ for equation (1.1) and $F_{u_x} \neq 0$ for equation (1.2). These conditions allows us to rewrite equation (1.1) in any of the following forms

$$u_{-1,-1} = \overline{F}(u, u_{-1,0}, u_{0,-1}), \tag{1.3}$$

$$u_{1,-1} = \hat{F}(u, u_{1,0}, u_{0,-1}), \tag{1.4}$$

$$u_{-1,1} = \tilde{F}(u, u_{-1,0}, u_{0,1}), \tag{1.5}$$

and equation (1.2) – in the form

$$(u_{-1})_x = \tilde{F}(x, u, u_{-1}, u_x). \tag{1.6}$$

Therefore, all "mixed shifts" $u_{m,n} := u_{i+m,j+n}$ (for both positive and negative non-zero n and m) can be expressed in terms of *dynamical variables* $u_{k,0}$, $u_{0,l}$ by virtue of equations (1.1), (1.3)–(1.5). (A more detailed explanation of the dynamical variables, the notation $u_{m,n}$ and the recursive procedure of the mixed shift elimination can be found, for example, in [11, 10].) Analogously, $u_m^{(n)} := \partial^n u_{i+m}/\partial x^n$ for any non-zero $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ can be expressed in terms of x and dynamical variables $u_l := u_{i+l}$, $u^{(k)} := \partial^k u_i/\partial x^k$ by virtue of equations (1.2), (1.6). The notation $g[u]$ means that the function g depends on a finite number of the dynamical variables (and x if we consider equation (1.2)). The considerations in this paper are local (for example, we use the local implicit function theorem to obtain (1.3)–(1.6)) and, for simplicity, all functions are assumed to be locally analytical.

In addition to the point transformations $v = g(u)$, some of the equations (1.1) and (1.2) admit non-point transformations $v = g[u]$ which are invertible in the sense of [16]. For example, the differential substitutions

$$v = \frac{u_x - \sin u}{2} \tag{1.7}$$

maps solutions of the differential-difference sine-Gordon equation [7, 14]

$$(u_1)_x - \sin u_1 = u_x + \sin u \tag{1.8}$$

into solutions of the equation

$$\frac{(v_1 - v)_x}{\sqrt{1 - (v_1 - v)^2}} = \pm(v_1 + v), \tag{1.9}$$

which is a semi-discrete version of the complex sine-Gordon equation. Here the sign of the right-hand side of equation (1.9) coincides with the sign of the $\cos u$ value[1]. Indeed, $v_1 = (u_x + \sin u)/2$ follows from equation (1.8) and, together with (1.7), gives us

$$u_x = v_1 + v, \quad \sin u = v_1 - v \quad \Longrightarrow \quad (v_1 - v)_x = u_x \cos u = \pm(v_1 + v)\sqrt{1 - (v_1 - v)^2}.$$

The inverse transformation can be found in [12]: the formula $u = \frac{\pi}{2} \pm (\arcsin(v_1 - v) - \frac{\pi}{2})$ maps any real solution of equation (1.9) into a solution of equation (1.8).

This example belongs to the following class of non-point invertible transformations introduced in [17]. Let functions $\varphi(x, y, z)$, $\psi(x, y, z)$ satisfy the condition $\varphi_y \psi_z - \varphi_z \psi_y \neq 0$ and equation (1.2) can be written in the form

$$\varphi(x, u_1, (u_1)_x) = \psi(x, u, u_x). \tag{1.10}$$

Then we rewrite (1.10) in the form of the system

$$v = \varphi(x, u, u_x), \qquad v_1 = \psi(x, u, u_x), \tag{1.11}$$

express u, u_x in terms of v, v_1 from (1.11) and obtain

$$u = p(x, v, v_1), \qquad u_x = q(x, v, v_1). \tag{1.12}$$

The system (1.12) is equivalent to the equation

$$D_x(p(x, v, v_1)) = q(x, v, v_1), \tag{1.13}$$

[1] A local transformation of an equation may, generally speaking, generate different equations for different domains of the "jet space". This is true for both point and non-point local transformations.

where D_x denotes the total derivative with respect to x. The substitution $v = \varphi(x, u, u_x)$ maps solutions of (1.10) into solutions of (1.13) and the transformation $u = p(x, v, v_1)$ maps solutions of (1.13) back into solutions of (1.10).

It is easy to see that the same scheme can be applied to the pure discrete equations of the form

$$\varphi(u_{0,1}, u_{1,1}) = \psi(u, u_{1,0}), \tag{1.14}$$

where $\varphi(y, z)$ and $\psi(y, z)$ are functionally independent. Indeed, expressing u and $u_{1,0}$ from

$$v = \varphi(u, u_{1,0}), \qquad v_{0,1} = \psi(u, u_{1,0}), \tag{1.15}$$

we obtain

$$u = p(v, v_{0,1}), \qquad u_{1,0} = q(v, v_{0,1}) \tag{1.16}$$

and rewrite (1.16) in the form of the equivalent equation

$$p(v_{1,0}, v_{1,1}) = q(v, v_{0,1}). \tag{1.17}$$

Thus, the transformation $v = \varphi(u, u_{1,0})$ maps solutions of (1.14) into solutions of (1.17) and the inverse transformation $u = p(v, v_{0,1})$ maps solutions of (1.17) back into solutions of (1.14). The transformations (1.14)–(1.17) were, in fact, used in [18] without explicit formulation of the above scheme.

The main result of this paper is the proof of the following statement: any invertible transformation of equations (1.1), (1.2) is a composition of shifts, point transformations and transformations (1.10)–(1.13), (1.14)–(1.17). Roughly speaking, equations (1.1) and (1.2) have no non-point invertible transformations other than (1.14)–(1.17) and (1.10)–(1.13), respectively. The proof is similar to that was used in [16] for continuous equations (hyperbolic PDEs).

The invertible transformations allow us to obtain objects associated with integrability of equations (1.13), (1.17) (such as conservation laws and higher symmetries) from the corresponding objects of equations (1.10), (1.14) because we can express shifts and derivatives of u in terms of shifts and derivatives of v. Therefore, the invertible transformations may be useful for constructing new examples of integrable equations of the form (1.1), (1.2). To illustrate this, in Section 4 we construct Darboux integrable equations related via invertible transformations to difference and differential-difference analogues of the Liouville equation. In addition, an example of constructing an equation possessing the higher symmetries is contained at the end of Section 2. In this section we also demonstrate that the scheme (1.14)–(1.17) generates autotransformations of the Hietarinta equation.

2 Invertible transformations of discrete equations

We let T_i and T_j denote the operators of the forward shifts in i and j by virtue of equation (1.1). These operators are defined by the following rules: $T_i(f(a, b, c, \dots)) = f(T_i(a), T_i(b), T_i(c), \dots)$ and $T_j(f(a, b, c, \dots)) = f(T_j(a), T_j(b), T_j(c), \dots)$ for any function f; $T_i(u_{m,0}) = u_{m+1,0}$ and $T_j(u_{0,n}) = u_{0,n+1}$; $T_i(u_{0,n}) = T_j^{n-1}(F)$ for positive n and $T_i(u_{0,n}) = T_j^{n+1}(\hat{F})$ for negative n, $T_j(u_{m,0}) = T_i^{m-1}(F)$ for positive m and $T_j(u_{m,0}) = T_i^{m+1}(\tilde{F})$ for negative m (i.e. mixed variables $u_{1,n}$ and $u_{m,1}$ are expressed in terms of the dynamical variables by virtue of equations (1.1), (1.4), (1.5)). The inverse (backward) shift operators T_i^{-1} and T_j^{-1} are defined in the similar way.

Definition 1. We say that a transformation $v = f[u]$ maps the equation (1.1) into an equation $v_{1,1} = G(v, v_{1,0}, v_{0,1})$ if

$$T_iT_j(f) = G(f, T_i(f), T_j(f)). \tag{2.1}$$

Definition 2. A transformation $v = f[u]$ of equation (1.1) is called invertible if any of the dynamical variables u, $u_{k,0}$, $u_{0,l}$, $k, l \in \mathbb{Z}$, can be expressed as a function of a finite subset of the variables

$$v := f, \qquad v_{r,0} := T_i^r(f), \qquad v_{0,s} := T_j^s(f), \qquad r, s \in \mathbb{Z}. \tag{2.2}$$

We exclude all mixed variables $v_{r,s}$, $rs \neq 0$, from (2.2) because we consider only the cases when the transformation maps (1.1) into an equation of the form

$$v_{1,1} = G(v, v_{1,0}, v_{0,1}), \qquad G_vG_{v_{1,0}}G_{v_{0,1}} \neq 0 \tag{2.3}$$

and the mixed variables can be expressed in terms of (2.2) by virtue of this equation.

It is easy to see that any shift $w = v_{r,s}$ maps equation (2.3) into equation (2.3) again and the composition of the shift and an invertible transformation $v = f[u]$ is invertible too. This leads to the following

Definition 3. Transformations $v = f[u]$ and $w = g[u]$ are called equivalent if there exist $r, s \in \mathbb{Z}$ such that $g = T_i^sT_j^r(f)$.

Proposition 1. *Let $v = g[u]$ be an invertible transformation that maps equation (1.1) into equation (2.3). Then this transformation is equivalent to either a transformation of the form*

$$w = f(u, u_{1,0}, u_{2,0}, \ldots, u_{m,0}), \tag{2.4}$$

or a transformation of the form

$$w = f(u, u_{0,1}, u_{0,2}, \ldots, u_{0,n}). \tag{2.5}$$

Proof. The transformation is equivalent to that of the form

$$v = h(u, u_{1,0}, \ldots, u_{k,0}, u_{0,1}, \ldots, u_{0,l}) \tag{2.6}$$

because we can eliminate "negative" variables $u_{r,0}$, $u_{0,s}$, $r, s < 0$ from the transformation by shifts of g. We can express u as

$$\begin{aligned} u &= P(v_{a,0}, v_{a+1,0}, \ldots, v_{b,0}, v_{0,c}, v_{0,c+1}, \ldots, v_{0,d}) \\ &:= P\big(T_i^a(h), T_i^{a+1}(h), \ldots, T_i^b(h), T_j^c(h), T_j^{c+1}(h), \ldots, T_j^d(h)\big) \end{aligned} \tag{2.7}$$

if the transformation is invertible. Differentiating equation (2.7) with respect to $u_{k+b,0}$, we obtain $P_{v_{b,0}}T_i^b(h_{u_{k,0}}) = 0 \Rightarrow P_{v_{b,0}} = 0$ if $b, k > 0$. The analogous reasoning gives $P_{v_{0,d}} = 0$ if $d, l > 0$. Thus, $b, d \leq 0$ if $kl \neq 0$.

Let $\big(T_i^{-1}(h)\big)_{u_{-1,0}} \big(T_j^{-1}(h)\big)_{u_{0,-1}} \neq 0$. Then $(T_i^a(h))_{u_{a,0}} \big(T_j^c(h)\big)_{u_{0,c}} \neq 0$ for any negative a and c, and we obtain $P_{v_{a,0}} = P_{v_{0,c}} = 0$ by differentiating equation (2.7) with respect to $u_{a,0}$ and $u_{0,c}$. Therefore, either $a = c = b = d = 0$ (i.e. $u = P(h)$ that is possible only if $k = l = 0$) or $\big(T_i^{-1}(h)\big)_{u_{-1,0}} \big(T_j^{-1}(h)\big)_{u_{0,-1}} = 0$. The latter equality means that either $T_i^{-1}(h) = \tilde{h}(u, u_{1,0}, \ldots, u_{k-1,0}, u_{0,1}, \ldots, u_{0,l})$ or $T_j^{-1}(h) = \tilde{h}(u, u_{1,0}, \ldots, u_{k,0}, u_{0,1}, \ldots, u_{0,l-1})$, i.e. any invertible transformation of the form (2.6) with $kl \neq 0$ is equivalent to a transformation $\tilde{v} = \tilde{h}(u, u_{1,0}, \ldots, u_{\tilde{k},0}, u_{0,1}, \ldots, u_{0,\tilde{l}})$ such that $\tilde{k}\tilde{l} < kl$. Applying this conclusion several times, we obtain that (2.6) is equivalent to a transformation $w = f(u, u_{1,0}, \ldots, u_{m,0}, u_{0,1}, \ldots, u_{0,n})$ with $mn = 0$. ∎

Definition 4. A transformation is called non-point if this transformation is not equivalent to any point transformation of the form $w = g(u)$.

Because the transformations $v = f(u_{m,0})$ and $v = f(u_{0,n})$ are equivalent to the point transformation $w = f(u)$, a transformation of the form (2.4) or (2.5) is non-point only if f depends on more than one variable. We use only this property of the non-point transformations in the proof of the following

Theorem 1. *Let a non-point invertible transformation of the form (2.4) map equation (1.1) into equation (2.3). Then equation (1.1) can be written in the form $\varphi(u_{0,1}, u_{1,1}) = \psi(u, u_{1,0})$, where $\varphi(y, z)$ and $\psi(y, z)$ are functionally independent, and the transformation is equivalent to the composition of the invertible transformation $w = \varphi(u, u_{1,0})$ and an invertible transformation of the form $v = h(w, w_{1,0}, w_{2,0}, \ldots, w_{m-1,0})$. In particular, any non-point invertible transformation of the form $v = f(u, u_{1,0})$ is equivalent to the composition of the transformation $w = \varphi(u, u_{1,0})$ and a point transformation $v = h(w)$.*

Proof. If $f_u = 0$ and s is the smallest integer for which $f_{u_{s,0}} \neq 0$, then the equivalent transformation $v = T_i^{-s}(f[u])$ depends on u. Therefore, we can, without loss of generality, assume that $f_u \neq 0$. We also can write

$$\begin{aligned} u_{l,0} &= P_l(v_{a,0}, v_{a+1,0}, \ldots, v_{b,0}, v_{0,c}, v_{0,c+1}, \ldots, v_{0,d}) \\ &= P_l\big(T_i^a(f), T_i^{a+1}(f), \ldots, T_i^b(f), T_j^c(f), T_j^{c+1}(f), \ldots, T_j^d(f)\big), \qquad l = \overline{0,m} \end{aligned}$$

because the transformation is invertible. Here the notation $l = \overline{0,m}$ means that l runs over all integers from 0 to m. Differentiating these equalities with respect to $u_{a,0}$, we obtain $(P_l)_{v_{a,0}} T_i^a(f_u) = 0 \Rightarrow (P_l)_{v_{a,0}} = 0$ if $a < 0$. The similar reasoning gives $(P_l)_{v_{b,0}} = 0$ if $b > 0$. Thus,

$$u_{l,0} = P_l\big(T_j^c(f), T_j^{c+1}(f), \ldots, T_j^d(f)\big), \qquad l = \overline{0,m}.$$

Let $c < 0$ and s be the biggest negative integer such that $(T_j^s(f))_{u_{0,-1}} \neq 0$. If $s \geq c$, then $T_j^c(f)$ depends on $u_{0,c-s-1}$ and $(P_l)_{u_{0,c-s-1}} = (P_l)_{v_{0,c}} (T_j^c(f))_{u_{0,c-s-1}} = 0 \Rightarrow (P_l)_{v_{0,c}} = 0$. Hence $s < c$, i.e. $(T_j^r(f))_{u_{0,-1}} = 0$ for all $r \geq c$. This implies $T_j^c(f) = g(u, u_{1,0}, \ldots, u_{m,0})$ and

$$u_{l,0} = P_l\big(g, T_j(g), \ldots, T_j^{\tilde{d}}(g)\big), \qquad l = \overline{0,m}. \tag{2.8}$$

If $c \geq 0$, then equations (2.8) holds too, with $g = f$ and $\tilde{d} = d$.

Repeating the above argumentation, we prove that $(T_j^r(g))_{u_{0,1}} = 0$ for all $r \leq \tilde{d}$. Let us consider the operators $X = T_j^{-1} \partial_{u_{0,1}} T_j$ (cf. [4]) and $Y = [\partial_{u_{0,-1}}, X]$, where $\partial_z := \frac{\partial}{\partial z}$. These operators have the form

$$X = \partial_u + \sum_{l=1}^{m} \xi_l \partial_{u_{l,0}}, \qquad Y = \sum_{l=1}^{m} \nu_l \partial_{u_{l,0}}$$

for functions of $u, u_{1,0}, \ldots, u_{m,0}$. According to equation (2.8), the set $\{g, T_j(g), \ldots, T_j^{\tilde{d}}(g)\}$ must contain $m+1$ functionally independent functions because $u_{l,0}$, $l = \overline{0,m}$, are functionally independent. Hence the system $X(z) = 0$, $Y(z) = 0$ has m functionally independent solutions depending on $u, u_{1,0}, \ldots, u_{m,0}$ and the vectors $(1, \xi_1, \ldots, \xi_m)$, $(0, \nu_1, \ldots, \nu_m)$ must be collinear. The latter is possible only if $\nu_l = 0$ for all $l = \overline{0,m}$. In particular,

$$\nu_1 = [T_j^{-1}(F_{u_{0,1}})]_{u_{0,-1}} = 0 \;\Rightarrow\; T_j^{-1}(F_{u_{0,1}}) = \alpha(u, u_{1,0}) \;\Rightarrow\; F_{u_{0,1}} = T_j(\alpha) = \alpha(u_{0,1}, F)$$

$$
\begin{aligned}
&\Rightarrow\ F_{uu_{0,1}} = \alpha_{u_{1,0}}(u_{0,1}, F)F_u, \quad F_{u_{1,0}u_{0,1}} = \alpha_{u_{1,0}}(u_{0,1}, F)F_{u_{1,0}} \\
&\Rightarrow\ (\ln(F_{u_{1,0}}) - \ln(F_u))_{u_{0,1}} = 0 \ \Rightarrow\ F_{u_{1,0}} - \beta(u, u_{1,0})F_u = 0 \\
&\Rightarrow\ F = E(\psi(u, u_{1,0}), u_{0,1}),
\end{aligned}
$$

where ψ is a solution of the equation $\psi_{u_{1,0}} - \beta(u, u_{1,0})\psi_u = 0$. Thus, equation (1.1) can be written in the form (1.14).

We can express g in terms of u, $\varphi(u, u_{1,0})$, $\varphi(u_{1,0}, u_{2,0})$, $\ldots$, $\varphi(u_{m-1,0}, u_{m,0})$:

$$
g = h(u, \varphi(u, u_{1,0}), \varphi(u_{1,0}, u_{2,0}), \ldots, \varphi(u_{m-1,0}, u_{m,0})).
$$

It is proved above that $X(g) = 0$. Taking this fact into account, we obtain $h_u = 0$ because $X(g) = X(h) = T_j^{-1}[h_u(u_{0,1}, \psi(u, u_{1,0}), \ldots, \psi(u_{m-1,0}, u_{m,0}))] = h_u$. This means that the transformation (2.4) is equivalent to the composition of the transformation $w = \varphi(u, u_{1,0})$ and the transformation $v = h(w, w_{1,0}, w_{2,0}, \ldots, w_{m-1,0})$. The latter transformation is invertible because

$$
w = \varphi(P_0, P_1) = \tilde{P}_0\big(g, T_j(g), \ldots, T_j^{\tilde{d}}(g)\big) = \tilde{P}_0\big(h, T_j(h), \ldots, T_j^{\tilde{d}}(h)\big)
$$

by virtue of equation (2.8). The expressions for other dynamical variables can be obtained by the formulas $w_{0,r} = T_j^r(\tilde{P}_0)$ and $w_{s,0} = T_i^s(\tilde{P}_0)$.

Let φ and ψ be functionally dependent. Under this assumption equation (1.14) has the form $\varphi(u_{0,1}, u_{1,1}) = E(\varphi(u, u_{1,0}))$ and all functions $T_j^r(g)$ can be expressed in terms of $\varphi(u, u_{1,0})$, $\varphi(u_{1,0}, u_{2,0})$, $\ldots$, $\varphi(u_{m-1,0}, u_{m,0})$ ($T_j(g) = h(E(\varphi(u, u_{1,0})), \ldots, E(\varphi(u_{m-1,0}, u_{m,0}))$ and so on). Hence the set $\{g, T_j(g), \ldots, T_j^{\tilde{d}}(g)\}$ contains no more than m functionally independent functions. But we prove above that this set must contain $m+1$ functionally independent functions if the transformation is invertible. Therefore, φ and ψ must be functionally independent if equation (1.14) admits an invertible transformation of the form $v = f(u, u_{1,0}, u_{2,0}, \ldots, u_{m,0})$. ■

It is not always easy to see whether equation (1.1) can be represented in the form (1.14). For example, at first glance it seems that the equation

$$
v_{1,1} = \frac{v(v_{1,0} + 1)}{v(v_{0,1} - v_{1,0}) + v_{0,1} + 1} \tag{2.9}
$$

does not admit an invertible transformation of the form $u = \varphi(v, v_{1,0})$. But in reality we can rewrite this equation as

$$
\frac{v_{1,1} + 1}{v_{0,1}v_{1,1} - 1} = \frac{v + 1}{v_{1,0}v - 1}
$$

and relate it to the equation

$$
(u_{1,1} - 1)(u_{0,1} + 1) = (u_{1,0} + 1)(u - 1) \tag{2.10}
$$

via the invertible transformation

$$
u = -2\frac{v_{1,0} + 1}{vv_{1,0} - 1} - 1, \qquad v = \frac{u_{0,1} - 1}{u + 1}.
$$

Therefore, it is useful to reformulate our result in the following form.

Corollary 1. *The equation (1.1) admits a non-point invertible transformation of the form (2.4) into an equation of the form (2.3) if and only if both the conditions*

$$
\left(\frac{F_{u_{1,0}}}{F_u}\right)_{u_{0,1}} = 0, \qquad F_u + F_{u_{1,0}}T_j^{-1}(F_{u_{0,1}}) \neq 0
$$

are satisfied.

Proof. If equation (1.1) is represented in the form (1.14), then the right-hand side F of (1.1) is determined as an implicit function from the identity

$$\varphi(u_{0,1}, F) = \psi(u, u_{1,0}). \tag{2.11}$$

Differentiating this identity with respect to u and $u_{1,0}$, we obtain

$$T_j(\varphi_{u_{1,0}}(u, u_{1,0}))F_u = \psi_u(u, u_{1,0}), \qquad T_j(\varphi_{u_{1,0}}(u, u_{1,0}))F_{u_{1,0}} = \psi_{u_{1,0}}(u, u_{1,0}). \tag{2.12}$$

Therefore, $F_{u_{1,0}}/F_u$ does not depend on $u_{0,1}$. Conversely, if $F_{u_{1,0}}/F_u = \beta(u, u_{1,0})$, then $F = E(\psi(u, u_{1,0}), u_{0,1})$ and (1.1) can be rewritten in the form (1.14).

Differentiating (2.11) with respect to $u_{0,1}$, we obtain $F_{u_{0,1}} = -T_j\left(\varphi_u/\varphi_{u_{1,0}}\right)$. This expression and equation (2.12) allow us to rewrite the functional independence condition for φ, ψ in the following way

$$\begin{aligned} &\varphi_{u_{1,0}}(u, u_{1,0})\psi_u(u, u_{1,0}) - \varphi_u(u, u_{1,0})\psi_{u_{1,0}}(u, u_{1,0}) \\ &\quad = \varphi_{u_{1,0}}(\psi_u + T_j^{-1}(F_{u_{0,1}})\psi_{u_{1,0}}) = T_j(\varphi_{u_{1,0}})\varphi_{u_{1,0}}(F_u + T_j^{-1}(F_{u_{0,1}})F_{u_{1,0}}) \neq 0. \qquad \blacksquare \end{aligned}$$

Naturally, the propositions analogous to Theorem 1 and Corollary 1 are valid for invertible transformations of the form $v = f(u, u_{0,1}, u_{0,2}, \ldots, u_{0,n})$ too.

Returning to equations (2.9), (2.10), we note that equation (2.10) was introduced in [13] in a slightly different form. This equation has also been used in [10] as an example of an equation which is inconsistent around the cube (in the sense of [1]) but possesses the higher symmetries. Therefore, we can obtain symmetries of equation (2.9) from symmetries of equation (2.10).

Indeed, if a transformation $v = f(u, u_{0,1})$ maps equation (1.1) into equation (2.3), then differentiation of (2.1) with respect to τ by virtue of a symmetry $u_\tau = \xi[u]$ of equation (1.1) gives us

$$L_G\left((f_{u_{0,1}}T_j + f_u)(\xi[u])\right) = (\lambda[u]T_j + \mu[u])\big(L_F(\xi[u])\big),$$

where

$$L_G = T_iT_j + G_{v_{1,0}}T_i + G_{v_{0,1}}T_j + G_v, \qquad L_F = T_iT_j + F_{u_{1,0}}T_i + F_{u_{0,1}}T_j + F_u.$$

Because $L_F(\xi[u]) = 0$ by definition of symmetry, we see that $v_\tau = f_{u_{0,1}}T_j(\xi[u]) + f_u\xi[u]$ (after rewriting in terms of v and its shifts) is a symmetry of equation (2.3). Applying this, for example, to the three-point symmetries

$$u_\tau = (u^2 - 1)(u_{1,0} - u_{-1,0}), \qquad u_\tau = (u^2 - 1)\left(\frac{1}{u_{0,1} + u} - \frac{1}{u + u_{0,-1}}\right)$$

of equation (2.10), we obtain the symmetries

$$v_\tau = (v + 1)^2\left(\frac{1}{vv_{1,0} - 1} - \frac{1}{vv_{-1,0} - 1}\right), \qquad v_\tau = v\left(\frac{1}{v_{0,1} + 1} - \frac{1}{v_{0,-1} + 1}\right)$$

of equation (2.9).

The Hietarinta [6] equation[2]

$$u_{1,1}(u + \beta)(u_{0,1} + \alpha) = u_{0,1}(u + \alpha)(u_{1,0} + \beta) \tag{2.13}$$

[2]We write this equation in the form used in [15].

is another interesting example. The invertible transformations

$$v = \frac{u_{1,0}(u+\alpha)}{u} - \alpha, \qquad w = \frac{\beta u_{0,1}}{\beta + u - u_{0,1}}$$

map this equation into equation (2.13) again. In addition, the Hietarinta equation is linearizable [15]. We note that the above properties of equation (2.13) are similar to those of the continuous equation

$$u_{xy} = (\alpha(x,y)e^u)_x + (\beta(x,y)e^{-u})_y + \gamma(x,y)$$

which was considered in [16].

3 Invertible transformations of differential-difference equations

We let T denote the operator of the forward shift in i by virtue of equation (1.2). This operator is defined by the following rules: $T(f(a,b,c,\dots)) = f(T(a),T(b),T(c),\dots)$ for any function f; $T(u_m) = u_{m+1}$; $T(u^{(n)}) = D_x^{n-1}(F)$ (mixed variables $u_1^{(n)}$ are expressed in terms of the dynamical variables by virtue of equation (1.2)). Here

$$D_x = \frac{\partial}{\partial x} + u^{(1)}\frac{\partial}{\partial u} + \sum_{k=1}^{\infty}\left(u^{(k+1)}\frac{\partial}{\partial u^{(k)}} + T^{(k-1)}(F)\frac{\partial}{\partial u_k} + T^{(1-k)}(\tilde{F})\frac{\partial}{\partial u_{-k}}\right),$$

i.e. D_x is the total derivative with respect to x by virtue of equations (1.2), (1.6). The inverse (backward) shift operator T^{-1} is defined in the similar way.

Definition 5. We say that a transformation $v = f[u]$ maps equation (1.2) into an equation

$$(v_1)_x = G(x, v, v_1, v_x), \qquad G_{v_x} \neq 0 \tag{3.1}$$

if $D_xT(f) = G(x, f, T(f), D_x(f))$.

Definition 6. A transformation $v = f[u]$ of equation (1.2) is called invertible if any of the dynamical variables u, u_k, $k \in \mathbb{Z}$, $u^{(l)}$, $l \in \mathbb{N}$ can be expressed as a function of a finite subset of the variables

$$x, \quad v := f, \quad v_r := T^r(f), \quad v^{(s)} := D_x^s(f), \quad r \in \mathbb{Z}, \quad s \in \mathbb{N}.$$

Definition 7. Transformations $v = f[u]$ and $w = g[u]$ are called equivalent if there exists $r \in \mathbb{Z}$ such that $g = T^r(f)$.

Proposition 2. *Let a transformation of the form $v = g[u]$ be invertible and map equation (1.2) into equation (3.1). Then this transformation is equivalent to either a transformation of the form*

$$w = f(x, u, u_1, u_2, \dots, u_m), \tag{3.2}$$

or a transformations of the form

$$w = f\big(x, u, u^{(1)}, u^{(2)}, \dots, u^{(n)}\big). \tag{3.3}$$

Definition 8. A transformation is called non-point if this transformation is not equivalent to any point transformation of the form $w = g(x,u)$.

It is easy to see that a transformation of the form (3.2) or (3.3) is non-point only if f depends on more than one of the variables $u, u_1, \ldots, u_m$ or on at least one of the variables $u^{(1)}, \ldots, u^{(n)}$, respectively.

Theorem 2. *Let a non-point invertible transformation of the form (3.3) map equation (1.2) into equation (3.1). Then equation (1.2) can be written in the form* $\varphi(x, u_1, (u_1)_x) = \psi(x, u, u_x)$, *where* $\varphi(x, y, z)$ *and* $\psi(x, y, z)$ *satisfy the condition* $\varphi_y\psi_z - \varphi_z\psi_y \neq 0$, *and the transformation is equivalent to the composition of the invertible transformation* $w = \varphi(x, u, u_x)$ *and an invertible transformation of the form* $v = h(x, w, w^{(1)}, w^{(2)}, \ldots, w^{(n-1)})$. *In particular, any non-point invertible transformation of the form* $v = f(x, u, u_x)$ *is equivalent to the composition of the transformation* $w = \varphi(x, u, u_x)$ *and a point transformation* $v = h(x, w)$.

Corollary 2. *The equation (1.2) admits a non-point invertible transformation of the form (3.3) into an equation of the form (3.1) if and only if both the conditions*

$$F_u F_{u_x u_1} - F_{u u_1} F_{u_x} = 0, \qquad F_u + F_{u_x} T^{-1}(F_{u_1}) \neq 0$$

are satisfied.

Theorem 3. *Let a non-point invertible transformation of the form (3.2) map equation (1.2) into equation (3.1). Then equation (1.2) can be written in the form* $D_x(p(x, u, u_1)) = q(x, u, u_1)$, *where* $p(x, y, z)$ *and* $q(x, y, z)$ *satisfy the condition* $p_y q_z - p_z q_y \neq 0$, *and the transformation is the composition of the transformation* $w = p(x, u, u_1)$ *and an invertible transformation of the form* $v = h(x, w, w_1, w_2, \ldots, w_{m-1})$. *In particular, any non-point invertible transformation of the form* $v = f(x, u, u_1)$ *is the composition of the transformation* $w = p(x, u, u_1)$ *and a point transformation* $v = h(x, w)$.

Corollary 3. *The equation (1.2) admits a non-point invertible transformation of the form (3.2) into an equation of the form (3.1) if and only if equation (1.2) has the form*

$$(u_1)_x = a(x, u, u_1)u_x + b(x, u, u_1),$$

where a *and* b *satisfy the condition* $a_x + a_{u_1} b - a b_{u_1} - b_u \neq 0$.

For brevity, we omit the proofs of the above propositions because they are very similar to the proofs for discrete equations.

4 Examples: the transformations of Liouville equation analogues

A special class of integrable equations of the form

$$u_{xy} = F(x, y, u, u_x, u_y) \tag{4.1}$$

consists of equations for which there exist both a differential substitution of the form $v = X(x, y, u_x, u_{xx}, \ldots)$ and a substitution of the form $w = Y(x, y, u_y, u_{yy}, \ldots)$ that map (4.1) into the equations $v_y = 0$ and $w_x = 0$, respectively. Such equations are called Darboux integrable or equations of the Liouville type. They not only are C-integrable (in accordance with the term of [3]) but also possess infinitely many symmetries of arbitrary high order [19, 20]. The complete classification of the Darboux integrable equations (4.1) has been performed in [20]. Equations with the analogous properties exist among equations of the form (1.1) and (1.2) too, but the classification of such equations is completed for a special case of equation (1.2) only [5]. Therefore, deriving new examples of discrete and semi-discrete Darboux integrable equations from already known equations may be useful (for example, to check the completeness of a future classification).

4.1 Discrete equations

The first example is the discrete Liouville equation

$$u_{1,1} = \frac{(u_{1,0}-1)(u_{0,1}-1)}{u} \tag{4.2}$$

from [9]. According to [2], this equation has the integrals

$$I[u] = \left(\frac{u_{2,0}}{u_{1,0}-1}+1\right)\left(\frac{u-1}{u_{1,0}}+1\right), \qquad J[u] = \left(\frac{u_{0,2}}{u_{0,1}-1}+1\right)\left(\frac{u-1}{u_{0,1}}+1\right), \tag{4.3}$$

i.e. functions $I[u]$, $J[u]$ such that $T_j(I[u]) = I[u]$, $T_i(J[u]) = J[u]$. In addition, equation (4.2) is linearizable: the substitution

$$u = \frac{z_{0,1}z_{1,0}}{(z_{1,0}-z)(z_{0,1}-z)} \tag{4.4}$$

maps solutions of the equation

$$z_{1,1} = z_{1,0} + z_{0,1} - z \tag{4.5}$$

into solutions of (4.2).

It is easy to see that (4.2) can be written in the form $T_j(\varphi(u,u_{1,0})) = \psi(u,u_{1,0})$ and the scheme (1.14)–(1.17) is applicable to this equation:

$$\begin{aligned}
&v = \varphi = \frac{u_{1,0}}{u-1}, \qquad v_{0,1} = \psi = \frac{u_{1,0}-1}{u},\\
&u = p = \frac{v+1}{v-v_{0,1}}, \qquad u_{1,0} = q = v\frac{v_{0,1}+1}{v-v_{0,1}},\\
&\frac{v_{1,0}+1}{v_{1,0}-v_{1,1}} = \frac{v_{0,1}+1}{v-v_{0,1}}v.
\end{aligned}$$

Thus, we obtain the equation

$$v\frac{v_{1,1}-v_{1,0}}{v_{0,1}-v} = \frac{v_{1,0}+1}{v_{0,1}+1} \tag{4.6}$$

that is related to the discrete Liouville equation via the invertible transformation $v = u_{1,0}/(u-1)$. Substituting the expressions of $u, u_{0,1}, u_{1,0}, \ldots$ in terms of $v, v_{0,1}, v_{1,0}, \ldots$ into (4.3), we obtain the integrals of equation (4.6):

$$I[v] = v_{1,0} + \frac{v_{1,0}+1}{v}, \qquad J[v] = \frac{(v_{0,3}-v_{0,1})(v_{0,2}-v)}{(v_{0,3}-v_{0,2})(v_{0,1}-v)}.$$

The composition

$$v = \frac{z_{0,2}(z_{1,0}-z)}{z(z_{2,0}-z_{1,0})}$$

of the transformation $v = u_{1,0}/(u-1)$ and (4.4) allows us to construct the solution

$$v = \frac{(\alpha_{i+2}+\beta_j)(\alpha_{i+1}-\alpha_i)}{(\alpha_i+\beta_j)(\alpha_{j+2}-\alpha_{i+1})}$$

of equation (4.6) from the general solution $z = \alpha_i + \beta_j$ of (4.5), where α_i and β_j are arbitrary.

Equation (4.6) can be written in the form $T_j(\varphi) = \psi$ but φ and ψ are functionally dependent ($\varphi = \psi = I[v]$). According to Theorem 1, this fact implies that (4.6) has no non-point invertible transformation of the form $\tilde{v} = f(v, v_{1,0}, \dots, v_{n,0})$ and hence equation (4.2) admits, up to equivalence, only the first order invertible transformations ($v = f(u_{1,0}/(u-1))$ and $w = g(u_{0,1}/(u-1))$ only).

Applying Corollary 1, we see that the other discrete version [8] of the Liouville equation

$$v_{1,1} = \frac{v_{1,0}v_{0,1} - 1}{v}$$

does not admit a non-point invertible transformation. This equation is mapped into (4.2) via the non-invertible transformation $u = v_{1,0}v_{0,1}$ and has the integrals

$$I[v] = \left(\frac{v_{3,0}}{v_{1,0}} + 1\right)\left(\frac{v}{v_{2,0}} + 1\right), \qquad J[v] = \left(\frac{v_{0,3}}{v_{0,1}} + 1\right)\left(\frac{v}{v_{0,2}} + 1\right).$$

4.2 Differential-difference equations

Let us consider the following analogue of the Liouville equation:

$$(u_1)_x = u_1\left(u_1 + \frac{u_x}{u} + u\right). \tag{4.7}$$

This equation has the integrals

$$X[u] = 2\frac{u_{xx}}{u} - 3\frac{u_x^2}{u^2} - u^2, \qquad I[u] = \left(1 + \frac{u_1}{u_2}\right)\left(1 + \frac{u_1}{u}\right),$$

i.e. functions $X[u]$, $I[u]$ such that $T(X) = X$, $D_x(I) = 0$. Like the discrete and continuous Liouville equations, equation (4.7) is linearizable: the substitution

$$u = \frac{(z_1 - z)z_x}{z_1 z} \tag{4.8}$$

maps solutions of the equation

$$(z_1)_x = z_x \tag{4.9}$$

into solutions of (4.7). The above information and some other details about equation (4.7) can be found in [2].

Equation (4.7) can be written as

$$\frac{(u_1)_x}{u_1} - u_1 = \frac{u_x}{u} + u.$$

Applying the scheme (1.10)–(1.13), we obtain

$$\begin{aligned} &v = \frac{1}{2}\left(\frac{u_x}{u} - u\right), \qquad v_1 = \frac{1}{2}\left(\frac{u_x}{u} + u\right), \\ &u = v_1 - v, \qquad u_x = v_1^2 - v^2, \\ &(v_1 - v)_x = v_1^2 - v^2. \end{aligned} \tag{4.10}$$

Thus, the invertible transformation $v = (u_x/u - u)/2$ maps equation (4.7) into the sequence of the coupled Riccati equations (4.10). Expressing $X[u]$ and $I[u]$ in terms of $v, v_1, v_x, \dots$, we obtain the integrals

$$X[v] = v_x - v^2, \qquad I[v] = \frac{(v_3 - v_1)(v_2 - v)}{(v_3 - v_2)(v_1 - v)}.$$

of equation (4.10). The composition

$$v = \frac{z_{xx}}{2z_x} - \frac{z_x}{z}$$

of the invertible transformation and (4.8) generates the solution

$$v = \frac{\beta_{xx}}{2\beta_x} - \frac{\beta_x}{\alpha_i + \beta} \tag{4.11}$$

of equation (4.10) from the general solution $z = \alpha_i + \beta(x)$ of (4.9), where α_i and $\beta(x)$ are arbitrary. Equation (4.10) was used in [2] as an example of an equation admitting the integrals and the solution (4.11) was constructed in this article by another method (directly form the equation $X[v] = \xi(x)$).

Moreover, equation (4.7) can be represented in the form (1.13) too. Applying the scheme (1.10)–(1.13) in the reverse order, we get

$$\begin{aligned} &w = p = \frac{u_1}{u}, \qquad w_x = q = \frac{u_1^2}{u} + u_1, \\ &u = \frac{w_x}{(w+1)w}, \qquad u_1 = \frac{w_x}{w+1}, \\ &\frac{(w_1)_x}{(w_1+1)w_1} = \frac{w_x}{w+1} \end{aligned}$$

and see that the invertible transformation $w = u_1/u$ maps (4.7) into the equation

$$(w_1)_x = w_x w_1 \frac{w_1 + 1}{w + 1}\,. \tag{4.12}$$

As above, we construct the integrals

$$X[w] = 2\frac{w_{xxx}}{w_x} - 3\frac{w_{xx}^2}{w_x^2}, \qquad I[w] = \frac{(w_1+1)(w+1)}{w_1}$$

of equation (4.12) by expressing $X[u]$ and $I[u]$ in terms of $w, w_1, w_x, \ldots$, and obtain its solution

$$w = \frac{(\alpha_{i+2} - \alpha_{i+1})(\alpha_i + \beta(x))}{(\alpha_{i+1} - \alpha_i)(\alpha_{i+2} + \beta(x))}$$

with arbitrary α_i and $\beta(x)$ by applying the composition

$$w = \frac{(z_2 - z_1)z}{(z_1 - z)z_2}$$

of the transformations $w = u_1/u$ and (4.8) to the general solution $z = \alpha_i + \beta(x)$ of equation (4.9).

The semi-discrete Liouville equation (4.7) is a special case of the Darboux integrable equation

$$(u_1)_x = u_x + \sqrt{Ce^{2u_1} + Be^{(u_1+u)} + Ce^{2u}}. \tag{4.13}$$

that was introduced in [5]. Indeed, replacing u in (4.7) by $\exp(u)$, we obtain equation (4.13) with $C = 1$, $B = 2$. Without loss of generality, we can assume that the constant C in equation (4.13) equals 1 or 0 because C can be scaled via the point transformation $u = \tilde{u} + \gamma$. Applying Corollary 2, we see that equation (4.13) admits an invertible transformation of the form $v = f(x, u, u_x)$ only if $B = 2C$. The invertible transformations

$$w = e^{u_1 - u}, \qquad e^u = \frac{w_x}{w\sqrt{Cw^2 + Bw + C}}$$

relate (4.13) to the equation

$$(w_1)_x = w_1 w_x \sqrt{\frac{Cw_1^2 + Bw_1 + C}{Cw^2 + Bw + C}}\,. \tag{4.14}$$

The later equation has the integrals

$$X[w] = 2\frac{w_{xxx}}{w_x} - 3\left(\frac{w_{xx}}{w_x}\right)^2 + \frac{3w_x^2(B^2 - 4C^2)}{4(Cw^2 + Bw + C)},$$
$$I[w] = \int^{w_1} \frac{ds}{s\sqrt{Cs^2 + Bs + C}} - \int^{w} \frac{ds}{\sqrt{Cs^2 + Bs + C}}$$

and can not be reduced to equation (4.12) via a point transformation because equation (4.14), in contrast to equation (4.12), does not admit an invertible transformation of the form $v = f(x, w, w_x, w_{xx})$ if $B \neq 2C$.

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