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Classical and Quantum Dilogarithm Identities

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Abstract. Using the quantum cluster algebra formalism of Fock and Goncharov, we present several forms of quantum dilogarithm identities associated with periodicities in quantum cluster algebras, namely, the tropical, universal, and local forms. We then demonstrate how classical dilogarithm identities naturally emerge from quantum dilogarithm identities in local form in the semiclassical limit by applying the saddle point method.

Key words: dilogarithm; quantum dilogarithm; cluster algebra

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1 Introduction

1.1 Pentagon relations

The *Euler dilogarithm* $\mathrm{Li}_2(x)$ and its variant, the *Rogers dilogarithm* $L(x)$ have appeared in several branches of mathematics (e.g., [47, 41, 65]). See (2.1) and (2.2) for the definition. The most important property of the functions is the *pentagon relation*. For $L(x)$, it takes the following form

$$L(x) + L(y) = L\left(\frac{x(1-y)}{1-xy}\right) + L(xy) + L\left(\frac{y(1-x)}{1-xy}\right), \qquad 0 \le x, y \le 1. \tag{1.1}$$

The *quantum dilogarithm* appears also in several branches of mathematics, e.g., discrete quantum systems [2, 17, 15, 14, 13, 16, 3, 4, 37], hyperbolic geometry and Teichmüller theory [34, 35, 20, 30], quantum topology [33, 1], Donaldson–Thomas invariants [43, 44, 45, 39, 50, 49], string theory [26, 27, 9], representation theory of algebras [56], etc., and it accumulates much attention recently.

Actually, there are at least two variants of the quantum dilogarithm.

The first one $\mathbf{\Psi}_q(x)$, where q is a parameter, is simply called the *quantum dilogarithm* here. See (3.1) for the definition. The study of the function as 'quantum exponential' goes back to [58], but the recognition as 'quantum dilogarithm' was made more recently [17, 15]. The following properties explain why it is considered as a quantum analogue of the dilogarithm [17, 15, 36].

(*a*) *Asymptotic behavior: In the limit* $q \to 1^-$,

$$\mathbf{\Psi}_q(x) \sim \exp\left(-\frac{\mathrm{Li}_2(-x)}{2\log q}\right). \tag{1.2}$$

(*b*) *Pentagon relation: If* $UV = q^2VU$, *then*

$$\mathbf{\Psi}_q(U)\mathbf{\Psi}_q(V) = \mathbf{\Psi}_q(V)\mathbf{\Psi}_q\big(q^{-1}UV\big)\mathbf{\Psi}_q(U). \tag{1.3}$$

Moreover, in the limit $q \to 1^-$, *the relation* (1.3) *reduces to the relation* (1.1).

The second variant of the quantum dilogarithm $\mathbf{\Phi}_b(z)$, where b is a parameter, was introduced by [14, 13]. Here we call it *Faddeev's quantum dilogarithm* (also known as the *noncompact quantum dilogarithm*). See (4.2) for the definition. The function $\mathbf{\Phi}_b(z)$ also satisfies properties parallel to the ones for $\mathbf{\Psi}_q(x)$ [14, 13, 64, 16].

(*a*) *Asymptotic behavior: In the limit $b \to 0$,*

$$\mathbf{\Phi}_b\left(\frac{z}{2\pi b}\right) \sim \exp\left(-\frac{\mathrm{Li}_2(-e^z)}{2\pi b^2\sqrt{-1}}\right). \tag{1.4}$$

(*b*) *Pentagon relation: If $[\hat{P}, \hat{Q}] = (2\pi\sqrt{-1})^{-1}$, then*

$$\mathbf{\Phi}_b(\hat{Q})\mathbf{\Phi}_b(\hat{P}) = \mathbf{\Phi}_b(\hat{P})\mathbf{\Phi}_b(\hat{P}+\hat{Q})\mathbf{\Phi}_b(\hat{Q}). \tag{1.5}$$

Moreover, in the limit $b \to 0$, the relation (1.5) *reduces to the relation* (1.1).

Despite the appearance of the Euler dilogarithm $\mathrm{Li}_2(x)$ in (1.2) and (1.4), we have the Rogers dilogarithm $L(x)$ in (1.1) when we take the limits of (1.3) and (1.5). Namely, the limits of (1.3) and (1.5) are not so trivial as termwise limit. The two functions $L(x)$ and $\mathrm{Li}_2(x)$ differ by logarithms (see (2.3) and (2.4)), and the noncommutativity of U, V and P, Q 'magically' turns $\mathrm{Li}_2(x)$ into $L(x)$. To clarify this phenomenon in a (much) wider situation is the main theme of this paper.

1.2 Classical and quantum dilogarithm identities from cluster algebras

In [52], based on cluster algebras by [21, 24], an identity of the Rogers dilogarithm was associated with any period of seeds of a cluster algebra. It looks as follows

$$\sum_{t=1}^{L} \varepsilon_t L\left(\frac{y_{k_t}(t)^{\varepsilon_t}}{1+y_{k_t}(t)^{\varepsilon_t}}\right) = 0. \tag{1.6}$$

A precise account will be given in Section 2.5. Here we only mention that $\varepsilon_1, \dots, \varepsilon_L$ is a certain sequence of signs called the *tropical sign-sequence*. The simplest case of the cluster algebra of type A_2 yields the pentagon relation (1.1). Thus, it provides a vast generalization of (1.1). Here we call this family the *classical dilogarithm identities*.

Cluster algebras have the quantum counterparts, called *quantum cluster algebras* [7, 18]. Here we use the formulation by [18]. Any period of seeds of a classical (nonquantum) cluster algebra is also a period of seeds of the corresponding quantum cluster algebra and *vice versa*. Recently, in parallel with the classical case, an identity of the quantum dilogarithm $\mathbf{\Psi}_q(x)$ was associated with any period of seeds of a quantum cluster algebra by [39] (see also [56, 50, 49]). Moreover, as a pleasant surprise, we simultaneously obtain at least *four variations* of quantum dilogarithm identities as follows.

1) *Identities in tropical form for $\mathbf{\Psi}_q(x)$*. This is the form presented by [39], and it looks as follows

$$\mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_1\alpha_1})^{\varepsilon_1} \cdots \mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_L\alpha_L})^{\varepsilon_L} = 1. \tag{1.7}$$

A precise account will be given in Section 3.4. The simplest case of the quantum cluster algebra of type A_2 yields the pentagon relation (1.3).

2) *Identities in universal form for $\mathbf{\Psi}_q(x)$*. This is the form presented by [63, 62], and it looks as follows

$$\mathbf{\Psi}_q(Y_{k_L}(L)^{\varepsilon_L})^{\varepsilon_L} \cdots \mathbf{\Psi}_q(Y_{k_1}(1)^{\varepsilon_1})^{\varepsilon_1} = 1. \tag{1.8}$$

A precise account will be given in Section 3.5. The simplest case of type A_2 yields the new variation of the pentagon relation for $\mathbf{\Psi}_q(x)$ recently found by [63] with a suitable identification of variables. In general, they are obtained from the identities in tropical form (1.7) by the 'shuffle method' due to A.Yu. Volkov [62].

3) *Identities in tropical form for* $\mathbf{\Phi}_b(z)$. This is the counterpart of the form (1.7), and it looks as follows

$$\mathbf{\Phi}_b(\varepsilon_1\alpha_1\hat{\mathsf{D}})^{\varepsilon_1}\cdots\mathbf{\Phi}_b(\varepsilon_L\alpha_L\hat{\mathsf{D}})^{\varepsilon_L}=1. \tag{1.9}$$

A precise account will be given in Section 4.5. The simplest case of type A_2 yields the pentagon relation (1.5).

4) *Identities in local form for* $\mathbf{\Phi}_b(z)$. This is the form presented by [20, 30]. In general they are specified not only by a period of seeds but also by any choice of sign-sequence. The case of tropical sign-sequence is important for our purpose, and in that case it looks as follows

$$\mathbf{\Phi}_b(\varepsilon_1\hat{\mathsf{D}}_{k_1}(1))^{\varepsilon_1}\rho^*_{k_1,\varepsilon_1}\cdots\mathbf{\Phi}_b(\varepsilon_L\hat{\mathsf{D}}_{k_L}(L))^{\varepsilon_L}\rho^*_{k_L,\varepsilon_L}\nu^*=1. \tag{1.10}$$

A precise account will be given in Section 4.6.

We call these identities (1.7)–(1.10) together the *quantum dilogarithm identities.*

With these classical and the corresponding quantum dilogarithm identities, it is natural to ask *how the latter reduce to the former in the limit* $q\to 1$ *or* $b\to 0$. In this paper we address this question. More precisely, we demonstrate how in the limit $b\to 0$ the classical dilogarithm identities (1.6) emerge as the leading term in the asymptotic expansion from the quantum dilogarithm identities in the form (1.10), that is, *the local form with tropical sign-sequence.* To do it, we apply the saddle point method (see, e.g., [59, p. 95]), also known as the stationary phase method, à la [15]. In particular, we show transparently how the aforementioned logarithmic gap between the Euler and Rogers dilogarithms is filled. See Section 5.5 for the bottom line.

Three remarks follow. First, the variables of quantum cluster algebras admit a natural quantum-mechanical formulation, where the limit $b\to 0$ corresponds to the limit $\hbar\to 0$ of the Planck constant $\hbar$. See (4.7) and (4.10). Furthermore, the classical dilogarithm identities appear as the leading terms of the quantum dilogarithm identities for the asymptotic expansion in $\hbar$. Therefore, following the standard terminology of quantum mechanics, we call the limiting procedure the *semiclassical limit.*

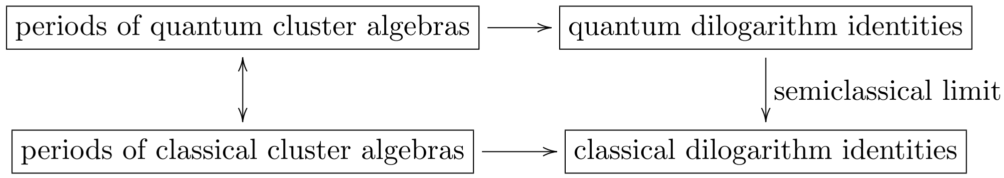
Second, even though our treatment of the saddle point method here is standard in quantum mechanics, we admit and stress that we did not pursue the complete, functional-analytic rigorousness. Namely, the validity of the method in total and specific details, for example, the uniqueness of the solution of the saddle point equations, the specification of the integration contour through the saddle point, etc., are not argued. Our objective here is not to prove the classical dilogarithm identities by this method, but to make a direct bridge between the classical and the quantum dilogarithm identities.

Third, there is actually the *fifth* form of quantum dilogarithm identities, namely, the *identities in local form for* $\mathbf{\Psi}_q(x)$ *with tropical sign-sequence.* This is the counterpart of the form (1.10), and it looks as follows

$$\mathbf{\Psi}_q(\hat{\mathsf{Y}}_{k_1}(1)^{\varepsilon_1})^{\varepsilon_1}\rho^*_{k_1,\varepsilon_1}\cdots\mathbf{\Psi}_q(\hat{\mathsf{Y}}_{k_L}(L)^{\varepsilon_L})^{\varepsilon_L}\rho^*_{k_L,\varepsilon_L}\nu^*=1. \tag{1.11}$$

One can also obtain the classical dilogarithm identities (1.6) from (1.11) in the semiclassical limit. However, the relevant differential operators are not self-adjoint. Therefore, the semiclassical limit is more natural for $\mathbf{\Phi}_b(z)$ *from the quantum-mechanical point of view.* For the completeness, we also present it in Appendix A.

In summary, our result establishes the following scheme



The organization of the paper is the following. In Section 2 we present the classical dilogarithm identities obtained from periods of cluster algebras. In Section 3 we present the quantum dilogarithm identities for the quantum dilogarithm $\mathbf{\Psi}_q(x)$. In Section 4 we present the quantum dilogarithm identities for the Faddeev's quantum dilogarithm $\mathbf{\Phi}_b(z)$. In Section 5 we demonstrate how Rogers dilogarithm identities naturally emerge from the quantum dilogarithm identities in local form in the semiclassical limit by applying the saddle point method. This is the main part of the paper. In Appendix A, we present the quantum dilogarithm identities in local form for $\mathbf{\Psi}_q(x)$. Then, we derive the classical dilogarithm identities from them in the semiclassical limit.

2 Classical dilogarithm identities

In this section we present the classical dilogarithm identities obtained from periods of cluster algebras following [52].

2.1 Euler and Rogers dilogarithms

Let $\mathrm{Li}_2(x)$ and $L(x)$ be the Euler and Rogers dilogarithm functions, respectively [47],

$$\mathrm{Li}_2(x) = -\int_0^x \left\{ \frac{\log(1-y)}{y} \right\} dy, \qquad x \le 1, \tag{2.1}$$

$$L(x) = -\frac{1}{2} \int_0^x \left\{ \frac{\log(1-y)}{y} + \frac{\log y}{1-y} \right\} dy, \qquad 0 \le x \le 1. \tag{2.2}$$

Two functions are related as follows

$$L(x) = \mathrm{Li}_2(x) + \frac{1}{2} \log x \log(1-x), \qquad 0 \le x \le 1, \tag{2.3}$$

$$-L\left(\frac{x}{1+x}\right) = \mathrm{Li}_2(-x) + \frac{1}{2} \log x \log(1+x), \qquad 0 \le x. \tag{2.4}$$

The function $L(x)$ satisfies the property (1.1) and also the following ones

$$L(0) = 0, \qquad L(1) = \frac{\pi^2}{6},$$

$$L(x) + L(1-x) = \frac{\pi^2}{6}, \qquad 0 \le x \le 1. \tag{2.5}$$

2.2 y-variables in cluster algebras

In this subsection we recall some definitions and properties of the cluster algebras with coefficients [21, 22], following the convention of [24] with slight change of notation and terminology. Here, we concentrate on the 'coefficients' or 'y-variables', since we do not explicitly use the 'cluster variables' or 'x-variables'.

Let I be a finite set, and fix the *initial y-seed* (B, y), which is a pair of a skew-symmetric (integer) matrix $B = (b_{ij})_{i,j\in I}$ and an I-tuple of commutative variables $y = (y_i)_{i\in I}$. Let $\mathbb{P}_{\mathrm{univ}}(y)$ be the *universal semifield of* y, which consists of all nonzero rational functions of y having subtraction-free expressions. It is a semifield, i.e., the Abelian multiplicative group with addition (but not with subtraction), by the ordinary multiplication and addition of rational functions.

Let (B', y') be any pair of a skew-symmetric matrix $B' = (b'_{ij})_{i,j\in I}$ and an I-tuple $y' = (y'_i)_{i\in I}$ with $y'_i \in \mathbb{P}_{\mathrm{univ}}(y)$. For each $k \in I$, we define another pair $(B'', y'') = \mu_k(B', y')$ of a skew-symmetric matrix $B'' = (b''_{ij})_{i,j\in I}$ and an I-tuple $y'' = (y''_i)_{i\in I}$ with $y''_i \in \mathbb{P}_{\mathrm{univ}}(y)$, called the *mutation of* (B', y') *at* k, by the following rule:

(*i*) *Mutation of matrix:*

$$b''_{ij} = \begin{cases} -b'_{ij}, & i = k \text{ or } j = k, \\ b'_{ij} + [-b'_{ik}]_+ b'_{kj} + b'_{ik}[b'_{kj}]_+ & \\ = b'_{ij} + [b'_{ik}]_+ b'_{kj} + b'_{ik}[-b'_{kj}]_+, & \text{otherwise.} \end{cases} \tag{2.6}$$

(*ii*) *Exchange relation of y-variables:*

$$y''_i = \begin{cases} {y'_k}^{-1}, & i = k, \\ y'_i {y'_k}^{[b'_{ki}]_+}(1 + y'_k)^{-b'_{ki}} & \\ = y'_i {y'_k}^{[-b'_{ki}]_+}(1 + {y'_k}^{-1})^{-b'_{ki}}, & i \neq k. \end{cases} \tag{2.7}$$

Here, $[a]_+ = a$ for $a \geq 0$ and 0 for $a < 0$. Starting from the initial y-seed (B, y), repeat the mutations. Each resulting pair (B', y') is called a *y-seed of* (B, y).

Remark 2.1. The convention of [24] adopted here is related with the convention of [18, 20, 39] by exchanging the matrix B' with its transposition.

2.3 Tropical y-variables

Let $\mathbb{P}_{\mathrm{trop}}(y)$ be the *tropical semifield* of $y = (y_i)_{i\in I}$, which is the Abelian multiplicative group freely generated by y endowed with the addition $\oplus$

$$\prod_{i\in I} y_i^{a_i} \oplus \prod_{i\in I} y_i^{b_i} = \prod_{i\in I} y_i^{\min(a_i,b_i)}.$$

There is a canonical surjective semifield homomorphism $\pi_{\mathbf{T}}$ (the *tropical evaluation*) from $\mathbb{P}_{\mathrm{univ}}(y)$ to $\mathbb{P}_{\mathrm{trop}}(y)$ defined by $\pi_{\mathbf{T}}(y_i) = y_i$ and $\pi_{\mathbf{T}}(\alpha) = 1$ ($\alpha \in \mathbb{Q}_+$). For any y-variable y'_i of a y-seed (B', y') of (B, y), let us write $[y'_i] := \pi_{\mathbf{T}}(y'_i)$ for simplicity. We call $[y'_i]$'s the *tropical y-variables* (the *principal coefficients* in [24]). They satisfy the exchange relation (2.7) by replacing y'_i and $+$ with $[y'_i]$ and $\oplus$.

We say that a Laurent monomial $[y'_i]$ is *positive* (resp. *negative*) if it is not 1 and all the exponents are nonnegative (resp. nonpositive).

Proposition 2.2 (Sign-coherence [11, 55, 48])**.** *For any y-seed (B', y') of (B, y), the Laurent monomial $[y'_i]$ in y is either positive or negative.*

Based on Proposition 2.2, for any y-seed (B', y') of (B, y), let $\varepsilon(y'_i)$ be 1 (resp. -1) if $[y'_i]$ is positive (resp. negative). We call it the *tropical sign of* y'_i by identifying ± 1 with the signs $\pm$.

Using the tropical sign $\varepsilon(y'_i)$, the tropical exchange relation is written as follows:

$$[y''_i] = \begin{cases} [y'_k]^{-1}, & i = k, \\ [y'_i][y'_k]^{[\varepsilon(y'_k) b'_{ki}]_+}, & i \neq k. \end{cases} \tag{2.8}$$

2.4 Periodicity of y-seeds

For any I-sequence $(k_1, k_2, \dots, k_L)$, set $(B(1), y(1)) := (B, y)$, and consider the sequence of mutations of y-seeds of (B, y),

$$(B(1), y(1)) \overset{\mu_{k_1}}{\longleftrightarrow} (B(2), y(2)) \overset{\mu_{k_2}}{\longleftrightarrow} \cdots \overset{\mu_{k_L}}{\longleftrightarrow} (B(L+1), y(L+1)). \tag{2.9}$$

Definition 2.3. Let $\nu : I \to I$ be any bijection. We say that an I-sequence $(k_1, k_2, \dots, k_L)$ is a *ν-period of* (B, y) if the following holds

$$b_{\nu(i)\nu(j)}(L+1) = b_{ij}(1), \qquad y_{\nu(i)}(L+1) = y_i(1), \qquad i, j \in I. \tag{2.10}$$

See [23, 38, 31, 32, 53] for various examples of periodicity.

Remarkably, the periodicity of y-seeds reduces to the periodicity of tropical y-variables, which is much simpler.

Proposition 2.4 ([31, 55])**.** *The condition* (2.10) *holds if and only if*

$$[y_{\nu(i)}(L+1)] = [y_i(1)], \qquad i \in I.$$

For $\tilde{I} \supset I$ and a skew-symmetric matrix $\tilde{B} = (\tilde{b}_{ij})_{i,j \in \tilde{I}}$, we say that $\tilde{B}$ *is an $\tilde{I}$-extension of* B if $\tilde{b}_{ij} = b_{ij}$ for any $i, j \in I$.

Example 2.5. For any skew-symmetric matrix B with index set I, which may be degenerate, let $I' = \{i' \mid i \in I\}$ be a copy of I and let $\tilde{I} = I \sqcup I'$. Define the skew-symmetric matrix $\tilde{B} = (\tilde{b}_{ij})_{i,j \in \tilde{I}}$ by

$$\tilde{b}_{ij} = \begin{cases} b_{ij}, & i, j \in I, \\ 1, & j \in I, i = j', \\ -1, & i \in I, j = i', \\ 0, & \text{otherwise}. \end{cases}$$

Then, $\tilde{B}$ is an $\tilde{I}$-extension of B; furthermore, $\tilde{B}$ is *nondegenerate*. The matrix $\tilde{B}$ is called the *principal extension of* B.

Proposition 2.6 (Extension Theorem [52])**.** *Suppose that an I-sequence $(k_1, \dots, k_L)$ is a ν-period of (B, y). Then, for any $\tilde{I}$-extension $\tilde{B}$ of B, $(k_1, \dots, k_L)$ is also a ν-period of $(\tilde{B}, \tilde{y})$.*

In Proposition 2.6 the periodicity of the 'external' variables $\tilde{y}_i$ $(i \in \tilde{I} \setminus I)$ is nontrivial.

2.5 Classical dilogarithm identities

Let $(k_1, \dots, k_L)$ be a ν-period of (B, y). For the mutation sequence (2.9), let N_+ and N_- be the numbers of the positive and negative monomials among $[y_{k_1}(1)]$, $\dots$, $[y_{k_L}(L)]$, respectively, so that $N_+ + N_- = L$.

The following is a generalization of the identities [28, 29, 25, 10, 51, 31, 32, 53] originated from the central charge identities in conformal field theory [40, 42, 5, 46].

Theorem 2.7 (Classical dilogarithm identities [52])**.** *The following identities hold*

$$\frac{6}{\pi^2} \sum_{t=1}^{L} L\left(\frac{y_{k_t}(t)}{1 + y_{k_t}(t)}\right) = N_-, \tag{2.11}$$

$$\frac{6}{\pi^2} \sum_{t=1}^{L} L\left(\frac{1}{1 + y_{k_t}(t)}\right) = N_+, \tag{2.12}$$

where the initial variables y_i $(i \in I)$ arbitrarily take values in positive real numbers.

Two identities (2.11) and (2.12) are equivalent due to (2.5).

Remark 2.8. In [52, Theorems 4.3 & 6.4], Proposition 2.6 and Theorem 2.7 are stated only for $\nu = \mathrm{id}$. However, the proofs therein are also applicable to a general ν.

We introduce the sign-sequence $(\varepsilon_1, \dots, \varepsilon_L)$ so that ε_t is the tropical sign of $y_{k_t}(t)$. We call it the *tropical sign-sequence* of (2.9). Using (2.5), one can also rewrite (2.11) and (2.12) in the following way.

Theorem 2.9. *For the tropical sign-sequence* $(\varepsilon_1, \dots, \varepsilon_L)$*,*

$$\sum_{t=1}^{L} \varepsilon_t L\left(\frac{y_{k_t}(t)^{\varepsilon_t}}{1 + y_{k_t}(t)^{\varepsilon_t}}\right) = 0. \tag{2.13}$$

2.6 Example of type A_1

Consider the simplest case, $I = \{1\}$ and

$$B = (0).$$

Let $(k_1, k_2) = (1, 1)$, and consider the sequence of mutations of y-seeds of (B, y),

$$(B(1), y(1)) \overset{\mu_1}{\longleftrightarrow} (B(2), y(2)) \overset{\mu_1}{\longleftrightarrow} (B(3), y(3)).$$

Then,

$$y_1(1) = y_1, \qquad y_1(2) = y_1^{-1}, \qquad y_1(3) = y_1.$$

Thus, (k_1, k_2) is a ν-period with $\nu = \mathrm{id}$, which is nothing but the involution property of the mutation. Also

$$[y_1(1)] = y_1, \qquad [y_1(2)] = y_1^{-1}, \qquad [y_1(3)] = y_1$$

and

$$\varepsilon_1 = 1, \qquad \varepsilon_2 = -1.$$

The classical dilogarithm identity (2.13) is

$$L\left(\frac{y_1}{1 + y_1}\right) - L\left(\frac{y_1}{1 + y_1}\right) = 0,$$

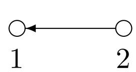
which is trivial.

2.7 Example of type A_2

Consider the simplest nontrivial case

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which is also represented by the quiver of type A_2



Let $(k_1,\dots,k_5)=(1,2,1,2,1)$, and consider the sequence of mutations of y-seeds of (B,y),

$$(B(1),y(1)) \overset{\mu_1}{\longleftrightarrow} (B(2),y(2)) \overset{\mu_2}{\longleftrightarrow} \cdots \overset{\mu_1}{\longleftrightarrow} (B(6),y(6)).$$

Then,

$$\begin{cases} y_1(1)=y_1, \\ y_2(1)=y_2, \end{cases} \qquad \begin{cases} y_1(2)=y_1^{-1}, \\ y_2(2)=y_2(1+y_1), \end{cases} \qquad \begin{cases} y_1(3)=y_1^{-1}(1+y_2+y_1y_2), \\ y_2(3)=y_2^{-1}(1+y_1)^{-1}, \end{cases}$$
$$\begin{cases} y_1(4)=y_1(1+y_2+y_1y_2)^{-1}, \\ y_2(4)=y_1^{-1}y_2^{-1}(1+y_2), \end{cases} \qquad \begin{cases} y_1(5)=y_2^{-1}, \\ y_2(5)=y_1y_2(1+y_2)^{-1}, \end{cases} \qquad \begin{cases} y_1(6)=y_2, \\ y_2(6)=y_1. \end{cases}$$

Thus, $(k_1,\dots,k_5)$ is a ν-period, where $\nu=(12)$ is the permutation of 1 and 2. Also

$$[y_1(1)]=y_1, \qquad [y_2(2)]=y_2, \qquad [y_1(3)]=y_1^{-1}, \qquad [y_2(4)]=y_1^{-1}y_2^{-1}, \qquad [y_1(5)]=y_2^{-1},$$

and

$$\varepsilon_1=\varepsilon_2=1, \qquad \varepsilon_3=\varepsilon_4=\varepsilon_5=-1.$$

The classical dilogarithm identity (2.13) is

$$L\left(\frac{y_1}{1+y_1}\right)+L\left(\frac{y_2(1+y_1)}{1+y_2+y_1y_2}\right) \\ -L\left(\frac{y_1}{(1+y_1)(1+y_2)}\right)-L\left(\frac{y_1y_2}{1+y_2+y_1y_2}\right)-L\left(\frac{y_2}{1+y_2}\right)=0.$$

By identifying $x=y_1/(1+y_1)$, $y=y_2(1+y_1)/(1+y_2+y_1y_2)$, it coincides with the pentagon relation (1.1).

3 Quantum dilogarithm identities for $\mathbf{\Psi}_q(x)$

In this section we present the quantum dilogarithm identities for $\mathbf{\Psi}_q(x)$. The content heavily relies on [18, 20, 39].

3.1 Quantum dilogarithm

Following [17, 15], define the *quantum dilogarithm* $\mathbf{\Psi}_q(x)$, for $|q|<1$ and $x\in\mathbb{C}$, by

$$\mathbf{\Psi}_q(x)=\sum_{n=0}^{\infty}\frac{(-qx)^n}{(q^2;q^2)_n}=\frac{1}{(-qx;q^2)_\infty}, \qquad (a;q)_n=\prod_{k=0}^{n-1}\left(1-aq^k\right). \tag{3.1}$$

We have the properties (1.2) and (1.3), and also the following recursion relations

$$\mathbf{\Psi}_q\left(q^{\pm 2}x\right)=\left(1+q^{\pm 1}x\right)^{\pm 1}\mathbf{\Psi}_q(x). \tag{3.2}$$

3.2 Quantum y-variables

So far, two kinds of *quantum cluster algebras* are known in the literature. The first one was introduced earlier by [7], where the x-variables are noncommutative and the y-variables are noncommutative but restricted to the tropical one. The second one was introduced by [18, 20], where the y-variables are noncommutative and the universal one but x-variables are commutative. For the relation between them, see [20, Section 2.7] and also [60]. Here we use the second one by [18, 20], and concentrate on the quantum y-variables only.

Let I be a finite set, and q be an indeterminate. We start from the *initial quantum y-seed* (B, Y), which is a pair of a skew-symmetric (integer) matrix $B = (b_{ij})_{i,j\in I}$ and an I-tuple of *noncommutative* variables $Y = (Y_i)_{i\in I}$ with

$$Y_i Y_j = q^{2b_{ji}} Y_j Y_i. \tag{3.3}$$

Accordingly, let $\mathbb{T}(B, \mathsf{Y})$ be the associated *quantum torus*, which is the $\mathbb{Q}(q)$-algebra generated by the noncommutative variables Y^α $(\alpha \in \mathbb{Z}^I)$ with the relations

$$q^{\langle\alpha,\beta\rangle}\mathsf{Y}^\alpha\mathsf{Y}^\beta = \mathsf{Y}^{\alpha+\beta}, \qquad \langle\alpha,\beta\rangle = -\langle\beta,\alpha\rangle = {}^t\alpha B\beta.$$

Thus, we have $\mathsf{Y}^\alpha\mathsf{Y}^\beta = q^{2\langle\beta,\alpha\rangle}\mathsf{Y}^\beta\mathsf{Y}^\alpha$. Set $\mathsf{Y}_i := \mathsf{Y}^{e_i}$ for the standard unit vector e_i $(i \in I)$. Then, by identifying Y_i with Y_i, we recover (3.3).

Following [39], let $\mathbb{A}(B, \mathsf{Y})$ be the associated *quantum affine space*, which is the $\mathbb{Q}(q)$-subalgebra of $\mathbb{T}(B, \mathsf{Y})$ generated by Y^α's with $\alpha \in (\mathbb{Z}_{\geq 0})^I$. Let $\hat{\mathbb{A}}(B, \mathsf{Y})$ be the completion of $\mathbb{A}(B, \mathsf{Y})$, which consists of the noncommutative formal power series of Y_i's. The *complete quantum torus* $\hat{\mathbb{T}}(B, \mathsf{Y})$ is the localization of $\hat{\mathbb{A}}(B, \mathsf{Y})$ at Y^α's with $\alpha \in (\mathbb{Z}_{\geq 0})^I$. Let $\mathsf{Frac}(\mathbb{A}(B, \mathsf{Y}))$ be the noncommutative fraction field of the algebra $\mathbb{A}(B, \mathsf{Y})$, which is viewed as a subskewfield of $\hat{\mathbb{T}}(B, \mathsf{Y})$ [7].

Let (B', Y') be any pair of a skew-symmetric matrix $B' = (b'_{ij})_{i,j\in I}$ and an I-tuple $Y' = (Y'_i)_{i\in I}$ with $Y'_i \in \mathsf{Frac}(\mathbb{A}(B, \mathsf{Y}))$ satisfying the relations (3.3) where everything is primed. For each $k \in I$, we define another same kind of pair $(B'', Y'') = \mu_k(B', Y')$, called the *mutation of* (B', Y') *at* k, where $B'' = (b''_{ij})_{i,j\in I}$, which is the same as (2.6), and $Y'' = (Y''_i)_{i\in I}$, $Y''_i \in \mathsf{Frac}(\mathbb{A}(B, \mathsf{Y}))$ is given by the following rule [18, 20]:

Exchange relation of quantum y-variables

$$Y''_i = \begin{cases} Y'^{-1}_k, & i = k,\\ q^{b'_{ik}[b'_{ki}]_+} Y'_i Y'^{[b'_{ki}]_+}_k \displaystyle\prod_{m=1}^{|b'_{ki}|} \left(1 + q^{-\mathrm{sgn}(b'_{ki})(2m-1)} Y'_k\right)^{-\mathrm{sgn}(b'_{ki})} & \\ = q^{b'_{ik}[-b'_{ki}]_+} Y'_i Y'^{[-b'_{ki}]_+}_k \displaystyle\prod_{m=1}^{|b'_{ki}|} \left(1 + q^{\mathrm{sgn}(b'_{ki})(2m-1)} Y'^{-1}_k\right)^{-\mathrm{sgn}(b'_{ki})}, & i \neq k. \end{cases} \tag{3.4}$$

Formally setting $q = 1$, it reduces to (2.7).

Now, starting from the quantum initial y-seed (B, Y), repeat the mutations. Each resulting pair (B', Y') is called a *quantum y-seed of* (B, Y).

3.3 Decomposition of mutations

Let (B', Y') and (B'', Y'') be a pair of quantum y-seeds of (B, Y) such that $(B'', Y'') = \mu_k(B', Y')$. Following [18], we decompose the mutation (3.4) into two parts, namely, the monomial part and the automorphism part.

(a) Monomial part. Define the isomorphisms $\tau_{k,\varepsilon}$ for each $\varepsilon = \pm 1$ by

$$\begin{aligned} \tau_{k,\varepsilon} :\ & \mathsf{Frac}(\mathbb{A}(B'', \mathsf{Y}'')) \to \mathsf{Frac}(\mathbb{A}(B', \mathsf{Y}')),\\ & \mathsf{Y}''_i \mapsto \begin{cases} \mathsf{Y}'^{-1}_k, & i = k,\\ \mathsf{Y}'^{e_i + [\varepsilon b'_{ki}]_+ e_k}, & i \neq k. \end{cases} \end{aligned} \tag{3.5}$$

The dependence of the map $\tau_{k,\varepsilon}$ on its source (B'', Y'') and target (B', Y') is suppressed for the notational simplicity and should be understood in the context. One can check that they are indeed homomorphisms using (2.6); furthermore, they are isomorphisms because the inverses are given by $\tau_{k,-\varepsilon}$ with b'_{ki} being replaced by $b''_{ki} = -b'_{ki}$ in (3.5). Compare with the exchange relation of tropical y-variables (2.8). Also note that, in $\mathbb{A}(B', \mathsf{Y}')$,

$$\mathsf{Y}'^{e_i + [\varepsilon b'_{ki}]_+ e_k} = q^{b'_{ik}[\varepsilon b'_{ki}]_+} \mathsf{Y}'_i \mathsf{Y}'^{[\varepsilon b'_{ki}]_+}_k. \tag{3.6}$$

(b) Automorphism part. It follows from (3.2) that, for $\mathsf{Y}'_k \in \mathbb{A}(B', \mathsf{Y}')$, the adjoint action $\mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}'_k))$ is defined on $\mathsf{Frac}(\mathbb{A}(B', \mathsf{Y}'))$ by

$$\begin{aligned}\mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}'_k))(\mathsf{Y}'_i) &:= \mathbf{\Psi}_q(\mathsf{Y}'_k)\mathsf{Y}'_i\mathbf{\Psi}_q(\mathsf{Y}'_k)^{-1} = \mathsf{Y}'_i\mathbf{\Psi}_q\big(q^{-2b'_{ki}}\mathsf{Y}'_k\big)\mathbf{\Psi}_q(\mathsf{Y}'_k)^{-1} \\ &= \mathsf{Y}'_i \prod_{m=1}^{|b'_{ki}|} \big(1 + q^{-\operatorname{sgn}(b'_{ki})(2m-1)}\mathsf{Y}'_k\big)^{-\operatorname{sgn}(b'_{ki})},\end{aligned} \tag{3.7}$$

and similarly,

$$\begin{aligned}\mathsf{Ad}\big(\mathbf{\Psi}_q(\mathsf{Y}'^{-1}_k)^{-1}\big)(\mathsf{Y}'_i) &:= \mathbf{\Psi}_q\big(\mathsf{Y}'^{-1}_k\big)^{-1}\mathsf{Y}'_i\mathbf{\Psi}_q\big(\mathsf{Y}'^{-1}_k\big) = \mathsf{Y}'_i\mathbf{\Psi}_q\big(q^{2b'_{ki}}\mathsf{Y}'^{-1}_k\big)^{-1}\mathbf{\Psi}_q\big(\mathsf{Y}'^{-1}_k\big) \\ &= \mathsf{Y}'_i \prod_{m=1}^{|b'_{ki}|} \big(1 + q^{\operatorname{sgn}(b'_{ki})(2m-1)}\mathsf{Y}'^{-1}_k\big)^{-\operatorname{sgn}(b'_{ki})}.\end{aligned} \tag{3.8}$$

By combining (3.5)–(3.8), we have the following intrinsic description of the exchange relation (3.4).

Proposition 3.1 ([18, 39])**.** *We have the equality*

$$\big(\mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}'_k))\tau_{k,+}\big)(\mathsf{Y}''_i) = \big(\mathsf{Ad}\big(\mathbf{\Psi}_q\big(\mathsf{Y}'^{-1}_k\big)^{-1}\big)\tau_{k,-}\big)(\mathsf{Y}''_i), \tag{3.9}$$

and either side of (3.9) *coincides with the right hand side of the exchange relation* (3.4) *with Y'_i replaced with Y'_i.*

Remark 3.2. In [18] the case $\varepsilon = 1$ was employed as the definition of the exchange relation. The importance of the use of *both* the descriptions by $\varepsilon = \pm 1$ for quantum dilogarithm identities was clarified by [39]. We use this refinement throughout the paper.

Example 3.3. Consider the sequence of mutations of quantum y-seeds of (B, Y),

$$(B(1), Y(1)) := (B, Y) \overset{\mu_{k_1}}{\longleftrightarrow} (B(2), Y(2)) \overset{\mu_{k_2}}{\longleftrightarrow} (B(3), Y(3)).$$

Then, for any sign-sequence $(\varepsilon_1, \varepsilon_2)$, we have

$$\begin{aligned}Y_i(2) &= \big(\mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}_{k_1}(1)^{\varepsilon_1})^{\varepsilon_1})\tau_{k_1,\varepsilon_1}\big)(\mathsf{Y}_i(2)), \\ Y_i(3) &= \big(\mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}_{k_1}(1)^{\varepsilon_1})^{\varepsilon_1})\tau_{k_1,\varepsilon_1}\mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}_{k_2}(2)^{\varepsilon_2})^{\varepsilon_2})\tau_{k_2,\varepsilon_2}\big)(\mathsf{Y}_i(3)).\end{aligned} \tag{3.10}$$

3.4 Quantum dilogarithm identities in tropical form

For any I-sequence $(k_1, k_2, \dots, k_L)$, set $(B(1), Y(1)) := (B, Y)$, and consider the sequence of mutations of quantum y-seeds of (B, Y),

$$(B(1), Y(1)) \overset{\mu_{k_1}}{\longleftrightarrow} (B(2), Y(2)) \overset{\mu_{k_2}}{\longleftrightarrow} \cdots \overset{\mu_{k_L}}{\longleftrightarrow} (B(L+1), Y(L+1)). \tag{3.11}$$

We say that an I-sequence $(k_1, k_2, \dots, k_L)$ is a *ν-period of* (B, Y) if the following condition holds for the sequence (3.11)

$$b_{\nu(i)\nu(j)}(L+1) = b_{ij}(1), \qquad Y_{\nu(i)}(L+1) = Y_i(1), \qquad i, j \in I. \tag{3.12}$$

The following theorem, essentially due to [20], tells that the periodicities of quantum y-seeds and (classical) y-seeds coincide.

Proposition 3.4 ([20])**.** *The condition* (2.10) *holds for the sequence* (2.9) *if and only if the condition* (3.12) *holds for the sequence* (3.11)*.*

Proof. The 'if' part immediately follows by formally setting $q = 1$ in the exchange relation (3.4) for quantum y-seeds. The 'only if' part is proved by [20, Lemma 2.22] using [7, Theorem 6.1], when the matrix B is nondegenerate. When B is degenerate, thanks to Example 2.5 and Proposition 2.6, it is reduced to the nondegenerate case. ∎

Now suppose that $(k_1, k_2, \dots, k_L)$ is a ν-period of (B, Y). Due to the periodicity of $B_{\nu(i)\nu(j)}(L + 1) = B_{ij}(1)$, we have the isomorphism

$$\begin{aligned}&\mathsf{Frac}(\mathbb{A}(B(1), \mathsf{Y}(1))) \to \mathsf{Frac}(\mathbb{A}(B(L + 1), \mathsf{Y}(L + 1))),\\ &\mathsf{Y}_i(1) \mapsto \mathsf{Y}_{\nu(i)}(L + 1).\end{aligned}$$

Let ν also denote this isomorphism by abusing the notation. For any sign-sequence $(\varepsilon_1, \dots, \varepsilon_k)$, the periodicity for (3.11) is expressed as follows [39].

$$\mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}_{k_1}(1)^{\varepsilon_1})^{\varepsilon_1})\tau_{k_1,\varepsilon_1} \cdots \mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}_{k_L}(L)^{\varepsilon_L})^{\varepsilon_L})\tau_{k_L,\varepsilon_L}\nu = \mathrm{id}_{\mathsf{Frac}(\mathbb{A}(B(1),\mathsf{Y}(1)))}. \tag{3.13}$$

To extract the identity involving only the quantum dilogarithm $\boldsymbol{\Psi}_q(y)$, we have to set $(\varepsilon_1, \dots, \varepsilon_L)$ to be the tropical sign-sequence of (2.9). (We also call it the tropical sign-sequence of (3.11).)

The following theorem is due to [39, Theorem 5.6]. The case of simply laced finite type for certain periods was obtained by [56] with a different method. See also [50, Comments (a), p. 5], [49] for the connection to the Donaldson–Thomas invariants. We include a proof because the argument therein will be used also later.

Theorem 3.5 (Quantum dilogarithm identities in tropical form [56, 39])**.** *Suppose that* $(k_1, \dots, k_L)$ *is a* ν*-period of* (B, Y)*, and let* $(\varepsilon_1, \dots, \varepsilon_L)$ *be the tropical sign-sequence of* (3.11)*. Let* $y_i(t)$ *be the corresponding* (*classical*) y*-variables in* (2.9)*, and let* $\alpha_t \in \mathbb{Z}^I$ $(t = 1, \dots, L)$ *be the vectors such that* $[y_{k_t}(t)] = y^{\alpha_t}$*.* (*The vector* α_t *is called the* **c***-vector of* $y_{k_t}(t)$ *in* [24].) *Then, the following identity holds*

$$\boldsymbol{\Psi}_q(\mathsf{Y}^{\varepsilon_1\alpha_1})^{\varepsilon_1} \cdots \boldsymbol{\Psi}_q(\mathsf{Y}^{\varepsilon_L\alpha_L})^{\varepsilon_L} = 1, \tag{3.14}$$

where $\mathsf{Y}^{\varepsilon_1\alpha_1}, \dots, \mathsf{Y}^{\varepsilon_L\alpha_L} \in \mathbb{A}(B, \mathsf{Y})$*.*

Proof. For the choice of ε_t above, the periodicity of tropical y-variables implies

$$\tau_{k_1,\varepsilon_1} \cdots \tau_{k_L,\varepsilon_L}\nu = \mathrm{id}. \tag{3.15}$$

Also note that $\mathsf{Y}_{k_1}(1)^{\varepsilon_1} = \mathsf{Y}^{\varepsilon_1\alpha_1}$ with $\alpha_1 = e_{k_1}$ and $\varepsilon_1 = 1$. Then, push out all τ_{k_t,ε_t}'s to the right in (3.13) as follows

$$\begin{aligned}&\mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}^{\varepsilon_1\alpha_1})^{\varepsilon_1})\tau_{k_1,\varepsilon_1}\mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}_{k_2}(2)^{\varepsilon_2})^{\varepsilon_2})\tau_{k_2,\varepsilon_2}\mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}_{k_3}(3)^{\varepsilon_3})^{\varepsilon_3}) \cdots \nu = \mathrm{id},\\ &\mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}^{\varepsilon_1\alpha_1})^{\varepsilon_1})\mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}^{\varepsilon_2\alpha_2})^{\varepsilon_2})\tau_{k_1,\varepsilon_1}\tau_{k_2,\varepsilon_2}\mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}_{k_3}(3)^{\varepsilon_3})^{\varepsilon_3}) \cdots \nu = \mathrm{id},\\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ &\mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}^{\varepsilon_1\alpha_1})^{\varepsilon_1}) \cdots \mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}^{\varepsilon_L\alpha_L})^{\varepsilon_L})\tau_{k_1,\varepsilon_1} \cdots \tau_{k_L,\varepsilon_L}\nu = \mathrm{id}.\end{aligned}$$

Thus, thanks to (3.15), we have for any $i \in I$

$$\mathsf{Ad}(\boldsymbol{\Psi}_q(\mathsf{Y}^{\varepsilon_1\alpha_1})^{\varepsilon_1} \cdots \boldsymbol{\Psi}_q(\mathsf{Y}^{\varepsilon_L\alpha_L})^{\varepsilon_L}))(\mathsf{Y}_i(1)) = \mathsf{Y}_i(1). \tag{3.16}$$

If B is nondegenerate, by considering the canonical form of B, one can easily show that the only Y^α which commutes with all Y_i's is 1. Therefore, (3.16) implies the identity (3.14). If B is degenerate, again thanks to Example 2.5 and Proposition 2.6, it is reduced to the nondegenerate case. ∎

3.5 Quantum dilogarithm identities in universal form

Let us rewrite the identity (3.14) with genuine 'nontropical', or universal, quantum y-variables $Y_{k_t}(t)$. This generalizes the new variation of the pentagon relation (3.21) recently found by [63] and its generalization to any simply laced finite type [62]. To be more precise, the pentagon relation of [63] is expressed by the *initial variables of the Y-system*, while our version is expressed by the *initial y-variables*, so that they have different expressions. However, they coincide under a suitable identification of variables as shown in Section 3.6. A.Yu. Volkov explained us how to derive his pentagon relation and its generalization to any simply laced finite type from the tropical one using the 'shuffle method' in the Y-system setting [62]. Below we apply his shuffle method adapted in our cluster algebraic setting.

Lemma 3.6. *Under the same assumption of Theorem 3.5, the following formulas hold for $t = 2, \dots, L$ (we call (3.18) the shuffle formula)*

$$\mathbf{\Psi}_q(Y_{k_t}(t)^{\varepsilon_t})^{\varepsilon_t} = \mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_1\alpha_1})^{\varepsilon_1}\cdots\mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_{t-1}\alpha_{t-1}})^{\varepsilon_{t-1}})(\mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_t\alpha_t})^{\varepsilon_t}), \tag{3.17}$$

$$\mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_1\alpha_1})^{\varepsilon_1}\cdots\mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_t\alpha_t})^{\varepsilon_t} = \mathbf{\Psi}_q(Y_{k_t}(t)^{\varepsilon_t})^{\varepsilon_t}\cdots\mathbf{\Psi}_q(Y_{k_1}(1)^{\varepsilon_1})^{\varepsilon_1}. \tag{3.18}$$

Proof. Let us prove (3.17) for $t = 3$, for example. By setting $i = k_3$ in (3.10) and repeating the argument in the proof of Theorem 3.5, we have

$$\begin{aligned} Y_{k_3}(3) &= (\mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}_{k_1}(1)^{\varepsilon_1})^{\varepsilon_1})\tau_{k_1,\varepsilon_1}\mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}_{k_2}(2)^{\varepsilon_2})^{\varepsilon_2})\tau_{k_2,\varepsilon_2})(\mathsf{Y}_{k_3}(3)) \\ &= \mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_1\alpha_1})^{\varepsilon_1}\mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_2\alpha_2})^{\varepsilon_2})(\mathsf{Y}^{\alpha_3}). \end{aligned}$$

Then, by extending the map $\mathsf{Ad}(\mathbf{\Psi}_q(\mathsf{Y}^{\varepsilon_t\alpha_t})^{\varepsilon_t})$ to $\hat{\mathbb{T}}(B,\mathsf{Y})$, we obtain (3.17) for $t = 3$. The general case is similar. Then, (3.18) follows from (3.17) by induction. ■

Applying (3.18) with $t = L$ to the identity (3.14), we immediately obtain the universal counterpart of (3.14).

Corollary 3.7 (Quantum dilogarithm identities in universal form ([63, 62]))**.** *Under the same assumption of Theorem 3.5, the following identity holds*

$$\mathbf{\Psi}_q(Y_{k_L}(L)^{\varepsilon_L})^{\varepsilon_L}\cdots\mathbf{\Psi}_q(Y_{k_1}(1)^{\varepsilon_1})^{\varepsilon_1} = 1. \tag{3.19}$$

Since (3.18) actually holds irrespective with the periodicity of the sequence (3.11), one can say that the two identities (3.14) and (3.19) are equivalent.

3.6 Example of type A_2

We continue to use the data in Section 2.7. For the initial quantum y-seed (B,Y), we have

$$Y_1Y_2 = q^2Y_2Y_1.$$

Consider the sequence of mutations of quantum y-seeds of (B,Y):

$$(B(1),Y(1)) \overset{\mu_1}{\longleftrightarrow} (B(2),Y(2)) \overset{\mu_2}{\longleftrightarrow} \cdots \overset{\mu_1}{\longleftrightarrow} (B(6),Y(6)).$$

Then,

$$\begin{cases} Y_1(1) = Y_1, \\ Y_2(1) = Y_2, \end{cases} \quad \begin{cases} Y_1(2) = Y_1^{-1}, \\ Y_2(2) = Y_2(1+qY_1), \end{cases} \quad \begin{cases} Y_1(3) = Y_1^{-1}(1+qY_2+Y_1Y_2), \\ Y_2(3) = Y_2^{-1}(1+q^{-1}Y_1)^{-1}, \end{cases}$$

$$\begin{cases} Y_1(4) = Y_1(1+qY_2+Y_1Y_2)^{-1}, \\ Y_2(4) = q^{-1}Y_1^{-1}Y_2^{-1}(1+qY_2), \end{cases} \quad \begin{cases} Y_1(5) = Y_2^{-1}, \\ Y_2(5) = q^{-1}Y_1Y_2(1+q^{-1}Y_2), \end{cases} \quad \begin{cases} Y_1(6) = Y_2, \\ Y_2(6) = Y_1. \end{cases}$$

The quantum dilogarithm identity in tropical form (3.14) is

$$\boldsymbol{\Psi}_q\left(Y_1\right)\boldsymbol{\Psi}_q\left(Y_2\right)\boldsymbol{\Psi}_q\left(Y_1\right)^{-1}\boldsymbol{\Psi}_q\left(q^{-1}Y_1Y_2\right)^{-1}\boldsymbol{\Psi}_q\left(Y_2\right)^{-1}=1,$$

where we used $Y^{e_1+e_2}=q^{-1}Y_1Y_2$. It coincides with the pentagon relation (1.3). The quantum dilogarithm identity in universal form (3.19) is

$$\boldsymbol{\Psi}_q\left(Y_2\right)^{-1}\boldsymbol{\Psi}_q\left(q(1+qY_2)^{-1}Y_2Y_1\right)^{-1}\boldsymbol{\Psi}_q\left((1+qY_2+Y_1Y_2)^{-1}Y_1\right)^{-1}$$
$$\times\boldsymbol{\Psi}_q\left(Y_2(1+qY_1)\right)\boldsymbol{\Psi}_q\left(Y_1\right)=1. \tag{3.20}$$

Meanwhile, the pentagon relation in [63] reads, in our convention of $\boldsymbol{\Psi}_q$,

$$\boldsymbol{\Psi}_q\left(X(1+qY)^{-1}\right)^{-1}\boldsymbol{\Psi}_q\left(qX(1+qX+qY)^{-1}Y\right)^{-1}\boldsymbol{\Psi}_q\left((1+qX)^{-1}Y\right)^{-1}$$
$$\times\boldsymbol{\Psi}_q\left(X\right)\boldsymbol{\Psi}_q\left(Y\right)=1, \tag{3.21}$$

with $YX=q^2XY$. Two relations (3.20) and (3.21) coincide by identifying $X=Y_2(1+qY_1)$, $Y=Y_1$.

Remark 3.8. The relation (3.21) should be compared with the quantum pentagon relation *at Nth roots of unity* [15], where Nth powers of the operators are central and they enter the relation as parameters. As was remarked by Bazhanov and Reshetikhin in [6], these parameters are related in exactly the same way as the arguments in the classical pentagon relation; see [6, equation (3.18)]. The quantum pentagon relation at roots of unity plays a central role in the construction of invariants of links in arbitrary 3-manifolds by using the combinatorics of triangulations [33] and in solvable 3-dimensional lattice models of Bazhanov and Baxter [2] (it is called the restricted star-triangle identity there).

4 Quantum dilogarithm identities for $\boldsymbol{\Phi}_b(z)$

In this section we present the quantum dilogarithm identities for $\boldsymbol{\Phi}_b(x)$. The content heavily relies on [20, 19].

4.1 Faddeev's quantum dilogarithm

Let b be a complex number with nonzero real part. Set

$$c_b=(b+b^{-1})\sqrt{-1}/2,\qquad q=e^{\pi b^2\sqrt{-1}},\qquad q^\vee=e^{\pi b^{-2}\sqrt{-1}},\qquad \overline{q}=(q^\vee)^{-1}=e^{-\pi b^{-2}\sqrt{-1}}. \tag{4.1}$$

Following [13, 14], define the *Faddeev's quantum dilogarithm* $\boldsymbol{\Phi}_b(z)$ for $z\in\mathbb{C}$ in the strip $|\mathrm{Im}\,z|<|\mathrm{Im}\,c_b|$ by

$$\boldsymbol{\Phi}_b(z)=\exp\left(-\frac{1}{4}\int_{-\infty}^{\infty}\frac{e^{-2zx\sqrt{-1}}}{\sinh(xb)\sinh(x/b)}\frac{dx}{x}\right), \tag{4.2}$$

where the singularity at $x=0$ is circled from above. It is analytically continued to a meromorphic function on the entire complex plane. We have the properties (1.4) and (1.5), and also the following ones (see, e.g., [57, 64, 16, 61] for more information).

(*i*) Symmetries:

$$\boldsymbol{\Phi}_b(z)=\boldsymbol{\Phi}_{b^{-1}}(z)=\boldsymbol{\Phi}_{-b}(z). \tag{4.3}$$

(ii) Recurrence relation:

$$\mathbf{\Phi}_b(z \pm b\sqrt{-1}) = \left(1 + e^{2\pi bz} q^{\pm 1}\right)^{\pm 1} \mathbf{\Phi}_b(z),$$
$$\mathbf{\Phi}_b(z \pm b^{-1}\sqrt{-1}) = \left(1 + e^{2\pi b^{-1}z} (q^{\vee})^{\pm 1}\right)^{\pm 1} \mathbf{\Phi}_b(z). \tag{4.4}$$

(iii) Unitarity: If b is real or $|b| = 1$, then

$$|\mathbf{\Phi}_b(z)| = 1, \qquad z \in \mathbb{R}. \tag{4.5}$$

(iv) Relation to $\mathbf{\Psi}_q(x)$: If $\operatorname{Im} b^2 > 0$, then

$$\mathbf{\Phi}_b(z) = \frac{\mathbf{\Psi}_q(e^{2\pi bz})}{\mathbf{\Psi}_{\overline{q}}(e^{2\pi b^{-1}z})}. \tag{4.6}$$

Note that, if $\operatorname{Im} b^2 > 0$, then $|q|, |\overline{q}| < 1$.

4.2 Representation of quantum y-variables

Let us recall a representation of quantum y-variables as differential operators in [20, 19]. We continue to use the data (4.1). In view of (4.1), we further set

$$\hbar = \pi b^2, \qquad q = e^{\hbar\sqrt{-1}}. \tag{4.7}$$

To any quantum y-seed (B', Y') of (B, Y) we associate operators $\hat{u}' = (\hat{u}'_i)_{i\in I}$ and $\hat{p}' = (\hat{p}'_i)_{i\in I}$ satisfying the relations

$$[\hat{u}'_i, \hat{u}'_j] = [\hat{p}'_i, \hat{p}'_j] = 0, \qquad [\hat{p}'_i, \hat{u}'_j] = \frac{\hbar}{\sqrt{-1}}\delta_{ij}. \tag{4.8}$$

The algebra of operators $\hat{u}'$ and $\hat{p}'$ has a natural representation on the Hilbert space $L^2(\mathbb{R}^I)$:

$$(\hat{u}'_i f)(u') = u'_i f(u'), \qquad (\hat{p}'_i f)(u') = \frac{\hbar}{\sqrt{-1}} \frac{\partial f(u')}{\partial u'_i}, \qquad u' \in \mathbb{R}^I. \tag{4.9}$$

Using Dirac's notation $f(u') = \langle u'|f\rangle$, we have formally

$$\langle u'|\hat{u}'_i|f\rangle = u'_i\langle u'|f\rangle, \qquad \langle u'|\hat{p}'_i|f\rangle = \frac{\hbar}{\sqrt{-1}} \frac{\partial}{\partial u'_i}\langle u'|f\rangle,$$

or simply

$$\langle u'|\hat{u}'_i = u'_i\langle u'|, \qquad \langle u'|\hat{p}'_i = \frac{\hbar}{\sqrt{-1}} \frac{\partial}{\partial u'_i}\langle u'|.$$

The set of generalized vectors $\{\langle u'|\}_{u'\in\mathbb{R}^I}$ will be called the *local coordinates* of (B', Y').

Define

$$\hat{w}'_i = \sum_{j\in I} b'_{ji}\hat{u}'_j, \qquad \hat{D}'_i = \hat{p}'_i + \hat{w}'_i, \qquad \hat{\mathsf{Y}}'_i = \exp \hat{D}'_i. \tag{4.10}$$

The following relations hold

$$[\hat{D}'_i, \hat{D}'_j] = 2\hbar\sqrt{-1}b'_{ji}, \qquad \hat{\mathsf{Y}}'_i\hat{\mathsf{Y}}'_j = q^{2b'_{ji}}\hat{\mathsf{Y}}'_j\hat{\mathsf{Y}}'_i. \tag{4.11}$$

Also recall the following general fact, which is a special case of the Baker–Campbell–Hausdorff formula: For any noncommutative variables A and B such that $[A, B] = C$ and $[C, A] = [C, B] = 0$, we have

$$e^A e^B = e^{C/2} e^{A+B}.$$

Thus, we have a representation of $\mathbb{T}(B, \mathsf{Y})$ on $L^2(\mathbb{R}^I)$ with

$$\mathsf{Y}'^{\alpha} \mapsto \hat{\mathsf{Y}}'^{\alpha} := \exp\left(\alpha\hat{D}'\right), \qquad \alpha\hat{D}' := \sum_{i\in I}\alpha_i\hat{D}'_i. \tag{4.12}$$

4.3 Decomposition of mutations

Here we present a result which is analogous to that of Section 3.3. Let (B',Y') and (B'',Y'') be a pair of quantum y-seeds of (B,Y) such that $(B'',Y'')=\mu_k(B',Y')$.

(a) Monomial part. For each $\varepsilon=\pm1$, consider the following map

$$\begin{aligned}\rho_{k,\varepsilon}:\ & \mathbb{R}^I\to\mathbb{R}^I,\\ & (u')\mapsto(u''),\\ & u''_i=\begin{cases}-u'_k+\sum\limits_{j\in I}[-\varepsilon b'_{jk}]_+u'_j, & i=k,\\ u'_i, & i\neq k.\end{cases}\end{aligned}\tag{4.13}$$

Let $\rho^*_{k,\varepsilon}$ be the induced map in the space of functions $L^2(\mathbb{R}^I)$,

$$\begin{aligned}\rho^*_{k,\varepsilon}:\ & L^2(\mathbb{R}^I)\to L^2(\mathbb{R}^I),\\ & f\mapsto f\circ\rho_{k,\varepsilon},\end{aligned}$$

or, formally,

$$\langle u'|\rho^*_{k,\varepsilon}=\langle\rho_{k,\varepsilon}(u')|=\langle u''|,$$

by which we relate the local coordinates of (B',Y') and (B'',Y'').

For any linear operator $\hat{O}$ acting on $L^2(\mathbb{R}^I)$, let

$$\mathsf{Ad}(\rho^*_{k,\varepsilon})(\hat{O}):=\rho^*_{k,\varepsilon}\hat{O}(\rho^*_{k,\varepsilon})^{-1}.$$

In other words, it is defined by the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}^I) & \xrightarrow{\rho^*_{k,\varepsilon}} & L^2(\mathbb{R}^I)\\ \hat{O}\big\downarrow & & \big\downarrow \mathsf{Ad}(\rho^*_{k,\varepsilon})(\hat{O})\\ L^2(\mathbb{R}^I) & \xrightarrow{\rho^*_{k,\varepsilon}} & L^2(\mathbb{R}^I).\end{array}$$

Then, we have

$$\mathsf{Ad}(\rho^*_{k,\varepsilon})(\hat{u}''_i)=\begin{cases}-\hat{u}'_k+\sum\limits_{j\in I}[-\varepsilon b'_{jk}]_+\hat{u}'_j, & i=k,\\ \hat{u}'_i, & i\neq k,\end{cases}\tag{4.14}$$

$$\mathsf{Ad}(\rho^*_{k,\varepsilon})(\hat{w}''_i)=\begin{cases}-\hat{w}'_k, & i=k,\\ \hat{w}'_i+[\varepsilon b'_{ki}]_+\hat{w}'_k, & i\neq k,\end{cases}\tag{4.15}$$

$$\mathsf{Ad}(\rho^*_{k,\varepsilon})(\hat{p}''_i)=\begin{cases}-\hat{p}'_k, & i=k,\\ \hat{p}'_i+[\varepsilon b'_{ki}]_+\hat{p}'_k, & i\neq k,\end{cases}\tag{4.16}$$

$$\mathsf{Ad}(\rho^*_{k,\varepsilon})(\hat{D}''_i)=\begin{cases}-\hat{D}'_k, & i=k,\\ \hat{D}'_i+[\varepsilon b'_{ki}]_+\hat{D}'_k, & i\neq k,\end{cases}\tag{4.17}$$

where (4.17) follows from (4.16) and (4.15). It follows from (4.17) that

$$\mathsf{Ad}(\rho^*_{k,\varepsilon})(\hat{\mathsf{Y}}''_i)=\begin{cases}\hat{\mathsf{Y}}'^{-1}_k, & i=k,\\ \hat{\mathsf{Y}}'^{e_i+[\varepsilon b'_{ki}]_+e_k}, & i\neq k,\end{cases}$$

which coincides with (3.5).

Remark 4.1. The transformation of (4.13) is the one for the *g-vectors* in [24] if ε is the tropical sign of y'_k. Similarly, for $w'_i = \sum_{j\in I} b'_{ji}u'_j$, the induced transformation

$$w''_i = \begin{cases} -w'_k, & i = k, \\ w'_i + [\varepsilon b'_{ki}]_+ w'_k, & i \neq k, \end{cases}$$

is the one for the *c-vectors* in [24], and it is the logarithmic form of the tropical exchange relation (2.8). They are known to be dual in the following sense [18, 54]

$$\sum_{i\in I} u''_i w''_i = \sum_{i\in I} u'_i w'_i.$$

(b) Automorphism part. We set

$$\hat{\mathsf{D}}'_i = \frac{1}{2\pi b}\hat{D}'_i. \tag{4.18}$$

Then, we have

$$\hat{\mathsf{Y}}'^{-1}_i \boldsymbol{\Phi}_b(\varepsilon\hat{\mathsf{D}}'_j)\hat{\mathsf{Y}}'_i = \boldsymbol{\Phi}_b\big(\varepsilon\hat{\mathsf{D}}'_j - \varepsilon\sqrt{-1}bb'_{ji}\big).$$

Thus, thanks to the recurrence relation (4.4), one obtains, for each $\varepsilon = \pm 1$,

$$\mathsf{Ad}(\boldsymbol{\Phi}_b(\varepsilon\hat{\mathsf{D}}'_k)^{\varepsilon})(\hat{\mathsf{Y}}'_i) = \hat{\mathsf{Y}}'_i \prod_{m=1}^{|b'_{ki}|} \big(1 + q^{-\varepsilon \mathrm{sgn}(b'_{ki})(2m-1)}\hat{\mathsf{Y}}'^{\,\varepsilon}_k\big)^{-\varepsilon \mathrm{sgn}(b'_{ki})}$$

by an analogous calculation to (3.7) and (3.8).

In summary, we have a parallel statement to Proposition 3.1.

Proposition 4.2 ([20, 19])**.** *We have the equality*

$$(\mathsf{Ad}(\boldsymbol{\Phi}_b(\hat{\mathsf{D}}'_k))\mathsf{Ad}(\rho^*_{k,+}))(\hat{\mathsf{Y}}''_i) = \big(\mathsf{Ad}\big(\boldsymbol{\Phi}_b(-\hat{\mathsf{D}}'_k)^{-1}\big)\mathsf{Ad}(\rho^*_{k,-})\big)(\hat{\mathsf{Y}}''_i), \tag{4.19}$$

and either side of (4.19) *coincides with the right hand side of the exchange relation* (3.4) *with* Y'_i *replaced with* $\hat{\mathsf{Y}}'_i$*.*

4.4 Dual operators

Following [13] and [20], we define the operators $\hat{\mathsf{Z}}'_i$ which are 'dual' to $\hat{\mathsf{Y}}'_i$ in the sense of the first equality of (4.3). In the situation in (4.10), we define

$$\hat{\mathsf{Z}}'_i = \exp(b^{-2}\hat{D}'_i).$$

Then, the following relations hold

$$\begin{aligned} &\hat{\mathsf{Z}}'_i\hat{\mathsf{Z}}'_j = (q^\vee)^{2b'_{ji}}\hat{\mathsf{Z}}'_j\hat{\mathsf{Z}}'_i, \\ &\hat{\mathsf{Y}}'_i\hat{\mathsf{Z}}'_j = \hat{\mathsf{Z}}'_j\hat{\mathsf{Y}}'_i. \end{aligned} \tag{4.20}$$

Remark 4.3. The duality between $\hat{\mathsf{Y}}'_i$ and $\hat{\mathsf{Z}}'_i$ is not manifest because of our preference for b over b^{-1} in (4.10) through (4.7). To see it manifestly, we set

$$\hat{\mathcal{D}}'_i = \frac{1}{\gamma}\left(\frac{\gamma^2}{4\pi\sqrt{-1}}\hat{\partial}'_i + \hat{w}'_i\right), \qquad \hat{\mathsf{Y}}'_i = \exp\big(2\pi b\hat{\mathcal{D}}'_i\big), \qquad \hat{\mathsf{Z}}'_i = \exp\big(2\pi b^{-1}\hat{\mathcal{D}}'_i\big),$$

where γ is an arbitrary nonzero real number and $\hat{\partial}'_i$ satisfy $[\hat{\partial}'_i, \hat{\partial}'_j] = 0$ and $[\hat{\partial}'_i, \hat{u}'_j] = \delta_{ij}$. The following relations hold irrespective of γ

$$[\hat{\mathcal{D}}'_i, \hat{\mathcal{D}}'_j] = \frac{\sqrt{-1}}{2\pi} b'_{ji}, \qquad \hat{\mathsf{Y}}'_i \hat{\mathsf{Y}}'_j = (q)^{2b'_{ji}} \hat{\mathsf{Y}}'_j \hat{\mathsf{Y}}'_i, \qquad \hat{\mathsf{Z}}'_i \hat{\mathsf{Z}}'_j = (q^\vee)^{2b'_{ji}} \hat{\mathsf{Z}}'_j \hat{\mathsf{Z}}'_i, \qquad \hat{\mathsf{Y}}'_i \hat{\mathsf{Z}}'_j = \hat{\mathsf{Z}}'_j \hat{\mathsf{Y}}'_i.$$

Now the duality $b \leftrightarrow b^{-1}$ is manifest. Further setting $\gamma = 2\pi b$, we have $\hat{\mathcal{D}}'_i = \hat{\mathsf{D}}'_i$ and we recover the operators $\hat{\mathsf{Y}}'_i$ and $\hat{\mathsf{Z}}'_i$ in the main text.

Due to the symmetry $b \leftrightarrow b^{-1}$ in (4.3) and the above remark, we immediately obtain the following from Proposition 4.2.

Proposition 4.4. *We have the equality*

$$(\mathsf{Ad}(\mathbf{\Phi}_b(\hat{\mathsf{D}}'_k))\mathsf{Ad}(\rho^*_{k,+}))(\hat{\mathsf{Z}}''_i) = \big(\mathsf{Ad}\big(\mathbf{\Phi}_b(-\hat{\mathsf{D}}'_k)^{-1}\big)\mathsf{Ad}(\rho^*_{k,-})\big)(\hat{\mathsf{Z}}''_i), \tag{4.21}$$

and either side of (4.21) *coincides with the right hand side of the exchange relation* (3.4) *with* Y'_i *and* q *replaced with* $\hat{\mathsf{Z}}'_i$ *and* $q^\vee$*, respectively.*

4.5 Quantum dilogarithm identities in tropical form

Suppose that $(k_1, k_2, \dots, k_L)$ is a ν-period of (B, Y) as in Section 3.4.

The identities parallel to (3.14) are available for $\mathbf{\Phi}_b(z)$ directly from (3.14).

Theorem 4.5 (Quantum dilogarithm identities in tropical form)**.** *Under the same assumption of Theorem* 3.5 *(in particular,* $(\varepsilon_1, \dots, \varepsilon_L)$ *is the tropical sign-sequence of the mutation sequence), the following identity holds.*

$$\mathbf{\Phi}_b(\varepsilon_1 \alpha_1 \hat{\mathsf{D}})^{\varepsilon_1} \cdots \mathbf{\Phi}_b(\varepsilon_L \alpha_L \hat{\mathsf{D}})^{\varepsilon_L} = 1, \tag{4.22}$$

where $\alpha_t \hat{\mathsf{D}} = \sum_{i \in I} \alpha_i(t) \hat{\mathsf{D}}_i$ *and* $\hat{\mathsf{D}}_i$ *is the operator in* (4.18) *for* (B, Y)*.*

Proof. Due to the symmetry $b \leftrightarrow b^{-1}$ in (4.3), one can assume that $\operatorname{Im} b^2 \geq 0$ without losing generality. By the continuity of $\mathbf{\Phi}_b$ with respect to b, it is enough to show the claim for $\operatorname{Im} b^2 > 0$. Then, by (4.6), we have

$$\mathbf{\Phi}_b(\varepsilon_t \alpha_t \hat{\mathsf{D}}) = \frac{\mathbf{\Psi}_q(\hat{\mathsf{Y}}^{\varepsilon_t \alpha_t})}{\mathbf{\Psi}_{\overline{q}}(\hat{\mathsf{Z}}^{\varepsilon_t \alpha_t})}.$$

Then, thanks to the commutativity (4.20), the relation (4.22) factorizes into two identities

$$\mathbf{\Psi}_q(\hat{\mathsf{Y}}^{\varepsilon_1 \alpha_1})^{\varepsilon_1} \cdots \mathbf{\Psi}_q(\hat{\mathsf{Y}}^{\varepsilon_L \alpha_L})^{\varepsilon_L} = 1, \tag{4.23}$$

$$\mathbf{\Psi}_{\overline{q}}(\hat{\mathsf{Z}}^{\varepsilon_L \alpha_L})^{\varepsilon_L} \cdots \mathbf{\Psi}_{\overline{q}}(\hat{\mathsf{Z}}^{\varepsilon_1 \alpha_1})^{\varepsilon_1} = 1, \tag{4.24}$$

where (4.23) is a specialization of (3.14), while (4.24) is equivalent to

$$\mathbf{\Psi}_{q^\vee}(\hat{\mathsf{Z}}^{\varepsilon_1 \alpha_1})^{\varepsilon_1} \cdots \mathbf{\Psi}_{q^\vee}(\hat{\mathsf{Z}}^{\varepsilon_L \alpha_L})^{\varepsilon_L} = 1,$$

which is another specialization of (3.14). ∎

4.6 Quantum dilogarithm identities in local form

Let $\langle u(t)|$ and $\hat{D}_i(t)$ denote the local coordinates and the operator in (4.10) for $(B(t), Y(t))$, respectively. Let $L^2(\mathbb{R}^I)_t$ be the Hilbert space together with the local coordinate $\langle u(t)|$, so that $\rho^*_{k_t,\varepsilon_t} : L^2(\mathbb{R}^I)_{t+1} \to L^2(\mathbb{R}^I)_t$.

For the bijection ν, we apply the same formalism as ρ. Namely, let $\nu : \mathbb{R}^I \to \mathbb{R}^I$ be the coordinate transformation defined by $(u(L+1)) \mapsto (u(1))$ with $u_i(1) = u_{\nu(i)}(L+1)$. Define $\nu^* : L^2(\mathbb{R}^I)_1 \to L^2(\mathbb{R}^I)_{L+1}$, $f \mapsto f \circ \nu$, and $\mathsf{Ad}(\nu^*)(\hat{O}) := \nu^* \hat{O} (\nu^*)^{-1}$ for any linear operator $\hat{O}$ acting on $L^2(\mathbb{R}^I)$. Then, $\mathsf{Ad}(\nu^*)(\hat{D}_i(1)) = \hat{D}_{\nu(i)}(L+1)$.

Let us recall the result of [20, Theorem 5.4]. *Suppose that b is a nonzero real number.* Note that, $\mathbf{\Phi}_b(\varepsilon_t \hat{\mathsf{D}}'_{k_t}(t))$ is a *unitary* operator by (4.5). By the periodicity assumption and Propositions 4.2 and 4.4, we have the following equalities for any sign-sequence $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_L)$

$$\mathsf{Ad}(\mathbf{\Phi}_b(\varepsilon_1 \hat{\mathsf{D}}_{k_1}(1))^{\varepsilon_1} \rho^*_{k_1,\varepsilon_1} \cdots \mathbf{\Phi}_b(\varepsilon_L \hat{\mathsf{D}}_{k_L}(L))^{\varepsilon_L} \rho^*_{k_L,\varepsilon_L} \nu^*)(\hat{\mathsf{Y}}_{\mathsf{i}}(1)) = \hat{\mathsf{Y}}_{\mathsf{i}}(1),$$
$$\mathsf{Ad}(\mathbf{\Phi}_b(\varepsilon_1 \hat{\mathsf{D}}_{k_1}(1))^{\varepsilon_1} \rho^*_{k_1,\varepsilon_1} \cdots \mathbf{\Phi}_b(\varepsilon_L \hat{\mathsf{D}}_{k_L}(L))^{\varepsilon_L} \rho^*_{k_L,\varepsilon_L} \nu^*)(\hat{\mathsf{Z}}_{\mathsf{i}}(1)) = \hat{\mathsf{Z}}_{\mathsf{i}}(1).$$

This is equivalent to saying that the operator

$$\hat{O}_{\vec{\varepsilon},b} = \mathbf{\Phi}_b(\varepsilon_1 \hat{\mathsf{D}}_{k_1}(1))^{\varepsilon_1} \rho^*_{k_1,\varepsilon_1} \cdots \mathbf{\Phi}_b(\varepsilon_L \hat{\mathsf{D}}_{k_L}(L))^{\varepsilon_L} \rho^*_{k_L,\varepsilon_L} \nu^*$$

commutes with $\hat{\mathsf{Y}}_{\mathsf{i}}(1)$ and $\hat{\mathsf{Z}}_{\mathsf{i}}(1)$ for any $i \in I$. It was shown in [20] that, when b^2 is irrational, such $\hat{O}_{\vec{\varepsilon},b}$ is the identity operator up to a complex scalar multiple $\lambda_{\vec{\varepsilon},b}$ by generalizing the result of [13]; furthermore, the claim holds for rational b^2 as well by continuity. Since $\hat{O}_{\vec{\varepsilon},b}$ is unitary, we have $|\lambda_{\vec{\varepsilon},b}| = 1$. Therefore, one obtains the following *local form* of the quantum dilogarithm identities. We call it so, since it is described by the family of local coordinates $\langle u(1)|, \dots, \langle u(L)|$ associated with the mutation sequence.

Theorem 4.6 (Quantum dilogarithm identities in local form [20])**.** *Let b be a nonzero real number. For any sign-sequence $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_L)$, the following identity holds.*

$$\mathbf{\Phi}_b(\varepsilon_1 \hat{\mathsf{D}}_{k_1}(1))^{\varepsilon_1} \rho^*_{k_1,\varepsilon_1} \cdots \mathbf{\Phi}_b(\varepsilon_L \hat{\mathsf{D}}_{k_L}(L))^{\varepsilon_L} \rho^*_{k_L,\varepsilon_L} \nu^* = \lambda_{\vec{\varepsilon},b}, \qquad |\lambda_{\vec{\varepsilon},b}| = 1. \tag{4.25}$$

For the tropical sign-sequence, we have a stronger version of Theorem 4.6. One can obtain it as a direct corollary of Theorem 4.5, and not via Theorem 4.6. So the assumption that b is real is not necessary here. *This is the identity we use to derive the corresponding classical dilogarithm identity.*

Theorem 4.7. *For the tropical sign-sequence $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_L)$, the following identity holds*

$$\mathbf{\Phi}_b(\varepsilon_1 \hat{\mathsf{D}}_{k_1}(1))^{\varepsilon_1} \rho^*_{k_1,\varepsilon_1} \cdots \mathbf{\Phi}_b(\varepsilon_L \hat{\mathsf{D}}_{k_L}(L))^{\varepsilon_L} \rho^*_{k_L,\varepsilon_L} \nu^* = 1. \tag{4.26}$$

In particular, $\lambda_{\vec{\varepsilon},b} = 1$ for the tropical sign sequence.

Proof. By the duality in Remark 4.1, the periodicity of tropical y-variables (3.15) is equivalent to

$$\rho^*_{k_1,\varepsilon_1} \cdots \rho^*_{k_L,\varepsilon_L} \nu^* = \mathrm{id}.$$

Multiply it from the right of (4.22). Then, repeat the argument in the proof of Theorem 3.5 in the inverse way. ■

In summary, for the tropical sign-sequence we have four forms of quantum dilogarithm identities (3.14), (3.19), (4.25), and (4.26). The first three identities are obtained from each other without referring to the seed periodicity of (3.11). The last one is obtained from the rest by assuming the tropical periodicity (3.15).

4.7 Example of type A_2

We continue to use the data in Sections 2.7 and 3.6.

The quantum dilogarithm identity in tropical form (4.22) is

$$\mathbf{\Phi}_b(\hat{\mathsf{D}}_1)\mathbf{\Phi}_b(\hat{\mathsf{D}}_2)\mathbf{\Phi}_b(\hat{\mathsf{D}}_1)^{-1}\mathbf{\Phi}_b(\hat{\mathsf{D}}_1+\hat{\mathsf{D}}_2)^{-1}\mathbf{\Phi}_b(\hat{\mathsf{D}}_2)^{-1}=1,$$
$$[\hat{\mathsf{D}}_1,\hat{\mathsf{D}}_2]=\frac{\sqrt{-1}}{2\pi}.$$

By identifying $\hat{Q}=\hat{\mathsf{D}}_1$, $\hat{P}=\hat{\mathsf{D}}_2$, it coincides with the pentagon relation (1.5).

Let us also write the relevant data for the identity (4.26) explicitly

$$\hat{\mathsf{D}}_{k_t}(t)=\begin{cases}\dfrac{1}{2\pi b}\left(\hat{p}_1(t)+\hat{u}_2(t)\right), & t=1,3,5,\\ \dfrac{1}{2\pi b}\left(\hat{p}_2(t)+\hat{u}_1(t)\right), & t=2,4.\end{cases}$$

The images of $\hat{u}_1(t+1)$, $\hat{u}_2(t+1)$, $\hat{w}_1(t+1)$, $\hat{w}_2(t+1)$ by the map $\mathsf{Ad}(\rho^*_{k_t,\varepsilon_t})$ are given in the order

$$\begin{array}{lllll}
t=1: & -\hat{u}_1(1), & \hat{u}_2(1), & -\hat{w}_1(1), & \hat{w}_2(1),\\
t=2: & \hat{u}_1(2), & -\hat{u}_2(2), & \hat{w}_1(2), & -\hat{w}_2(2),\\
t=3: & -\hat{u}_1(3)+\hat{u}_2(3), & \hat{u}_2(3), & -\hat{w}_1(3), & \hat{w}_2(3)+\hat{w}_1(3),\\
t=4: & \hat{u}_1(4), & -\hat{u}_2(4)+\hat{u}_1(4), & \hat{w}_1(4)+\hat{w}_2(4), & -\hat{w}_2(4),\\
t=5: & -\hat{u}_1(5)+\hat{u}_2(5), & \hat{u}_2(5), & -\hat{w}_1(5), & \hat{w}_2(5)+\hat{w}_1(5).
\end{array}$$

5 From quantum to classical dilogarithm identities

In this section we demonstrate how the classical quantum dilogarithm identities (2.13) emerge from the quantum dilogarithm identities in local form (4.26) in the semiclassical limit. This is the main part of the paper.

5.1 Position and momentum bases

We are going to evaluate the operator in the left hand side of (4.26), which is actually the identity operator, by the standard quantum physics method.

Throughout Section 5 we assume that b is a nonzero real number.

Recall that we set $\hbar=\pi b^2$ in (4.7). The asymptotic property (1.4) is written as

$$\mathbf{\Phi}_b\left(\frac{z}{2\pi b}\right)\sim\exp\left(\frac{\sqrt{-1}}{\hbar}\frac{1}{2}\mathrm{Li}_2(-e^z)\right),\qquad \hbar\to 0, \tag{5.1}$$

where and in the rest $\sim$ means the leading term for the asymptotic expansion in $\hbar$.

Let $(B(t),Y(t))$ be the quantum Y-seed of $(B(1),Y(1))=(B,Y)$ in (3.11). Let $L^2(\mathbb{R}^I)_t$ be the Hilbert space together with the local coordinate $\langle u(t)|$ associated with $(B(t),Y(t))$ in the previous section.

Let $\{|u(t)\rangle \mid u(t)\in\mathbb{R}^I\}$ and $\{|p(t)\rangle \mid p(t)\in\mathbb{R}^I\}$ be the standard position and the momentum bases of $L^2(\mathbb{R}^I)_t$, respectively. They satisfy the following properties, where $n=|I|$,

$$\hat{u}_i(t)|u(t)\rangle=u_i(t)|u(t)\rangle,\qquad \hat{p}_i|p(t)\rangle=p_i(t)|p(t)\rangle,$$
$$\langle u(t)|u'(t)\rangle=\prod_{i\in I}\delta(u_i(t)-u'_i(t)),\qquad \langle p(t)|p'(t)\rangle=(2\pi\hbar)^n\prod_{i\in I}\delta(p_i(t)-p'_i(t)),$$
$$\langle u(t)|p(t)\rangle=\exp\left(\frac{\sqrt{-1}}{\hbar}u(t)p(t)\right),\qquad \langle p(t)|u(t)\rangle=\exp\left(-\frac{\sqrt{-1}}{\hbar}u(t)p(t)\right),$$

where $u(t)p(t) := \sum_{i\in I} u_i(t)p_i(t)$,

$$1 = \int du(t)|u(t)\rangle\langle u(t)|, \qquad 1 = \int \frac{dp(t)}{(2\pi\hbar)^n}|p(t)\rangle\langle p(t)|. \tag{5.2}$$

In particular, we have

$$\frac{\langle u(t)|\hat{D}_i(t)|p(t)\rangle}{\langle u(t)|p(t)\rangle} = p_i(t) + w_i(t), \qquad w_i(t) := \sum_{j=1} b_{ji}(t)u_j(t). \tag{5.3}$$

Let $\hat{O}$ be the composition of the operators in the left hand side of (4.26), namely,

$$\hat{O} = \mathbf{\Phi}_b(\varepsilon_1 \hat{\mathsf{D}}_{k_1}(1))^{\varepsilon_1}\rho^*_{k_1,\varepsilon_1}\cdots \mathbf{\Phi}_b(\varepsilon_L \hat{\mathsf{D}}_{k_L}(L))^{\varepsilon_L}\rho^*_{k_L,\varepsilon_L}\nu^* \, (= 1),$$

where $(\varepsilon_1,\dots,\varepsilon_L)$ is the tropical sign-sequence. Choose any position eigenvector $|u(1)\rangle$. Then, set the momentum eigenvector $|\tilde{p}(1)\rangle$ such that its eigenvalues are given by

$$\tilde{p}_i(1) = w_i(1) := \sum_{j\in I} b_{ji}(1)u_j(1), \tag{5.4}$$

where the notation $\tilde{p}(1)$ is used for later convenience. The condition (5.4) will be used only at the last stage when we construct the solution of the saddle point equations in Section 5.4.

The main idea of our consideration is to study the semiclassical behavior of the quantum identity by using *q*-*p* symbols of operators, see for example [8]. By Dirac's argument [12], the semiclassical limit of a *q*-*p* symbol of a unitary operator $\mathcal{O}$ is given by the exponential of the generating function of the canonical transformation, which quantum mechanically corresponds to the unitary inner transformation generated by $\mathcal{O}$. In our case, the *q*-*p* symbol corresponds to the '*u*-*p*' symbol defined by

$$F(u(1),\tilde{p}(1)) := \frac{\langle u(1)|\hat{O}|\tilde{p}(1)\rangle}{\langle u(1)|\tilde{p}(1)\rangle}.$$

Below we show that the leading term of $\log F(u(1),\tilde{p}(1))$ in the limit $\hbar\to 0$ yields the left hand side of (2.13) up to a multiplicative constant. We know *a priori* that its value is 0, which yields the right hand side of (2.13).

5.2 Integral expression

By inserting the intermediate complete states (5.2), we obtain the following integral expression

$$\begin{aligned} F(u(1),\tilde{p}(1)) = (2\pi\hbar)^{-n(2L-1)}\int & dp(1)d\tilde{p}(2)du(2)dp(2)d\tilde{p}(3)\cdots d\tilde{p}(L)du(L)dp(L) \\ & \times \langle\tilde{p}(1)|u(1)\rangle \frac{\langle u(1)|\mathbf{\Phi}_b(\varepsilon_1\hat{\mathsf{D}}_{k_1}(1))^{\varepsilon_1}|p(1)\rangle}{\langle u(1)|p(1)\rangle}\langle u(1)|p(1)\rangle\langle p(1)|\rho^*_{k_1,\varepsilon_1}|\tilde{p}(2)\rangle \\ & \times \langle\tilde{p}(2)|u(2)\rangle \frac{\langle u(2)|\mathbf{\Phi}_b(\varepsilon_2\hat{\mathsf{D}}_{k_2}(2))^{\varepsilon_2}|p(2)\rangle}{\langle u(2)|p(2)\rangle}\langle u(2)|p(2)\rangle\langle p(2)|\rho^*_{k_2,\varepsilon_2}|\tilde{p}(3)\rangle\cdots \\ & \times \langle\tilde{p}(L)|u(L)\rangle \frac{\langle u(L)|\mathbf{\Phi}_b(\varepsilon_L\hat{\mathsf{D}}_{k_L}(L))^{\varepsilon_L}|p(L)\rangle}{\langle u(L)|p(L)\rangle}\langle u(L)|p(L)\rangle\langle p(L)|\rho^*_{k_L,\varepsilon_L}\nu^*|\tilde{p}(1)\rangle. \end{aligned}$$

The integration over $p(L)$ is done by (4.16), and it yields the relation

$$\tilde{p}_i(1) = \begin{cases} -p_{k_L}(L), & \nu(i) = k_L, \\ p_{\nu(i)}(L) + [\varepsilon_L b'_{k_L\nu(i)}(L)]_+ p_{k_L}(L), & \nu(i)\neq k_L. \end{cases} \tag{5.5}$$

Similarly, the integration over $\tilde{p}(t+1)$ $(t = 1, \dots, L-1)$ yields the relation

$$\tilde{p}_i(t+1) = \begin{cases} -p_{k_t}(t), & i = k_t, \\ p_i(t) + [\varepsilon_t b'_{k_t i}(t)]_+ p_{k_t}(t), & i \neq k_t. \end{cases} \tag{5.6}$$

Thus, $\tilde{p}(t+1)$ is now a dependent variable of $p(t)$ by (5.6).

In view of (4.10) it is natural to introduce new dependent variables

$$y_{k_t}(t) = \exp(p_{k_t}(t) + w_{k_t}(t)), \qquad t = 1, \dots, L, \tag{5.7}$$

where the notation $y_i(t)$ anticipates the identification with classical y-variables eventually. Then, by (5.3), we have

$$\frac{\langle u(t)|\hat{\mathsf{D}}_{k_t}(t)|p(t)\rangle}{\langle u(t)|p(t)\rangle} = \frac{1}{2\pi b} \log y_{k_t}(t), \tag{5.8}$$

and the remaining integration has the following form

$$\begin{aligned} F(u(1), \tilde{p}(1)) = (2\pi\hbar)^{-n(L-1)} \int & dp(1) \cdots dp(L-1) du(2) \cdots du(L) \\ & \times \mathbf{\Phi}_b \left(\frac{\log y_{k_1}(1)^{\varepsilon_1}}{2\pi b} \right)^{\varepsilon_1} \exp\left(\frac{\sqrt{-1}}{\hbar} u(1)(p(1) - \tilde{p}(1)) \right) \\ & \times \mathbf{\Phi}_b \left(\frac{\log y_{k_2}(2)^{\varepsilon_2}}{2\pi b} \right)^{\varepsilon_2} \exp\left(\frac{\sqrt{-1}}{\hbar} u(2)(p(2) - \tilde{p}(2)) \right) \cdots \\ & \times \mathbf{\Phi}_b \left(\frac{\log y_{k_L}(L)^{\varepsilon_L}}{2\pi b} \right)^{\varepsilon_L} \exp\left(\frac{\sqrt{-1}}{\hbar} u(L)(p(L) - \tilde{p}(L)) \right). \end{aligned}$$

Using (5.1), we have

$$\begin{aligned} F(u(1), \tilde{p}(1)) \sim (2\pi\hbar)^{-n(L-1)} \int & dp(1) \cdots dp(L-1) du(2) \cdots du(L) \\ & \exp\left(\frac{\sqrt{-1}}{\hbar} \sum_{t=1}^{L} \left\{ \frac{1}{2} \varepsilon_t \mathrm{Li}_2(-y_{k_t}(t)^{\varepsilon_t}) + u(t)(p(t) - \tilde{p}(t)) \right\} \right). \end{aligned} \tag{5.9}$$

To evaluate the integral expression (5.9) in the semiclassical limit, we apply the saddle point method. It consists of three steps.

Step 1. Write the saddle point equations, that is, the extremum condition of the integrand of (5.9) for the independent variables $p(1), \dots, p(L-1)$ and $u(2), \dots, u(L)$.

Step 2. Find a solution of the saddle point equations.

Step 3. Evaluate the integrand at the solution.

5.3 Saddle point equations

Let us derive the saddle point equations for (5.9). We use the following formulas, which are obtained from (2.1), (4.10), (5.7), and (5.8),

$$\begin{aligned} \frac{\partial}{\partial p_i(t)} \left(\frac{1}{2} \varepsilon_t \mathrm{Li}_2(-y_{k_t}(t)^{\varepsilon_t}) \right) &= \delta_{i k_t} \log(1 + y_{k_t}(t)^{\varepsilon_t})^{-1/2}, \\ \frac{\partial}{\partial u_i(t)} \left(\frac{1}{2} \varepsilon_t \mathrm{Li}_2(-y_{k_t}(t)^{\varepsilon_t}) \right) &= -\log(1 + y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2}. \end{aligned}$$

(a) Extremum conditions with respect to $u_i(t)$ $(t = 2, \dots, L)$.

By differentiating the integrand of (5.9) by $u_i(t)$, we have

$$-\log(1+y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2}+p_i(t)-\tilde{p}_i(t)=0, \tag{5.10}$$

or, equivalently,

$$e^{p_i(t)}=e^{\tilde{p}_i(t)}(1+y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2}.$$

Combining it with (5.6), we also have

$$e^{\tilde{p}_i(t+1)}=\begin{cases}(e^{\tilde{p}_{k_t}(t)})^{-1}, & i=k_t,\\ e^{\tilde{p}_i(t)}(e^{\tilde{p}_{k_t}(t)})^{[\varepsilon_t b_{k_t i}(t)]_+}(1+y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2}, & i\neq k_t.\end{cases} \tag{5.11}$$

(b) Extremum conditions with respect to $p_i(t)$ $(t=1,\dots,L-1)$.
By differentiating the integrand of (5.9) by $p_i(t)$, we have

$$\log(1+y_{k_t}(t)^{\varepsilon_t})^{-1/2}+u_{k_t}(t)-\sum_{j\in I}[\varepsilon_t b_{k_t j}(t)]_+u_j(t+1)+u_{k_t}(t+1)=0, \qquad i=k_t, \tag{5.12}$$

$$u_i(t)-u_i(t+1)=0, \qquad i\neq k_t, \tag{5.13}$$

or, equivalently,

$$e^{u_i(t+1)}=\begin{cases}(e^{u_{k_t}(t)})^{-1}\prod_{j\in I}(e^{u_j(t)})^{[-\varepsilon_t b_{jk_t}(t)]_+}(1+y_{k_t}(t)^{\varepsilon_t})^{1/2} & i=k_t,\\ e^{u_i(t)}, & i\neq k_t.\end{cases} \tag{5.14}$$

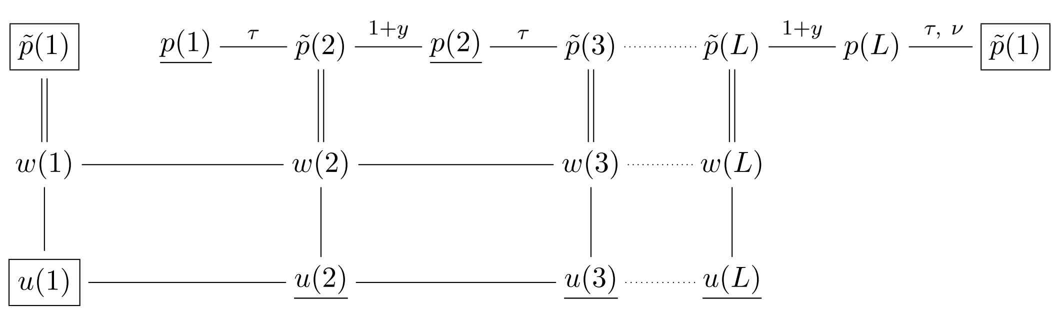
With (2.6), this also implies the following equations for $w_i(u)=\sum_{j\in I}b_{ji}(t)u_j(t)$

$$e^{w_i(t+1)}=\begin{cases}(e^{w_{k_t}(t)})^{-1}, & i=k_t,\\ e^{w_i(t)}(e^{w_{k_t}(t)})^{[\varepsilon_t b_{k_t i}(t)]_+}(1+y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2}, & i\neq k_t,\end{cases} \tag{5.15}$$

which is identical to (5.11).

5.4 Solution

Let us summarize the relevant variables and their relations schematically



Here, the framed variables are the initial variables and the underlined variables are the remaining integration variables which should be determined to solve the saddle point equations. This is a highly complicated systems of equations, but the relevance to the y-seed mutations of (2.9) is rather clear. To see it quickly, set

$$y_i(t):=e^{\tilde{p}_i(t)}e^{w_i(t)}.$$

Note that $\tilde{p}_i(t) = p_i(t)$ if $i = k_t$ by (5.10), therefore, it agrees with the previous definition (5.7). Then, multiply two identities (5.11) and (5.15), we have

$$y_i(t+1) = \begin{cases} y_{k_t}(t)^{-1}, & i = k_t, \\ y_i(t) y_k(t)^{[\varepsilon_t b_{k_t i}(t)]_+} (1 + y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)}, & i \neq k_t. \end{cases}$$

This is nothing but (2.7). Furthermore, (5.11) and (5.15) suggest that

$$y_i(t)^{1/2} = e^{\tilde{p}_i(t)} = e^{w_i(t)}.$$

Having this observation in mind, let us describe the construction of the solution more clearly.

(*i*) (y-variables) We have $u_i(1)$ as initial data, from which $w_i(1)$ is uniquely determined. Temporarily forgetting (5.7), set $y_i(1) = e^{2w_i(1)}$, from which $y_i(t)$ ($t = 2, \dots, L$) are determined by the mutation sequence (2.9).

(*ii*) (u-variables) Set $u_i(t)$ ($t = 2, \dots, L$) by (5.13) and (5.12). Then, (5.15) is also satisfied.

(*iii*) (p-variables) Set $\tilde{p}_i(t)$ by $e^{\tilde{p}_i(t)} = y_i(t)^{1/2}$. This forces the relation $\tilde{p}_i(1) = w_i(1)$, which is guaranteed by the assumption (5.4). Then, $p_i(t)$ are determined by (5.5) and (5.6). Since $\tilde{p}_i(t)$ satisfies (5.11) by definition, (5.10) is also satisfied.

(*iv*) (compatibility) The only thing to be checked is (5.7). Since $p_{k_t}(t) = \tilde{p}_{k_t}(t)$ by (5.10), it is enough to show

$$e^{\tilde{p}_i(t)} = y_i(t)^{1/2}, \qquad e^{w_i(t)} = y_i(t)^{1/2}. \tag{5.16}$$

The first equality is by definition. The second equality is true for $t = 1$ by definition. Then, the rest is shown by (2.7) and the square of (5.15).

Thus, we obtain the desired solution of the saddle point equations. We do not argue on the uniqueness of the solution here as stated in Section 1.2.

Remark 5.1. Since (5.14) is the square half of the exchange relation of the x-variables of the corresponding cluster algebras [18, Proposition 2.3], the variable $e^{u_i(t)}$ is regarded as the square half of the x-variable $x_i(t)$.

5.5 Result

As the final step, we evaluate the logarithm of the integrand in (5.9) at the solution of the saddle point equations in Section 5.4. Using (5.1) and ignoring the common factor, it is given by

$$\sum_{t=1}^{L} \left\{ \frac{1}{2} \varepsilon_t \mathrm{Li}_2(-y_{k_t}(t)^{\varepsilon_t}) + \sum_{i \in I} u_i(t)(p_i(t) - \tilde{p}_i(t)) \right\}. \tag{5.17}$$

Recall that

$$p_i(t) - \tilde{p}_i(t) = \log(1 + y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2},$$
$$w_i(t) = \frac{1}{2} \log y_i(t)$$

by (5.10) and (5.16). Then, the second term of (5.17) is rewritten as

$$\begin{aligned} \sum_{i \in I} u_i(t)(p_i(t) - \tilde{p}_i(t)) &= \sum_{i \in I} u_i(t) \log(1 + y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2} \\ &= \frac{1}{2} \left(\sum_{i \in I} b_{i k_t}(t) u_i(t) \right) \log(1 + y_{k_t}(t)^{\varepsilon_t}) \end{aligned}$$

$$= \frac{1}{2} w_{k_t}(t) \log(1 + y_{k_t}(t)^{\varepsilon_t}) = \frac{1}{4} \varepsilon_t \log y_{k_t}(t)^{\varepsilon_t} \log(1 + y_{k_t}(t)^{\varepsilon_t}). \quad (5.18)$$

Therefore, by (2.4), (5.17) is equal to

$$-\frac{1}{2} \sum_{t=1}^{L} \varepsilon_t L\left(\frac{y_{k_t}(t)^{\varepsilon_t}}{1 + y_{k_t}(t)^{\varepsilon_t}}\right),$$

but we know it is 0 from the beginning. This is the classical dilogarithm identity (2.13).

A Quantum dilogarithm identities in local form for $\boldsymbol{\Psi}_q(x)$ and their semiclassical limits

In this section we present the quantum dilogarithm identities in local form for $\boldsymbol{\Psi}_q(x)$ with tropical sign-sequence. Then, we derive the classical dilogarithm identities from them in the semiclassical limits. The treatment is parallel to the one in Sections 4 and 5 with slight complication.

A.1 Representation of quantum y-variables

We consider a representation of quantum y-variables as differential operators which are quite similar to the one in Section 4.2 but slightly different.

Throughout the section, let $\hbar$ be a positive real number, and λ be a complex number such that

$$\operatorname{Im} \lambda^2 > 0.$$

We reset

$$q = e^{\lambda^2 \hbar \sqrt{-1}}. \quad (A.1)$$

By the assumption, we have $|q| < 1$. Compare with q in (4.7), where $|q| = 1$ when b is real. This difference is due to the fact that $\boldsymbol{\Psi}_q(x)$ is convergent only for $|q| < 1$, while $\boldsymbol{\Phi}_b(z)$ is well-defined also for $|q| = 1$. The phase λ is the main difference between the two cases and the source of extra complication for $\boldsymbol{\Psi}_q(x)$ which persists throughout the section.

The asymptotic property (1.4) is written as

$$\boldsymbol{\Psi}_q(x) \sim \exp\left(\frac{\sqrt{-1}}{\lambda^2 \hbar} \frac{1}{2} \mathrm{Li}_2(-x)\right), \qquad \hbar \to 0. \quad (A.2)$$

Because of λ, the argument x of the dilogarithms $\mathrm{Li}_2(x)$ and $L(x)$ eventually take values in $\mathbb{C}$ in the semiclassical limit. They are defined by analytic continuation of (2.1) and (2.2) along the integration path. To avoid the ambiguity of the branches, we assume that $\operatorname{Im} \lambda$ *is sufficiently small* (or, q is sufficiently close to the unit circle $|q| = 1$) so that the resulting argument x in this section is in a neighborhood of the interval $(-\infty, 1]$ for $\mathrm{Li}_2(x)$ or $[0, 1]$ for $L(x)$.

To any quantum y-seed (B', Y') of (B, Y) we associate operators $\hat{u}' = (\hat{u}'_i)_{i \in I}$ and $\hat{p}' = (\hat{p}'_i)_{i \in I}$, and the local coordinates $\{\langle u'|\}_{u' \in \mathbb{R}^I}$ as in (4.8) and (4.9).

We reset $\hat{\mathsf{Y}}'_i$ in (4.10) as

$$\hat{w}'_i = \sum_{j \in I} b'_{ji} \hat{u}'_j, \qquad \hat{D}'_i = \hat{p}'_i + \hat{w}'_i, \qquad \hat{\mathsf{Y}}'_i = \exp(\lambda \hat{D}'_i). \quad (A.3)$$

The relations in (4.11) still hold with q in (A.1), and we have a representation of $\mathbb{T}(B, \mathsf{Y})$ of (4.12).

Let (B', Y') and (B'', Y'') be a pair of quantum y-seeds of (B, Y) such that $(B'', Y'') = \mu_k(B', Y')$. Let $\rho_{k,\varepsilon}$ be the map in (4.13). Then, repeating the argument in Section 4.3, we obtain

$$\mathsf{Ad}(\rho^*_{k,\varepsilon})(\hat{\mathsf{Y}}''_i) = \begin{cases} \hat{\mathsf{Y}}'^{\,-1}_k, & i = k, \\ \hat{\mathsf{Y}}'^{e_i + [\varepsilon b'_{ki}]_+ e_k}, & i \neq k. \end{cases}$$

A.2 Quantum dilogarithm identities in local form for $\boldsymbol{\Psi}_q(x)$

Under the same assumption and notation for Theorem 4.7, we obtain the counterpart of Theorem 4.7 for $\boldsymbol{\Psi}_q(x)$ by repeating its proof.

Theorem A.1. *For the tropical sign-sequence $\vec{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_L)$, the following identity holds*

$$\boldsymbol{\Psi}_q(\hat{\mathsf{Y}}_{k_1}(1)^{\varepsilon_1})^{\varepsilon_1} \rho^*_{k_1,\varepsilon_1} \cdots \boldsymbol{\Psi}_q(\hat{\mathsf{Y}}_{k_L}(L)^{\varepsilon_L})^{\varepsilon_L} \rho^*_{k_L,\varepsilon_L} \nu^* = 1. \tag{A.4}$$

Let $\hat{O}$ be the composition of the operators in the left hand side of (A.4). Again, choose any position eigenvector $|u(1)\rangle$ and set the momentum eigenvector $|\tilde{p}(1)\rangle$ by (5.4). Set

$$F(u(1), \tilde{p}(1)) := \frac{\langle u(1)|\hat{O}|\tilde{p}(1)\rangle}{\langle u(1)|\tilde{p}(1)\rangle}.$$

Below we show that the leading term of $\log F(u(1), \tilde{p}(1))$ in the limit $\hbar \to 0$ yields the left hand side of (2.13) up to a multiplicative constant.

A.3 Integral expression

Repeating the argument in Section 5.2, we obtain the following integral expression

$$\begin{aligned} F(u(1), \tilde{p}(1)) = (2\pi\hbar)^{-n(L-1)} \int & dp(1) \cdots dp(L-1) du(2) \cdots du(L) \\ & \times \boldsymbol{\Psi}_q\left(y_{k_1}(1)^{\varepsilon_1}\right)^{\varepsilon_1} \exp\left(\frac{\sqrt{-1}}{\hbar} u(1)(p(1) - \tilde{p}(1))\right) \\ & \times \boldsymbol{\Psi}_q\left(y_{k_2}(2)^{\varepsilon_2}\right)^{\varepsilon_2} \exp\left(\frac{\sqrt{-1}}{\hbar} u(2)(p(2) - \tilde{p}(2))\right) \cdots \\ & \times \boldsymbol{\Psi}_q\left(y_{k_L}(L)^{\varepsilon_L}\right)^{\varepsilon_L} \exp\left(\frac{\sqrt{-1}}{\hbar} u(L)(p(L) - \tilde{p}(L))\right), \end{aligned}$$

where $\tilde{p}(t)$ is the one in Section 5.2, while we reset

$$y_{k_t}(t) = \exp\left(\lambda(p_{k_t}(t) + w_{k_t}(t))\right). \tag{A.5}$$

Using (A.2), we have

$$\begin{aligned} F(u(1), \tilde{p}(1)) \sim (2\pi\hbar)^{-n(L-1)} \int & dp(1) \cdots dp(L-1) du(2) \cdots du(L) \\ & \times \exp\left(\frac{\sqrt{-1}}{\hbar} \sum_{t=1}^{L} \left\{ \frac{1}{2\lambda^2} \varepsilon_t \mathrm{Li}_2(-y_{k_t}(t)^{\varepsilon_t}) + u(t)(p(t) - \tilde{p}(t)) \right\}\right). \end{aligned} \tag{A.6}$$

A.4 Saddle point equations

The saddle point equations for (A.6) are obtained in the same manner as in Section 5.3. We use the following formulas, which are obtained from (2.1), (A.3), and (A.5)

$$\frac{\partial}{\partial p_i(t)}\left(\frac{1}{2\lambda^2}\varepsilon_t \mathrm{Li}_2(-y_{k_t}(t)^{\varepsilon_t})\right) = \delta_{ik_t}\frac{1}{\lambda}\log(1+y_{k_t}(t)^{\varepsilon_t})^{-1/2},$$
$$\frac{\partial}{\partial u_i(t)}\left(\frac{1}{2\lambda^2}\varepsilon_t \mathrm{Li}_2(-y_{k_t}(t)^{\varepsilon_t})\right) = -\frac{1}{\lambda}\log(1+y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2}.$$

(a) Extremum conditions with respect to $u_i(t)$ $(t=2,\dots,L)$.
By differentiating the integrand of (A.6) by $u_i(t)$, we have

$$-\frac{1}{\lambda}\log(1+y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2} + p_i(t) - \tilde{p}_i(t) = 0. \tag{A.7}$$

Combining it with (5.6), we also have

$$e^{\lambda\tilde{p}_i(t+1)} = \begin{cases} (e^{\lambda\tilde{p}_{k_t}(t)})^{-1}, & i = k_t, \\ e^{\lambda\tilde{p}_i(t)}(e^{\lambda\tilde{p}_{k_t}(t)})^{[\varepsilon_t b_{k_t i}(t)]_+}(1+y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2}, & i \neq k_t. \end{cases} \tag{A.8}$$

(b) Extremum conditions with respect to $p_i(t)$ $(t=1,\dots,L-1)$.
By differentiating the integrand of (A.6) by $p_i(t)$, we have

$$\frac{1}{\lambda}\log(1+y_{k_t}(t)^{\varepsilon_t})^{-1/2} + u_{k_t}(t) \\ \qquad - \sum_{j\in I}[\varepsilon_t b_{k_t j}(t)]_+ u_j(t+1) + u_{k_t}(t+1) = 0, \quad i = k_t, \tag{A.9}$$

$$u_i(t) - u_i(t+1) = 0, \qquad i \neq k_t. \tag{A.10}$$

With (2.6), this also implies the following equations for $w_i(u) = \sum_{j\in I} b_{ji}(t)u_j(t)$.

$$e^{\lambda w_i(t+1)} = \begin{cases} (e^{\lambda w_{k_t}(t)})^{-1}, & i = k_t, \\ e^{\lambda w_i(t)}(e^{\lambda w_{k_t}(t)})^{[\varepsilon_t b_{k_t i}(t)]_+}(1+y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2}, & i \neq k_t. \end{cases} \tag{A.11}$$

A.5 Solution

The (complex) solution of the saddle point equations is constructed in the same manner as in Section 5.4 and given as follows.

(*i*) (y-variables) We have $u_i(1)$ as initial data, from which $w_i(1)$ is uniquely determined. Temporarily forgetting (A.5), set $y_i(1) = e^{2\lambda w_i(1)}$, from which $y_i(t)$ $(t=2,\dots,L)$ are determined by the mutation sequence (2.9).

(*ii*) (u-variables) Set $u_i(t)$ $(t=2,\dots,L)$ by (A.10) and (A.9). Then, (A.11) is also satisfied.

(*iii*) (p-variables) Set $\tilde{p}_i(t)$ by $e^{\lambda\tilde{p}_i(t)} = y_i(t)^{1/2}$. This forces the relation $\tilde{p}_i(1) = w_i(1)$, which is guaranteed by the assumption (5.4). Then, $p_i(t)$ are determined by (5.5) and (5.6). Since $\tilde{p}_i(t)$ satisfies (A.8) by definition, (A.7) is also satisfied.

A.6 Result

The evaluation of the logarithm of the integrand in (A.6) at the solution of the saddle point equations is done in the same manner as in Section 5.5. Using (A.2) and ignoring the common factor, it is given by

$$\sum_{t=1}^{L}\left\{\frac{1}{2}\varepsilon_t \mathrm{Li}_2(-y_{k_t}(t)^{\varepsilon_t}) + \lambda^2\sum_{i\in I}u_i(t)(p_i(t)-\tilde{p}_i(t))\right\}.$$

Recall that

$$\lambda(p_i(t) - \tilde{p}_i(t)) = \log(1 + y_{k_t}(t)^{\varepsilon_t})^{-b_{k_t i}(t)/2},$$
$$\lambda w_i(t) = \frac{1}{2}\log y_i(t).$$

Then, repeating the calculation in (5.18), we obtain the classical dilogarithm identity (2.13) with complex argument. Taking the limit $\lambda \to 1$ further, we recover the identity (2.13) with real argument.

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