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Another New Solvable Many-Body Model of Goldfish Type*

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Abstract. A new *solvable* many-body problem is identified. It is characterized by nonlinear *Newtonian* equations of motion ("acceleration equal force") featuring one-body and two-body velocity-dependent forces "of goldfish type" which determine the motion of an arbitrary number N of unit-mass point-particles in a plane. The N (generally *complex*) values $z_n(t)$ at time t of the N coordinates of these moving particles are given by the N eigenvalues of a time-dependent $N \times N$ matrix $U(t)$ explicitly known in terms of the $2N$ initial data $z_n(0)$ and $\dot{z}_n(0)$. This model comes in two different variants, one featuring 3 arbitrary coupling constants, the other only 2; for special values of these parameters *all* solutions are completely periodic with the same period independent of the initial data ("*isochrony*"); for other special values of these parameters this property holds up to corrections vanishing exponentially as $t \to \infty$ ("*asymptotic isochrony*"). Other *isochronous* variants of these models are also reported. Alternative formulations, obtained by changing the dependent variables from the N zeros of a monic polynomial of degree N to its N coefficients, are also exhibited. Some mathematical findings implied by some of these results – such as *Diophantine* properties of the zeros of certain polynomials – are outlined, but their analysis is postponed to a separate paper.

Key words: nonlinear discrete-time dynamical systems; integrable and solvable maps; isochronous discrete-time dynamical systems; discrete-time dynamical systems of goldfish type

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1 Introduction

The technique used in this paper to identify a new *solvable* many-body problem has become by now standard. Its more convenient version starts from the identification of a *solvable* matrix problem characterized by two *first-order*, generally *autonomous*, matrix ODEs defining the time evolution of two $N \times N$ matrices $U \equiv U(t)$ and $V \equiv V(t)$:

$$\dot{U} = F(U,V), \qquad \dot{V} = G(U,V), \tag{1.1a}$$

where the two functions $F(U,V)$, $G(U,V)$ may depend on several scalar parameters but on no other matrix besides U and V. Here and hereafter superimposed dots denote of course differentiations with respect to the independent variable t ("time"). The *solvable* character of this system amounts to the possibility to obtain the solution of its initial-value problem,

$$U(t) = U(t;U_0,V_0), \qquad V(t) = V(t;U_0,V_0), \qquad U_0 \equiv U(0), \qquad U_0 \equiv V(0), \tag{1.1b}$$

*This paper is a contribution to the Special Issue "Geometrical Methods in Mathematical Physics". The full collection is available at http://www.emis.de/journals/SIGMA/GMMP2012.html with the two matrix functions $U(t; U_0, V_0)$, $V(t; U_0, V_0)$ *explicitly* known. For instance, in this paper we shall focus (see (3.3) below) on a system (1.1a) which can be related to a *linear* (third-order, matrix) evolution equation (see (3.3) below), so that its initial-value problem can be *explicitly solved*.

One then introduces the eigenvalues $z_n(t)$ of one of these two matrices, by setting, say

$$U(t) = R(t)Z(t)[R(t)]^{-1}, \qquad Z(t) = \mathrm{diag}[z_n(t)]. \tag{1.2}$$

Remark 1.1. The diagonalizing matrix $R(t)$ is identified by this formula up to right-multiplication by an *arbitrary diagonal* $N \times N$ matrix $D(t)$, $R(t) \Rightarrow \tilde{R}(t) \equiv R(t)D(t)$.

Then one introduces a new $N \times N$ matrix $Y(t)$ by setting

$$V(t) = R(t)Y(t)[R(t)]^{-1}, \qquad Y(t) = [R(t)]^{-1}V(t)R(t), \tag{1.3a}$$

where of course the $N \times N$ matrix $R(t)$ is that defined above, see (1.2). This matrix $Y(t)$ is of course generally *nondiagonal*:

$$Y_{nm}(t) = \delta_{nm} y_n(t) + (1 - \delta_{nm}) Y_{nm}(t). \tag{1.3b}$$

Notation 1.1. Indices such as n, m, ℓ run from 1 to N (unless otherwise explicitly indicated), and N is an *arbitrary positive integer* (indeed generally $N \geq 2$). δ_{nm} is the Kronecker symbol, $\delta_{nm} = 1$ if $n = m$, $\delta_{nm} = 0$ if $n \neq m$. Note that hereafter we use the notation Y_{nm} to denote the $N\,(N-1)$ *off-diagonal* elements of the $N \times N$ matrix Y.

It often turns out that the time evolution of the eigenvalues $z_n\,(t)$ is then characterized by a system of N second-order ODEs which read as follows:

$$\ddot{z}_n = f(z_n, \dot{z}_n) + \sum_{\ell=1,\,\ell=n}^{N} \left[Y_{n\ell} Y_{\ell n} \frac{g^{(1)}(z_n, \dot{z}_n) g^{(2)}(z_\ell, \dot{z}_\ell)}{z_n - z_\ell} \right], \tag{1.4}$$

where the three functions $f(z, \dot{z})$, $g^{(1)}(z, \dot{z})$ and $g^{(2)}(z, \dot{z})$ can be computed from the two matrix functions $F(U, V)$, $G(U, V)$ (see below). It is then natural to try and interpret this system of ODEs as the *Newtonian* equations of motion ("acceleration equal force") of a many-body problem characterizing the motion of N particles whose coordinates coincide with the N eigenvalues $z_n(t)$; an N-body problem generally featuring one-body and two-body velocity-dependent forces, with the $N(N-1)$ quantities $Y_{n\ell} Y_{\ell n}$ playing the role of "coupling constants". But these quantities are not time-independent, nor can they be arbitrarily assigned: they are the *off-diagonal* elements of the $N \times N$ matrix Y, hence they should be themselves considered as *dependent variables*, the time evolution of which is characterized by the system of $N(N-1)$ ODEs implied for them by (1.1a) via (1.2) and (1.3).

Two options are then open to provide nonetheless a "physical" interpretation for the equations of motion (1.4).

One option that we do not pursue here is to provide some kind of "physical" interpretation for these $N(N-1)$ quantities $Y_{n\ell} Y_{\ell n}$ as additional (internal) degrees of freedom of the moving particles.

A second option – the one which we pursue below – is to find a (time-independent) *ansatz* expressing the $N(N-1)$ quantities $Y_{n\ell}$ in terms of the N coordinates z_m, or possibly of the $2N$ quantities z_m, $\dot{z}_m$; an *ansatz* consistent with the $N(N-1)$ equations of motion satisfied by the $N(N-1)$ quantities $Y_{n\ell}$, which satisfies these equations either *identically* (i.e., independently from the time evolution of the N coordinates $z_m(t)$) or, as it were, *self-consistently* (i.e., thanks to the time evolution (1.4) of the N coordinates $z_m(t)$ with the $N(N-1)$ quantities $Y_{n\ell}$ assigned according to the *ansatz*). Given a matrix system of type (1.1a) no technique is known to assess *a priori* whether or not such an *ansatz* exist. However the experience accumulated over time suggests that, if such an *ansatz* does exist, it has one of the following two forms:

$$\text{ansatz 1}: \quad Y_{n\ell} = \frac{g^{(1)}(z_n)g^{(2)}(z_\ell)}{z_n - z_\ell}; \tag{1.5a}$$

$$\text{ansatz 2}: \quad Y_{n\ell} = \left\{g^{(1)}(z_n)g^{(2)}(z_\ell)\left[\dot{z}_n + f^{(1)}(z_n)\right]\left[\dot{z}_\ell + f^{(2)}(z_\ell)\right]\right\}^{1/2}; \tag{1.5b}$$

of course with the functions appearing in the right-hand side of these formulas chosen appropriately. And as a rule the *ansatz 1* should work *identically*, i.e. independently of the time evolution of the coordinates $z_m(t)$; while to ascertain the validity of *ansatz 2* the equations of motion (1.4) should be used *self-consistently*.

For instance in the very simple case of the equations of motion (1.1a) with $F(U,V) = V$ and $G(U,V) = 0$ implying $\ddot{U} = 0$ and $U(t) = U_0 + V_0 t$, $V(t) = V_0$, both *ansätze* exist: the *ansatz 1* reads in this case $Y_{n\ell} = ig/(z_n - z_\ell)$ with i the imaginary unit (introduced for convenience) and g an *arbitrary constant*, and it yields the prototypical "CM" model characterized by the equations of motion

$$\ddot{z}_n = 2g^2 \sum_{\ell=1,\,\ell\neq n}^{N} (z_n - z_\ell)^{-3}; \tag{1.6}$$

while the *ansatz 2* in this case reads $Y_{n\ell} = (\dot{z}_n \dot{z}_\ell)^{1/2}$, and it yields the prototypical "goldfish" model characterized by the equations of motion

$$\ddot{z}_n = \sum_{\ell=1,\,\ell\neq n}^{N} \left(\frac{2\dot{z}_n \dot{z}_\ell}{z_n - z_\ell}\right). \tag{1.7a}$$

Nomenclature and historical remarks. The model characterized by the Newtonian equations of motion (1.6) – which obtain from the Hamiltonian

$$H_{\mathrm{CM}}(\underline{z},\underline{p}) = \frac{1}{2}\sum_{n=1}^{N} p_n^2 + \frac{1}{2}g^2 \sum_{n,m=1,\,n\neq m}^{N} (z_n - z_m)^{-2}$$

– is usually associated with the names of those who first demonstrated the possibility to treat this many-body problem exactly, respectively in the *quantal* [1] and in the *classical* [2] contexts; accordingly, we usually call "many-body problems of CM-type" those featuring in the right-hand ("forces") side of their Newtonian equations of motion a term such as that appearing in the right-hand side of (1.6) (in addition of course to other terms). Such models are typically produced by the *ansatz 1*.

The *solvable* character of the many-body problem characterized by the Newtonian equations of motion (1.7a) – which is also Hamiltonian, for instance with the Hamiltonian [3, 4, 5]

$$H_{\mathrm{gold}}(\underline{z},\underline{p}) = \sum_{n=1}^{N}\left[\exp(p_n)\prod_{m=1,\,m\neq n}^{N}(z_n - z_m)^{-1}\right]$$

– is demonstrated by the following neat *Prescription* [5, 6, 7]: the N values of the coordinates $z_n(t)$ providing the solution of the initial-value problem of the equations of motion (1.7a) are the N zeros of the following algebraic equation for the variable z:

$$\sum_{n=1}^{N}\left[\frac{\dot{z}_n(0)}{z - z_n(0)}\right] = \frac{1}{t}. \tag{1.7b}$$

(Note that this formula amounts to a polynomial equation of degree N in z, as it is immediately seen by multiplying it by the polynomial $\prod_{m=1}^{N}[z - z_m(0)]$). In [7] it was suggested that this model, in view of its neat character, be considered a "goldfish" (meaning, in Russian traditional lore, a very remarkable item, endowed with magical properties); accordingly, we usually call "many-body problems of goldfish-type" those featuring in the right-hand ("forces") side of their Newtonian equations of motion a term such as that appearing in the right-hand side of (1.7a) (in addition of course to other terms). Such models are typically produced by the *ansatz 2*.

The original idea of the approach described above is due to Olshanetsky and Perelomov, who introduced it to solve, in the classical context, the many-body model characterized by the Newtonian equations of motion (1.6) [8]. For a more detailed description of their work see the review paper [9], the book [10], Section 2.1.3.2 (entitled "The technique of solution of Olshanetsky and Perelomov") in [11], and other references cited in these books. In Section 4.2.2 (entitled "Goldfishing") of [5] several many-body problems, mainly "of goldfish type", are reviewed, the *solvable* character of which has been ascertained by this approach; and several other such models are discussed in more recent papers [12, 13, 14, 15].

The present paper provides two further additions to the list of *solvable* many-body problems "of goldfish type"; and we expect that other items will be added to this list in the future, possibly by a continuation of the case-by-case search of *solvable* matrix evolution equations allowing – via the route outlined above and described in more detail, in a specific case, below (see Section 3) – the identification of working *ansätze* leading to *new* systems of Newtonian equations (thereby shown to be themselves *solvable*, inasmuch as their solution is reduced to the *algebraic* task of computing the N eigenvalues of an explicitly known $N \times N$ matrix). The identification of a *new* model of this kind constitutes – in our opinion – an interesting finding (even if several analogous models have been previously discovered): we view these many-body problems as gems embedded in the magma of the generic many-body problems which are not amenable to exact treatments (although the latter include of course more examples of applicative interest and are also mathematically interesting to investigate the emergence and phenomenology of chaotic behaviors).

In the following Section 2 the main findings of this paper are reported, including in particular a description of two new *solvable* many-body problems (one featuring 3, the other only 2, *a priori arbitrary* parameters), of the algebraic solution of their initial-value problems, and of the variety of behaviors (including *isochrony* and *asymptotic isochrony*) featured by them for certain assignments of their parameters; two additional *isochronous* systems are moreover exhibited in Section 2.1. In Section 3 these results are proven. Section 4 provides the alternative formulations of these models, obtained by changing the dependent variables from the N zeros of a monic polynomial of degree N to its N coefficients. A final Section 5 entitled "Outlook" outlines further developments, including in particular the identification of *Diophantine* properties of the zeros of certain polynomials; but their detailed discussion is postponed to a separate paper.

2 Main findings

The two models treated in this paper are characterized by the following two sets of *Newtonian* equations of motion "of goldfish type".

Model (i):

$$\ddot{z}_n = -3\dot{z}_n z_n + \gamma \dot{z}_n - z_n^3 + \gamma z_n^2 + [-a + b(\gamma + b)]z_n + a(\gamma + b) + \sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{2\left(\dot{z}_n + a + bz_n + z_n^2\right)\left(\dot{z}_\ell + a + bz_\ell + z_\ell^2\right)}{z_n - z_\ell}\right], \tag{2.1a}$$

where a, b and γ are 3 *a priori arbitrary* parameters.

Model (ii):

$$\ddot{z}_n = -3\dot{z}_n z_n - 3b\dot{z}_n - z_n^3 - 3bz_n^2 - 2\left(a+b^2\right)z_n - 2ab + \sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{2\left(\dot{z}_n + a + bz_n + z_n^2/2\right)\left(\dot{z}_\ell + a + bz_\ell + z_\ell^2/2\right)}{z_n - z_\ell}\right], \tag{2.1b}$$

where a and b are 2 *a priori arbitrary* parameters.

Notation 2.1. These models describe the motion of an *arbitrary* number N (generally $N \geq 2$) of points moving in the *complex* z-plane (but see below Remark 2.1). Their positions are identified by the *complex* dependent variables $z_n \equiv z_n(t)$. The independent variable t ("time") is instead *real*. Superimposed dots denote of course time-differentiations. The parameters featured by these models are generally *arbitrary complex* numbers; unless otherwise specified when discussing special cases.

Remark 2.1. It is possible to reformulate these models so that they describe the motion of pointlike "physical particles" moving in a *real* – say, horizontal – plane: see for instance, in [11], Section 4.1 entitled "How to obtain by complexification rotation-invariant many-body models in the plane from certain many-body models on the line". This task is left to the interested reader. But hereafter we feel free to refer to the models identified by the *Newtonian* equations of motion (2.1) as many-body problems characterizing the motion of N particles in a plane.

Remark 2.2. Additional parameters could be inserted in these models by shifting or rescaling the dependent variables z_n or the independent variable t. We will not indulge in such trivial exercises (see also Remark 3.1 below).

The *solvable* character of these two models is demonstrated by the following

Proposition 2.1. *The solution of the initial-value problems of the two many-body models characterized by the Newtonian equations of motion "of goldfish type"* (2.1) *are given by the eigenvalues of the $N \times N$ matrix $U(t)$, the explicit expression of which in terms of the $2N$ initial data $z_n(0)$, $\dot{z}_n(0)$ and of the time t is given by the following formulas* (*see* (3.6a) *with* (3.7))*:*

$$U(t) = i\left\{I + A\exp[i(\omega_2-\omega_1)t] + B\exp[i(\omega_3-\omega_1)t]\right\}^{-1} \times \left\{\omega_1 I + \omega_2 A\exp[i(\omega_2-\omega_1)t] + \omega_3 B\exp[i(\omega_3-\omega_1)t]\right\}, \tag{2.2a}$$

with the two constant $N \times N$ matrices A and B defined as follows:

$$A = -(\omega_1-\omega_3)(\omega_2-\omega_3)^{-1}\left[I - i(\omega_1+\omega_3)\left(V_0+\omega_3^2\right)^{-1}(U_0 - i\omega_3)^{-1}\right] \times \left[I - i(\omega_2+\omega_3)\left(V_0+\omega_3^2\right)^{-1}(U_0 - i\omega_3)^{-1}\right]^{-1}, \tag{2.2b}$$

$$B = (\omega_1-\omega_2)(\omega_2-\omega_3)^{-1}\left[I - i(\omega_1+\omega_2)\left(V_0+\omega_2^2\right)^{-1}(U_0 - i\omega_2)^{-1}\right] \times \left[I - i(\omega_2+\omega_3)\left(V_0+\omega_2^2\right)^{-1}(U_0 - i\omega_2)^{-1}\right]^{-1}. \tag{2.2c}$$

Here and hereafter I is the $N \times N$ unit matrix, the $N \times N$ matrix $U_0 \equiv U(0)$ is diagonal and is given in terms of the initial particle positions $z_n(0)$ as follows,

$$U_0 = \mathrm{diag}[z_n(0)], \tag{2.3}$$

while the $N \times N$ matrix $V_0 \equiv V(0)$ is the sum of a diagonal and a dyadic matrix, being given componentwise by the following formulas in terms of the initial particle positions $z_n(0)$ and velocities $\dot{z}_n(0)$:

$$(V_0)_{nm} = -\delta_{nm}\left[a + bz_n(0) + (c-1)z_n^2(0)\right] + V_n V_m, \tag{2.4a}$$

$$V_n = \left[\dot{z}_n(0) + a + b z_n(0) + c z_n^2(0)\right]^{1/2}, \tag{2.4b}$$

with $c = 1$ *for model* (i) *and* $c = 1/2$ *for model* (ii) (*see* (2.6)). *As for the* 3 *constants* ω_j *appearing in* (2.2), *they are defined by the following formula* (*see* (3.5b)):

$$\omega^3 + i\gamma\omega^2 + \beta\omega - i\alpha = (\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3) \tag{2.5a}$$

implying

$$\alpha = -i\omega_1\omega_2\omega_3, \qquad \beta = \omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1, \qquad \gamma = i(\omega_1 + \omega_2 + \omega_3), \tag{2.5b}$$

with α *and* β *respectively* α, β *and* γ *given by the following expressions for model* (i) *respectively for model* (ii) (*see* (3.21a) *respectively* (3.21b)):

model (i):

$$c = 1, \qquad \alpha = a(\gamma + b), \qquad \beta = -a + b(\gamma + b); \tag{2.6a}$$

model (ii):

$$c = \frac{1}{2}, \qquad \alpha = -2ab, \qquad \beta = -2\left(a + b^2\right), \qquad \gamma = -3b. \tag{2.6b}$$

Note that the 3 *a priori arbitrary* parameters ω_j have the dimension of an *inverse time*; above and hereafter we assume for simplicity that they are *different* among each other (except in the following Subsection 2.1, where they are all assumed to vanish).

It is plain (see (2.2)) that, if the 3 constants ω_j are *integer* multiples of a single *real* constant ω,

$$\omega_j = k_j\omega, \qquad j = 1, 2, 3, \tag{2.7a}$$

then the matrix $U(t)$ is periodic,

$$U(t) = U(t \pm T), \tag{2.7b}$$

with period

$$T = \frac{2\pi}{|\omega|}. \tag{2.8}$$

Here and throughout the 3 parameters k_j are *integer numbers* (positive, negative or vanishing, but different among themselves); their definition, as well as that of the *positive* parameter ω, is made unequivocal (up to permutations; once the 3 parameters α, β, γ are assigned, compatibly via (2.5b) with (2.7a)) by the requirement that this *positive* parameter ω be assigned the *largest* value for which (2.7a) holds.

More generally, if the *real parts* of the 3 constants ω_j are *integer* multiples of a single *real* constant ω and the *imaginary parts* of 2 of them coincide while the *imaginary part* of the third is larger, say,

$$\mathrm{Re}(\omega_j) = k_j\omega, \quad j = 1, 2, 3; \qquad \mathrm{Im}(\omega_1) = \mathrm{Im}(\omega_2) < \mathrm{Im}(\omega_3), \tag{2.9a}$$

then the matrix $U(t)$ is *asymptotically periodic* with period T, namely it becomes periodic with period T in the remote future *up to exponentially vanishing corrections*, so that

$$\lim_{t\to\infty} |U(t) - U(t \pm T)| = 0. \tag{2.9b}$$

While for *generic* values of the parameters, implying via (2.5) that the *imaginary parts* of the 3 quantities ω_j are *different* among themselves,

$$\mathrm{Im}(\omega_1) \neq \mathrm{Im}(\omega_2), \qquad \mathrm{Im}(\omega_2) \neq \mathrm{Im}(\omega_3), \qquad \mathrm{Im}(\omega_3) \neq \mathrm{Im}(\omega_1), \tag{2.10a}$$

then clearly $U(t)$ tends to a time-independent matrix as $t \to \pm\infty$,

$$U(t) \underset{t\to\pm\infty}{\rightarrow} U(\pm\infty). \tag{2.10b}$$

Let us emphasize that these outcomes, (2.7b) or (2.9b) or (2.10b), obtain – provided the 3 parameters α, β, γ satisfy (2.5b) with (2.7a) or (2.9a) or (2.10a) – for *arbitrary initial data*, and that the period T is as well *independent of the initial data*. Let us also note that these properties hold unless $U(t)$ is *singular*; clearly a *nongeneric* circumstance, in which case $U(t)$ would in fact still feature the properties indicated above, but only in the sense in which, for instance, the function $\tan[\omega(t-t_0)/2]$ (with ω and t_0 two *real* numbers) is periodic with period T.

These properties of the $N \times N$ matrix $U(t)$ carry of course over to its eigenvalues $z_n(t)$, hence to the generic solutions of the many-body problems "of goldfish type" (2.1); these models are therefore *isochronous* respectively *asymptotically isochronous* if their parameters satisfy the relevant conditions, see (2.5b) with (2.7a) respectively (2.9a).

Remark 2.3. Let us however recall that the periods of the time evolution of individual eigenvalues of a periodic matrix may be a (generally small) *positive integer multiple* of the period of the matrix, due to the possibility that different eigenvalues exchange their roles over the time evolution (for a discussion of this possibility see [16], where a justification is also provided of the statement made above that the relevant *positive integer multiple* is "generally small").

2.1 Two additional *isochronous* many-body models

Additional *isochronous* models obtain by applying, to the special cases of the two many-body models (2.1) with all parameters vanishing,

$$\ddot{z}_n = -3\dot{z}_n z_n - z_n^3 + + \sum_{\ell=1,\,\ell\neq n}^{N} \left[\frac{2(\dot{z}_n + c z_n^2)(\dot{z}_\ell + c z_\ell^2)}{z_n - z_\ell}\right], \tag{2.11}$$

with $c = 1$ respectively $c = 1/2$, the standard "isochronizing" trick, see for instance Section 2.1 (entitled "The trick") of [5]. It amounts in these cases to the following change of dependent and independent variables,

$$\tilde{z}_n(t) = \exp(i\omega t) z_n(\tau), \qquad \tau = \frac{\exp(i\omega t) - 1}{i\omega}, \tag{2.12a}$$

implying

$$\tilde{z}_n(0) = z_n(0), \qquad \dot{\tilde{z}}_n(0) = z_n'(0) + i\omega z_n(0). \tag{2.12b}$$

Here and below ω is again an arbitrary *real* constant to which we associate the period T, see (2.8), and of course appended primes denote differentiations with respect to the argument of the function they are appended to (hence $z_n'(\tau) = dz_n(\tau)/d\tau$). Hence clearly the Newtonian equations of motion of these two models read as follows:

$$\ddot{\tilde{z}}_n = 3i\omega\dot{\tilde{z}}_n + 2\omega^2\tilde{z}_n - 3\dot{\tilde{z}}_n\tilde{z}_n + 3i\omega\tilde{z}_n^2 - \tilde{z}_n^3$$

$$+ \sum_{\ell=1,\, \ell\neq n}^{N} \left[\frac{2(\dot{\tilde{z}}_n - i\omega\tilde{z}_n + c\tilde{z}_n^2)(\dot{\tilde{z}}_\ell - i\omega\tilde{z}_\ell + c\tilde{z}_\ell^2)}{\tilde{z}_n - \tilde{z}_\ell} \right], \tag{2.13}$$

again with $c = 1$ respectively $c = 1/2$. The solutions of the corresponding initial-value problems are clearly given by the following variant (of the case with $\omega_j = 0$) of Proposition 2.1:

Proposition 2.2. *The solution of the initial-value problems of the two many-body models characterized by the Newtonian equations of motion "of goldfish type" (2.13) are given by the eigenvalues of the $N \times N$ matrix $\tilde{U}(t)$ given by the following formulas in terms of the $2N$ initial data $\tilde{z}_n(0)$, $\dot{\tilde{z}}_n(0)$ and of the time t:*

$$\tilde{U}(t) = \exp(i\omega t)\big(I + \tilde{A}\tau + \tilde{B}\tau^2\big)^{-1}(\tilde{A} + 2\tilde{B}\tau), \tag{2.14a}$$

with the two constant $N \times N$ matrices $\tilde{A}$ and $\tilde{B}$ defined as follows:

$$\tilde{A} = \operatorname{diag}[\tilde{z}_n(0)], \qquad \tilde{B}_{nm} = \frac{1}{2}v_n v_m, \qquad v_n = \dot{\tilde{z}}_n(0) - i\omega\tilde{z}_n(0) + c\tilde{z}_n^2(0), \tag{2.14b}$$

of course always with $c = 1$ respectively $c = 1/2$ and $\tau \equiv \tau(t)$ defined in terms of the time t by (2.12a). Note that the $N \times N$ matrix $\tilde{A}$ is diagonal and the $N \times N$ matrix $\tilde{B}$ is now dyadic.

It is plain that the matrix $\tilde{U}(t)$ is *isochronous* with period T (see (2.14a), (2.12a) and (2.8)),

$$\tilde{U}(t \pm T) = \tilde{U}(t), \tag{2.15}$$

and the same property of *isochrony* holds therefore for the generic solutions of the two many-body models (2.13), up to the observation made above (see Remark 2.3).

3 Proofs

The starting point of our treatment is the following system of two coupled matrix ODEs satisfied by the two $N \times N$ matrices $U \equiv U(t)$ and $V \equiv V(t)$:

$$\dot{U} = -U^2 + V, \qquad \dot{V} = -UV + \alpha I + \beta U + \gamma V. \tag{3.1}$$

Notation 3.1. The 3 scalars α, β, γ are 3, *a priori* arbitrary, constant parameters; I is the unit $N \times N$ matrix; and we trust the rest of the notation to be self-evident (see also Sections 1 and 2).

Remark 3.1. Additional parameters could of course be introduced by scalar shifts or rescalings of the dependent variables U, V or of the independent variable t ("time"). We forsake any discussion of such trivial transformations (see Remark 2.2).

To solve this matrix system we introduce the $N \times N$ matrix $W \equiv W(t)$ by setting

$$U(t) = [W(t)]^{-1}\dot{W}(t), \qquad V(t) = [W(t)]^{-1}\ddot{W}(t). \tag{3.2}$$

It is then easily seen that the system (3.1) entails that the matrix W satisfy the following *linear third-order* matrix ordinary differential equation (ODE):

$$\dddot{W} = \alpha W + \beta\dot{W} + \gamma\ddot{W}; \tag{3.3}$$

and the converse is as well true (in fact, perhaps easier to verify), namely if W satisfies this *linear third-order* matrix ODE, then the two matrices U and V defined by (3.2) satisfy the system (3.1).

Clearly the *general* solution of this *linear third-order* matrix ODE reads

$$W(t) = \sum_{j=1}^{3} \left[W^{(j)} \exp(i\omega_j t) \right], \tag{3.4}$$

where the 3 *constant* matrices $W^{(j)}$ are *arbitrary*, i is the *imaginary unit* ($i^2 = -1$; introduced here for notational convenience), and the 3 scalars ω_j are the 3 roots of the following cubic equation in ω:

$$\omega^3 + i\gamma\omega^2 + \beta\omega - i\alpha = 0, \tag{3.5a}$$

$$\omega^3 + i\gamma\omega^2 + \beta\omega - i\alpha = (\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3), \tag{3.5b}$$

so that

$$\alpha = -i\omega_1\omega_2\omega_3, \qquad \beta = \omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1, \qquad \gamma = i(\omega_1 + \omega_2 + \omega_3). \tag{3.5c}$$

It is then easily seen that the *general* solution of the system (3.1) can be written as follows:

$$\begin{aligned} U(t) = {} & i\big\{ I + A \exp[i(\omega_2 - \omega_1)t] + B \exp[i(\omega_3 - \omega_1)t] \big\}^{-1} \\ & \times \big\{ \omega_1 I + \omega_2 A \exp[i(\omega_2 - \omega_1)t] + \omega_3 B \exp[i(\omega_3 - \omega_1)t] \big\}, \end{aligned} \tag{3.6a}$$

$$\begin{aligned} V(t) = {} & -\big\{ I + A \exp[i\omega_2 - \omega_1)t] + B \exp[i(\omega_3 - \omega_1)t] \big\}^{-1} \\ & \times \big\{ \omega_1^2 I + \omega_2^2 A \exp[i(\omega_2 - \omega_1)t] + \omega_3^2 B \exp[i(\omega_3 - \omega_1)t] \big\}, \end{aligned} \tag{3.6b}$$

where A and B are two, *a priori* arbitrary, *constant* $N \times N$ matrices. And a trivial if tedious computation shows that these formulas provide the solution of the *initial-value* problem for the system (3.1) if the two matrices A and B are expressed in terms of the *initial values* $U_0 \equiv U(0)$, $V_0 \equiv V(0)$ as follows:

$$\begin{aligned} A = {} & -(\omega_1 - \omega_3)(\omega_2 - \omega_3)^{-1} \big[I - i(\omega_1 + \omega_3)\big(V_0 + \omega_3^2\big)^{-1}(U_0 - i\omega_3)^{-1} \big] \\ & \times \big[I - i(\omega_2 + \omega_3)\big(V_0 + \omega_3^2\big)^{-1}(U_0 - i\omega_3)^{-1} \big]^{-1}, \end{aligned} \tag{3.7a}$$

$$\begin{aligned} B = {} & (\omega_1 - \omega_2)(\omega_2 - \omega_3)^{-1} \big[I - i(\omega_1 + \omega_2)\big(V_0 + \omega_2^2\big)^{-1}(U_0 - i\omega_2)^{-1} \big] \\ & \times \big[I - i(\omega_2 + \omega_3)\big(V_0 + \omega_2^2\big)^{-1}(U_0 - i\omega_2)^{-1} \big]^{-1}. \end{aligned} \tag{3.7b}$$

To derive, from the *solvable* matrix system (3.1), the *solvable* many-body problem reported in the preceding section, we follow the procedure outlined in Section 1. This requires that we introduce – in addition to the *diagonal* $N \times N$ matrix Z respectively the *nondiagonal* $N \times N$ matrix Y associated to U respectively V via (1.2) respectively (1.3a) – the auxiliary $N \times N$ matrix $M \equiv M(t)$ defined as follows in terms of the *diagonalizing* matrix $R(t)$, see (1.2):

$$M(t) = [R(t)]^{-1} \dot{R}(t). \tag{3.8a}$$

In the following we indicate as $\mu_n \equiv \mu_n(t)$ respectively $M_{nm} \equiv M_{nm}(t)$ the *diagonal* respectively *off-diagonal* elements of this matrix:

$$M_{nm} = \delta_{nm}\mu_n + (1 - \delta_{nm})M_{nm}. \tag{3.8b}$$

Remark 3.2. As implied by Remark 1.1, the *diagonal* elements μ_n can be assigned freely, since the transformation $R(t) \Rightarrow \tilde{R}(t) = R(t)D(t)$ with $D(t) = \mathrm{diag}[d_n(t)]$ implies, for the *diagonal* elements $\tilde{\mu}_n(t)$ of the matrix $\tilde{M} = [\tilde{R}(t)]^{-1}\dot{\tilde{R}}(t)$, the expression $\tilde{\mu}_n(t) = \mu_n(t) + \dot{d}_n(t)/d_n(t)$, with a corresponding change of the *off-diagonal* elements of the matrix M, $M_{nm} \Rightarrow \tilde{M}_{nm} = \delta_n^{-1} M_{nm} \delta_m$. Note that here and hereafter we denote as M_{nm} the *off-diagonal* elements of the matrix M.

It is then easily seen that the equations (1.2) characterizing the time evolution of U and V imply the following equations characterizing the time evolution of Z and Y:

$$\dot{Z} + [M, Z] = -Z^2 + Y, \qquad \dot{Y} + [M, Y] = -ZY + \alpha I + \beta Z + \gamma Y. \tag{3.9}$$

Notation 3.2. The notation $[A, B]$ denotes the commutator of the two matrices A, B: $[A, B] \equiv AB - BA$.

Let us now look separately at the *diagonal* and *off-diagonal* parts of these two matrix equations, (3.9).

The *diagonal* part of the first of these two equations reads (see (1.2) and (1.3b))

$$\dot{z}_n = -z_n^2 + y_n, \tag{3.10a}$$

implying

$$y_n = \dot{z}_n + z_n^2. \tag{3.10b}$$

Likewise, the *off-diagonal* part of the first of these two equations reads

$$-(z_n - z_m)M_{nm} = Y_{nm}, \qquad n \neq m, \tag{3.11a}$$

implying

$$M_{nm} = -\frac{Y_{nm}}{z_n - z_m}, \qquad n \neq m. \tag{3.11b}$$

The *diagonal* part of the second of these two equations reads (see (1.2) and (1.3b))

$$\dot{y}_n = -z_n y_n + \alpha + \beta z_n + \gamma y_n + \sum_{\ell=1,\, \ell \neq n}^{N} (Y_{n\ell} M_{\ell n} - M_{n\ell} Y_{\ell n}), \tag{3.12a}$$

implying, via (3.10b) and (3.11b),

$$\dot{y}_n = -\dot{z}_n z_n + \gamma \dot{z}_n - z_n^3 + \gamma z_n^2 + \beta z_n + \alpha + 2 \sum_{\ell=1,\, \ell \neq n}^{N} \left(\frac{Y_{n\ell} Y_{\ell n}}{z_n - z_\ell} \right). \tag{3.12b}$$

We now note that, via this equation, time-differentiation of (3.10a) yields the following set of Newtonian-like equations of motion:

$$\ddot{z}_n = -3\dot{z}_n z_n + \gamma \dot{z}_n - z_n^3 + \gamma z_n^2 + \beta z_n + \alpha + 2 \sum_{\ell=1,\, \ell \neq n}^{N} \left(\frac{Y_{n\ell} Y_{\ell n}}{z_n - z_\ell} \right), \tag{3.13}$$

confirming the treatment outlined in Section 1, see in particular (1.4).

Finally we consider the *off-diagonal* elements of the second of the matrix equations (3.9). The relevant equations read, componentwise, as follows:

$$\dot{Y}_{nm} = -(z_n - \gamma)Y_{nm} + \sum_{k=1}^{N} (Y_{nk} M_{km} - M_{nk} Y_{km}), \qquad n \neq m, \tag{3.14a}$$

namely, via (1.3b) and (3.8b),

$$\dot{Y}_{nm} = -(z_n - \gamma)Y_{nm} - (\mu_n - \mu_m)Y_{nm} + (y_n - y_m)M_{nm}$$

$$+ \sum_{\ell=1,\, \ell\neq n,m}^{N} (Y_{n\ell} M_{\ell m} - M_{n\ell} Y_{\ell m}), \qquad n \neq m. \tag{3.14b}$$

And via (3.10b) and (3.11b) (and a tiny bit of algebra) this becomes

$$\frac{\dot{Y}_{nm}}{Y_{nm}} = -2z_n - z_m + \gamma - \mu_n + \mu_m - \frac{\dot{z}_n - \dot{z}_m}{z_n - z_m} + \sum_{\ell=1,\, \ell\neq n,m}^{N} \left[\frac{Y_{n\ell} Y_{\ell m}}{Y_{nm}} \left(\frac{1}{z_n - z_\ell} + \frac{1}{z_m - z_\ell} \right) \right], \qquad n \neq m. \tag{3.14c}$$

The next step is to try out the *ansätze* (1.5), to see if one can thereby get rid of the quantities Y_{nm}.

We leave the (rather easy but unfortunately unproductive) task to verify that the *ansatz* (1.5a) does not work, i.e. that it does not allow to eliminate the quantities Y_{nm} by finding an assignment of the functions $g^{(1)}(z)$ and $g^{(2)}(z)$ which, when inserted in (1.5a), yield $N(N-1)$ quantities Y_{nm} that satisfy the $N(N-1)$ ODEs (3.14c) (even by taking into account the possibility to assign freely – see Remark 3.2 – the N quantities μ_n).

We show that instead the *ansatz* (1.5b) allows the elimination of the quantities Y_{nm} and leads to the *Newtonian* equations of motion "of goldfish type" (2.1). Indeed the insertion in (3.14c) of (1.5b) with

$$g^{(1)}(z) = g^{(2)}(z) = g(z), \qquad f^{(1)}(z) = f^{(2)}(z) = f(z), \tag{3.15a}$$

and the assignment (see Remark 3.2)

$$\mu_n = -\frac{z_n}{2} \tag{3.15b}$$

entails that the $N(N-1)$ equations (3.14c) can be re-formulated as follows:

$$\frac{1}{2}\left\{ \frac{\ddot{z}_n + \dot{z}_n f'(z_n)}{\dot{z}_n + f(z_n)} + \frac{g'(z_n)}{g(z_n)} + 3z_n - \gamma + ((n \Rightarrow m)) \right\} = -\frac{\dot{z}_n - \dot{z}_m}{z_n - z_m} + \sum_{\ell=1,\, \ell\neq n,m}^{N} \left\{ g(z_\ell)[\dot{z}_\ell + f(z_\ell)] \left(\frac{1}{z_n - z_\ell} + \frac{1}{z_m - z_\ell} \right) \right\}, \qquad n \neq m. \tag{3.16a}$$

Notation 3.3. Here and below primes indicate differentiations with respect to the argument of the functions they are appended to; and the convenient shorthand notation $+((n \Rightarrow m))$ denotes addition of whatever comes before it, with the index n replaced by the index m.

It is easily seen that these equations can be re-written as follows:

$$\frac{\ddot{z}_n + \dot{z}_n f'(z_n)}{\dot{z}_n + f(z_n)} + \frac{g'(z_n)}{g(z_n)} + 3z_n - \gamma - \sum_{\ell=1,\, \ell\neq n}^{N} \left\{ \frac{g(z_\ell)[\dot{z}_\ell + f(z_\ell)]}{z_n - z_\ell} \right\} + ((n \Rightarrow m)) = 2\left\{ \frac{[g(z_n) - 1]\dot{z}_n - [g(z_m) - 1]\dot{z}_m + g(z_n)f(z_n) - g(z_m)f(z_m)}{z_n - z_m} \right\}, \qquad n \neq m. \tag{3.16b}$$

This suggests the assignments

$$g(z) = 1, \qquad f(z) = a + bz + cz^2, \tag{3.17}$$

with the 3 parameters a, b, c *a priori arbitrary*, since thereby the $N(N-1)$ equations (3.16b) get reduced to the N equations

$$\frac{\ddot{z}_n + \dot{z}_n(b+2cz_n)}{\dot{z}_n + a + bz_n + cz_n^2} = (2c-3)z_n + \gamma + b + 2\sum_{\ell=1,\,\ell\neq n}^{N}\left(\frac{\dot{z}_\ell + a + bz_\ell + cz_\ell^2}{z_n - z_\ell}\right), \tag{3.18a}$$

or equivalently (in Newtonian form)

$$\begin{aligned}\ddot{z}_n &= -3\dot{z}_n z_n + \gamma\dot{z}_n + (2c-3)cz_n^3 + [\gamma c + 3b(c-1)]z_n^2 \\ &\quad + [a(2c-3) + b(\gamma+b)]z_n + a(\gamma+b) \\ &\quad + \sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{2\big(\dot{z}_n + a + bz_n + cz_n^2\big)\big(\dot{z}_\ell + a + bz_\ell + cz_\ell^2\big)}{z_n - z_\ell}\right].\end{aligned} \tag{3.18b}$$

Consistency requires now that this set of N Newtonian equations of motions coincide with the N analogous equations (3.13), which, via (1.5b) with (3.15a) and (3.17), now read

$$\begin{aligned}\ddot{z}_n &= -3\dot{z}_n z_n + \gamma\dot{z}_n - z_n^3 + \gamma z_n^2 + \beta z_n + \alpha \\ &\quad + \sum_{\ell=1,\,\ell\neq n}^{N}\left[\frac{2\big(\dot{z}_n + a + bz_n + cz_n^2\big)\big(\dot{z}_\ell + a + bz_\ell + cz_\ell^2\big)}{z_n - z_\ell}\right].\end{aligned} \tag{3.19}$$

This clearly requires that the following 4 constraints on the 6 parameters α, β, γ, a, b, c be satisfied (note that, somewhat miraculously, the two velocity-dependent one-body terms in the right-hand sides of the last two equations match automatically, as well as the two-body terms):

$$(2c-3)c = -1, \tag{3.20a}$$

$$\gamma c + 3b(c-1) = \gamma, \tag{3.20b}$$

$$a(2c-3) + b(\gamma+b) = \beta, \tag{3.20c}$$

$$a(\gamma+b) = \alpha. \tag{3.20d}$$

And it is easily seen that this entails two alternative possibilities:

model (i): a, b, γ arbitrary and

$$c = 1, \qquad \beta = -a + b(\gamma+b), \qquad \alpha = a(\gamma+b); \tag{3.21a}$$

model (ii): a, b arbitrary and

$$c = \frac{1}{2}, \qquad \gamma = -3b, \qquad \beta = -2\big(a+b^2\big), \qquad \alpha = -2ab. \tag{3.21b}$$

(Note that, in *case (i)*, another miracle occurred: the solution $c = 1$ of the first, (3.20a), of the 4 constraints (3.20) entailed that the second, (3.20b), of these 4 equations hold identically).

Clearly these two possibilities correspond to the two *solvable* many-body models “of goldfish type” (2.1).

Next, we must justify the assertions made in the preceding section (see Proposition 2.1) concerning the solution of the initial-value problems for the many-body models characterized by the Newtonian equations of motion (2.1). The treatment given above (in this section) entails that these solutions are provided by the eigenvalues of the $N\times N$ matrix $U(t)$ evolving according to the explicit formula (3.6a) with (3.7); the missing detail is to express the two initial $N\times N$ matrices $U_0 \equiv U(0)$ and $V_0 \equiv V(0)$ appearing in the right-hand side of (3.7) in terms of the $2N$ initial data, $z_n(0)$ and $\dot{z}_n(0)$, of the many-body problems (2.1).

To simplify the derivation of these formulas it is convenient to make the assumption (allowed by the treatment given above) that the matrix $U(t)$ be *initially diagonal*, namely (see (1.2)) that

$$R(0) = I. \tag{3.22}$$

This entails (see (1.2) and (1.3a)) that

$$U(0) = \mathrm{diag}[z_n(0)], \qquad V(0) = Y(0). \tag{3.23}$$

The formula (2.3) for U_0 is thereby immediately implied.

The formula (2.4) for $V_0 \equiv V(0) = Y(0)$ is also easily obtained, since the expression (3.10b) of the *diagonal* elements of the matrix Y and the *ansatz* (1.5b) with (3.15a) and (3.17) for the *off-diagonal* elements of Y entail that, componentwise,

$$\begin{aligned} Y_{nm}(0) = {} & \delta_{nm}[\dot{z}_n(0) + z_n^2(0)] + (1-\delta_{nm})\left[\dot{z}_n(0) + a + bz_n(0) + cz_n^2(0)\right]^{1/2} \\ & \times \left[\dot{z}_m(0) + a + bz_m(0) + cz_m^2(0)\right]^{1/2}, \end{aligned} \tag{3.24}$$

and this clearly yields (2.4).

This completes the proof of the results of the preceding Section 2. As for the findings reported in Section 2.1, we consider their derivation sufficiently obvious – for instance via the treatment detailed in Section 2.1 of [5]; and by repeating the relevant treatment as given above, especially in the last part of this section – to justify us to dispense here from any further elaboration.

4 Additional findings

In this section we introduce the time-dependent (monic) polynomial $\psi(z,t)$ whose zeros are the N eigenvalues $z_n(t)$ of the $N \times N$ matrix $U(t)$:

$$\psi(z,t) = \det[zI - U(t)], \tag{4.1a}$$

$$\psi(z,t) = \prod_{n=1}^{N} [z - z_n(t)] = z^N + \sum_{m=1}^{N} \left[c_m(t) z^{N-m}\right]. \tag{4.1b}$$

The last of these formulas introduces the N coefficients $c_m \equiv c_m(t)$ of the monic polynomial $\psi(z,t)$; of course it implies that these coefficients are related to the zeros $z_n(t)$ as follows:

$$c_1 = -\sum_{n=1}^{N} z_n, \qquad c_2 = \sum_{n,m=1,\, n>m}^{N} z_n z_m, \tag{4.1c}$$

and so on.

The fact that the initial-value problem associated with the time evolution (2.1) of the N coordinates z_n can be *solved* by *algebraic* operations implies that the same *solvable* character holds for the time evolution of the monic polynomial $\psi(z,t)$ and of the N coefficients $c_m(t)$. In this section we display explicitly the equations that characterize these time evolutions. The procedure to obtain these equations from the equations of motion (2.1) is quite tedious but standard; a key role in this development are the identities reported, for instance, in Appendix A of [5] (but there are 2 misprints in these formulas: in equation (A.8k) the term $(N+1)$ inside the square bracket should instead read $(N-3)$; in equation (A.8l) the term N^2 inside the square bracket should instead read $N(N-2)$). Here we limit our presentation to reporting the final result.

The equation characterizing the time evolution of the monic polynomial $\psi(z,t)$ implied by the Newtonian equations of motion (2.1) reads as follows:

$$\begin{aligned}&\psi_{tt}+\left(\eta_0+\eta_1 z+\eta_2 z^2\right)\psi_{zt}+(\theta_0+\theta_1 z)\psi_t\\&\quad+\left(\alpha_0+\alpha_1 z+\alpha_2 z^2+\alpha_3 z^3+\alpha_4 z^4\right)\psi_{zz}+\left(\beta_0+\beta_1 z+\beta_2 z^2+\beta_3 z^3\right)\psi_z\\&\quad+\left(\gamma_0+\gamma_1 z+\gamma_2 z^2\right)\psi=0.\end{aligned}\tag{4.2a}$$

Here the subscripted variables denote partial differentiations, and the 17 coefficients appearing in this equation are defined in terms of the quantities $c_1 \equiv c_1(t)$, $\dot{c}_1 \equiv \dot{c}_1(t)$ and $c_2 \equiv c_2(t)$, see (4.1c), of the *arbitrary positive integer* N and of the 3 free parameters a, b and γ characterizing *model* (i), respectively of the 2 free parameters a and b characterizing *model* (ii), as follows:

model (i):

$$\begin{aligned}&\eta_0=-2a,\qquad \eta_1=-2b,\qquad \eta_2=-2;\\&\theta_0=-2c_1+2\,(N-1)\,b-\gamma,\qquad \theta_1=2N-1;\\&\alpha_0=a^2,\qquad \alpha_1=2ab,\qquad \alpha_2=2a+b^2,\qquad \alpha_3=2b,\qquad \alpha_4=1;\\&\beta_0=[2c_1-(2N-3)b+\gamma]a,\qquad \beta_1=2bc_1-(2N-3)\left(a+b^2\right)+b\gamma,\\&\beta_2=2c_1-2(2N-3)b+\gamma,\qquad \beta_3=-(2N-3);\\&\gamma_0=-\dot{c}_1+c_1^2-2(N-1)bc_1+\gamma c_1-Na+N(N-2)b^2-Nb\gamma,\\&\gamma_1=-(2N-1)c_1+2N(N-2)b-N\gamma,\qquad \gamma_2=N(N-2);\end{aligned}\tag{4.2b}$$

model (ii):

$$\begin{aligned}&\eta_0=-2a,\qquad \eta_1=-2b,\qquad \eta_2=-1;\\&\theta_0=-c_1+2(N+1)b,\qquad \theta_1=N+1;\\&\alpha_0=a^2,\qquad \alpha_1=2ab,\qquad \alpha_2=a+b^2,\qquad \alpha_3=b,\qquad \alpha_4=\frac{1}{4};\\&\beta_0=(c_1-2Nb)a,\qquad \beta_1=bc_1-Na-2Nb^2,\qquad \beta_2=\frac{c_1}{2}-2Nb,\qquad \beta_3=-\frac{N}{2};\\&\gamma_0=-2\dot{c}_1+c_1^2-(N+2)bc_1-\frac{3c_2}{2}+Na+N(N+1)b^2,\\&\gamma_1=-\frac{(N+1)c_1}{2}+N(N+1)b,\qquad \gamma_2=\frac{N(N+1)}{4}.\end{aligned}\tag{4.2c}$$

Remark 4.1. The equation (4.2a) characterizing the time evolution of the monic polynomial ψ looks like a *linear partial differential equation*, but it is in fact a *nonlinear functional equation*, because some of its coefficients depend on the quantities c_1 and c_2 which themselves depend on ψ, indeed clearly (see (4.1))

$$c_1 \equiv c_1(t)=\frac{\psi^{(N-1)}(0,t)}{(N-1)!},\qquad c_2 \equiv c_2(t)=\frac{\psi^{(N-2)}(0,t)}{(N-2)!},\tag{4.3a}$$

where we used the shorthand notation $\psi^{(j)}(z,t)$ to indicate the j-th partial derivative with respect to the variable z of $\psi(z,t)$,

$$\psi^{(j)}(z,t)\equiv\frac{\partial^j\psi(z,t)}{\partial z^j},\qquad j=1,2,\ldots.\tag{4.3b}$$

As for the system of N nonlinear autonomous second-order ODEs of *Newtonian* type characterizing the time evolution of the N coefficients c_m, for *model* (i) they read as follows:

$$\begin{aligned}
&\ddot{c}_m + p_m^{(-1)}\dot{c}_{m-1} + p_m^{(0)}\dot{c}_m + p_m^{(1)}\dot{c}_{m+1} \\
&\qquad + q_m^{(-2)}c_{m-2} + q_m^{(-1)}c_{m-1} + q_m^{(0)}c_m + q_m^{(1)}c_{m+1} + q_m^{(2)}c_{m+2} \\
&\quad = 2c_1\dot{c}_m - 2(N-m+1)ac_1c_{m-1} \\
&\qquad + \left[\dot{c}_1 - c_1^2 + 2(m-1)bc_1 - \gamma c_1\right]c_m + (2m+1)c_1c_{m+1},
\end{aligned} \tag{4.4a}$$

with the $3N$ (time-independent) coefficients $p_m^{(j)}$ and the $5N$ (time-independent) coefficients $q_m^{(j)}$ defined here in terms of the 3 *free* parameters a, b and γ (and of the numbers m and N) as follows:

$$\begin{aligned}
&p_m^{(-1)} = -2(N-m+1)a, \qquad p_m^{(0)} = 2(m-1)b - \gamma, \qquad p_m^{(1)} = 2m+1; \\
&q_m^{(-2)} = (N-m+2)(N-m+1)a^2, \\
&q_m^{(-1)} = (N-m+1)[-(2m-3)b + \gamma]a, \\
&q_m^{(0)} = -m(2N-2m+1)a + m(m-2)b^2 - mb\gamma, \\
&q_m^{(1)} = 2\left(m^2-1\right)b - (m+1)\gamma, \qquad q_m^{(2)} = m(m+2).
\end{aligned} \tag{4.4b}$$

The analogous equations for *model* (ii) read as follows:

$$\begin{aligned}
&\ddot{c}_m + p_m^{(-1)}\dot{c}_{m-1} + p_m^{(0)}\dot{c}_m + p_m^{(1)}\dot{c}_{m+1} \\
&\qquad + q_m^{(-2)}c_{m-2} + q_m^{(-1)}c_{m-1} + q_m^{(0)}c_m + q_m^{(1)}c_{m+1} + q_m^{(2)}c_{m+2} \\
&\quad = c_1\dot{c}_m - (N-m+1)ac_1c_{m-1} \\
&\qquad + \left[2\dot{c}_1 + \frac{3c_2}{2} - c_1^2 + (m+2)bc_1\right]c_m + \left(\frac{m}{2}+1\right)c_1c_{m+1},
\end{aligned} \tag{4.5a}$$

with the $3N$ (time-independent) coefficients $p_m^{(j)}$ and the $5N$ (time-independent) coefficients $q_m^{(j)}$ defined here in terms of the 2 *free* parameters a and b (and of the numbers m and N) as follows:

$$\begin{aligned}
&p_m^{(-1)} = -2(N-m+1)a, \qquad p_m^{(0)} = 2(m+1)b, \qquad p_m^{(1)} = m+2; \\
&q_m^{(-2)} = (N-m+2)(N-m+1)a^2, \qquad q_m^{(-1)} = -2m(N-m+1)ab, \\
&q_m^{(0)} = -m(N-m-1)a + m(m+1)b^2, \\
&q_m^{(1)} = (m+1)(m+2)b, \qquad q_m^{(2)} = \frac{(m+2)(m+3)}{4}.
\end{aligned} \tag{4.5b}$$

Of course in these equations of motion, (4.4a) and (4.5a), it is understood that, for $n < 0$ and for $n > N$, the coefficients c_n vanish identically, $c_n = 0$, while $c_0 = 1$ (see (4.1)).

It is plain that these equations of motion, (4.4) and (4.5), inherit the properties of the original many-body models: they clearly are as well *solvable* by algebraic operations, and of course if the original many-body model is *isochronous* the corresponding model for the coefficients c_m is as well *isochronous*, namely

$$c_m(t \pm T) = c_m(t), \tag{4.6a}$$

and likewise if the original many-body model is *asymptotically isochronous*, the corresponding model for the coefficients c_m is as well *asymptotically isochronous*, namely

$$\lim_{t\to\infty}[c_m(t \pm T) - c_m(t)] = 0. \tag{4.6b}$$

This of course entails that the conditions on the 3 *a priori free* parameters a, b and γ of the version (4.4) of *model* (i), or on the 2 *a priori free* parameters a and b of the version (4.5) of *model* (ii), which are necessary and sufficient to imply that these models be *isochronous* respectively *asymptotically isochronous*, are those indicated in Section 1, namely are those implied by (2.6) with (2.5b) and (2.7a) respectively (2.9a); while in the *generic* case, see (2.10), the coefficients $c_m(t)$ tend asymptotically to time-independent values,

$$\lim_{t\to\pm\infty}[c_m(t)] = c_m(\pm\infty). \tag{4.6c}$$

4.1 Two additional *isochronous* models

Here we report formulas analogous to those reported above, but related to the many-body models discussed in Section 2.1 rather than to those reported in Section 2. We consider the derivation of these results sufficiently straightforward not to require any detailed elaboration here beyond the terse hints provided below.

The starting point are the two special cases of the two systems (4.4a) and (4.5a) which obtain by setting all the free parameters to vanish. We write them in compact form as follows:

$$\begin{aligned}\ddot{c}_m &= 2cc_1\dot{c}_m - [2c(m-1)+3]\dot{c}_{m+1} + \left[(3-2c)\dot{c}_1 - c_1^2 + 2\left(1-c^2\right)c_2\right]c_m \\ &\quad + \left(2c^2m+1\right)c_1c_{m+1} - (m+2)\left[c^2(m-1)+1\right]c_{m+2}\end{aligned} \tag{4.7}$$

with $c = 1$ respectively $c = 1/2$. It is remarkable that the number N does not appear in these equations; although we always assume that these equations hold for $m = 1, 2, \ldots, N$ and that the dependent variables c_n vanish identically for $n > N$, with N an *arbitrary positive integer*. The diligent reader may also check that this equation also holds identically for $m = 0$ with $c_0 = 1$ (see (4.1)).

We then make the following change of dependent and independent variables:

$$\tilde{c}_m(t) = \exp(im\omega t)c_m(\tau), \qquad \tau = \frac{\exp(i\omega t)-1}{i\omega}, \tag{4.8}$$

with ω again a *real* arbitrary constant to which we associate the period T (see (2.8)). One thereby easily sees that the new dependent variables $\tilde{c}_m(t)$ satisfy the following system of *autonomous* Newtonian equations of motion:

$$\begin{aligned}\ddot{\tilde{c}}_m &= \left[(2m+1)i\omega + 2c\tilde{c}_1\right]\dot{\tilde{c}}_m - [2c(m-1)+3]\dot{\tilde{c}}_{m+1} \\ &\quad + \left\{m(m+1)\omega^2 - [2c(m-1)+3]i\omega\tilde{c}_1 + (3-2c)\dot{\tilde{c}}_1 - \tilde{c}_1^2 + 2\left(1-c^2\right)\tilde{c}_2\right\}\tilde{c}_m \\ &\quad + \left\{(m+1)[2c(m-1)+3]i\omega + \left(2c^2m+1\right)\tilde{c}_1\right\}\tilde{c}_{m+1} \\ &\quad - (m+2)\left[c^2(m-1)+1\right]\tilde{c}_{m+2}.\end{aligned} \tag{4.9}$$

It is plain from the way these two models (with $c = 1$ or $c = 1/2$) have been derived that they are *isochronous*, namely the generic solutions of these nonlinear Newtonian equations of motion satisfy the condition

$$\tilde{c}_m(t+T) = \tilde{c}_m(t). \tag{4.10}$$

5 Outlook

It is clearly far from trivial that the Newtonian many-body models introduced in this paper – see (2.1), (2.13), (4.4), (4.5), and (4.9) – can be *solved*, for arbitrary initial data, by *algebraic* operations. Also remarkable is that, for special assignments of their parameters, the systems of *autonomous nonlinear* ODEs (2.1), (4.4) and (4.5) are *isochronous* or *asymptotically isochronous*, and the systems of *autonomous nonlinear* ODEs (2.13) and (4.9) are *isochronous*.

Let us end this paper by pointing out that *Diophantine* findings can be obtained from a *nonlinear autonomous isochronous* dynamical system by investigating its behavior in the *infinitesimal vicinity* of its equilibria. The relevant equations of motion become then generally *linear*, but they of course retain the properties to be *autonomous* and *isochronous*. For a system of *linear autonomous* ODEs, the property of *isochrony* implies that *all* the eigenvalues of the matrix of its coefficients are *integer numbers* (up to a common rescaling factor). When the *linear* system describes the behavior of a *nonlinear autonomous* system in the *infinitesimal vicinity* of its equilibria, these matrices can generally be *explicitly* computed in terms of the values at equilibrium of the dependent variables of the original, *nonlinear* model. In this manner nontrivial *Diophantine* findings and conjectures have been discovered and proposed: see for instance the review of such developments in Appendix C (entitled "Diophantine findings and conjectures") of [5]. Analogous results obtained by applying this approach to the *isochronous* systems of *autonomous nonlinear* ODEs introduced above will be reported in a separate paper if they turn out to be novel and interesting.

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