Four-Dimensional Spin Foam Perturbation Theory

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Abstract. We define a four-dimensional spin-foam perturbation theory for the BF-theory with a $B \wedge B$ potential term defined for a compact semi-simple Lie group G on a compact orientable 4-manifold M. This is done by using the formal spin foam perturbative series coming from the spin-foam generating functional. We then regularize the terms in the perturbative series by passing to the category of representations of the quantum group $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G and q is a root of unity. The Chain–Mail formalism can be used to calculate the perturbative terms when the vector space of intertwiners $\Lambda \otimes \Lambda \to A$, where A is the adjoint representation of $\mathfrak{g}$, is 1-dimensional for each irrep Λ. We calculate the partition function Z in the dilute-gas limit for a special class of triangulations of restricted local complexity, which we conjecture to exist on any 4-manifold M. We prove that the first-order perturbative contribution vanishes for finite triangulations, so that we define a dilute-gas limit by using the second-order contribution. We show that Z is an analytic continuation of the Crane–Yetter partition function. Furthermore, we relate Z to the partition function for the $F \wedge F$ theory.

Key words: spin foam models; BF-theory; spin networks; dilute-gas limit; Crane–Yetter invariant; spin-foam perturbation theory

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1   Introduction

Spin foam models are state-sum representations of the path integrals for BF theories on simplicial complexes. Spin foam models are used to define topological quantum field theories and quantum gravity theories, see [1]. However, there are also perturbed BF theories in various dimensions, whose potential terms are powers of the B field, see [10]. The corresponding spin-foam perturbation theory generating functional was formulated in [10], but further progress was hindered by the lack of the regularization procedure for the corresponding perturbative expansion and the problem of implementation of the triangulation independence.

The problem of implementation of the triangulation independence for general spin foam perturbation theory was studied in [2], and a solution was proposed, in the form of calculating the perturbation series in a special limit. This limit was called the dilute-gas limit, and it was given by $\lambda \to 0$, $N \to \infty$, such that $g = \lambda N$ is a fixed constant, where λ is the perturbation theory parameter, also called the coupling constant, N is the number of d-simplices in a simplicial decomposition of a d-dimensional compact manifold M and g is the effective perturbation

parameter, also called the renormalized coupling constant. However, the dilute-gas limit could be used in a concrete example only if one knew how to regularize the perturbative contributions.

The regularization problem has been solved recently in the case of three-dimensional (3d) Euclidean quantum gravity with a cosmological constant [9], following the approach of [3, 8]. The 3d Euclidean classical gravity theory is equivalent to the SU(2) BF-theory with a B^3 perturbation, and the corresponding spin foam perturbation expansion can be written by using the Ponzano–Regge model. The terms in this series can be regularized by replacing all the spin-network evaluations with the corresponding quantum spin-network evaluations at a root of unity. By using the Chain–Mail formalism [23] one can calculate the quantum group perturbative corrections, and show that the first-order correction vanishes [9]. Consequently, the dilute-gas limit has to be modified so that $g = \lambda^2 N$ is the effective perturbation parameter [9].

Another result of [9] was to show that the dilute gas limit cannot be defined for an arbitrary class of triangulations of the manifold. One needs a restricted class of triangulations such that the number of possible isotopy classes of a graph defined from the perturbative insertions is bounded. In 3d this can be achieved by using the triangulations coming from the barycentric subdivisions of a regular cubulation of the manifold [9].

In this paper we are going to define the four-dimensional (4d) spin-foam perturbation theory by using the same approach and the techniques as in the 3d case. We start from a BF-theory with a $B \wedge B$ potential term defined for a compact semi-simple Lie group G on a compact 4-manifold M. In Section 2 we define the formal spin foam perturbative series by using the spin-foam generating functional method. We then regularize the terms in the series by passing to the category of representations for the quantum group $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G and q is a root of unity. In Sections 4 and 5 we then use the Chain–Mail formalism to calculate the perturbative contributions. The first-order perturbative contribution vanishes, so that we define the dilute-gas limit in Section 6 by using the second-order contribution. We calculate the partition function Z in the dilute-gas limit for a class of triangulations of a 4-dimensional manifold which are arbitrarily fine and have a controllable local complexity. We conjecture that such a class of triangulations always exists for any 4-dimensional manifold, and can be given by the triangulations corresponding to the barycentric subdivisions of a fixed cubulation of the manifold. We then show that Z is given as an analytic continuation of the Crane–Yetter partition function. In Section 7 we relate the path-integral for the $F \wedge F$ theory with the spin foam partition function and in Section 8 we present our conclusions.

2   Spin foam perturbative expansion

Let $\mathfrak{g}$ be the Lie algebra of a semisimple compact Lie group G. The action for a perturbed BF-theory in 4d can be written as

$$S = \int_M \left(B^I \wedge F_I + \lambda \, g_{IJ} \, B^I \wedge B^J \right), \tag{2.1}$$

where $B = B^I L_I$ is a $\mathfrak{g}$-valued two-form, L_I is a basis of $\mathfrak{g}$, $F = dA + \frac{1}{2}[A, A]$ is the curvature 2-form for the G-connection A on a principal G-bundle over M, $X^I = X_I$ and g_{IJ} is a symmetric G-invariant tensor. Here if X and Y are vector fields in the manifold M then $[A, A](X, Y) = [A(X), A(Y)]$.

We will consider the case when $g_{IJ} \propto \delta_{IJ}$, where δ_{IJ} is the Kronecker delta symbol. In the case of a simple Lie group, this is the only possibility, while in the case of a semisimple Lie group one can also have g_{IJ} which are not proportional to δ_{IJ}. For example, in the case of the $SO(4)$ group one can put $g_{ab,cd} = \epsilon_{abcd}$, where ϵ is the totally antisymmetric tensor and $1 \leq a, \ldots, d \leq 4$. We will also use the notation $\mathrm{Tr}\,(XY) = X^I Y_I$ and $\langle XY \rangle = g_{IJ} X^I Y^J$.

Consider the path integral

$$Z(\lambda, M) = \int \mathcal{D}A\mathcal{D}B e^{i \int_M \left(B^I \wedge F_I + \lambda \langle B \wedge B \rangle\right)}. \tag{2.2}$$

It can be evaluated perturbatively in λ by using the generating functional

$$Z_0(\mathcal{J}, M) = \int \mathcal{D}A\mathcal{D}B e^{i \int_M \left(B^I \wedge F_I + B^I \wedge \mathcal{J}_I\right)}, \tag{2.3}$$

(where $\mathcal{J}_I L^I$ is an arbitrary 2-form valued in $\mathfrak{g}$) and the formula

$$Z(\lambda, M) = \exp\left(-i\lambda \int_M g_{IJ} \frac{\delta}{\delta \mathcal{J}_I} \wedge \frac{\delta}{\delta \mathcal{J}_J}\right) Z_0(\mathcal{J}, M)\Big|_{\mathcal{J}=0}. \tag{2.4}$$

The path integrals (2.3) and (2.4) can be represented as spin foam state sums by discretizing the 4-manifold M, see [10]. This is done by using a simplicial decomposition (triangulation) of M, $T(M)$. It is useful to introduce the dual cell complex $T^*(M)$ [24] (a cell decomposition of M), and we will denote the vertices, edges and faces of $T^*(M)$ as v, l and f, respectively. A vertex v of $T^*(M)$ is dual to a 4-simplex σ of $T(M)$, an edge l of $T^*(M)$ is dual to a tetrahedron τ of $T(M)$ and a face f of $T^*(M)$ is dual to a triangle Δ of $T(M)$.

The action (2.1) then takes the following form on $T(M)$

$$S = \sum_\Delta \text{Tr}\left(B_\Delta F_f\right) + \frac{\lambda}{5} \sum_\sigma \sum_{\Delta', \Delta'' \in \sigma} \langle B_{\Delta'} B_{\Delta''} \rangle,$$

where Δ' and Δ'' are pairs of triangles in a four-simplex σ whose intersection is a single vertex of σ and $B_\Delta = \int_\Delta B$. The variable F_f is defined as

$$e^{F_f} = \prod_{l \in \partial f} g_l,$$

where f is the face dual to a triangle Δ, l's are the edges of the polygon boundary of f and g_l are the dual edge holonomies.

One can then show that

$$Z_0(\mathcal{J}, M) = \sum_{\Lambda_f, \iota_l} \prod_f \dim \Lambda_f \prod_v A_5(\Lambda_{f(v)}, \iota_{l(v)}, \mathcal{J}_{f(v)}), \tag{2.5}$$

where the amplitude $A_5(\Lambda_{f(v)}, \iota_{l(v)}, \mathcal{J}_{f(v)})$, also called the weight for the 4-simplex σ, is given by the evaluation of the four-simplex spin network whose edges are colored by ten $\Lambda_{f(v)}$ irreps and five $\iota_{l(v)}$ intertwiners, while each edge has a $D^{(\Lambda)}(e^{\mathcal{J}})$ insertion. Here $D^{(\Lambda)}(e^{\mathcal{J}})$ is the representation matrix for a group element $e^{\mathcal{J}}$ in the irreducible representation (irrep) Λ, see [10, 18]. Note that a vertex v is dual to a 4-simplex σ, so that the set of faces $f(v)$ intersecting at v is dual to the set of ten triangles of σ. Similarly, the set of five dual edges $l(v)$ intersecting at v is dual to the set of five tetrahedrons of σ. The sum in (2.5) is over all colorings Λ_f of the set of faces f of $T^*(M)$ by the irreps Λ_f of G, as well as over the corresponding intertwiners ι_l for the dual complex edges l. Equivalently, Λ_f label the triangles of $T(M)$, while ι_l label the tetrahedrons of $T(M)$.

In the case of the $SU(2)$ group and $\mathcal{J} = 0$, the amplitude A_5 gives the $15j$ symbol, see [1, 13]. For the general definition of $15j$-symbols $A_5(\Lambda_{f(v)}, \iota_{l(v)})$ see [16, 17]. Then $Z_0(\mathcal{J} = 0, M)$ can be written as

$$Z_0(M) = \sum_{\Lambda_f, \iota_l} \prod_f \dim \Lambda_f \prod_v A_5(\Lambda_{f(v)}, \iota_{l(v)}), \tag{2.6}$$

which after quantum group regularization (by passing to a root of unity) becomes a manifold invariant known as the Crane–Yetter invariant [6].

The formula (2.4) is now given by the discretized expression

$$Z(\lambda, M, T) = \exp\left(-\frac{i\lambda}{5} \sum_{\sigma} \sum_{\Delta,\Delta'\in\sigma} g^{IJ} \frac{\partial^2}{\partial \mathcal{J}_f^I \partial \mathcal{J}_{f'}^J}\right) Z_0(\mathcal{J}, M)\Big|_{\mathcal{J}=0}, \tag{2.7}$$

where T denotes the triangulation of M. The equation (2.7) can be rewritten as

$$Z(\lambda, M, T) = \sum_{\Lambda_f, \iota_l} \prod_f \dim \Lambda_f \exp\left(-i\lambda \sum_\sigma \hat{V}_\sigma\right) \prod_v A_5(\Lambda_{f(v)}, \iota_{l(v)}), \tag{2.8}$$

where the operator $\hat{V}_\sigma$ is given by

$$\hat{V}_\sigma = \frac{1}{5} \sum_{\Delta,\Delta'\in\sigma} g^{IJ} L_I^{(\Lambda)} \otimes L_J^{(\Lambda')} \equiv \frac{1}{5} \sum_{f,f';v\in f\cap f'} g^{IJ} L_I^{(\Lambda_f)} \otimes L_J^{(\Lambda_{f'})}. \tag{2.9}$$

This operator acts on the σ-spin network evaluation A_5 by inserting the Lie algebra basis element $L^{(\Lambda)}$ for an irrep Λ into the spin network edge carrying the irrep Λ. The expression (2.9) follows from (2.5), (2.7) and the relation

$$\frac{\partial D^{(\Lambda)}(e^{\mathcal{J}})}{\partial \mathcal{J}^I}\Big|_{\mathcal{J}=0} = L_I^{(\Lambda)}.$$

Following (2.9), let us define a g-edge in a 4-simplex spin network as a line connecting the middle points of two edges of the spin network, such that this line is labelled by the tensor g_{IJ}. We associate to a g-edge the linear map

$$\sum_{IJ} g^{IJ} L_I^{(\Lambda_f)} \otimes L_J^{(\Lambda_{f'})},$$

where Λ_f and $\Lambda_{f'}$ are the labels of the spin network edges connected by the g-edge and $L_I^{(\Lambda_f)}$ denotes the action of the basis element L_I of $\mathfrak{g}$ in the representation Λ_f.

The action of the operator $g^{IJ} L_I^{(\Lambda_f)} \otimes L_J^{(\Lambda_{f'})}$ in a single 4-simplex of the $\mathcal{J} = 0$ spin foam state sum (2.5) can be represented as the evaluation of a spin network $\Gamma_{\sigma,g}$ obtained from the 4-simplex spin network Γ_σ by adding a g-edge between the two edges of Γ_σ labeled by Λ_f and $\Lambda_{f'}$.

When $g_{IJ} \propto \delta_{IJ}$ and the intertwiners $C^{\Lambda\Lambda'A}$, from $\Lambda \otimes \Lambda'$ to A, where A is the adjoint representation, form a one-dimensional vector space, $\Gamma_{\sigma,g}$ becomes the 4-simplex spin network with an insertion of an edge labeled by the adjoint irrep, see Fig. 1. This simplification happens because the matrix elements of $L_I^{(\Lambda)}$ can be identified with the components of the intertwiner $C^{\Lambda\Lambda'A}$, since these intertwiners are one-dimensional vector spaces, i.e.

$$\left(L_I^{(\Lambda)}\right)_{\alpha\beta} = C_{\alpha\beta I}^{\Lambda\Lambda A}, \tag{2.10}$$

so that

$$\sum_I \left(L_I^{(\Lambda)}\right)_{\alpha\beta} \left(L_I^{(\Lambda')}\right)_{\alpha'\beta'} = \sum_I C_{\alpha\beta I}^{\Lambda\Lambda A} C_{I\alpha'\beta'}^{A\Lambda'\Lambda'}. \tag{2.11}$$

Then the right-hand side of the equation (2.11) represents the evaluation of the spin network in Fig. 2. The condition (2.10) is not too restrictive since it includes the $SU(2)$ and $SO(4)$ groups. We need to consider this particular case in order to be able to use the Chain–Mail techniques.
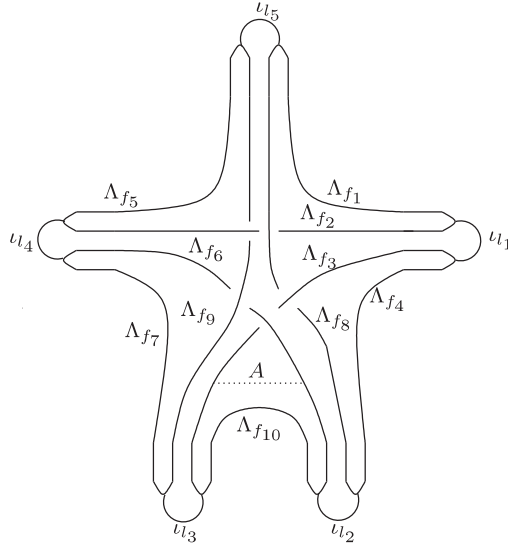
Figure 1. A $15j$ symbol (4-simplex spin network) with a g-edge insertion (dashed line). Here A is the adjoint representation.
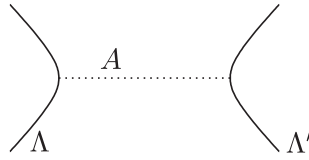

Figure 2. Spin network form of equation (2.11).

The action of $(\hat{V}_\sigma)^n$ in A_5 is given by the evaluation of a $\Gamma_{\sigma,n}$ spin network which is obtained from the Γ_σ spin network by inserting n g-edges labeled by the adjoint irrep. These additional edges connect the edges of Γ_σ which correspond to the triangles of the 4-simplex σ where the operators $L_I^{(\Lambda)} \otimes L_I^{(\Lambda')}$ from $\hat{V}_\sigma$ act.

Let

$$Z(M,T) = \sum_{n=0}^\infty i^n \lambda^n Z_n(M,\lambda,T),$$

then

$$Z_0(M) = Z_0(\mathcal{J}, M)\Big|_{\mathcal{J}=0}.$$

The state sum Z_0 is infinite, unless it is regularized. The usual way of regularization is by using the representations of the quantum group $U_q(\mathfrak{g})$ at a root of unity, which, by passing to a finite-dimensional quotient, yields a modular Hopf algebra [26]. There are only finitely many irreps with non-zero quantum dimension in this case, and the corresponding state sum Z_0 has the same form as in the Lie group case, except that the usual spin network evaluation used for the spin-foam amplitudes has to be replaced by the quantum spin network evaluation. In this way one obtains a finite and triangulation independent Z_0, usually known as the Crane–Yetter invariant [6]. This 4-manifold invariant is determined by the signature of the manifold [23, 26]. The same procedure of passing to the quantum group at a root of unity can be applied to the perturbative corrections Z_n, but in order to obtain triangulation independent results, the dilute gas limit has to be implemented [2, 9].

2.1   The Chain–Mail formalism and observables of the Crane–Yetter invariant

The Chain–Mail formalism for defining the Turaev–Viro invariant and the Crane–Yetter invariant was introduced by Roberts in [23]. In the four-dimensional case, the construction of the related manifold invariant $Z_0(M)$ had already been implemented by Broda in [4]. However, the equality with the Crane–Yetter invariant, as well as the relation of $Z_0(M)$ with the signature of M appears only in the work of Roberts [23].

We will follow the conventions of [3]. Let M be a four-dimensional manifold. Suppose we have a handle decomposition [12, 14, 24] of M, with a unique 0-handle, and with h_i handles of order i (where $i = 1, 2, 3, 4$). This gives rise to the link CHL_M in the three-sphere S^3, with $h_2 + h_1$ components (the "Chain–Mail link"), which is the Kirby diagram of the handle decomposition [12, 14]. Namely, we have a dotted unknotted (and 0-framed) circle for each 1-handle of M, determining the attaching of the 1-handle along the disjoint union of two balls, and we also have a framed knot for each 2-handle, along which we attach it. This is the four-dimensional counterpart of the three-dimensional Chain–Mail link of Roberts, see [23, 3].

The Crane–Yetter invariant $Z_0(M)$, which coincides with the invariant $Z_0(\mathcal{J}, M)_{\mathcal{J}=0}$, defined in the introduction, see equation (2.6), can be represented as a multiple of the spin-network evaluation of the chain mail link CHL_M, colored with the following linear combination of quantum group irreps (the Ω-element):

$$\Omega = \sum_{\Lambda}(\dim_q \Lambda)\Lambda,$$

see [23]. Explicitly, by using the normalizations of [3]:

$$Z_0(M) = \eta^{-\frac{1}{2}(h_2+h_1+h_3-h_4+1)}\langle \mathrm{CHL}_M, \Omega^{h_2+h_1}\rangle, \tag{2.12}$$

where

$$\eta = \sum_{\Lambda}(\dim_q \Lambda)^2.$$

Roberts also proved in [23] that $Z_0(M) = \kappa^{s(M)}$, where κ is the evaluation of a 1-framed unknot colored with the Ω-element and $s(M)$ denotes the signature of M.

Given a triangulated manifold (M, T), consider the natural handle decomposition of M obtained from thickening the dual cell decomposition of M; see [23, 24]. Then a handle decomposition of M (with a single 0-handle), such that (2.12) explicitly gives the formula for $Z_0(M) = Z_0(\mathcal{J}, M)_{\mathcal{J}=0}$, appearing in equation (2.5), is obtained from this handle decomposition by canceling pairs of 0- and 1-handles [12, 14, 24], until a single 0-handle is left; in this case, in the vicinity of each 4-simplex, the Chain–Mail link has the form given in Fig. 3. This explicit calculation appears in [23, 3] and essentially follows from the Lickorish encircling lemma [13, 15]: the spin-network evaluation of a graph containing a unique strand (colored with the representation Λ) passing through a zero framed unknot colored with Ω vanishes unless Λ is the trivial representation.

A variant of the Crane–Yetter model Z_0 in (2.12) is achieved by inspecting its observables, addressed in [3]. Consider a triangulated 4-manifold M. Consider the handle decomposition of M obtained from thickening the dual complex of the triangulation, and eliminating pairs of 0- and 1-handles until a single 0-handle is left. Any triangle of the triangulation of M therefore yields a 2-handle of M.

Now choose a set S with n_S triangles of M, which will span a (possibly singular) surface Σ_2 of M. Color each $t \in S$ with a representation Λ_t. The associated observable of the Crane–Yetter invariant is:

$$Z_0(M, S, \Lambda_S) = \eta^{-\frac{1}{2}(h_2+h_1+h_3-h_4+1)}\langle \mathrm{CHL}_M; \Omega^{h_2+h_1-n_S}, \Lambda_S\rangle \prod_{t\in S}\dim_q(\Lambda_t), \tag{2.13}$$
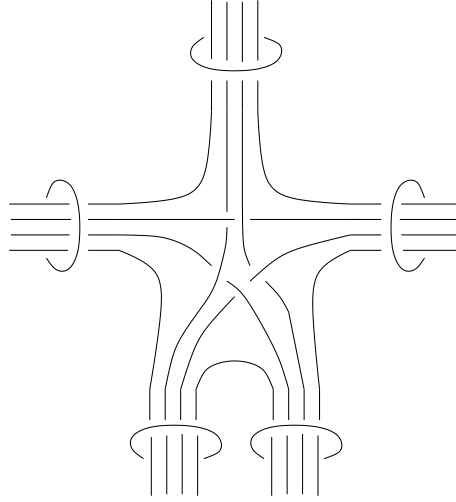
Figure 3. Portion of the chain-mail link corresponding to a 4-simplex; this may have additional meridian circles (corresponding to 1-handles) since we also eliminate pairs of 0- and 1-handles, until a single 0-handle is left.

where $\langle \mathrm{CHL}_M; \Omega^{h_2+h_1-n_S}, \Lambda_S \rangle$ denotes the spin-network evaluation of the Chain–Mail link CHL_M, where the components associated with the triangles $t \in S$ are colored by Λ_s and the remaining components with Ω. We can see CHL_M as a pair (L_S, K_S), where K_S denotes the components of the Chain–Mail link given by the triangles of S and L_S the remaining components of the Chain–Mail link. We thus have $\mathrm{CHL}_M = L_S \cup K_S$.

Let $Z_{\mathrm{WRT}}(N, \Gamma)$ denote the Witten–Reshetikhin–Turaev invariant of the colored graph Γ embedded in the 3-manifold N, in the normalization of [3]. Then Theorem 2 of [3] says that:

$$Z_0(M, S, \Lambda_S) = Z_{\mathrm{WRT}}\big(\partial(M \setminus \hat{\Sigma}_2), K'_S\big)\kappa^{s(M\setminus\hat{\Sigma}_2)} \eta^{\frac{\chi(\Sigma_2)}{2}-n_S} \prod_{t\in S} \dim_q(\Lambda_t). \qquad (2.14)$$

Here $\hat{\Sigma}_2$ is an open regular neighborhood of Σ_2 in M, s denotes the signature of the manifold and χ denotes the Euler characteristic. The link K'_S is the link in $\partial(M \setminus \hat{\Sigma}_2)$ along which the 2-handles associated to the triangles of Σ_2 would be attached, in order to obtain M. This theorem of [3] essentially follows from the fact that the pair (L_S, K_S) is a surgery presentation of the pair $\big(\partial(M \setminus \hat{\Sigma}_2), K'_S\big)$, a link embedded in a manifold, apart from connected sums with $S^1 \times S^2$.

3 The first-order correction

Recall that there are two possible ways of representing the Crane–Yetter invariant Z_0: as a state sum invariant (2.6) and as the evaluation of a Chain–Mail link (2.12). It follows from (2.8) that Z_1 can be written as NZ'_0 where N is the number of 4-simplices of $T(M)$ and Z'_0 is the state sum given by a modification of the state sum Z_0 where a single 4-simplex is perturbed by the operator $\hat{V}_\sigma$.

In order to calculate Z'_0 consider a 4-manifold M with a triangulation T whose dual complex is T^*. Given a 4-simplex $\sigma \in T$ we define an insertion $\mathcal{I}$ as being a choice of a pair of triangles of σ which do not belong to the same tetrahedron of σ and have therefore a single vertex in common (following (2.7) we will distinguish the order in which the triangles appear). Given the colorings Λ_f of the triangles of σ (or of the dual faces f) and the colorings ι_l of the tetrahedrons of σ (or of the dual edges l), then $A_5(\Lambda_f, \iota_l, \mathcal{I})$ is the evaluation of the spin network of Fig. 1.
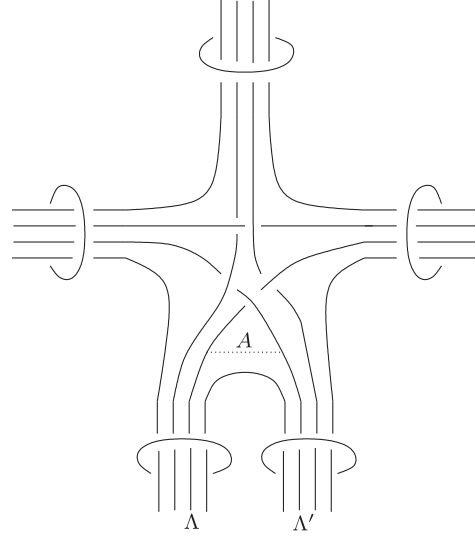
Figure 4. Portion of the graph $CML_M^{\mathcal{I}}$ corresponding to a 4-simplex with an insertion. All strands are to be colored with Ω, unless they intersect the insertion, as indicated.

We then have:

$$Z_1(M,T) = \frac{1}{5} \sum_{\sigma} \sum_{\mathcal{I}_\sigma} \sum_{\Lambda_f, \iota_l} A_5(\Lambda_{f(\sigma)}, \iota_{l(\sigma)}, \mathcal{I}_\sigma) \prod_f \dim \Lambda_f \prod_{v \neq v(\sigma)} A_5(\Lambda_{f(v)}, \iota_{l(v)}), \qquad (3.1)$$

where $v(\sigma)$ is the vertex of T^* corresponding to σ. This sum is over the set of all 4-simplices σ of $T(M)$, as well as over the set $\mathcal{I}_\sigma$ of all insertions of σ and over the set of all colorings (Λ_f, ι_l) of the faces and the edges of $T^*(M)$ (or equivalently, a sum over the colorings of the triangles and the tetrahedrons of $T(M)$.)

The infinite sum in (3.1) is regularized by passing to the category of representations of the quantum group $U_q(g)$, where q is a root of unity. In order to calculate $Z_1(M,T)$ in this case, let us represent it as an evaluation of the Chain–Mail link $\mathrm{CHL}(M,T)$ [23] in the three-sphere S^3.

As explained in Subsection 2.1, the invariant $Z_0(M)$ can be represented as a multiple of the evaluation of the chain mail link CHL_M colored with the linear combination of the quantum group irreps $\Omega = \sum_\Lambda (\dim_q \Lambda)\Lambda$, see equation (2.12). Analogously, by extending the 3-dimensional approach of [9], a chain-mail formulation for the equation (3.1) can be given. Consider the handle decomposition of M obtained by thickening the dual cell decomposition of M associated to the triangulation T of M. One can cancel pairs of 0- and 1-handles, until a single 0-handle is left. Let CHL_M be the associated chain-mail link (the Kirby diagram of the handle decomposition). We then have

$$Z_1(M,T) = \frac{1}{5} \sum_{\mathcal{I}} \sum_{\Lambda,\Lambda'} \eta^{-\frac{1}{2}(h_2+h_1+h_3-h_4+1)} \dim_q \Lambda \dim_q \Lambda' \langle \mathrm{CHL}_M^{\mathcal{I}}, \Omega^{h_2+h_1-2}, \Lambda, \Lambda' \rangle, \quad (3.2)$$

where, as before, an insertion $\mathcal{I}$ is the choice of a pair of triangles t_1 and t_2 in some 4-simplex of M, such that t_1 and t_2 have only one vertex in common. Given an insertion $\mathcal{I}$, the graph $\mathrm{CHL}_M^{\mathcal{I}}$ is obtained from the chain-mail link CHL_M by inserting a single edge (colored with the adjoint representation of $\mathfrak{g}$) connecting the components of CHL_M (colored with Λ and Λ') corresponding to t_1 and t_2; see Fig. 4. $\mathrm{CHL}_M^{\mathcal{I}}$ can be considered as a pair $(L_{\mathcal{I}}, \Gamma_{\mathcal{I}})$ where $L_{\mathcal{I}}$ denotes the components of $\mathrm{CHL}_M^{\mathcal{I}}$ not incident to the insertion $\mathcal{I}$ (which are exactly $h_2 + h_2 - 2$) and $\Gamma_{\mathcal{I}}$ denotes the component of $\mathrm{CHL}_M^{\mathcal{I}}$ containing the insertion $\mathcal{I}$. Hence we use the notation $\langle \mathrm{CHL}_M^{\mathcal{I}}, \Omega^{h_2+h_1-2}, \Lambda, \Lambda' \rangle$ to mean the evaluation of the pair $(L_{\mathcal{I}}, \Gamma_{\mathcal{I}})$ where all components
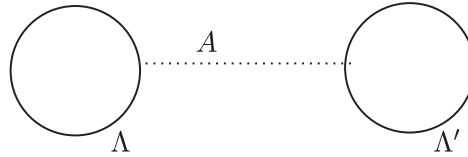
Figure 5. Spin network $\Gamma_1(\Lambda, \Lambda')$. Here A is the adjoint representation.

of $L_{\mathcal{I}}$ are colored with Ω and the two circles of $\Gamma_{\mathcal{I}}$ are colored with Λ and Λ', with an extra edge connecting them, colored with the adjoint representation A.

Consider an insertion $\mathcal{I}$ connecting the triangles t_1 and t_2, which intersect at a single vertex. Equation (3.2) coincides, apart from the inclusion of the single insertion, with the observables of the Crane–Yetter invariant [3] (Subsection 2.1) for the pair of triangles colored with Λ and Λ'; see equation (2.13). By using the discussion in Section 6 and Theorem 2 of [3] (see equation (2.14)) one therefore obtains, for each insertion $\mathcal{I}$ and each pair of irreps Λ, Λ':

$$\eta^{-\frac{1}{2}(h_2+h_1+h_3-h_4+1)}\langle \mathrm{CHL}_M^{\mathcal{I}}, \Omega^{h_2+h_1-2}, \Lambda, \Lambda'\rangle = Z_{\mathrm{WRT}}(S^3, \Gamma(\Lambda, \Lambda'))\eta^{-\frac{3}{2}}Z_0(M), \qquad (3.3)$$

where $\Gamma(\Lambda, \Lambda')$ is the colored link of Fig. 5. In addition Z_{WRT} denotes the Witten–Reshetikhin–Turaev invariant [27, 22] of graphs in manifolds, in the normalization of [3, 8]. Note that in the notation of equation (2.14), $\Sigma_2 = t_1 \cup t_2$ is two triangles which intersect at a vertex, thus $\chi(\Sigma_2) = 1$ and also its regular neighborhood $\hat{\Sigma}_2$ is homeomorphic to the 4-disk, thus $s(M) = s(M \setminus \hat{\Sigma}_2)$.

Equation (3.3) follows essentially from the fact that the pair $\mathrm{CHL}_M^{\mathcal{I}} = (L_{\mathcal{I}}, \Gamma_{\mathcal{I}})$ is a surgery presentation [14, 12] of the pair $(S^3, \Gamma(\Lambda, \Lambda'))$, apart from connected sums with $S^1 \times S^2$; c.f. Theorem 3, below.

To see this, note that (after turning the circles associated with the 1-handles of M into dotted circles) the link $L_{\mathcal{I}}$ is a Kirby diagram for the manifold M minus an open regular neighborhood Σ' of the 2-complex Σ made from the vertices and edges of the triangulation of M, together with the triangles t_1 and t_2. Since t_1 and t_2 intersect at a single vertex, any regular neighborhood $\hat{\Sigma}_2$ of the (singular) surface Σ_2 spanned by t_1 and t_2 is homeomorphic to the 4-disk D^4. Therefore Σ' is certainly homeomorphic to the boundary connected sum $\left(\natural_{i=1}^k (D^3 \times S^1)\right)\natural D^4$, whose boundary is $\left(\#_{i=1}^k (S^2 \times S^1)\right)\# S^3$, for some positive integer k. Here $\#$ denotes the connected sum of manifolds and $\natural$ denotes the boundary connected sum of manifolds. The circles $c_1, c_2 \subset \Gamma_{\mathcal{I}}$ associated with the triangles t_1 and t_2 define a link which lives in $\partial \hat{\Sigma}_2 \cong S^3 \subset \left(\#_{i=1}^k (S^2 \times S^1)\right)\# S^3$.

The two circles c_1 and c_2 define a 0-framed unlink in S^3, with each individual component being unknotted. Let us see why this is the case. We will turn the underlying handle decomposition of M upside down, by passing to the dual handle decomposition of M, where each i-simplex of the triangulation of M yields an i-handle of M; see [12, p. 107]. Consider the bit $P \subset M$ of the handle-body yielded by the 2-complex Σ, thus P is (like Σ') a regular neighborhood of Σ. Maintaining the 0-handle generated by the vertex $t_1 \cap t_2$, eliminate some pairs of 0- and 1-handles, in the usual way, until a single 0-handle of P is left. Clearly $\partial P^* = \partial(M \setminus \Sigma')$, where $*$ denotes the orientation reversal. The circles c_1 and c_2, in ∂P^*, correspond now (since we considered the dual handle decomposition) to the belt-spheres of the 2-handles of P (attached along c_{t_1} and c_{t_2}) and associated with the triangles t_1 and t_2. Since c_1 and c_2 are 0-framed meridians going around c_{t_1} and c_{t_2} (see [12, Example 1.6.3]) it therefore follows that these circles are unlinked and are also, individually, unknotted; see Fig. 6. Given this and the fact that the insertion $\mathcal{I}$ colored with A also lives in S^3, it follows that $\mathrm{CHL}_M^{\mathcal{I}} = (L_{\mathcal{I}}, \Gamma_{\mathcal{I}})$ is a surgery presentation of the pair $(S^3, \Gamma(\Lambda, \Lambda'))$, apart from the connected sums, distant from $\Gamma(\Lambda, \Lambda')$, with $S^1 \times S^2$.

Since the evaluation of the tadpole spin network is zero, it follows that $Z_{\mathrm{WRT}}(S^3, \Gamma(\Lambda, \Lambda')) = 0$, and consequently
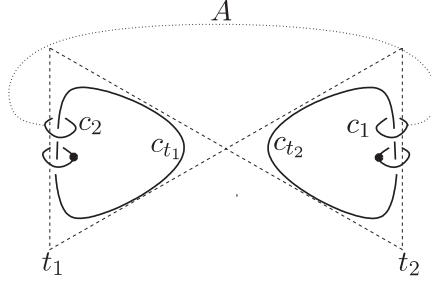
Figure 6. The Kirby diagram for P in the vicinity of the triangles t_1 and t_2. We show the belt-spheres c_1 and c_2 of the 2-handles of P (attaching along c_{t_1} and c_{t_2}) associated with the triangles t_1 and t_2.

Theorem 1. *For any triangulation T of M we have $Z_1(M,T) = 0$.*

4   The second-order correction

Since $Z_1 = 0$, we have to calculate Z_2 in an appropriate limit such that the partition function Z is different from Z_0 and such that Z is independent of the triangulation [9]. Let N be the number of 4-simplices. From (2.8) we obtain

$$Z_2(M) = \frac{1}{2} \sum_{\Lambda_f, \iota_l} \prod_f \dim \Lambda_f \left(\sum_\sigma \hat{V}_\sigma^2 + \sum_{\sigma \neq \sigma'} \hat{V}_\sigma \hat{V}_{\sigma'} \right) \prod_v A_5(\Lambda_{f(v)}, \iota_{l(v)}), \qquad (4.1)$$

where

$$\hat{V}_\sigma A_5(\Lambda_{f(v)}, \iota_{l(v)}) = \frac{1}{5} \sum_{\text{insertions } \mathcal{I} \text{ of } \sigma} A_5(\Lambda_{f(v)}, \iota_{l(v)}, \mathcal{I}), \qquad (4.2)$$

if σ is dual to v, see Fig. 1. On the other hand, $\hat{V}_\sigma A_5(\Lambda_{f(v)}, \iota_{l(v)}) = A_5(\Lambda_{f(v)}, \iota_{l(v)})$ if v is not dual to σ. In order to to solve the possible framing and crossing ambiguities arising from the equation (4.1), a method analogous to the one used in [9] can be employed. Note that there are exactly 30 insertions in a 4-simplex σ, corresponding to pairs of triangles of σ with a single vertex in common. This is because there are exactly three triangles of σ having only one vertex in common with a given triangle of σ.

Analogously to the first-order correction, Z_2 can be written as

$$Z_2(M,T) = \frac{1}{2} \eta^{-\frac{1}{2}(h_2 + h_1 + h_3 - h_4 + 1)} \sum_{k=1}^N \langle \text{CHL}_M, \Omega^{h_2 + h_1}, \hat{V}_k^2 \rangle$$

$$+ \frac{1}{2} \eta^{-\frac{1}{2}(h_2 + h_1 + h_3 - h_4 + 1)} \sum_{1 \leq k \neq l \leq N} \langle \text{CHL}_M, \Omega^{h_2 + h_2}, \hat{V}_k, \hat{V}_l \rangle, \qquad (4.3)$$

where the first sum denotes the contributions from two insertions $\hat{V}$ in the same 4-simplex σ_k and the second sum represents the contributions when the two insertions $\hat{V}$ act in different 4-simplices σ_k and σ_l. As in the previous section, we will use the handle decomposition of M with an unique 0-handle naturally obtained from the thickening of $T^*(M)$.

Note that each $\langle \text{CHL}_M, \Omega^{h_2 + h_1}, \hat{V}_k^2 \rangle$ corresponds to a sum over all the possible choices of pairs of insertions in the 4-simplex σ_k. The value of $\langle \text{CHL}_M, \Omega^{h_2 + h_1}, \hat{V}_k^2 \rangle$ is obtained from the evaluation of the chain-mail link CHL_M colored with Ω, which contains g-edges carrying the adjoint representation, as in the calculation of the first-order correction.
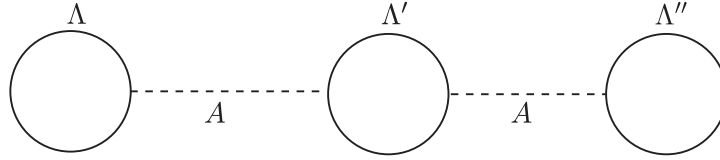
Figure 7. Colored graph $\Gamma_2'(\Lambda, \Lambda', \Lambda'')$, a wedge graph. Here A is the adjoint representation.

A configuration C is, by definition, a choice of insertions distributed along a set of 4-simplices of M. Given a positive integer n and a set R of 4-simplices of M, we denote by $\mathcal{C}_R^n$ the set of configurations with n insertions distributed along R. By expanding each V into a sum of insertions, the equation (4.3) can be written as:

$$Z_2(M,T) = \frac{1}{50}\eta^{-\frac{1}{2}(h_2+h_1+h_3-h_4+1)} \sum_{k=1}^{N} \sum_{C \in \mathcal{C}_{\{\sigma_k\}}^2} \langle \mathrm{CHL}_M^C, \Omega^{h_2+h_1} \rangle$$

$$+ \frac{1}{50}\eta^{-\frac{1}{2}(h_2+h_1+h_3-h_4+1)} \sum_{1 \leq k \neq l \leq N} \sum_{C \in \mathcal{C}_{\{\sigma_k,\sigma_l\}}^2} \langle \mathrm{CHL}_M^C, \Omega^{h_2+h_2} \rangle. \tag{4.4}$$

Note that each graph CHL_M^C splits naturally as (L_C, Γ_C), where the first component contains the circles non incident to any insertion of C.

The second sum in equation (4.4) vanishes, because it is the sum of terms proportional to:

$$\left(\sum_{\Lambda,\Lambda'} \langle \Gamma_1(\Lambda, \Lambda') \rangle \right)^2 Z_0(M) \tag{4.5}$$

and to

$$\sum_{\Lambda,\Lambda',\Lambda''} \langle \Gamma_2'(\Lambda, \Lambda', \Lambda'') \rangle Z_0(M), \tag{4.6}$$

where Γ_1 is the dumbbell spin network of Fig. 5, and Γ_2' is a three-loop spin network, see Fig. 7.

These spin networks arise from the cases when the pair (L_C, Γ_C) is a surgery presentation of the disjoint union $(S^3, \Gamma_1) \sqcup (S^3, \Gamma_1)$ and of (S^3, Γ_2'), respectively, apart from connected sums with $S^1 \times S^2$; see Theorem 3 below. The former case corresponds to a situation where the two insertions act in pairs of triangles without a common triangle, and the latter corresponds to a situation where the two pairs of triangles have a triangle in common, which necessarily is a triangle in the intersection $\sigma_k \cap \sigma_l$ of the 4-simplices σ_k and σ_l. The evaluations in (4.5) and (4.6) vanish since the corresponding spin networks have tadpole subdiagrams.

The first sum in (4.4) also gives the terms proportional to the ones in equations (4.5) and (4.6). These terms correspond to two insertions connecting two pairs made from four distinct triangles of σ_k and to two insertions connecting two pairs of triangles made from three distinct triangles of σ_k, respectively. All these terms vanish.

The non-vanishing terms in equation (4.4) arise from a pair of insertions connecting the same two triangles in a 4-simplex. There are exactly 30 of these. Therefore, by using Theorem 3 of Section 5, we obtain:

$$Z_2(M,T) = \frac{3N}{5}\eta^{-2} \sum_{\Lambda,\Lambda'} \dim_q \Lambda \dim_q \Lambda' \langle \Gamma_2(\Lambda, \Lambda') \rangle Z_0(M),$$

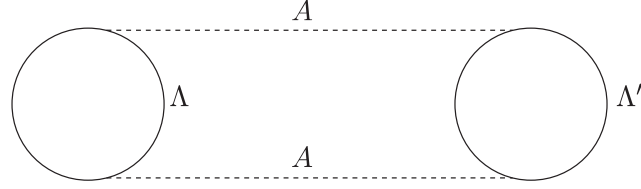where Γ_2 is a two-handle dumbbell spin network, see Fig. 8. We thus have:

Figure 8. Colored graph $\Gamma_2(\Lambda, \Lambda')$, a double dumbbell graph. As usual A is the adjoint representation.

Theorem 2. *The second-order perturbative correction $Z_2(M, T)$ divided by the number of N of 4-simplices of the manifold is triangulation independent. In fact:*

$$\frac{Z_2(M, T)}{N} = \frac{3}{5}\eta^{-2} \sum_{\Lambda, \Lambda'} \dim_q(\Lambda) \dim_q(\Lambda') \langle \Gamma_2(\Lambda, \Lambda') \rangle Z_0(M).$$

Here $\langle \Gamma_2(\Lambda, \Lambda') \rangle$ denotes the spin-network evaluation of the colored graph $\Gamma_2(\Lambda, \Lambda')$. Note that

$$\langle \Gamma_2(\Lambda, \Lambda') \rangle = \frac{\theta(A, \Lambda, \Lambda)\theta(A, \Lambda', \Lambda')}{\dim_q A},$$

which is is obviously non-zero, and therefore $Z_2(M, T) \neq 0$.

5   Higher-order corrections

For $n > 2$, the contributions to the partition function will be of the form

$$\eta^{-\frac{1}{2}(h_2 + h_1 + h_3 - h_4 + 1)} \langle \mathrm{CHL}_M, \Omega^{h_1 + h_2}, (\hat{V}_1)^{k_1} \cdots (\hat{V}_N)^{k_N} \rangle,$$

where $k_1 + \cdots + k_N = n$. By using the equation (4.2), each of these terms splits as a sum of terms of the form:

$$\frac{1}{5^n} \eta^{-\frac{1}{2}(h_2 + h_1 + h_3 - h_4 + 1)} \langle \mathrm{CHL}_M^C, \Omega^{h_1 + h_2} \rangle = \langle C \rangle, \tag{5.1}$$

where C is a set of n insertions (a "configuration") distributed among the N 4-simplices of the chosen triangulation of M, such that the 4-simplex σ_i has k_i insertions. Insertions are added to the chain-mail link CHL_M as in Fig. 4, forming a graph CHL_M^C (for framing and crossing ambiguities we refer to [9]).

Note that each graph CHL_M^C splits as (L_C, Γ_C) where L_C contains the components of CHL_M not incident to any insertion. As in the $n = 1$ and $n = 2$ cases, equation (5.1) coincides, apart from the extra insertions, with the observables of the Crane–Yetter invariant defined in [3]; see Subsection 2.1. Therefore, by using the same argument that proves Theorem 2 of [3] we have:

Theorem 3. *Given a configuration C consider the 2-complex Σ_C spanned by the k triangles of M incident to the insertions of C. Let Σ_C' be a regular neighborhood of Σ_C in M. Then M can be obtained from $M \setminus \Sigma_C'$ by adding the 2-handles corresponding to the faces of Σ (and some further 3- and 4-handles, corresponding to the edges and vertices of Σ). These 2-handles attach along a framed link K in $\partial(M \setminus \Sigma')$, a manifold diffeomorphic to $\partial(\Sigma_C')$ with the reverse orientation. The insertions of C can be transported to this link K defining a graph K_C in $\partial(\Sigma_C'^*) = \partial(M \setminus \Sigma_C')$. We have:*

$$\langle C \rangle = \sum_{\Lambda_1, \ldots, \Lambda_k} Z_{\mathrm{WRT}}\big(\partial(\Sigma_C'^*), K_C; \Lambda_1, \ldots, \Lambda_k\big) \kappa^{s(M \setminus \Sigma')} \eta^{\frac{\chi(\Sigma)}{2} - k} \dim_q \Lambda_1 \cdots \dim_q \Lambda_k,$$

where $s(M \setminus \Sigma')$ denotes the signature of the manifold $M \setminus \Sigma'$ and χ denotes the Euler characteristic.

Note that, up to connected sums with $S^1 \times S^2$, the pair $\text{CHL}_M^C = (L_C, \Gamma_C)$ is a surgery presentation of $(\partial(M \setminus \Sigma_C'), K_C) = (\partial(\Sigma_C'^*), K_C)$. Unlike the $n = 1$ and $n = 2$ cases, it is not possible to determine the pair $(\partial(\Sigma_C'^*), K_C)$ for $n \geq 3$ without having an additional information about the configuration. In fact, considering the set of all triangulations of M, an infinite number of diffeomorphism classes for $(M \setminus \Sigma', K_C)$ is in general possible for a fixed n; see [9] for the three dimensional case. This makes it complicated to analyze the triangulation independence of the formula for $Z_n(M, T)$ for $n \geq 3$.

Since

$$Z(M, T) = \sum_n \lambda^n Z_n(M, T),$$

where

$$Z_n(M, T) = \sum_{k_1 + \cdots + k_N = n} \frac{1}{k_1! \cdots k_N!} \eta^{-\frac{1}{2}(h_2 + h_1 + h_3 - h_4 + 1)} \langle \text{CHL}_M, \Omega^{h_1 + h_2}, (\hat{V}_1)^{k_1} \cdots (\hat{V}_N)^{k_N} \rangle,$$

in order to resolve the triangulation dependence of Z_n, let us introduce the quantities

$$z_n = \lim_{N \to \infty} \frac{Z_n(M, T)}{N^{\frac{n}{2}} Z_0(M)}; \tag{5.2}$$

see [9, 2]. The limit is to be extended to the set of all triangulations T of M, with N being the number of 4-simplices of M, in a sense to be made precise; see [9]. From Sections 3, 4 and Theorem 2 it follows that

$$z_1 = 0, \qquad z_2 = \frac{3\eta^{-2}}{5 \dim_q A} \left(\sum_\Lambda \dim_q \Lambda \theta(A, \Lambda, \Lambda) \right)^2.$$

Note that the values of z_1 and z_2 are universal for all compact 4-manifolds. The expression for z_2 is finite because there are only finitely many irreps for the quantum group $U_q(\mathfrak{g})$, of non-zero quantum dimension, when q is a root of unity.

6   Dilute-gas limit

We will now show how to define and calculate the limit in the equation (5.2). Let M be a 4-manifold and let us consider a set $\mathcal{S}$ of triangulations of M, such that for any given $\epsilon > 0$ there exists a triangulation $T \in \mathcal{S}$ such that the diameter of the biggest 4-simplex is smaller than ϵ, i.e. the triangulations in $\mathcal{S}$ can be chosen to be arbitrarily fine. We want to calculate the limit in equation (5.2) only for triangulations belonging to the set $\mathcal{S}$.

Furthermore, we suppose that $\mathcal{S}$ is such that (c.f. [9]):

Restriction 1 (Control of local complexity-I)**.** *Together with the fact that the triangulations in $\mathcal{S}$ are arbitrarily fine we suppose that:*

There exists a positive integer L such that any 4-simplex of any triangulation $T \in \mathcal{S}$ intersects at most L 4-simplices of T.

Let us fix n and consider $Z_{2n}(M, T)$ when $N \to \infty$. The value of Z_{2n} will be given as a sum of contributions of configurations C such that n_1 insertions of $\hat{V}$ act in a 4-simplex σ_1, n_2 of insertions of $\hat{V}$ act in the 4-simplex $\sigma_2 \neq \sigma_1$ and so on, such that $n_1 + n_2 + \cdots + n_N = 2n$ and $n_k \geq 0$.

A configuration for which any 4-simplex has either zero or two insertions, with all 4-simplices which have insertions being disjoint will be called a dilute-gas configuration. There will be

$$15^n \frac{N!}{n!(N-n)!} - \delta(N, T),$$

dilute-gas configurations, where δ is the number of pairs of 4-simplices in T with non-empty intersection. From Restriction 1 it follows that $\delta(N, T) = O(N)$ as $N \to \infty$.

Each dilute-gas configuration contributes $z_2^n Z_0(M)$ to $Z_{2n}(M, T)$ and we can write

$$\frac{Z_{2n}}{Z_0} = \left(\frac{N!}{n!(N-n)!} - O(N) \right) z_2^n + \sum_{\text{non-dilute } C} \frac{\langle C \rangle}{Z_0}, \tag{6.1}$$

where $\langle C \rangle = \frac{1}{5^n} \eta^{-\frac{1}{2}(h_2 + h_1 + h_3 - h_4 + 1)} \langle \mathrm{CHL}_M^C, \Omega^{h_1 + h_2} \rangle$, denotes the contribution of the configuration C.

Let us describe the contribution of the non-dilute configurations C more precisely. Recall that a configuration C is given by a choice of n insertions connecting n pairs of triangles of M, where each pair belongs to the same 4-simplex of M and the triangles have only one common vertex.

Given a configuration C with n insertions, consider a (combinatorial) graph γ_C with a vertex for each triangle appearing in C and for each insertion and edge connecting the corresponding vertices. The graph γ_C is obtained from Γ_C by collapsing the circles of Γ_C of it into vertices. However note that γ_C is merely a combinatorial graph, whereas Γ_C is a graph in S^3, which can have a complicated embedding.

If γ_C has a connected component homeomorphic to the graph made from two vertices and an edge connecting them, then $\langle C \rangle$ vanishes, since in this case the embedded graph whose surgery presentation is given by $(\mathrm{CHL}_M^C, \Gamma_C)$ will have a tadpole. In fact, looking at Theorem 3, one of the connected components of $(\partial(\Sigma'^*), K_C)$ will be (S^3, Γ_1), where Γ_1 is the graph in Fig. 5.

Consider a manifold with a triangulation with N 4-simplices, satisfying Restriction 1. The number of possible configurations C with l insertions with make a connected graph γ_C is bounded by $N(10L)^{l-1}(l-1)!$. In particular the number of non-dilute configurations V with $2n$ insertions and yielding a non-zero contribution is bounded by

$$\max_{\substack{l_1 + \cdots + l_{2n} = 2n \\ l_i \neq 1 \\ \exists_i : l_i \geq 3}} b \prod_{i=1}^{2n} N(10L)^{l_k - 1}(l_k - 1)! = O(N^{n-1}). \tag{6.2}$$

This is simply the statement that if a graph γ_C has k connected components, then it has $O(N^k)$ possible configurations. Since $k \leq n - 1$ for a non-dilute configuration, the bound (6.2) follows.

We now need to estimate the value of $\langle C \rangle$ for a non-dilute configuration C. We will need to make the following restriction on the set $\mathcal{S}$. We refer to the notation introduced in Theorem 3.

Restriction 2 (Control of local complexity-II)**.** *The set of $\mathcal{S}$ of triangulations of M is such that given a positive integer n then the number of possible diffeomorphism classes for the pair $(\partial(\Sigma'_C), K_C)$ is finite as we vary the triangulation $T \in \mathcal{S}$ and the configuration C with n insertions.*

In the three-dimensional case a class of triangulations $\mathcal{S}$ satisfying Restrictions 1 and 2 was constructed by using a particular class of cubulations of 3-manifolds (which always exist, see [5]) and their barycentric subdivisions, see [9]. These cubulations have a simple local structure, with only three possible types of local configurations, which permits a case-by-case analysis as the cubulations are refined through barycentric subdivisions. In the case of four-dimensional

cubulations, no such list is known, although it has been proven that a finite (and probably huge) list exists. Therefore the approach used in the three-dimensional case cannot be directly applied to the four-dimensional case. However, it is reasonable to assume that triangulations coming from the barycentric subdivisons of a cubulation of M satisfy Restriction 2.

More precisely, given a cubulation $\square$ of M, let $\Delta_\square$ be the triangulation obtained from $\square$ by taking the cone of each i-face of each cube of $\square$, starting with the 2-dimensional faces. Consider the class $\mathcal{S} = \{\Delta_{\square^{(n)}}\}_{n=0}^\infty$, where $\square^{(n)}$ is the barycentric subdivision of order n of $\square$. Then we can see that Restriction 1 is satisfied by this example, and we conjecture that $\mathcal{S}$ also satisfies Restriction 2.

The Restriction 2 combined with Theorem 3 implies that the value of $\langle C \rangle$ in equation (6.1) is bounded for a fixed n, considering the set of all triangulations in the set $\mathcal{S}$ and all possible configurations with $2n$ insertions. Since the number of non-dilute configurations C which have a non-zero contribution is of $O(N^{n-1})$, it follows that

$$\sum_{\text{non-dilute } C} \frac{\langle C \rangle}{Z_0} = O\left(N^{n-1}\right).$$

Therefore

$$\lim_{N \to \infty} \frac{Z_{2n}(M, T)}{N^n Z_0(M)} = \frac{z_2^n}{n!},$$

or

$$\frac{Z_{2n}(M, T)}{Z_0(M)} \approx \frac{z_2^n}{n!} N^n, \tag{6.3}$$

for large N.

In the case of Z_{2n+3}, the dominant configurations for triangulations with a large number N of 4-simplices consist of configurations C (as before called dilute) whose associated combinatorial graph γ_C has as connected components a connected closed graph with three edges and $(n-1)$ connected closed graphs with two edges. We can write:

$$\frac{Z_{2n+3}}{Z_0} = \sum_{\text{dilute } C} \langle C \rangle + \sum_{\text{non-dilute } C} \langle C \rangle.$$

Since the number of dilute configurations is of $O(N^n)$, while the second sum is of $O(N^{n-1})$, due to the Restrictions 1 and 2, we then obtain for large N

$$\frac{Z_{2n+3}}{Z_0} = O(N^n).$$

More precisely

$$\frac{Z_{2n+3}}{Z_0} \approx z_3(z_2)^n N \frac{(N-1)!}{n!(N-1-n)!},$$

or

$$\frac{Z_{2n+3}}{Z_0} \approx z_3(z_2)^n \frac{N^n}{n!}, \tag{6.4}$$

for large N, where z_3 is the sum of two terms. The first term is

$$\frac{30}{6 \times 5^3} \sum_{\Lambda, \Lambda'} \dim_q \Lambda \dim_q \Lambda' \langle \Gamma_3(\Lambda, \Lambda') \rangle,$$
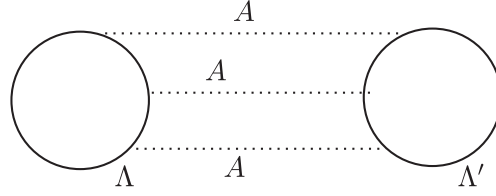
Figure 9. Colored graph $\Gamma_3(\Lambda, \Lambda')$, a triple dumbbell graph. As usual A is the adjoint representation.
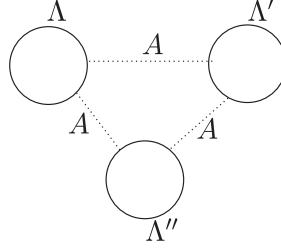

Figure 10. Colored graph $\Gamma'_3(\Lambda, \Lambda', \Lambda'')$. As usual A is the adjoint representation.

where Γ_3 is the triple dumbbell graph of Fig. 9, corresponding to three insertions connecting the same pair of triangles of the underlying 4-simplex (there are exactly 30 of these). The second term is

$$\frac{30}{6 \times 5^3} \sum_{\Lambda, \Lambda', \Lambda''} \dim_q \Lambda \dim_q \Lambda' \dim_q \Lambda'' \langle \Gamma'_3(\Lambda, \Lambda', \Lambda'') \rangle,$$

where Γ'_3 appears in Fig. 10. This corresponds to three insertions making a chain of triangles, pairwise having only a vertex in common (there are exactly 30 insertions like these for each 4-simplex).

6.1   Large-N asymptotics

Let us now study the asymptotics of $Z(M, T)$ for $N \to \infty$. We will denote $Z(M, T)$ as $Z(\lambda, N)$ and $Z_0(M)$ as $Z_0(\lambda_0)$, in order to highlight the fact that the Crane–Yetter state sum $Z_0(M)$ can be understood as a path integral for the BF-theory with a cosmological constant term $\lambda_0 \int_M \langle B \wedge B \rangle$, such that λ_0 is a certain function of an integer k_0, which specifies the quantum group at a root of unity whose representations are used to construct the Crane–Yetter state sum. In the case of a quantum $SU(2)$ group it has been conjectured that $\lambda_0 = 4\pi/k_0$, see [25]. Consequently

$$Z(\lambda) = \int \mathcal{D}A \, \mathcal{D}B \, e^{i \int_M \langle B \wedge F + \lambda B \wedge B \rangle} = \int \mathcal{D}A\mathcal{D}B e^{i \int_M \langle B \wedge F + \lambda_0 B \wedge B \rangle} e^{i(\lambda - \lambda_0) \int_M \langle B \wedge B \rangle},$$

which means that our perturbation parameter is $\lambda - \lambda_0$ instead of λ.

Let us consider the partial sums

$$Z_P(\lambda, N) = \sum_{n=0}^{P} i^n Z_n(N)(\lambda - \lambda_0)^n,$$

where $P = \sqrt{N}$. In this way we ensure that each perturbative order n in Z_P is much smaller than N when N is large. We can then use the estimates from the previous section, and from (6.3)

and (6.4) we obtain

$$\frac{Z_P(\lambda, N)}{Z_0(\lambda_0)} \approx \sum_{m=0}^{P/2} i^{2m}(\lambda - \lambda_0)^{2m} \frac{N^m}{m!} z_2^m + \sum_{m=0}^{(P-3)/2} i^{2m+3}(\lambda - \lambda_0)^{2m+3} z_3(z_2)^m \frac{N^m}{m!}$$

$$\approx \left[1 + iz_3(\lambda - \lambda_0)^3\right] \sum_{m=0}^{\infty} (-1)^m \frac{g^m}{m!} z_2^m = \left[1 + iz_3(\lambda - \lambda_0)^3\right] e^{-gz_2},$$

where

$$g = (\lambda - \lambda_0)^2 N. \tag{6.5}$$

Given that

$$Z(\lambda, N) = \lim_{P \to \infty} Z_P(\lambda, N)$$

for $|\lambda - \lambda_0| < r$, where r is the radius of convergence of

$$Z(\lambda, N) = \sum_{n=0}^{\infty} i^n (\lambda - \lambda_0)^n Z_n(N), \tag{6.6}$$

then

$$Z(\lambda, N) \approx Z_{\sqrt{N}}(\lambda, N)$$

for large N. Therefore

$$\frac{Z(\lambda, N)}{Z_0(\lambda_0)} \approx \left[1 + iz_3(\lambda - \lambda_0)^3\right] \exp\left(-gz_2\right), \tag{6.7}$$

for $\lambda \approx \lambda_0$, where $g = (\lambda - \lambda_0)^2 N$. In the limit $N \to \infty$, $\lambda \to \lambda_0$ and g constant we obtain

$$\frac{Z(\lambda, N)}{Z_0(\lambda_0)} \to \exp(-gz_2).$$

We can rewrite this result as

$$Z(M, g) = e^{-gz_2} Z_0(M), \tag{6.8}$$

where $Z(M, g)$ is the perturbed partition function in the dilute-gas limit. This value is triangulation independent and it depends on the renormalized coupling constant g.

Note that (6.7) can be rewritten as

$$\frac{Z^*(\lambda, g)}{Z_0(\lambda_0)} \approx \left[1 + iz_3(\lambda - \lambda_0)^3\right] \exp\left(-gz_2\right), \tag{6.9}$$

where we have changed the variable N to variable $g = N(\lambda - \lambda_0)^2$ and

$$Z^*(\lambda, g) = Z\left(\lambda, \frac{g}{(\lambda - \lambda_0)^2}\right). \tag{6.10}$$

The approximation (6.9) is valid for $\lambda \to \lambda_0$ and

$$\frac{g}{(\lambda - \lambda_0)^2} \to \infty.$$

The result (6.8) can be understood as the lowest order term in the asymptotic expansion (6.9) where g is a constant. However, one can have a more general situation where $g = f(\lambda - \lambda_0)$ such that

$$\frac{f(\lambda - \lambda_0)}{(\lambda - \lambda_0)^2} \to \infty, \tag{6.11}$$

for $\lambda \to \lambda_0$.

In this case

$$Z^*(\lambda, g) \approx e^{-z_2 f(\lambda - \lambda_0)} \left[1 + i z_3 (\lambda - \lambda_0)^3\right] Z_0(\lambda_0), \tag{6.12}$$

which opens the possibility that

$$Z^*(\lambda, g) \approx \left[Z_0(k)\right]_{k = \phi(\lambda)}, \tag{6.13}$$

where $Z_0(k)$ is the real number extension of the Crane–Yetter state sum and $k = \phi(\lambda)$ is the relation between k and λ.

In the case of a quantum $SU(2)$ group at a root of unity

$$Z_0(M, k) = e^{-i\pi s(M) R(k)}, \tag{6.14}$$

where $s(M)$ is the signature of M and $R(k) = k(2k + 1)/4(k + 2)$, see [23]. The value of λ which corresponds to k is conjectured to be $k = 4\pi/\lambda$, see [25], and this is an example of the function ϕ.

The relation $\lambda \propto 1/k$ could be checked by calculating the large-spin asymptotics of the quantum $15j$-symbol for the case of the root of unity, in analogy with the three-dimensional case, where by computing the asymptotics of the quantum $6j$-symbol one can find that $\lambda = 8\pi^2/k^2$, see [21]. The quantum $15j$-symbol asymptotics is not known, but for our purposes it is sufficient to know that $\lambda \to 0$ as $k \to \infty$.

Equations (6.12), (6.13) and (6.14) imply

$$f(\lambda - \lambda_0) \approx i \frac{\pi s(M)}{z_2} [h(\lambda) - h(\lambda_0)] + i \frac{z_3}{z_2} (\lambda - \lambda_0)^3, \tag{6.15}$$

where $h(\lambda) = R(\phi(\lambda))$. The solution (6.15) is consistent with the condition (6.11), since $h'(\lambda_0) \neq 0$. However, f has to be a complex function, although the original definition (6.5) suggests a real function. This means that λ has to take complex values in order for (6.13) to hold, i.e. we need to perform an analytic continuation of the function in (6.9).

Also note that the definition (6.6) and the fact that $Z_1(N) = 0$ (Theorem 1) imply that $Z'(\lambda_0, N) = 0$, but this does not imply that

$$\lim_{\lambda \to \lambda_0} \frac{dZ^*(\lambda, f(\lambda - \lambda_0))}{d\lambda} = 0$$

since $Z(\lambda, N)$ and $Z^*(\lambda, g)$ are different functions of λ due to (6.10) and $g = f(\lambda - \lambda_0)$. Therefore the approximation (6.13) is consistent. This also implies that we can define a triangulation independent $Z(\lambda, M)$ as the real number extension of the function $Z_0(k, M)$, see (6.14). Therefore

$$Z(\lambda, M) = Z_0(k, M)|_{k = \phi(\lambda)} = \tilde{Z}_0(\lambda, M). \tag{6.16}$$

7 Relation to $\langle F \wedge F \rangle$ theory

It is not difficult to see that the equations of motion for the action (2.1) are equivalent to the equations of motion for the action

$$\tilde{S} = \int_M \langle F \wedge F \rangle,$$

because the B field can be expressed algebraically as $B_I = -g_{IJ}F^J/(2\lambda)$.

At the path-integral level, this property is reflected by the following consideration. One can formally perform a Gaussian integration over the B field in the path integral (2.2), which gives the following path integral

$$Z = D(\lambda, M) \int \mathcal{D}A \exp\left(\frac{i}{4\lambda} \int_M \langle F \wedge F \rangle\right), \tag{7.1}$$

where $D(\lambda, M)$ is the factor coming from the determinant factor in the Gaussian integration formula.

More precisely, if we discretize M by using a triangulation T with n triangles, then the path integral (2.2) becomes a finite-dimensional integral

$$\int \prod_{l,I} dA_l^I \int \prod_{f,I} dB_f^I \exp\left(i \sum_{\Delta} Tr\left(B_\Delta F_f\right) + \frac{i\lambda}{5} \sum_\sigma \sum_{\Delta', \Delta'' \in \sigma} \langle B_{\Delta'} B_{\Delta''} \rangle\right). \tag{7.2}$$

The integral over the B variables in (7.2) can be written as

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{k,l} dB_{kl} e^{i\lambda(\mathcal{B}, Q\mathcal{B}) + i(\mathcal{B}, \mathcal{F})}, \tag{7.3}$$

where $m = \dim A$, $\mathcal{B} = (B_{11}, \ldots, B_{mn})$ and $\mathcal{F} = (F_{11}, \ldots, F_{mn})$ are vectors in $\mathbf{R}^{mn}$, (X, Y) is the usual scalar product in $\mathbf{R}^{mn}$ and Q is an $mn \times mn$ matrix.

The integral (7.3) can be defined as the analytic continuation $\lambda \to i\lambda$, $\mathcal{F} \to i\mathcal{F}$ of the formula

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \prod_{k,l} dB_{kl} e^{-\lambda(\mathcal{B}, Q\mathcal{B}) + (\mathcal{F}, \mathcal{B})} = \sqrt{\frac{\pi^{mn}}{\lambda^{mn} \det Q}} e^{(\mathcal{F}, Q^{-1}\mathcal{F})/4\lambda}, \tag{7.4}$$

so that when $n \to \infty$ such that the triangulations become arbitrarily fine, we can represent the limit as the path integral (7.1).

Since $\int_M \langle F \wedge F \rangle$ is a topological invariant of M, which is the second characteristic class number $c_2(M)$, see [7], we can write

$$Z(M, \lambda) = E(M, \lambda) e^{ic_2(M)/\lambda},$$

where

$$E(M, \lambda) = D(M, \lambda) \int \mathcal{D}A$$

and $D(M, \lambda)$ denotes the $(\lambda^{mn} \det Q)^{-1/2}$ factor from (7.4). As we have shown in the previous section, $Z(\lambda, M) = \tilde{Z}_0(M, \lambda)$, so that in the case of $SU(2)$

$$E(M, \lambda) = e^{-ic_2(M)/\lambda - i\pi s(M)h(\lambda)}. \tag{7.5}$$

Therefore one can calculate the volume of the moduli space of connections on a principal bundle provided that the relation $k = \phi(\lambda)$ is known for the corresponding quantum group.

8   Conclusions

The techniques developed for the 3d spin foam perturbation theory in [9] can be extended to the 4d case, and hence the 4d partition function has the same form as the corresponding 3d partition function in the dilute gas limit, see (6.8). The constant z_2 depends only on the group G and an integer, and z_2 is related to the second-order perturbative contribution, see Section 5. The constant z_2 appears because the constant z_1 vanishes for the same reason as in the 3d case, which is the vanishing of the tadpole spin network evaluation.

The result (6.8) implies that $Z(M, g)$ is not a new manifold invariant, but it is proportional to the Crane–Yetter invariant. Given that the renormalized coupling constant g is an arbitrary number, a more usefull way of representing our result is the asymptotic formula (6.12). This formula allowed us to conclude that $Z(M, \lambda)$ can be identified as the Crane–Yetter partition function evaluated at $k = \phi(\lambda)$, see (6.13) and (6.16). The formula (6.12) also applies to the spin foam perturbation expansion in 3d, where z_2 and z_3 are given as the state sums of the corresponding 3d graphs, see [9]. Therefore the formula (6.12) is the justification for the conjecture made in [9], where the triangulation independent $Z(\lambda, M)$ was identified with the Turaev–Viro partition function $Z_{TV}(M, k)$ for $k = 4\pi^2/\lambda^2$ in the $SU(2)$ case.

The relation (6.16) was useful for determining the volume of the moduli space of connections on the G-principal bundle for arbitrary values of λ, given that $Z(M, \lambda)$ is related to the path integral of the $\langle F \wedge F \rangle$ theory, see (7.5). However, it still remains to be proved the conjecture that $k \propto 1/\lambda$ for $G = SU(2)$, while for the other groups the function $k = \phi(\lambda)$ is not known.

Note that the result (6.8) depends on the existence of a class of triangulations of M which are arbitrarily fine, but having a finite degree of local complexity. As explained in Section 5 it is reasonable to assume that such a class exists, and can be constructed by considering the triangulations coming from the barycentric subdivisions of a fixed cubulation of M.

Our approach applies to Lie groups whose vector space of intertwiners $\Lambda \otimes \Lambda \to A$ is one-dimensional for each irreducible representation Λ. This is true for the $SU(2)$ and $SO(4)$ groups, but it is not true for the $SU(3)$ group. This can probably be fixed by adding extra information to the chain-mail link with insertions at the 3-valent vertices.

Also note that we only considered the $g_{IJ} \propto \delta_{IJ}$ case. This is sufficient for simple Lie groups, but in the case of semi-simple groups one can have non-trivial g_{IJ}. Especially interesting is the $SO(4)$ case, where $g_{IJ} \propto \epsilon_{abcd}$. In the general case one will have to work with spin networks which will have $L^{(\Lambda)}$ and g_{IJ} insertions, so that it would be very interesting to find out how to generalize the Chain–Mail formalism to this case.

One of the original motivations for developing a four-dimensional spin-foam perturbation theory was a possibility to obtain a nonperturbative definition of the four-dimensional Euclidean quantum gravity theory, see [10] and also [19, 20]. The reason for this is that general relativity with a cosmological constant is equivalent to a perturbed BF-theory given by the action (2.1), where $G = SO(4, 1)$ for a positive cosmological constant, while $G = SO(3, 2)$ for a negative cosmological constant and $g_{IJ} = \epsilon_{abcd}$ in both cases, see [19, 20]. However, the g_{IJ} in the gravity case is not a G-invariant tensor, since it is only invariant under a subgroup of G, which is the Lorentz group. Consequently this perturbed BF theory is not topological.

In the Euclidean gravity case one has $G = SO(5)$, and the invariant subgroup is $SO(4)$ since $g_{IJ} = \epsilon_{abcd}$. One can then formulate a spin foam perturbation theory along the lines of Section 3. However, the Chain–Mail techniques cannot be used, because g_{IJ} is not a G-invariant tensor and therefore one lacks an efficient way of calculating the perturbative contributions. In order to make further progress, a generalization of the Chain–Mail calculus has to be found in order to accommodate the case when g_{IJ} is invariant only under a subgroup of G.

Acknowledgments

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