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Orthogonality Relations for Multivariate Krawtchouk Polynomials

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Abstract. The orthogonality relations of multivariate Krawtchouk polynomials are discussed. In case of two variables, the necessary and sufficient conditions of orthogonality is given by Grünbaum and Rahman in [*SIGMA* **6** (2010), 090, 12 pages]. In this study, a simple proof of the necessary and sufficient condition of orthogonality is given for a general case.

Key words: multivariate orthogonal polynomial; hypergeometric function

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1 Introduction

Consider

$$X(n,N)=\{\boldsymbol{x}=(x_0,x_1,\dots,x_{n-1})\in\mathbb{N}_0^n \mid |\boldsymbol{x}|=N\},$$

where $\mathbb{N}_0$ is the set of nonnegative integers and $|\boldsymbol{x}|=x_0+x_1+\dots+x_{n-1}$. We recall the multinomial coefficient

$$\binom{N}{\boldsymbol{x}}=\binom{N}{x_0,\dots,x_{n-1}}=(-1)^{x_1+\dots+x_{n-1}}\frac{(-N)_{x_1+\dots+x_{n-1}}}{x_1!\cdots x_{n-1}!}$$

for $\boldsymbol{x}\in X(n,N)$. Let $M_n(R)$ be the set of all $n\times n$ matrices over a set R. We fix $\boldsymbol{x}\in X(n,N)$ and $A=(a_{ij})_{1\le i,j\le n-1}\in M_{n-1}(\mathbb{C})$. We define the functions $\phi_A(\boldsymbol{x};\boldsymbol{m})$ of $\boldsymbol{m}=(m_0,\dots,m_{n-1})\in X(n,N)$ by the following generating function

$$\Phi_N(A;\boldsymbol{x})=\prod_{i=0}^{n-1}\left(\sum_{j=0}^{n-1}a_{ij}t_j\right)^{x_i}=\sum_{\boldsymbol{m}\in X(n,N)}\binom{N}{\boldsymbol{m}}\phi_A(\boldsymbol{x};\boldsymbol{m})\boldsymbol{t}^{\boldsymbol{m}},\tag{1}$$

where $\boldsymbol{t}^{\boldsymbol{m}}=t_0^{m_0}t_1^{m_1}\cdots t_{n-1}^{m_{n-1}}$ and $a_{0j}=a_{i0}=1$ for $0\le i,j\le n-1$. We know a hypergeometric expression of $\phi_A(\boldsymbol{x};\boldsymbol{m})$

$$\phi_A(\boldsymbol{x};\boldsymbol{m})=\sum_{\substack{\sum_{i,j}c_{ij}\le N\\(c_{ij})\in M_{n-1}(\mathbb{N}_0)}}\frac{\prod\limits_{i=1}^{n-1}(-x_i)_{\sum\limits_{j=1}^{n-1}c_{ij}}\prod\limits_{j=1}^{n-1}(-m_j)_{\sum\limits_{i=1}^{n-1}c_{ij}}}{(-N)_{\sum_{i,j}c_{ij}}}\frac{\prod(1-a_{ij})^{c_{ij}}}{\prod c_{ij}!},\tag{2}$$

where $M_{n-1}(\mathbb{N}_0)$ is the set of square matrices of degree $n-1$ with nonnegative integer elements. We prove the formula (2) in the last section of this paper.

This type of hypergeometric functions was originally defined by Aomoto and Gel'fand for general parameters. We are interested in the aspects of discrete orthogonal polynomials of these functions with weights

$$b_n(\boldsymbol{x};N;\boldsymbol{\eta}_{(i)}) = \binom{N}{\boldsymbol{x}} \prod_{j=0}^{n-1} \eta_{ji}^{x_j}$$

for $\boldsymbol{\eta}_{(i)} = (\eta_{0i},\dots,\eta_{n-1i}) \in \mathbb{C}^{*n}$ $(i=1,2)$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. For a special case, when $n=2$, they are well known and called the Krawtchouk polynomials. For general values of n, the author shows that such orthogonal polynomials appear as the zonal spherical functions of Gel'fand pairs of complex reflection groups [3]. In general, the author and H. Tanaka give the orthogonality relation of $\phi_A(\boldsymbol{x};\boldsymbol{m})$s by using the character algebras [4]. R.C. Griffiths shows that the polynomials defined by the generating function (1) are mutually orthogonal [1]. In their paper [2], Grünbaum and Rahman discuss and determine the necessary and sufficient conditions of the orthogonality of $\phi_A(\boldsymbol{x};\boldsymbol{m})$s for $A \in M_2(\mathbb{C})$, which are proved by analytic methods. In these four literature, the authors consider the case that weights $b_n(\boldsymbol{x};N;\boldsymbol{\eta}_{(i)})$ are positive.

In this study, we give a linear algebraic proof of the Grünbaum and Rahman's condition for general values of n and for arbitrary weights including the complex case. Our proof of the sufficient condition is close to [1] and [4].

For $A \in M_{n-1}(\mathbb{C})$, we define a matrix $A_0 = (a_{ij})_{0\le i,j\le n-1} \in M_n(\mathbb{C})$ by substituting $a_{0j} = a_{i0} = 1$ $(0 \le \forall\, i,j \le n-1)$. Our main result is as follows:

Theorem 1. *The following are equivalent.*

(*a*) *The orthogonality relation*

$$\sum_{\boldsymbol{x}\in X(n,N)} b_n(\boldsymbol{x};N;\boldsymbol{\eta}_{(1)})\phi_A(\boldsymbol{x};\boldsymbol{m})\overline{\phi_A(\boldsymbol{x};\boldsymbol{m}')} = \delta_{\boldsymbol{m},\boldsymbol{m}'}\frac{\boldsymbol{\eta}_{(2)}^{\boldsymbol{m}}}{\binom{N}{\boldsymbol{m}}}$$

holds for $\forall\, \boldsymbol{m},\boldsymbol{m}' \in X(n,N)$.

(*b*) *A relation*

$$A_0^* D_1 A_0 = \zeta D_2 \tag{3}$$

holds for some N*th root of unity* ζ. *Here* A_0^* *is the conjugate transpose of* A_0 *and* $D_i = \mathrm{diag}(\eta_{0i},\eta_{1i},\dots,\eta_{n-1i}) \in \mathrm{GL}_n(\mathbb{C})$ *is a diagonal matrix* $(i=1,2)$.

Remark 1. We assume that the diagonal elements of D_1 and D_2 appearing in the above-mentioned theorem are real. Thus, one can recover the formula (1.18) of Grünbaum and Rahman's paper [2] by substituting $n=3$ and $A = \begin{bmatrix} 1-u_1 & 1-u_2 \\ 1-v_1 & 1-v_2 \end{bmatrix} \in M_2(\mathbb{C})$. In this case, since the diagonal elements of $A_0^*D_1A_0$ and D_2 are positive, $\zeta = 1$.

Remark 2 ([4])**.** We assume that a pair of finite groups (G,H) is a Gel'fand pair and A_0 is the table of the zonal spherical functions of (G,H). Let $D_0,\dots,D_{n-1}$ be the double cosets of H in G, and $d_0,\dots,d_n$ be the dimensions of the irreducible components of 1_H^G. Put $D_1 = \mathrm{diag}(|D_0|,\dots,|D_{n-1}|)$ and $D_2 = \mathrm{diag}(|G|/d_0,\dots,|G|/d_{n-1})$. Then (3) holds from the orthogonality relation of the zonal spherical functions. Furthermore $\phi_A(\boldsymbol{x};\boldsymbol{m})$'s are realized as the zonal spherical functions of a Gel'fand pair $(G\wr S_N, H\wr S_N)$. Therefore they satisfy the orthogonality relation (a) in the theorem. In general, A_0 is an eigenmatrix of a character algebra is considered in [4].

This paper organized as follows. First, we prove the main theorem in the next section. Second, we prove (2) in the last section. It seems to be the first explicit proof of this fact.

2 Proof of Theorem 1

Let $\mathbf{e}_i = (\delta_{0i}, \delta_{1i}, \ldots, \delta_{ni})$ be an ith unit vector $(0 \le i \le n-1)$. We put $\mathbf{s} = (s_0, s_1, \ldots, s_{n-1})$ and $\mathbf{t} = (t_0, t_1, \ldots, t_{n-1})$. We compute

$$\prod_{i=0}^{n-1} \left(s_i \mathbf{e}_i A_0 \, {}^t\mathbf{t}\right)^{x_i} = \prod_{i=0}^{n-1} \left(\sum_{j=0}^{n-1} a_{ij} t_j\right)^{x_i} \prod_{i=0}^{n-1} s_i^{x_i} = \Phi_N(A, \boldsymbol{x}) \prod_{i=0}^{n-1} s_i^{x_i}. \tag{4}$$

By multiplying (4) with multinomial coefficient, we have

$$\sum_{\boldsymbol{x} \in X(n,N)} \binom{N}{\boldsymbol{x}} \prod_{i=0}^{n-1} \left(s_i \mathbf{e}_i A_0 \, {}^t\mathbf{t}\right)^{x_i} = \left(\sum_{i=0}^{n-1} s_i \mathbf{e}_i A_0 \, {}^t\mathbf{t}\right)^N = \left(\mathbf{s} A_0 \, {}^t\mathbf{t}\right)^N. \tag{5}$$

First, we assume (3), and then change the variables, say

$$\mathbf{s} = \mathbf{u} A_0^* D_1,$$

where $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$. This change of variables is same as $s_i = \eta_{i1} \mathbf{e}_i \overline{A_0} \, {}^t\mathbf{u}$. We substitute $\mathbf{s}$ for (5). Then, the right side of (5) gives

$$\left(\mathbf{s} A \, {}^t\mathbf{t}\right)^N = \left(\mathbf{u} \zeta D_2 \, {}^t\mathbf{t}\right)^N = \left(\mathbf{u} D_2 \, {}^t\mathbf{t}\right)^N = \sum_{\boldsymbol{m} \in X(n,N)} \binom{N}{\boldsymbol{m}} \prod_{i=0}^{n} (\eta_{i2} u_i t_i)^{m_i}. \tag{6}$$

Under this substitution, we consider the left side of (5) and have

$$\begin{aligned}
\sum_{\boldsymbol{x} \in X(n,N)} \binom{N}{\boldsymbol{x}} \prod_{i=0}^{n-1} \left(s_i \mathbf{e}_i A_0 \, {}^t\mathbf{t}\right)^{x_i} &= \sum_{\boldsymbol{x} \in X(n,N)} \binom{N}{\boldsymbol{x}} \prod_{i=0}^{n-1} s_i^{x_i} \prod_{i=0}^{n-1} \left(\mathbf{e}_i A_0 \, {}^t\mathbf{t}\right)^{x_i} \\
&= \sum_{\boldsymbol{x} \in X(n,N)} \binom{N}{\boldsymbol{x}} \prod_{i=0}^{n-1} \left(\eta_{i1} \mathbf{e}_i \overline{A_0} \, {}^t\mathbf{u}\right)^{x_i} \prod_{i=0}^{n-1} \left(\mathbf{e}_i A_0 \, {}^t\mathbf{t}\right)^{x_i} \\
&= \sum_{\boldsymbol{x} \in X(n,N)} \binom{N}{\boldsymbol{x}} \prod_{i=0}^{n-1} {\eta_{i1}}^{x_i} \prod_{i=0}^{n-1} \left(\mathbf{e}_i \overline{A_0} \, {}^t\mathbf{u}\right)^{x_i} \prod_{i=0}^{n-1} \left(\mathbf{e}_i A_0 \, {}^t\mathbf{t}\right)^{x_i}.
\end{aligned}$$

We expand the last two products of the above-mentioned formula in terms of $\mathbf{u}^{\boldsymbol{m}} \mathbf{t}^{\boldsymbol{m}'}$'s;

$$\prod_{i=0}^{n-1} \left(\mathbf{e}_i \overline{A_0} \, {}^t\mathbf{u}\right)^{x_i} \prod_{i=0}^{n-1} \left(\mathbf{e}_i A_0 \, {}^t\mathbf{t}\right)^{x_i} = \sum_{\boldsymbol{m}, \boldsymbol{m}' \in X(n,N)} \binom{N}{\boldsymbol{m}} \binom{N}{\boldsymbol{m}'} \phi_A(\boldsymbol{x}; \boldsymbol{m}) \overline{\phi_A(\boldsymbol{x}; \boldsymbol{m}')} \mathbf{u}^{\boldsymbol{m}} \mathbf{t}^{\boldsymbol{m}'}.$$

Now, the left side of (5) gives

$$\sum_{\boldsymbol{m}, \boldsymbol{m}' \in X(n,N)} \binom{N}{\boldsymbol{m}} \binom{N}{\boldsymbol{m}'} \left(\sum_{\boldsymbol{x} \in X(n,N)} \prod_{i=0}^{n-1} \eta_{i1}^{x_i} \binom{N}{\boldsymbol{x}} \phi_A(\boldsymbol{x}; \boldsymbol{m}) \overline{\phi_A(\boldsymbol{x}; \boldsymbol{m}')}\right) \mathbf{u}^{\boldsymbol{m}} \mathbf{t}^{\boldsymbol{m}'}. \tag{7}$$

By comparing coefficients of $\mathbf{u}^k \mathbf{t}^{k'}$ of (6) with (7), we conclude that

$$\sum_{\boldsymbol{x} \in X(n,N)} \prod_{i=0}^{n-1} \eta_{i1}^{x_i} \binom{N}{\boldsymbol{x}} \phi_A(\boldsymbol{x}; \boldsymbol{m}) \overline{\phi_A(\boldsymbol{x}; \boldsymbol{m}')} = \frac{\prod_{i=0}^{n-1} \eta_{i2}^{m_i}}{\binom{N}{\boldsymbol{m}}} \delta_{\boldsymbol{m}\boldsymbol{m}'}. \tag{8}$$

Conversely, we assume (8) and substitute it for (7). Then, we reverse the above-mentioned computations and observe that (5) gives

$$\left(\mathbf{u}D_2\,{}^t\mathbf{t}\right)^N = \left(\mathbf{u}A_0^*D_1A_0\,{}^t\mathbf{t}\right)^N.$$

This means that $\mathbf{u}\zeta(\mathbf{u},\mathbf{t})D_2\,{}^t\mathbf{t} = \mathbf{u}A_0^*D_1A_0\,{}^t\mathbf{t}$ holds for some Nth root of unity $\zeta(\mathbf{u},\mathbf{t})$. We have $A_0^*D_1A_0 = \operatorname{diag}(\zeta(\mathbf{e}_0,\mathbf{e}_0)\eta_{02},\ldots,\zeta(\mathbf{e}_{n-1},\mathbf{e}_{n-1})\eta_{n-12})$. We put $\mathbf{e}_{ij}(\theta,\varepsilon) = \cos\theta\mathbf{e}_i + \varepsilon\sin\theta\mathbf{e}_j$ for $|\varepsilon| > 0$. We define

$$\zeta_\varepsilon(\theta) = \frac{\mathbf{e}_{ij}(\theta,\varepsilon)A_0^*D_1A_0\,{}^t\mathbf{e}_{ij}(\theta,\varepsilon)}{\mathbf{e}_{ij}(\theta,\varepsilon)D_2\,{}^t\mathbf{e}_{ij}(\theta,\varepsilon)} = \frac{\zeta(\mathbf{e}_i,\mathbf{e}_i)\cos^2\theta\eta_{i2} + \zeta(\mathbf{e}_j,\mathbf{e}_j)\varepsilon^2\sin^2\theta\eta_{j2}}{\cos^2\theta\eta_{i2} + \varepsilon^2\sin^2\theta\eta_{j2}}.$$

Since $\zeta_\varepsilon(\frac{\pi}{2}) = \zeta(\varepsilon\mathbf{e}_j,\varepsilon\mathbf{e}_j)$ does not depend on ε, we have $\zeta(\varepsilon\mathbf{e}_j,\varepsilon\mathbf{e}_j) = \zeta(\mathbf{e}_j,\mathbf{e}_j)$. By taking ε as $\operatorname{Arg}(\frac{-\varepsilon^2\eta_{j2}}{\eta_{i2}}) \neq 0$, we have that $\zeta_\varepsilon(\theta)$ is a continuous function from $[0,\frac{\pi}{2}]$ to the set of the Nth root of unities. Therefore $\zeta_\varepsilon(\theta)$ is a constant function, especially $\zeta_\varepsilon(0) = \zeta(\mathbf{e}_i,\mathbf{e}_i) = \zeta_\varepsilon(\frac{\pi}{2}) = \zeta(\varepsilon\mathbf{e}_j,\varepsilon\mathbf{e}_j)$ $(0 \le i < j \le n-1)$. Consequently we have

$$\zeta D_2 = A_0^*D_1A_0$$

for some Nth root of unity ζ.

3 Proof of (2)

Here we give a proof of the formula (2) through direct computations. We need the following lemma.

Lemma 1. *Put $\boldsymbol{p} = (p_0,\ldots,p_{n-1}) \in \mathbb{N}_0^n$ with $|\boldsymbol{p}| \le N$. For $\boldsymbol{m} = (m_0,\ldots,m_{n-1}) \in X(n,N)$ and $\boldsymbol{z} = (z_0,\ldots,z_{n-1}) \in X(n,N-|\boldsymbol{p}|)$ with $\boldsymbol{m}-\boldsymbol{z} \in \mathbb{N}_0^n$, we have*

$$\binom{N-|\boldsymbol{p}|}{\boldsymbol{z}} = \binom{N}{\boldsymbol{m}}\frac{\prod_{i=0}^{n-1}(-m_i)_{m_i-z_i}}{(-N)_{|\boldsymbol{p}|}}.$$

Proof. We compute

$$\begin{aligned}\binom{N-|\boldsymbol{p}|}{\boldsymbol{z}} &= \binom{N}{\boldsymbol{m}}\frac{(N-|\boldsymbol{p}|)!}{N!}\prod_{i=0}^{n-1}\binom{m_i}{m_i-z_i,\,z_i}(m_i-z_i)! = \binom{N}{\boldsymbol{m}}\frac{\prod_{i=0}^{n-1}(-1)^{m_i-z_i}(-m_i)_{m_i-z_i}}{\binom{N}{|\boldsymbol{p}|}|\boldsymbol{p}|!}\\ &= \binom{N}{\boldsymbol{m}}\frac{(-1)^{|\boldsymbol{p}|}\prod_{i=0}^{n-1}(-1)^{m_i-z_i}(-m_i)_{m_i-z_i}}{(-N)_{|\boldsymbol{p}|}} = \binom{N}{\boldsymbol{m}}\frac{\prod_{i=0}^{n-1}(-m_i)_{m_i-z_i}}{(-N)_{|\boldsymbol{p}|}}.\end{aligned}$$ ■

Now, we can prove the formula (2). We put $b_{ij} = 1-a_{ij}$ and $\boldsymbol{c}_i = (c_{i0},\ldots,c_{in-1})$. We compute

$$\begin{aligned}\Phi_N(A;\boldsymbol{x}) &= \prod_{i=0}^{n-1}\left(\sum_{j=0}^{n-1}a_{ij}t_j\right)^{x_i} = \prod_{i=0}^{n-1}\left\{\sum_{j=0}^{n-1}t_j - \sum_{j=0}^{n-1}b_{ij}t_j\right\}^{x_i}\\ &= \prod_{i=0}^{n-1}\left\{\sum_{p_i=0}^{x_i}(-1)^{p_i}\binom{x_i}{x_i-p_i,\,p_i}\left(\sum_{j=0}^{n-1}t_j\right)^{x_i-p_i}\left(\sum_{j=0}^{n-1}b_{ij}t_j\right)^{p_i}\right\}\end{aligned}$$

$$= \sum_{\substack{0\le p_i\le x_i\\(0\le i\le n-1)}} (-1)^{|\boldsymbol{p}|} \left(\sum_{j=0}^{n-1} t_j\right)^{N-|p|} \prod_{i=0}^{n-1} \binom{x_i}{x_i-p_i, p_i} \left(\sum_{j=0}^{n-1} b_{ij} t_j\right)^{p_i}$$

$$= \sum_{\substack{0\le p_i\le x_i\\(0\le i\le n-1)}} (-1)^{|\boldsymbol{p}|} \sum_{\substack{|\boldsymbol{c}_i|=p_i,\\|\boldsymbol{z}|=N-|p|}} \binom{N-|\boldsymbol{p}|}{\boldsymbol{z}} \boldsymbol{t}^{\boldsymbol{z}} \prod_{i=0}^{n-1} \binom{x_i}{x_i-p_i, p_i} \prod_{i=0}^{n-1} \binom{p_i}{\boldsymbol{c}_i} \prod_{i,j=0}^{n-1} b_{ij}^{c_{ij}} t_j^{c_{ij}}$$

$$= \sum_{0\le|\boldsymbol{p}|\le N} (-1)^{|\boldsymbol{p}|} \sum_{\substack{|\boldsymbol{c}_i|=p_i,\\|\boldsymbol{z}|=N-|p|}} \binom{N-|\boldsymbol{p}|}{\boldsymbol{z}} \frac{\prod_{i=0}^{n-1} (-1)^{|\boldsymbol{c}_i|}(-x_i)_{|\boldsymbol{c}_i|}}{\prod_{i,j=0}^{n-1} c_{ij}!} \boldsymbol{t}^{\boldsymbol{z}} \prod_{i,j=0}^{n-1} b_{ij}^{c_{ij}} t_j^{c_{ij}}$$

$$= \sum_{\boldsymbol{m}\in X(n,N)} \binom{N}{\boldsymbol{m}} \left[\sum_{\sum_{ij} c_{ij}\le N} \frac{\prod_{j=0}^{n-1} (-m_j)_{\sum_{i=0}^{n-1} c_{ij}}}{(-N)_{\sum_{ij} c_{ij}}} \frac{\prod_{i=1}^{n-1} (-x_i)_{|\boldsymbol{c}_i|}}{\prod_{i,j=0}^{n-1} c_{ij}!} \prod_{i,j=0}^{n-1} b_{ij}^{c_{ij}} \right] \boldsymbol{t}^{\boldsymbol{m}}.$$

In the last equation, we use $|\boldsymbol{p}| = \sum_{ij} c_{ij}$, $m_i - z_i = \sum_{i=0}^{n-1} c_{ij} \ge 0$ and Lemma 1. Since $b_{0j} = b_{i0} = 0$ for any i and j, we have the formula.

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