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Exact Solutions with Two Parameters for an Ultradiscrete Painlevé Equation of Type $A_6^{(1)}$ ⋆

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Abstract. An ultradiscrete system corresponding to the q-Painlevé equation of type $A_6^{(1)}$, which is a q-difference analogue of the second Painlevé equation, is proposed. Exact solutions with two parameters are constructed for the ultradiscrete system.

Key words: Painlevé equations; ultradiscrete systems

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1 Introduction

Discrete Painlevé equations are prototype integrable systems studied from various points of view [24, 28]. They are discrete equations which are reduced to the Painlevé equations in suitable limiting processes, and moreover, which pass the singularity confinement test [4]. Many results are already given about special solutions of discrete Painlevé equations [5, 11, 12, 13, 16, 25].

Ultradiscretization [30] is a limiting procedure transforming a given difference equation into a cellular automaton. In addition the cellular automaton constructed by this procedure preserves the essential properties of the original equation, such as the structure of exact solutions. In this procedure, we first replace a dependent variable x_n in a given equation by

$$x_n = \exp\left(\frac{X_n}{\varepsilon}\right),$$

where ε is a positive parameter. Then, we apply $\varepsilon \log$ to both sides of the equation and take the limit $\varepsilon \to +0$. Using identity

$$\lim_{\varepsilon\to+0} \varepsilon \log\left(e^{X/\varepsilon} + e^{Y/\varepsilon}\right) = \max(X, Y)$$

and exponential laws, we find that addition, multiplication, and division for the original variables are replaced by maximum, addition, and subtraction for the new ones, respectively. In this way the original difference equation is approximated to a piecewise linear equation which can be regarded as a time evolution rule for a cellular automaton.

It is an interesting problem to study ultradiscrete analogues of the Painlevé equations and the structure of their solutions. Some ultradiscrete Painlevé equations and their special solutions are studied in, for example, [3, 8, 9, 10, 22, 26, 29]. However the structure of the general solutions is completely unclear today.

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In this paper we propose a new ultradiscrete Painlevé equation of simultaneous type. With this purpose, we start with a q-Painlevé equation of type $A_6^{(1)}$ (q-$P(A_6)$) [5, 11, 12, 18, 19, 27, 28]

$$f_n f_{n-1} = 1 + g_{n-1}, \qquad g_n g_{n-1} = \frac{aq^{2n} f_n}{f_n + q^n}, \tag{1.1}$$

where a and q are parameters. Equation (1.1) is the simplest nontrivial q-Painlevé equation that admits a Bäcklund transformation. This equation is also referred to as q-analogue of the second Painlevé equation

$$(f_{n+1} f_n - 1)(f_n f_{n-1} - 1) = \frac{aq^{2n} f_n}{f_n + q^n}$$

and reduced to the second Painlevé equation

$$\frac{d^2 y}{ds^2} = 2y^3 + 2sy + c$$

in a continuous limit [23].

Furthermore, we propose an exact solution with two parameters for the ultradiscrete system. Although the Painlevé equations and the q-analogues of these are not generally solvable in terms of elementary functions [17, 18, 20, 31], it is an amazing fact that the ultradiscrete analogues of these are "solvable".

In Section 2, we present an ultradiscrete analogue of q-$P(A_6)$. In Section 3, we give an exact solution with two parameters of this ultradiscrete system. In Section 4, we construct an ultradiscrete Bäcklund transformation. The exact solutions with two parameters are also obtained from a "seed" solution. In Section 5, we give ultradiscrete hypergeometric solutions which are included in the solutions with two parameters. Finally concluding remarks are given in Section 6.

2 Ultradiscrete Painlevé equation

We construct an ultradiscrete analogue of q-$P(A_6)$ (1.1). Let us introduce

$$f_n = \exp(F_n/\varepsilon), \qquad g_n = \exp(G_n/\varepsilon), \qquad q = \exp(Q/\varepsilon), \qquad a = \exp(A/\varepsilon)$$

and take the limit $\varepsilon \to +0$. Then q-$P(A_6)$ (1.1) is reduced to an ultradiscrete analogue of q-$P(A_6)$ (ud-$P(A_6)$),

$$F_n + F_{n-1} = \max(0, G_{n-1}), \tag{2.1a}$$

$$G_n + G_{n-1} = A + 2nQ - \max(0, nQ - F_n). \tag{2.1b}$$

Because one cannot make a known second order single equation from this system, this ud-$P(A_6)$ is an essentially new ultradiscrete Painlevé system.

In [6], we have given another ud-$P(A_6)$ by means of ultradiscretization with parity variables, which is an extended version of ultradiscrete procedure. This procedure keeps track of the sign of original variables [15]. We have also presented its special solution that corresponds to the hypergeometric solution in the discrete system.

3 Solutions

In order to construct a solution of ud-$P(A_6)$, we take the following strategy. First we seek solutions for linear systems which are obtained from the piecewise linear system. These solutions satisfy ud-$P(A_6)$ in some restricted range of n. Next we connect these solutions together to ensure that they satisfy (2.1) for any n.

Theorem 1. *ud-$P(A_6)$ admits the following solution for $Q > 0$, $A = 2(m + r)Q$, $m \in \mathbb{N}$, $-1/2 < r \le 1/2$:*

$$F_n = d_1 (-1)^{n-m}, \qquad G_n = \frac{2n + 2m + 2r + 1}{2} Q + d_2 (-1)^{n-m},$$

for $n \le -m - 1$, where d_1 and d_2 satisfy

$$-(m + 2) Q \le d_1 \le (m + 1) Q, \qquad \frac{2r - 5}{2} Q \le d_2 \le \frac{3 - 2r}{2} Q;$$

$$F_n = \frac{n + m + r}{2} Q + e_1 (-1)^{n-m} - e_2 (n - m)(-1)^{n-m},$$

$$G_n = \frac{2n + 2m + 2r + 1}{2} Q + e_2 (-1)^{n-m},$$

for $-m \le n \le m - 1$, where e_1 and e_2 satisfy

$$-\frac{1 + 2r}{2} Q \le e_2 \le \frac{3 + 2r}{2} Q, \qquad e_1 + e_2 \le \frac{1 + r}{2} Q, \qquad e_1 + 2e_2 \ge -\frac{2 + r}{2} Q,$$

$$e_1 + (2m - 1) e_2 \le \frac{2m + r - 1}{2} Q, \qquad e_1 + 2me_2 \ge -\frac{2m + r}{2} Q,$$

and

$$F_n = \frac{n + 2m + 2r}{3} Q + h_1 \cos \frac{2}{3} \pi (n - m) + \frac{2h_2 - h_1}{\sqrt{3}} \sin \frac{2}{3} \pi (n - m),$$

$$G_n = \frac{2n + 4m + 4r + 1}{3} Q + h_2 \cos \frac{2}{3} \pi (n - m) + \frac{h_2 - 2h_1}{\sqrt{3}} \sin \frac{2}{3} \pi (n - m),$$

for $n \ge m$, where h_1 and h_2 satisfy

$$h_1 \le \frac{6 - 2r}{3} Q, \qquad h_2 \ge \frac{2r - 4}{3} Q, \qquad h_2 - h_1 \le \frac{2 - 2r}{3} Q.$$

Here the relations between d_1, d_2 and e_1, e_2 are

$$d_1 = \frac{r}{2} Q + e_1 + 2me_2 - 2 \max\left(0, \frac{2r - 1}{2} Q - e_2\right), \qquad d_2 = e_2,$$

and those between e_1, e_2 and h_1, h_2 are

$$h_1 = -\frac{r}{6} Q + e_1, \qquad h_2 = \frac{1 - 2r}{6} Q + e_2 - \max\left(0, -\frac{r}{2} Q - e_1\right).$$

Proof. We consider the case $A = 2(m + r)Q$, $m \in \mathbb{N}$ and $-1/2 < r \le 1/2$. If $G_{n-1} \le 0$ and $nQ - F_n \le 0$, then ud-$P(A_6)$ (2.1) can be written as the following system of linear equations:

$$F_n + F_{n-1} = 0, \qquad G_n + G_{n-1} = (2n + 2m + 2r) Q. \tag{3.1}$$

The general solution to the linear system (3.1) is

$$F_n = d_1 (-1)^{n-m}, \qquad G_n = \frac{2n + 2m + 2r + 1}{2} Q + d_2 (-1)^{n-m}, \tag{3.2}$$

where d_1 and d_2 are arbitrary constants. If $d_1 = d_2 = 0$, the particular solution (3.2) satisfies $G_{n-1} \le 0$ and $nQ - F_n \le 0$ for $n \le -m - 1$. The sufficient condition that the general solution (3.2) satisfies $G_{n-1} \le 0$ and $nQ - F_n \le 0$ for $n \le -m - 1$ is

$$-(m + 2) Q \le d_1 \le (m + 1) Q, \qquad \frac{2r - 5}{2} Q \le d_2 \le \frac{3 - 2r}{2} Q. \tag{3.3}$$

Therefore (3.2) that satisfies (3.3) is a solution to ud-$P(A_6)$ for $n \le -m-1$. If $G_{n-1} \ge 0$ and $nQ - F_n \le 0$, then ud-$P(A_6)$ (2.1) can be written as the following system of linear equations:

$$F_n + F_{n-1} = G_{n-1}, \qquad G_n + G_{n-1} = (2n+2m+2r)\,Q. \tag{3.4}$$

The general solution to the linear system (3.4) is

$$\begin{aligned} F_n &= \frac{n+m+r}{2}Q + e_1\,(-1)^{n-m} - e_2\,(n-m)\,(-1)^{n-m}, \\ G_n &= \frac{2n+2m+2r+1}{2}Q + e_2\,(-1)^{n-m}, \end{aligned} \tag{3.5}$$

where e_1 and e_2 are arbitrary constants. If $e_1 = e_2 = 0$, (3.5) satisfies $G_n \ge 0$ and $nQ - F_n \le 0$ for $-m \le n \le m-1$. The condition that the general solution (3.5) satisfies $G_n \ge 0$ and $nQ - F_n \le 0$ for $-m \le n \le m-1$ is

$$\begin{aligned} &-\frac{1+2r}{2}Q \le e_2 \le \frac{3+2r}{2}Q, \qquad e_1 + e_2 \le \frac{1+r}{2}Q, \qquad e_1 + 2e_2 \ge -\frac{2+r}{2}Q, \\ &e_1 + (2m-1)\,e_2 \le \frac{2m+r-1}{2}Q, \qquad e_1 + 2me_2 \ge -\frac{2m+r}{2}Q. \end{aligned} \tag{3.6}$$

Therefore (3.5) that satisfies (3.6) is a solution to ud-$P(A_6)$ for $-m \le n \le m-1$. If $G_{n-1} \ge 0$ and $nQ - F_n \ge 0$, then ud-$P(A_6)$ (2.1) can be written as the following system of linear equations:

$$F_n + F_{n-1} = G_{n-1}, \qquad G_n + G_{n-1} = (n+2m+2r)\,Q + F_n. \tag{3.7}$$

The general solution to the linear system (3.7) is

$$\begin{aligned} F_n &= \frac{n+2m+2r}{3}Q + h_1 \cos\frac{2}{3}\pi\,(n-m) + \frac{2h_2 - h_1}{\sqrt{3}} \sin\frac{2}{3}\pi\,(n-m), \\ G_n &= \frac{2n+4m+4r+1}{3}Q + h_2 \cos\frac{2}{3}\pi\,(n-m) + \frac{h_2 - 2h_1}{\sqrt{3}} \sin\frac{2}{3}\pi\,(n-m), \end{aligned} \tag{3.8}$$

where h_1 and h_2 are arbitrary constants. If $h_1 = h_2 = 0$, (3.8) satisfies $G_{n-1} \ge 0$ and $nQ - F_n \ge 0$ for $n \ge m+1$. The condition that the general solution (3.8) satisfies $G_{n-1} \ge 0$ and $nQ - F_n \ge 0$ for $n \ge m+1$ is

$$h_1 \le \frac{6-2r}{3}Q, \qquad h_2 \ge \frac{2r-4}{3}Q, \qquad h_2 - h_1 \le \frac{2-2r}{3}Q. \tag{3.9}$$

Therefore (3.8) that satisfies (3.9) is a solution to ud-$P(A_6)$ for $n \ge m+1$. The relations between d_1, d_2 and e_1, e_2 can be obtained from (2.1a) for $n = -m$:

$$F_{-m} + F_{-m-1} = \max\left(0, G_{-m-1}\right),$$

(3.2) for $n = -m-1$:

$$F_{-m-1} = -d_1, \qquad G_{-m-1} = \frac{2r-1}{2}Q - d_2,$$

and (3.5) for $n = -m, -m-1$ respectively:

$$F_{-m} = \frac{r}{2}Q + 2me_2 + e_1, \qquad G_{-m-1} = \frac{2r-1}{2}Q - e_2.$$

We have

$$d_1 = \frac{r}{2}Q + e_1 + 2me_2 - 2\max\left(0, \frac{2r-1}{2}Q - e_2\right), \qquad d_2 = e_2.$$

Moreover the relations between e_1, e_2 and h_1, h_2 can be obtained from (2.1b) for $n = m$:

$$G_m + G_{m-1} = (4m + 2r)\,Q - \max\left(0, mQ - F_m\right),$$

(3.5) for $n = m, m-1$ respectively:

$$F_m = \frac{2m + r}{2}Q + e_1, \qquad G_{m-1} = \frac{4m + 2r - 1}{2}Q - e_2,$$

and (3.8) for $n = m$:

$$F_m = \frac{3m + 2r}{3}Q + h_1, \qquad G_m = \frac{6m + 4r + 1}{3}Q + h_2.$$

And we have

$$h_1 = -\frac{r}{6}Q + e_1, \qquad h_2 = \frac{1 - 2r}{6}Q + e_2 - \max\left(0, -\frac{r}{2}Q - e_1\right).$$

When $|e_1|$ and $|e_2|$ are sufficiently small, we shall write "$e_1 \sim 0$, $e_2 \sim 0$" as an abbreviation, If $e_1 \sim 0$ and $e_2 \sim 0$, then we find that

$$d_1 \sim \frac{r}{2}Q, \qquad d_2 \sim 0$$

satisfy (3.3), and

$$h_1 \sim -\frac{r}{6}Q, \qquad h_2 \sim \frac{1 - 2r}{6}Q - \max\left(0, -\frac{r}{2}Q\right)$$

satisfy (3.9). Therefore we have Theorem 1 by connecting these solutions together. ■

Theorem 2. *ud-P(A_6) admits the following solution for $Q > 0$, $A = 2(m + r)Q$, $-m \in \mathbb{N}$, $0 < r \le 1/2$:*

$$F_n = d_1\,(-1)^n\,, \qquad G_n = \frac{2n + 2m + 2r + 1}{2}Q + d_2\,(-1)^n$$

for $n \le -1$, where d_1 and d_2 satisfy

$$-2Q \le d_1 \le Q, \qquad \frac{2m + 2r - 1}{2}Q \le d_2 \le \frac{-2m - 2r + 3}{2}Q;$$

$$F_n = e_1\,(-1)^n\,, \qquad G_n = \frac{2n + 4m + 4r + 1}{4}Q + e_1 n\,(-1)^n + e_2\,(-1)^n$$

for $0 \le n \le -2m - 1$, where e_1 and e_2 satisfy

$$-Q \le e_1 \le 2Q, \qquad e_2 \le -\frac{4m + 4r + 1}{4}Q, \qquad e_1 + e_2 \ge \frac{4m + 4r + 3}{4}Q,$$

$$-\left(2m + 2\right)e_1 + e_2 \le \frac{3 - 4r}{4}Q, \qquad -\left(2m + 3\right)e_1 + e_2 \ge \frac{4r - 5}{4}Q,$$

and

$$F_n = \frac{n + 2m + 2r}{3}Q + h_1 \cos\frac{2}{3}\pi\left(n + 2m\right) + \frac{2h_2 - h_1}{\sqrt{3}}\sin\frac{2}{3}\pi\left(n + 2m\right),$$

$$G_n = \frac{2n + 4m + 4r + 1}{3}Q + h_2 \cos\frac{2}{3}\pi\left(n + 2m\right) + \frac{h_2 - 2h_1}{\sqrt{3}}\sin\frac{2}{3}\pi\left(n + 2m\right)$$

for $n \geq -2m$, where h_1 and h_2 satisfy

$$h_1 \leq \frac{4r+3}{3}Q, \qquad h_2 \geq -\frac{4r+1}{3}Q, \qquad h_2 - h_1 \leq \frac{4r+5}{3}Q.$$

Here the relations between d_1, d_2 and e_1, e_2 are

$$d_1 = e_1, \qquad d_2 = -\frac{1}{4}Q + e_2 + \max\left(0, -e_1\right),$$

and those between e_1, e_2 and h_1, h_2 are

$$h_1 = -\frac{2r}{3}Q + e_1 + \max\left\{0, \frac{4r-1}{4}Q + \left(2m+1\right)e_1 - e_2\right\},$$
$$h_2 = -\frac{4r+1}{12}Q - 2me_1 + e_2 + \max\left\{0, \frac{4r-1}{4}Q + \left(2m+1\right)e_1 - e_2\right\}.$$

Theorem 3. *ud-$P(A_6)$ admits the following solution for $Q > 0$, $A = 2(m+r)Q$, $-m \in \mathbb{N}$, $-1/2 < r \leq 0$:*

$$F_n = d_1\left(-1\right)^n, \qquad G_n = \frac{2n+2m+2r+1}{2}Q + d_2\left(-1\right)^n$$

for $n \leq -1$, where d_1 and d_2 satisfy

$$-2Q \leq d_1 \leq Q, \qquad \frac{2m+2r-1}{2}Q \leq d_2 \leq \frac{-2m-2r+3}{2}Q;$$
$$F_n = e_1\left(-1\right)^n, \qquad G_n = \frac{2n+4m+4r+1}{4}Q + e_1 n\left(-1\right)^n + e_2\left(-1\right)^n$$

for $0 \leq n \leq -2m$, where e_1 and e_2 satisfy

$$-Q \leq e_1 \leq 2Q, \qquad e_2 \leq -\frac{4m+4r+1}{4}Q, \qquad e_1 + e_2 \geq \frac{4m+4r+3}{4}Q,$$
$$-\left(2m+1\right)e_1 + e_2 \geq \frac{4r-1}{4}Q, \qquad -\left(2m+2\right)e_1 + e_2 \leq \frac{3-4r}{4}Q,$$

and

$$F_n = \frac{n+2m+2r}{3}Q + h_1\cos\frac{2}{3}\pi\left(n+2m\right) + \frac{2h_2-h_1}{\sqrt{3}}\sin\frac{2}{3}\pi\left(n+2m\right),$$
$$G_n = \frac{2n+4m+4r+1}{3}Q + h_2\cos\frac{2}{3}\pi\left(n+2m\right) + \frac{h_2-2h_1}{\sqrt{3}}\sin\frac{2}{3}\pi\left(n+2m\right)$$

for $n \geq -2m+1$, where h_1 and h_2 satisfy

$$h_1 \leq \frac{4r+3}{3}Q, \qquad h_2 \geq -\frac{4r+7}{3}Q, \qquad h_2 - h_1 \leq \frac{4r+5}{3}Q.$$

Here the relations between d_1, d_2 and e_1, e_2 are

$$d_1 = e_1, \qquad d_2 = -\frac{1}{4}Q + e_2 + \max\left(0, -e_1\right),$$

and those between e_1, e_2 and h_1, h_2 are

$$h_1 = \frac{4r+3}{12}Q - \left(2m-1\right)e_1 + e_2 - \max\left(0, \frac{4r+1}{4}Q - 2me_1 + e_2\right),$$
$$h_2 = -\frac{4r+1}{12}Q - 2me_1 + e_2.$$

Proof. We consider the case $A = 2(m+r)Q$, $-m \in \mathbb{N}$ and $-1/2 < r \le 1/2$. If $G_{n-1} \le 0$ and $nQ - F_n \le 0$, then ud-$P(A_6)$ (2.1) can be written as the following system of linear equations:

$$F_n + F_{n-1} = 0, \qquad G_n + G_{n-1} = (2n + 2m + 2r)\, Q. \tag{3.10}$$

The general solution to the linear system (3.10) is

$$F_n = d_1 (-1)^n, \qquad G_n = \frac{2n + 2m + 2r + 1}{2} Q + d_2 (-1)^n, \tag{3.11}$$

where d_1 and d_2 are arbitrary constants. If $d_1 = d_2 = 0$, the particular solution (3.11) satisfies $G_n \le 0$ and $nQ - F_n \le 0$ for $n \le -1$. The condition that the general solution (3.11) satisfies $G_{n-1} \le 0$ and $nQ - F_n \le 0$ for $n \le -1$ is

$$-2Q \le d_1 \le Q, \qquad \frac{2m + 2r - 1}{2} Q \le d_2 \le \frac{-2m - 2r + 3}{2} Q. \tag{3.12}$$

Therefore (3.11) that satisfies (3.12) is a solution to ud-$P(A_6)$ for $n \le -1$. If $G_{n-1} \le 0$ and $nQ - F_n \ge 0$, then (2.1) can be written as the following system of linear equations:

$$F_n + F_{n-1} = 0, \qquad G_n + G_{n-1} = (n + 2m + 2r)\, Q + F_n. \tag{3.13}$$

The general solution to the linear system (3.13) is

$$F_n = e_1 (-1)^n, \qquad G_n = \frac{2n + 4m + 4r + 1}{4} Q + e_1 n (-1)^n + e_2 (-1)^n, \tag{3.14}$$

where e_1 and e_2 are arbitrary constants. If $e_1 = e_2 = 0$ and $0 < r \le 1/2$, (3.14) satisfies $G_{n-1} \le 0$ and $nQ - F_n \ge 0$ for $1 \le n \le -2m-1$. The condition that the general solution (3.14) satisfies $G_{n-1} \le 0$ and $nQ - F_n \ge 0$ for $1 \le n \le -2m - 1$ is

$$\begin{aligned}
&-Q \le e_1 \le 2Q, \qquad e_2 \le -\frac{4m + 4r + 1}{4} Q, \qquad e_1 + e_2 \ge \frac{4m + 4r + 3}{4} Q,\\
&-(2m+2)\, e_1 + e_2 \le \frac{3 - 4r}{4} Q, \qquad -(2m+3)\, e_1 + e_2 \ge \frac{4r - 5}{4} Q.
\end{aligned} \tag{3.15}$$

Therefore (3.14) that satisfies (3.15) is a solution to ud-$P(A_6)$ for $1 \le n \le -2m-1$. If $e_1 = e_2 = 0$ and $-1/2 < r \le 0$, then (3.14) satisfies $G_{n-1} \le 0$ and $nQ - F_n \ge 0$ for $1 \le n \le -2m$. The condition that the general solution (3.14) satisfies $G_{n-1} \le 0$ and $nQ - F_n \ge 0$ for $1 \le n \le -2m$ is

$$\begin{aligned}
&-Q \le e_1 \le 2Q, \qquad e_2 \le -\frac{4m + 4r + 1}{4} Q, \qquad e_1 + e_2 \ge \frac{4m + 4r + 3}{4} Q,\\
&-(2m+1)\, e_1 + e_2 \ge \frac{4r - 1}{4} Q, \qquad -(2m+2)\, e_1 + e_2 \le \frac{3 - 4r}{4} Q.
\end{aligned} \tag{3.16}$$

Therefore (3.14) that satisfies (3.16) is a solution to ud-$P(A_6)$ for $1 \le n \le -2m$. If $G_{n-1} \ge 0$ and $nQ - F_n \ge 0$, then ud-$P(A_6)$ (2.1) can be written as the following system of linear equations:

$$F_n + F_{n-1} = G_{n-1}, \qquad G_n + G_{n-1} = (n + 2m + 2r)\, Q + F_n. \tag{3.17}$$

The general solution to the linear system (3.17) is

$$\begin{aligned}
F_n &= \frac{n + 2m + 2r}{3} Q + h_1 \cos \frac{2}{3}\pi (n + 2m) + \frac{2h_2 - h_1}{\sqrt{3}} \sin \frac{2}{3}\pi (n + 2m),\\
G_n &= \frac{2n + 4m + 4r + 1}{3} Q + h_2 \cos \frac{2}{3}\pi (n + 2m) + \frac{h_2 - 2h_1}{\sqrt{3}} \sin \frac{2}{3}\pi (n + 2m),
\end{aligned} \tag{3.18}$$

where h_1 and h_2 are arbitrary constants. If $h_1 = h_2 = 0$ and $0 < r \leq 1/2$, (3.18) satisfies the conditions $G_n \geq 0$ and $nQ - F_n \geq 0$ for $n \geq -2m$. The condition that the general solution (3.18) satisfies $G_n \geq 0$ and $nQ - F_n \geq 0$ for $n \geq -2m$ is

$$h_1 \leq \frac{4r+3}{3}Q, \qquad h_2 \geq -\frac{4r+1}{3}Q, \qquad h_2 - h_1 \leq \frac{4r+5}{3}Q. \tag{3.19}$$

Therefore (3.18) that satisfies (3.19) is a solution to ud-$P(A_6)$ for $n \geq -2m$. If $h_1 = h_2 = 0$ and $-1/2 < r \leq 0$, (3.18) satisfies $G_n \geq 0$ and $nQ - F_n \geq 0$ for $n \geq -2m+1$. The condition that the general solution (3.18) satisfies $G_n \geq 0$ and $nQ - F_n \geq 0$ for $n \geq -2m+1$ is

$$h_1 \leq \frac{4r+3}{3}Q, \qquad h_2 \geq -\frac{4r+7}{3}Q, \qquad h_2 - h_1 \leq \frac{4r+5}{3}Q. \tag{3.20}$$

Therefore (3.18) that satisfies (3.20) is a solution to ud-$P(A_6)$ for $n \geq -2m+1$. The relations between d_1, d_2 and e_1, e_2 can be obtained from (2.1b) for $n = 0$:

$$G_0 + G_{-1} = (2m + 2r)\, Q - \max\left(0, -F_0\right),$$

(3.11) for $n = 0, -1$ respectively:

$$F_0 = d_1, \qquad G_{-1} = \frac{2m+2r-1}{2}Q - d_2,$$

and (3.14) for $n = 0$:

$$F_0 = e_1, \qquad G_0 = \frac{4m+4r+1}{4}Q + e_2.$$

We have

$$d_1 = e_1, \qquad d_2 = -\frac{1}{4}Q + e_2 + \max\left(0, -e_1\right).$$

Moreover in the case $0 < r \leq 1/2$, the relations between e_1, e_2 and h_1, h_2 can be obtained from (2.1a) for $n = -2m$:

$$F_{-2m} + F_{-2m-1} = \max\left(0, G_{-2m-1}\right),$$

(3.14) for $n = -2m-1$:

$$F_{-2m-1} = -e_1, \qquad G_{-2m-1} = \frac{4r-1}{4}Q + (2m+1)\, e_1 - e_2,$$

and (3.18) for $n = -2m, -2m-1$ respectively:

$$F_{-2m} = \frac{2r}{3}Q + h_1, \qquad G_{-2m-1} = \frac{4r-1}{3}Q + h_1 - h_2.$$

We have

$$h_1 = -\frac{2r}{3}Q + e_1 + \max\left\{0, \frac{4r-1}{4}Q + (2m+1)\, e_1 - e_2\right\},$$
$$h_2 = -\frac{4r+1}{12}Q - 2me_1 + e_2 + \max\left\{0, \frac{4r-1}{4}Q + (2m+1)\, e_1 - e_2\right\}.$$

In the case $-1/2 < r \leq 0$, the relations between e_1, e_2 and h_1, h_2 can be obtained from (2.1a) for $n = -2m+1$:

$$F_{-2m+1} + F_{-2m} = \max\left(0, G_{-2m}\right),$$

(3.14) for $n = -2m$:

$$F_{-2m} = e_1, \qquad G_{-2m} = \frac{4r+1}{4}Q - 2me_1 + e_2,$$

and (3.18) for $n = -2m+1, -2m$ respectively:

$$F_{-2m+1} = \frac{2r+1}{3}Q - h_1 + h_2, \qquad G_{-2m} = \frac{4r+1}{3}Q + h_2.$$

We have

$$h_1 = \frac{4r+3}{12}Q - (2m-1)\,e_1 + e_2 - \max\left(0, \frac{4r+1}{4}Q - 2me_1 + e_2\right),$$
$$h_2 = -\frac{4r+1}{12}Q - 2me_1 + e_2.$$

If $e_1 \sim 0$, $e_2 \sim 0$, then we find that

$$d_1 \sim 0, \qquad d_2 \sim -\frac{1}{4}Q$$

satisfy (3.12),

$$h_1 \sim -\frac{2r}{3}Q + \max\left(0, \frac{4r-1}{4}Q\right), \qquad h_2 \sim -\frac{4r+1}{12}Q + \max\left(0, \frac{4r-1}{4}Q\right)$$

satisfy (3.19), and

$$h_1 \sim \frac{4r+3}{12}Q - \max\left(0, \frac{4r+1}{4}Q\right), \qquad h_2 \sim -\frac{4r+1}{12}Q$$

satisfy (3.20). We have Theorem 2 and Theorem 3 by connecting these solutions together. ■

Theorem 4. *ud-P*(A_6) *admits the following solution for* $Q > 0$, $A = 2rQ$, $-1/2 < r \leq 1/2$:

$$F_n = d_1\,(-1)^n, \qquad G_n = \frac{2n+2r+1}{2}Q + d_2\,(-1)^n,$$

for $n \leq -1$, *where* d_1 *and* d_2 *satisfy*

$$-2Q \leq d_1 \leq Q, \qquad \frac{2r-5}{2}Q \leq d_2 \leq \frac{3-2r}{2}Q,$$

and

$$F_n = \frac{n+2r}{3}Q + h_1 \cos\frac{2}{3}\pi n + \frac{2h_2 - h_1}{\sqrt{3}} \sin\frac{2}{3}\pi n,$$
$$G_n = \frac{2n+4r+1}{3}Q + h_2 \cos\frac{2}{3}\pi n + \frac{h_2 - 2h_1}{\sqrt{3}} \sin\frac{2}{3}\pi n,$$

for $n \geq 1$, *where* h_1 *and* h_2 *satisfy*

$$h_1 \leq \frac{4r+3}{3}Q, \qquad h_2 \geq \frac{2r-4}{3}Q, \qquad h_2 - h_1 \leq \frac{2-2r}{3}Q.$$

Here the relations between d_1, d_2 *and* F_0, G_0 *are*

$$d_1 = F_0 - \max\{0, 2rQ - G_0 - \max(0, -F_0)\}, \qquad d_2 = -\frac{2r+1}{2}Q + G_0 + \max(0, -F_0),$$

and those between h_1, h_2 *and* F_0, G_0 *are*

$$h_1 = -\frac{2r}{3}Q + F_0 - \max(0, -G_0), \qquad h_2 = G_0 - \frac{4r+1}{3}Q.$$

Proof. We consider the case $A = 2rQ$ and $-1/2 < r \le 1/2$. If $G_{n-1} \le 0$ and $nQ - F_n \le 0$, then ud-$P(A_6)$ (2.1) can be written as the following system of linear equations:

$$F_n + F_{n-1} = 0, \qquad G_n + G_{n-1} = (2n + 2r)\, Q. \tag{3.21}$$

The general solution to the linear system (3.21) is

$$F_n = d_1 (-1)^n, \qquad G_n = \frac{2n + 2r + 1}{2} Q + d_2 (-1)^n, \tag{3.22}$$

where d_1 and d_2 are arbitrary constants. If $d_1 = d_2 = 0$, the particular solution (3.22) satisfies $G_{n-1} \le 0$ and $nQ - F_n \le 0$ for $n \le -1$. The sufficient condition that the general solution (3.22) satisfies $G_{n-1} \le 0$ and $nQ - F_n \le 0$ for $n \le -1$ is

$$-2Q \le d_1 \le Q, \qquad \frac{2r - 5}{2} Q \le d_2 \le \frac{3 - 2r}{2} Q. \tag{3.23}$$

Therefore (3.22) that satisfies (3.23) is a solution to ud-$P(A_6)$ for $n \le -1$. If $G_{n-1} \ge 0$ and $nQ - F_n \ge 0$, then ud-$P(A_6)$ (2.1) can be written as the following system of linear equations:

$$F_n + F_{n-1} = G_{n-1}, \qquad G_n + G_{n-1} = (n + 2r)\, Q + F_n. \tag{3.24}$$

The general solution to the linear system (3.24) is

$$\begin{aligned} F_n &= \frac{n + 2r}{3} Q + h_1 \cos \frac{2}{3}\pi n + \frac{2h_2 - h_1}{\sqrt{3}} \sin \frac{2}{3}\pi n, \\ G_n &= \frac{2n + 4r + 1}{3} Q + h_2 \cos \frac{2}{3}\pi n + \frac{h_2 - 2h_1}{\sqrt{3}} \sin \frac{2}{3}\pi n, \end{aligned} \tag{3.25}$$

where h_1 and h_2 are arbitrary constants. If $h_1 = h_2 = 0$, (3.25) satisfies $G_n \ge 0$ and $nQ - F_n \ge 0$ for $n \ge 1$. The condition that the general solution (3.25) satisfies $G_n \ge 0$ and $nQ - F_n \ge 0$ for $n \ge 1$ is

$$h_1 \le \frac{4r + 3}{3} Q, \qquad h_2 \ge \frac{2r - 4}{3} Q, \qquad h_2 - h_1 \le \frac{2 - 2r}{3} Q. \tag{3.26}$$

Therefore (3.25) that satisfies (3.26) is a solution to ud-$P(A_6)$ for $n \ge 2$. The relations between d_1, d_2 and F_0, G_0 can be obtained from (2.1) for $n = 0$:

$$F_0 + F_{-1} = \max(0, G_{-1}), \qquad G_0 + G_{-1} = 2rQ - \max(0, -F_0),$$

and (3.22) for $n = -1$:

$$F_{-1} = -d_1, \qquad G_{-1} = \frac{2r - 1}{2} Q - d_2.$$

We have

$$d_1 = F_0 - \max\{0, 2rQ - G_0 - \max(0, -F_0)\}, \qquad d_2 = -\frac{2r + 1}{2} Q + G_0 + \max(0, -F_0).$$

Moreover the relations between h_1, h_2 and F_0, G_0 can be obtained from (2.1a) for $n = 1$:

$$F_1 + F_0 = \max(0, G_0),$$

and (3.25) for $n = 1, 0$ respectively:

$$F_1 = \frac{2r + 1}{3} Q - h_1 + h_2, \qquad G_0 = \frac{4r + 1}{3} Q + h_2.$$

And we have

$$h_1 = -\frac{2r}{3}Q + F_0 - \max(0, -G_0), \qquad h_2 = G_0 - \frac{4r+1}{3}Q.$$

If $F_0 \sim 0$ and $G_0 \sim 0$, then we find that

$$d_1 \sim -\max(0, 2rQ), \qquad d_2 \sim -\frac{2r+1}{2}Q$$

satisfy (3.23), and

$$h_1 \sim -\frac{2r}{3}Q, \qquad h_2 \sim -\frac{4r+1}{3}Q$$

satisfy (3.26). Therefore we have Theorem 4 by connecting these solutions together. ■

The exact solutions with two parameters for any parameter A have been given in this section.

4 Bäcklund transformation

q-$P(A_6)$ have the Bäcklund transformation [5, 28]. That is, if f_n and g_n satisfy q-$P(A_6)$ (1.1), then

$$\mathsf{f}_n = \frac{q^n}{g_n}\frac{aq^{n+1}f_{n+1} + g_n}{q^n f_{n+1} + g_n}, \qquad \mathsf{g}_n = \frac{q^{n+1}}{f_{n+1}}\frac{aq^{n+1}f_{n+1} + g_n}{q^n f_{n+1} + g_n} \tag{4.1}$$

satisfy q-$P(A_6)$:

$$\mathsf{f}_n\mathsf{f}_{n-1} = 1 + \mathsf{g}_{n-1}, \qquad \mathsf{g}_n\mathsf{g}_{n-1} = \frac{aq^2q^{2n}\mathsf{f}_n}{\mathsf{f}_n + q^n},$$

and

$$\mathtt{f}_{n+1} = \frac{q^{n+1}}{g_n}\frac{aq^n f_n + g_n}{q^{n+1}f_n + g_n}, \qquad \mathtt{g}_n = \frac{q^n}{f_n}\frac{aq^n f_n + g_n}{q^{n+1}f_n + g_n} \tag{4.2}$$

also satisfy q-$P(A_6)$:

$$\mathtt{f}_n\mathtt{f}_{n-1} = 1 + \mathtt{g}_{n-1}, \qquad \mathtt{g}_n\mathtt{g}_{n-1} = \frac{aq^{-2}q^{2n}\mathtt{f}_n}{\mathtt{f}_n + q^n}.$$

So we apply the procedure of the ultradiscretization to (4.1) and (4.2). Then we have the following theorems.

Theorem 5. *If F_n and G_n satisfy ud-$P(A_6)$ (2.1), then*

$$\begin{aligned}\mathsf{F}_n &= \max\{F_{n+1} + (n+1)Q + A - G_n, 0\} - \max(F_{n+1}, G_n - nQ),\\ \mathsf{G}_n &= Q + \max\{(n+1)Q + A, G_n - F_{n+1}\} - \max(F_{n+1}, G_n - nQ)\end{aligned}$$

satisfy ud-$P(A_6)$:

$$\mathsf{F}_n + \mathsf{F}_{n-1} = \max(0, \mathsf{G}_{n-1}), \qquad \mathsf{G}_n + \mathsf{G}_{n-1} = A + 2Q + 2nQ - \max(0, nQ - \mathsf{F}_n).$$

Proof. We can obtain

$$\begin{aligned}\mathsf{F}_n &= \max\{F_{n+1}+(n+1)Q+A-G_n,0\}-\max(F_{n+1},G_n-nQ)\\ &= nQ-G_n+\max\{A+(n+1)Q+\max(0,G_n),F_n+G_n\}\\ &\quad -\max\{nQ+\max(0,G_n),F_n+G_n\},\\ \mathsf{G}_n &= Q+\max\{(n+1)Q+A,G_n-F_{n+1}\}-\max(F_{n+1},G_n-nQ)\\ &= (n+1)Q+F_n-\max(0,G_n)+\max\{A+(n+1)Q+\max(0,G_n),F_n+G_n\}\\ &\quad -\max\{nQ+\max(0,G_n),F_n+G_n\}\end{aligned}$$

by using (2.1a), and

$$\begin{aligned}\mathsf{F}_{n-1} &= \max(F_n+nQ+A-G_{n-1},0)-\max\{F_n,G_{n-1}-(n-1)Q\}\\ &= G_n-nQ+\max(F_n,nQ)-F_n+\max\{G_n+\max(F_n,nQ),nQ\}\\ &\quad -\max\{G_n+\max(F_n,nQ),A+(n+1)Q\},\\ \mathsf{G}_{n-1} &= Q+\max\{nQ+A,G_{n-1}-F_n\}-\max\{F_n,G_{n-1}-(n-1)Q\}\\ &= A+(n+1)Q-F_n+\max\{G_n+\max(F_n,nQ),nQ\}\\ &\quad -\max\{G_n+\max(F_n,nQ),A+(n+1)Q\}\end{aligned}$$

by using (2.1b). Thus we find

$$\begin{aligned}\mathsf{F}_n+\mathsf{F}_{n-1} &= \max(0,\mathsf{G}_{n-1})\\ &= \max(F_n,nQ)-F_n+\max\{A+(n+1)Q+\max(0,G_n),F_n+G_n\}\\ &\quad -\max\{G_n+\max(F_n,nQ),A+(n+1)Q\},\\ \mathsf{G}_n+\mathsf{G}_{n-1} &= A+2Q+2nQ-\max(0,nQ-\mathsf{F}_n)=A+(2n+2)Q-\max(0,G_n)\\ &\quad +\max\{A+(n+1)Q+\max(0,G_n),F_n+G_n\}\\ &\quad -\max\{G_n+\max(F_n,nQ),A+(n+1)Q\}.\end{aligned}$$

■

Theorem 6. *If F_n and G_n satisfy ud-$P(A_6)$ (2.1), then*

$$\begin{aligned}\mathtt{F}_{n+1} &= \max(nQ+A+F_n-G_n,0)-\max\{F_n,G_n-(n+1)Q\},\\ \mathtt{G}_n &= -Q+\max(nQ+A,G_n-F_n)-\max\{F_n,G_n-(n+1)Q\}\end{aligned}$$

satisfy ud-$P(A_6)$:

$$\mathtt{F}_n+\mathtt{F}_{n-1}=\max(0,\mathtt{G}_{n-1}),\qquad \mathtt{G}_n+\mathtt{G}_{n-1}=A-2Q+2nQ-\max(0,nQ-\mathtt{F}_n).$$

Proof. We can obtain

$$\begin{aligned}\mathtt{F}_{n-1} &= \max\{(n-2)Q+A+F_{n-2}-G_{n-2},0\}-\max\{F_{n-2},G_{n-2}-(n-1)Q\},\\ &= (n-1)Q-G_{n-2}+\max\{A+(n-2)Q+\max(0,G_{n-2}),F_{n-1}+G_{n-2}\}\\ &\quad -\max\{(n-1)Q+\max(0,G_{n-2}),F_{n-1}+G_{n-2}\}\end{aligned}$$

by using (2.1a), and

$$\begin{aligned}\mathtt{F}_n &= \max\{(n-1)Q+A+F_{n-1}-G_{n-1},0\}-\max(F_{n-1},G_{n-1}-nQ)\\ &= G_{n-2}-(n-1)Q+\max\{F_{n-1},(n-1)Q\}-F_{n-1}\\ &\quad +\max[G_{n-2}+\max\{F_{n-1},(n-1)Q\},(n-1)Q]\\ &\quad -\max[G_{n-2}+\max\{F_{n-1},(n-1)Q\},A+(n-2)Q],\\ \mathtt{G}_{n-1} &= -Q+\max\{(n-1)Q+A,G_{n-1}-F_{n-1}\}-\max(F_{n-1},G_{n-1}-nQ)\end{aligned}$$

$$
\begin{aligned}
&= A + (n-2)\,Q - F_{n-1} + \max\left[G_{n-2} + \max\left\{F_{n-1}, (n-1)\,Q\right\}, (n-1)\,Q\right] \\
&\quad - \max\left[G_{n-2} + \max\left\{F_{n-1}, (n-1)\,Q\right\}, A + (n-2)\,Q\right]
\end{aligned}
$$

by using (2.1b). Thus we find

$$
\begin{aligned}
\mathtt{F}_n + \mathtt{F}_{n-1} = \max\left(0, \mathtt{G}_{n-1}\right) &= \max\left\{F_{n-1}, (n-1)\,Q\right\} - F_{n-1} \\
&+ \max\left\{A + (n-2)\,Q + \max\left(0, G_{n-2}\right), F_{n-1} + G_{n-2}\right\} \\
&- \max\left[G_{n-2} + \max\left\{F_{n-1}, (n-1)\,Q\right\}, A + (n-2)\,Q\right].
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\mathtt{F}_n &= \max\left\{(n-1)\,Q + A + F_{n-1} - G_{n-1}, 0\right\} - \max\left(F_{n-1}, G_{n-1} - nQ\right) \\
&= nQ - G_{n-1} + \max\left\{A + (n-1)\,Q + \max\left(0, G_{n-1}\right), F_n + G_{n-1}\right\} \\
&\quad - \max\left\{nQ + \max\left(0, G_{n-1}\right), F_n + G_{n-1}\right\}, \\
\mathtt{G}_{n-1} &= -Q + \max\left\{(n-1)\,Q + A, G_{n-1} - F_{n-1}\right\} - \max\left(F_{n-1}, G_{n-1} - nQ\right) \\
&= (n-1)\,Q + F_n - \max\left(0, G_{n-1}\right) \\
&\quad + \max\left\{A + (n-1)\,Q + \max\left(0, G_{n-1}\right), F_n + G_{n-1}\right\} \\
&\quad - \max\left\{nQ + \max\left(0, G_{n-1}\right), F_n + G_{n-1}\right\}
\end{aligned}
$$

by using (2.1a), and

$$
\begin{aligned}
\mathtt{G}_n &= -Q + \max\left(nQ + A, G_n - F_n\right) - \max\left\{F_n, G_n - (n+1)\,Q\right\} \\
&= A + (n-1)\,Q - F_n + \max\left\{G_{n-1} + \max\left(F_n, nQ\right), nQ\right\} \\
&\quad - \max\left\{G_{n-1} + \max\left(F_n, nQ\right), A + (n-1)\,Q\right\}
\end{aligned}
$$

by using (2.1b). Thus we find

$$
\begin{aligned}
\mathtt{G}_n + \mathtt{G}_{n-1} = A - 2Q + 2nQ - \max\left(0, nQ - \mathtt{F}_n\right) &= A + (2n-2)\,Q - \max\left(0, G_{n-1}\right) \\
&+ \max\left\{A + (n-1)\,Q + \max\left(0, G_{n-1}\right), F_n + G_{n-1}\right\} \\
&- \max\left\{G_{n-1} + \max\left(F_n, nQ\right), A + (n-1)\,Q\right\}. \qquad \blacksquare
\end{aligned}
$$

So the exact solutions also can be obtained from the solution in Theorem 4 by using the Bäcklund transformation.

5 Special solutions

In [5], Hamamoto, Kajiwara and Witte constructed hypergeometric solutions to q-$P(A_6)$ by applying Bäcklund transformations to the "seed" solution which satisfies a Riccati equation. Their solutions have a determinantal form with basic hypergeometric function elements whose continuous limits are showed by them to be Airy functions, the hypergeometric solutions of the second Painlevé equation. In [18, 19], S. Nishioka proved that transcendental solutions of q-$P(A_6)$ in a decomposable extension may exist only for special parameters, and that each of them satisfies the Riccati equation mentioned above if we apply the Bäcklund transformations to it appropriate times. He also proved non-existence of algebraic solutions.

q-$P(A_6)$ (1.1) for $a = q^{2m+1}$ ($m \in \mathbb{Z}$) has the hypergeometric solution. The case of $A = (2m+1)\,Q$ in ud-$P(A_6)$ corresponds to $a = q^{2m+1}$ in the discrete system. It is hard to apply the ultradiscretization procedure to the hypergeometric series. However according to [22], an ultradiscrete hypergeometric solution is given in terms of nQ and $(-1)^n\,Q$. If $h_1 = h_2 = 0$ and

$r = 1/2$ in Theorem 4, then we obtain an ultradiscrete hypergeometric solution of ud-$P(A_6)$ for $A = Q$:

$$F_n = \begin{cases} \frac{1}{3}Q\,(-1)^n & (n \leq -1), \\ \frac{n+1}{3}Q & (n \geq 0), \end{cases} \qquad G_n = \begin{cases} (n+1)\,Q & (n \leq -1), \\ \frac{2n+3}{3}Q & (n \geq 0). \end{cases}$$

If $h_1 = h_2 = 0$ and $r = 1/2$ in Theorem 1, then we obtain an ultradiscrete hypergeometric solution of ud-$P(A_6)$ for $A = (2m+1)\,Q$ $(m \in \mathbb{N})$:

$$F_n = \begin{cases} \frac{1}{3}Q\,(-1)^{n+m} & (n \leq -m-1), \\ \frac{2n+2m+1}{4}Q + \frac{1}{12}Q\,(-1)^{n-m} & (-m \leq n \leq m-1), \\ \frac{n+2m+1}{3}Q & (n \geq m), \end{cases}$$

$$G_n = \begin{cases} (n+m+1)\,Q & (n \leq m-1), \\ \frac{2n+4m+3}{3}Q & (n \geq m). \end{cases}$$

If $h_1 = h_2 = 0$ and $r = 1/2$ in Theorem 2, then we have an ultradiscrete hypergeometric solution for $A = (2m+1)Q$ $(-m \in \mathbb{N})$:

$$F_n = \begin{cases} 0 & (n \leq -2m-1), \\ \frac{n+2m+1}{3}Q & (n \geq -2m), \end{cases}$$

$$G_n = \begin{cases} (n+m+1)\,Q & (n \leq -1), \\ \frac{2n+4m+3}{4}Q - \frac{1}{12}Q\,(-1)^n & (0 \leq n \leq -2m-1), \\ \frac{2n+4m+3}{3}Q & (n \geq -2m). \end{cases}$$

6 Concluding remarks

We have given the ultradiscrete analogue of q-$P(A_6)$. Moreover, we have presented the exact solutions with two parameters. These solutions are expressed by using linear functions and periodic functions. But the exact solution is only useful when the two parameters are in a limited range. If one wants to construct the exact solution for any initial values, then one needs to use a multitude of branches with respect to n in order to express a solution. We have also presented its special solutions that correspond to the hypergeometric solutions of q-$P(A_6)$. The ultradiscrete hypergeometric solutions are included in the resulting solutions with two parameters.

There are many studies on analytic properties of solutions to the Painlevé equations [1, 2, 7]. But there exist few studies on analytic properties of the q-Painlevé equations [14, 21]. We hope to study the q-Painlevé equations by employing the results in the ultradiscrete systems.

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