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On the Number of Real Roots of the Yablonskii–Vorob'ev Polynomials

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Abstract. We study the real roots of the Yablonskii–Vorob'ev polynomials, which are special polynomials used to represent rational solutions of the second Painlevé equation. It has been conjectured that the number of real roots of the nth Yablonskii–Vorob'ev polynomial equals $\left[\frac{n+1}{2}\right]$. We prove this conjecture using an interlacing property between the roots of the Yablonskii–Vorob'ev polynomials. Furthermore we determine precisely the number of negative and the number of positive real roots of the nth Yablonskii–Vorob'ev polynomial.

Key words: second Painlevé equation; rational solutions; real roots; interlacing of roots; Yablonskii–Vorob'ev polynomials

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1 Introduction

In this paper we study the real roots of the Yablonskii–Vorob'ev polynomials Q_n ($n \in \mathbb{N}$). Yablonskii and Vorob'ev found these polynomials while studying the hierarchy of rational solutions of the second Painlevé equation. The Yablonskii–Vorob'ev polynomials satisfy the defining differential-difference equation

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4\big(Q_nQ_n'' - (Q_n')^2\big),$$

with $Q_0 = 1$ and $Q_1 = z$.

The Yablonskii–Vorob'ev polynomials Q_n are monic polynomials of degree $\frac{1}{2}n(n+1)$, with integer coefficients. The first few are given in Table 1. Yablonskii [8] and Vorob'ev [7] expressed the rational solutions of the second Painlevé equation,

$$P_{\mathrm{II}}(\alpha): \ w''(z) = 2w(z)^3 + zw(z) + \alpha,$$

with complex parameter α, in terms of logarithmic derivatives of the Yablonskii–Vorob'ev polynomials, as summerized in the following theorem:

Theorem 1. *$P_{\mathrm{II}}(\alpha)$ has a rational solution iff $\alpha = n \in \mathbb{Z}$. For $n \in \mathbb{Z}$ the rational solution is unique and if $n \geq 1$, then it is equal to*

$$w_n = \frac{Q_{n-1}'}{Q_{n-1}} - \frac{Q_n'}{Q_n}.$$

The other rational solutions are given by $w_0 = 0$ and for $n \geq 1$, $w_{-n} = -w_n$.

In [5] we proved the irrationality of the nonzero real roots of the Yablonskii–Vorob'ev polynomials, in this article we determine precisely the number of real roots of these polynomials. Clarkson [1] conjectured that the number of real roots of Q_n equals $\left[\frac{n+1}{2}\right]$, where $[x]$ denotes the integer part of x for real numbers x. In Section 2 we prove this conjecture and obtain the following theorem, where Z_n is defined as the set of real roots of Q_n for $n \in \mathbb{N}$.

Table 1.

Yablonskii–Vorob'ev polynomials
$Q_2 = 4 + z^3$
$Q_3 = -80 + 20z^3 + z^6$
$Q_4 = z\big(11200 + 60z^6 + z^9\big)$
$Q_5 = -6272000 - 3136000z^3 + 78400z^6 + 2800z^9 + 140z^{12} + z^{15}$
$Q_6 = -38635520000 + 19317760000z^3 + 1448832000z^6 - 17248000z^9 + 627200z^{12} + 18480z^{15} + 280z^{18} + z^{21}$
$Q_7 = z\big(-3093932441600000 - 49723914240000z^6 - 828731904000z^9 + 13039488000z^{12} + 62092800z^{15} + 5174400z^{18} + 75600z^{21} + 504z^{24} + z^{27}\big)$
$Q_8 = -991048439693312000000 - 743286329769984000000z^3 + 371643164884992000000z^6 + 176972935659520000z^9 + 1266965334835200000z^{12} + 40773609676800z^{15} - 6629855232000z^{18} + 124309785600z^{21} + 2018016000z^{24} + 32771200z^{27} + 240240z^{30} + 840z^{33} + z^{36}$

Theorem 2. *For every $n \in \mathbb{N}$, the number of real roots of Q_n equals*

$$|Z_n| = \left[\frac{n+1}{2}\right]. \tag{1}$$

Furthermore for $n \geq 2$,

$$\min(Z_{n-1}) > \min(Z_{n+1}), \qquad \max(Z_{n-1}) < \max(Z_{n+1}). \tag{2}$$

The argument is inductive and an important ingredient is the fact that the real roots of Q_{n-1} and Q_{n+1} interlace, which is proven by Clarkson [1].

Kaneko and Ochiai [4] found a direct formula for the lowest degree coefficients of the Yablonskii–Vorob'ev polynomials Q_n depending on n. In particular the sign of $Q_n(0)$ can be determined for $n \in \mathbb{N}$. In Section 3 we use this to determine precisely the number of positive and the number of negative real roots of Q_n, which yields to the following theorem.

Theorem 3. *Let $n \in \mathbb{N}$, then the number of negative real roots of Q_n is equal to*

$$|Z_n \cap (-\infty, 0)| = \left[\frac{n+1}{3}\right].$$

The number of positive real roots of Q_n is equal to

$$|Z_n \cap (0, \infty)| = \begin{cases} \left[\dfrac{n}{6}\right] & \text{if } n \text{ is even,} \\ \left[\dfrac{n+3}{6}\right] & \text{if } n \text{ is odd.} \end{cases}$$

As a consequence, for every $n \in \mathbb{N}$, we can calculate the number of positive real poles of the rational solution w_n with residue 1 and with residue -1, and the number of negative real poles of the rational solution w_n with residue 1 and with residue -1.

2 Number of real roots

Let P and Q be polynomials with no common real roots. We say that the real roots of P and Q interlace if and only if in between any two real roots of P, Q has a real root and in between any two real roots of Q, P has a real root. Throughout this paper we use the convention $\mathbb{N} = \{0, 1, 2, \ldots\}$ and define $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

Theorem 4. *For every $n \in \mathbb{N}$, Q_n has only simple roots. Furthermore for $n \geq 1$, Q_{n-1} and Q_{n+1} have no common roots and Q_{n-1} and Q_n have no common roots.*

Proof. See Fukutani, Okamoto and Umemura [3]. ■

Theorem 5. *For every $n \geq 1$, the real roots of Q_{n-1} and Q_{n+1} interlace.*

Proof. See Clarkson [1]. ■

Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions and $x \in \mathbb{R}$. We say that f crosses g positively at x if and only if $f(x) = g(x)$ and there is a $\delta > 0$ such that $f(y) < g(y)$ for $x - \delta < y < x$ and $f(y) > g(y)$ for $x < y < x + \delta$. We say that f crosses g negatively at x if and only if $f(x) = g(x)$ and there is a $\delta > 0$ such that $f(y) > g(y)$ for $x - \delta < y < x$ and $f(y) < g(y)$ for $x < y < x + \delta$. So f crosses g negatively at x if and only if g crosses f positively at x.

Let $m \in \mathbb{N}$ and suppose that f is m times differentiable, then we denote the mth derivative of f by $f^{(m)}$ with convention $f^{(0)} = f$.

Proposition 1. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be analytic functions and $x \in \mathbb{R}$. Then f crosses g positively at x if and only if there is an $m \geq 1$ such that $f^{(i)}(x) = g^{(i)}(x)$ for $0 \leq i < m$ and $f^{(m)}(x) > g^{(m)}(x)$.*

Similarly f crosses g negatively at x if and only if there is a $m \geq 1$ such that $f^{(i)}(x) = g^{(i)}(x)$ for $0 \leq i < m$ and $f^{(m)}(x) < g^{(m)}(x)$.

Proof. This is proven easily using Taylor's theorem. ■

Lemma 1. *For every $n \in \mathbb{N}^*$ we have*

$$Q'_{n+1}Q_{n-1} - Q_{n+1}Q'_{n-1} = (2n+1)Q_n^2, \tag{3a}$$

$$Q''_{n+1}Q_{n-1} - Q_{n+1}Q''_{n-1} = 2(2n+1)Q_nQ'_n, \tag{3b}$$

$$Q'''_{n+1}Q_{n-1} - Q_{n+1}Q'''_{n-1} = 2(2n+1)\left(Q'_n\right)^2 + (2n+1)Q_nQ''_n. \tag{3c}$$

Proof. See Fukutani, Okamoto and Umemura [3]. ■

The following proposition contains some well-known properties of the Yablonskii–Vorob'ev polynomials, see for instance Clarkson and Mansfield [2].

Proposition 2. *For every $n \in \mathbb{N}$, Q_n is a monic polynomial of degree $\frac{1}{2}n(n+1)$ with integer coefficients. As a consequence, for $n \geq 1$,*

$$\lim_{x\to\infty} Q_n(x) = \infty, \qquad \lim_{x\to-\infty} Q_n(x) = \begin{cases} -\infty & \text{if } n \equiv 1,2 \pmod 4, \\ \infty & \text{if } n \equiv 0,3 \pmod 4. \end{cases}$$

By Proposition 2, Q_n has real coefficients and hence we can consider Q_n as a real-valued function defined on the real line, that is, we consider

$$Q_n : \mathbb{R} \to \mathbb{R}.$$

Proposition 3. *Let $n \in \mathbb{N}^*$, if $x \in \mathbb{R}$ is such that Q_{n+1} crosses Q_{n-1} positively at x, then*

$$Q_{n+1}(x) = Q_{n-1}(x) > 0.$$

Similarly if $x \in \mathbb{R}$ is such that Q_{n+1} crosses Q_{n-1} negatively at x, then

$$Q_{n+1}(x) = Q_{n-1}(x) < 0.$$

Proof. Let $n \in \mathbb{N}^*$. Suppose $x \in \mathbb{R}$ is such that Q_{n+1} crosses Q_{n-1} positively at x. If

$$Q_{n+1}(x) = Q_{n-1}(x) = 0,$$

then Q_{n+1} and Q_{n-1} have a common root, which contradicts Theorem 4.

Let us assume

$$Q_{n+1}(x) = Q_{n-1}(x) < 0. \tag{4}$$

Then by Proposition 1,

$$Q'_{n+1}(x) - Q'_{n-1}(x) \geq 0. \tag{5}$$

Therefore, by equation (3a),

$$\begin{aligned} 0 \leq (2n+1)Q_n(x)^2 &= Q'_{n+1}(x)Q_{n-1}(x) - Q_{n+1}(x)Q'_{n-1}(x) \\ &= Q_{n+1}(x)\left(Q'_{n+1}(x) - Q'_{n-1}(x)\right) \leq 0, \end{aligned}$$

where in the last inequality we used equation (4) and equation (5).

We conclude

$$(2n+1)Q_n(x)^2 = Q_{n+1}(x)\left(Q'_{n+1}(x) - Q'_{n-1}(x)\right) = 0,$$

so $Q_n(x) = 0$ and $Q'_{n+1}(x) = Q'_{n-1}(x)$. Therefore by equation (3b),

$$\begin{aligned} Q_{n+1}(x)\left(Q''_{n+1}(x) - Q''_{n-1}(x)\right) &= Q''_{n+1}(x)Q_{n-1}(x) - Q_{n+1}(x)Q''_{n-1}(x) \\ &= 2(2n+1)Q_n(x)Q'_n(x) = 0. \end{aligned}$$

We conclude $Q''_{n+1}(x) = Q''_{n-1}(x)$. Since $Q_n(x) = 0$ and, by Theorem 4, Q_n has only simple roots, we have $Q'_n(x) \neq 0$. Therefore by (3c),

$$\begin{aligned} Q_{n+1}(x)\left(Q'''_{n+1}(x) - Q'''_{n-1}(x)\right) &= Q'''_{n+1}(x)Q_{n-1}(x) - Q_{n+1}(x)Q'''_{n-1}(x) \\ &= 2(2n+1)\left(Q'_n(x)\right)^2 + (2n+1)Q_n(x)Q''_n(x) \\ &= 2(2n+1)\left(Q'_n(x)\right)^2 > 0. \end{aligned}$$

Since $Q_{n+1}(x) < 0$ we conclude $Q'''_{n+1}(x) < Q'''_{n-1}(x)$. So $Q'_{n+1}(x) = Q'_{n-1}(x)$, $Q''_{n+1}(x) = Q''_{n-1}(x)$ but $Q'''_{n+1}(x) < Q'''_{n-1}(x)$. Therefore by Proposition 1, Q_{n+1} does not cross Q_{n-1} positively at x and we have obtained a contradiction. We conclude that

$$Q_{n+1}(x) = Q_{n-1}(x) > 0.$$

The second part of the proposition is proven similar. ■

We prove theorem 2, using Theorem 5 and Proposition 3.

Proof of Theorem 2. Observe that (1) is correct for $n = 0, 1, 2, 3, 4$. Furthermore it is easy to see that (2) is true for $n = 1, 2, 3$. We proceed by induction, suppose $n \geq 4$ and

$$|Z_{n-1}| = \left[\frac{n}{2}\right].$$

Then Q_{n-1} has at least 2 real roots. By Theorem 5 the real roots of Q_{n-1} and Q_{n+1} interlace, hence Q_{n+1} has a real root. Let us define

$$z := \min(Z_{n+1}), \qquad z_1 := \min(Z_{n-1}), \qquad z_2 := \min(Z_{n-1} \setminus \{z_1\}),$$

so z is the smallest real root of Q_{n+1} and z_1 and z_2 are the smallest and second smallest real root of Q_{n-1} respectively.

By Theorem 5 the real roots of Q_{n-1} and Q_{n+1} interlace, hence either $z < z_1$ or $z_1 < z < z_2$. We prove that $z_1 < z < z_2$ can not be the case. Suppose $z_1 < z < z_2$ and suppose $n \equiv 0, 1 \pmod 4$, then by Proposition 2,

$$\lim_{x \to -\infty} Q_{n-1}(x) = \infty.$$

Hence $Q_{n-1}(x) > 0$ for $x < z_1$. Since $Q_{n-1}(z_1) = 0$, this implies $Q'_{n-1}(z_1) \leq 0$. By Theorem 4, Q_{n-1} has only simple roots, hence $Q'_{n-1}(z_1) \neq 0$, so $Q'_{n-1}(z_1) < 0$. Therefore by Proposition 1, Q_{n-1} crosses 0 negatively at z_1. Hence $Q_{n-1}(x) < 0$ for $z_1 < x < z_2$, in particular

$$Q_{n-1}(z) < 0. \tag{6}$$

Since $n \equiv 0, 1 \pmod 4$, we have by Proposition 2,

$$\lim_{x \to -\infty} Q_{n+1}(x) = -\infty.$$

Therefore $Q_{n+1}(x) < 0$ for $x < z$, in particular

$$Q_{n+1}(z_1) < 0.$$

Define the polynomial $P := Q_{n+1} - Q_{n-1}$, then

$$P(z_1) = Q_{n+1}(z_1) - Q_{n-1}(z_1) = Q_{n+1}(z_1) - 0 < 0,$$

and by equation (6),

$$P(z) = Q_{n+1}(z) - Q_{n-1}(z) = 0 - Q_{n-1}(z) > 0.$$

So P is a polynomial with $P(z_1) < 0$, $P(z) > 0$ and $z_1 < z$. Hence there is a $z_1 < x < z$ such that P crosses 0 positively at x, for instance

$$x := \inf \{t \in (z_1, z) \mid P(t) > 0\},$$

has the desired properties.

Since $P = Q_{n+1} - Q_{n-1}$ crosses 0 positively at x, Q_{n+1} crosses Q_{n-1} positively at x. But $z_1 < x < z_2$, hence

$$Q_{n+1}(x) = Q_{n-1}(x) < 0. \tag{7}$$

This contradicts Proposition 3.

If $n \equiv 2, 3 \pmod 4$, then by a similar argument, there is a $z_1 < x < z$ such that Q_{n+1} crosses Q_{n-1} negatively at x with

$$Q_{n+1}(x) = Q_{n-1}(x) > 0,$$

which again contradicts Proposition 3.

We conclude that $z_1 < z < z_2$ can not be the case and hence $z < z_1$, that is,

$$\min(Z_{n-1}) > \min(Z_{n+1}).$$

Let us define

$$w := \max(Z_{n+1}), \qquad w_1 := \max(Z_{n-1}), \qquad w_2 := \max(Z_{n-1} \setminus \{w_1\}),$$

so w is the largest real root of Q_{n+1} and w_1 and w_2 are the largest and second largest real root of Q_{n-1} respectively.

Suppose $w_1 > w$, then by a similar argument as the above, there is a $w_2 < x < w$ such that Q_{n+1} crosses Q_{n-1} positively at x with

$$Q_{n+1}(x) = Q_{n-1}(x) < 0.$$

This is in contradiction with Proposition 3, so $w_1 < w$, that is

$$\max(Z_{n-1}) < \max(Z_{n+1}). \tag{8}$$

Let $z_1 < z_2 < \cdots < z_k$ be the real roots of Q_{n-1} with $k = \left[\frac{n}{2}\right]$ and $z'_1 < z'_2 < \cdots < z'_m$ be the real roots of Q_{n+1}. Then by equations (7) and (8), $z'_1 < z_1$, $z_k < z'_m$ and since by Theorem 5 the real roots of Q_{n-1} and Q_{n+1} interlace, we have

$$z'_1 < z_1 < z'_2 < z_2 < z'_3 < z_3 < \cdots < z'_{k-1} < z_{k-1} < z'_k < z_k < z'_{k+1} = z'_m.$$

Hence $m = k + 1$, that is,

$$|Z_{n+1}| = m = k + 1 = \left[\frac{n}{2}\right] + 1 = \left[\frac{n+2}{2}\right].$$

The theorem follows by induction. ■

3 Number of positive and negative real roots

For a polynomial P we denote the set of real roots of P by Z_P.

Lemma 2. *Let P and Q be polynomials with real coefficients, both a positive leading coefficient and only simple roots. Assume that the real roots of P and Q interlace. Furthermore suppose both P and Q have a real root and*

$$\min(Z_P) > \min(Z_Q), \qquad \max(Z_P) < \max(Z_Q).$$

Then we have the following relations between the number of negative and positive real roots of P and Q,

$$|Z_Q \cap (-\infty, 0)| = |Z_P \cap (-\infty, 0)| + \begin{cases} 1 & \text{if } P(0) = 0, \\ 0 & \text{if } Q(0) = 0, \\ 1 & \text{if } P(0) > 0 \text{ and } Q(0) > 0, \\ 0 & \text{if } P(0) > 0 \text{ and } Q(0) < 0, \\ 0 & \text{if } P(0) < 0 \text{ and } Q(0) > 0, \\ 1 & \text{if } P(0) < 0 \text{ and } Q(0) < 0, \end{cases}$$

$$|Z_Q \cap (0, \infty)| = |Z_P \cap (0, \infty)| + \begin{cases} 1 & \text{if } P(0) = 0, \\ 0 & \text{if } Q(0) = 0, \\ 0 & \text{if } P(0) > 0 \text{ and } Q(0) > 0, \\ 1 & \text{if } P(0) > 0 \text{ and } Q(0) < 0, \\ 1 & \text{if } P(0) < 0 \text{ and } Q(0) > 0, \\ 0 & \text{if } P(0) < 0 \text{ and } Q(0) < 0. \end{cases}$$

Proof. Let $z_1 > z_2 > \cdots > z_n$ be the real roots of P and $z'_1 > z'_2 > \cdots > z'_m$ be the real roots of Q. Observe that

$$z_n = \min(Z_P) > \min(Z_Q) = z'_m, \qquad z_1 = \max(Z_P) < \max(Z_Q) = z'_1.$$

Therefore, since the real roots of P and Q interlace, we have

$$z'_1 > z_1 > z'_2 > z_2 > \cdots > z'_n > z_n > z'_{n+1} = z'_m, \tag{9}$$

In particular $m = n + 1$.

Suppose $P(0) = 0$. Then there is an unique $1 \le k \le n$ such that $z_k = 0$. So equation (9) implies

$$z'_1 > z_1 > z'_2 > z_2 > \cdots > z'_{k-1} > z_{k-1} > z'_k > z_k = 0 > z'_{k+1} > z_{k+1} > \cdots > z'_n > z_n > z'_{n+1}.$$

Therefore

$$\begin{aligned}
&|Z_Q \cap (-\infty, 0)| = n + 1 - (k + 1) + 1 = n - k + 1 = |Z_P \cap (-\infty, 0)| + 1,\\
&|Z_Q \cap (0, \infty)| = k = |Z_P \cap (0, \infty)| + 1.
\end{aligned}$$

The case $Q(0) = 0$ is proven similarly.

Suppose $P(0) > 0$ and $Q(0) > 0$. Since P has a positive leading coefficient and is not constant, we have

$$\lim_{x\to\infty} P(x) = \infty.$$

Therefore, since z_1 is the largest real root of P, $P(x) > 0$ for $x > z_1$. Since P has only simple roots, P crosses 0 positively at z_1, so $P(x) < 0$ for $z_2 < x < z_1$. Again since P has only simple roots, P crosses 0 negatively at z_2, so $P(x) > 0$ for $z_3 < x < z_2$. Inductively we see that when $1 \le i < n$ is even, $P(x) > 0$ for $z_{i+1} < x < z_i$, and when $1 \le i < n$ is odd, $P(x) < 0$ for $z_{i+1} < x < z_i$. Furthermore $P(x) > 0$ for $x < z_n$ if n is even and $P(x) < 0$ for $x < z_n$ if n is odd.

Similarly we have, for $1 \le i < n + 1$ even, $Q(x) > 0$ for $z'_{i+1} < x < z'_i$, and for $1 \le i < n + 1$ odd, $Q(x) < 0$ for $z'_{i+1} < x < z'_i$. Furthermore $Q(x) < 0$ for $x < z'_{n+1}$, if n is even and $Q(x) > 0$ for $x < z'_{n+1}$, if n is odd.

There are three cases to consider: $z_1 > 0 > z_n$, $z_1 < 0$ and $z_n > 0$.

We first assume $z_1 > 0 > z_n$. Then there is an unique $1 \le k \le n$ such that $z_k > 0 > z_{k+1}$. Since $z_k > 0 > z_{k+1}$ and $P(0) > 0$, we conclude that k is even. By equation (9),

$$z'_k > z_k > 0 > z_{k+1} > z'_{k+2}.$$

Since k is even, $Q(x) > 0$ for $z'_{k+1} < x < z'_k$ and $Q(x) < 0$ for $z'_{k+2} < x < z'_{k+1}$. But $z'_{k+2} < 0 < z'_k$ and $Q(0) > 0$, hence $z'_{k+1} < 0 < z'_k$. Therefore

$$\begin{aligned}
&|Z_Q \cap (-\infty, 0)| = n + 1 - (k + 1) + 1 = |Z_P \cap (-\infty, 0)| + 1,\\
&|Z_Q \cap (0, \infty)| = k = |Z_P \cap (0, \infty)|.
\end{aligned}$$

Let us assume $z_1 < 0$, then P has no positive real roots. Observe $Q(x) < 0$ for $z'_2 < x < z'_1$. Suppose $z'_1 > 0$, then $z'_2 > 0$ since $Q(0) > 0$. Hence by equation (9), $z'_1 > z_1 > z'_2 > 0$, so $z_1 > 0$ and we have a contradiction. So $z'_1 < 0$, hence all the real roots of Q are negative and we have

$$\begin{aligned}
&|Z_Q \cap (-\infty, 0)| = m = n + 1 = |Z_P \cap (-\infty, 0)| + 1,\\
&|Z_Q \cap (0, \infty)| = 0 = |Z_P \cap (0, \infty)|.
\end{aligned}$$

Finally let us assume $z_n > 0$, then P has no negative real roots. By equation (9), $z'_n > 0$. Since $P(0) > 0$, $P(x) > 0$ for $x < z_n$, therefore n must be even. Hence $Q(x) > 0$ for $z'_{n+1} < x < z'_n$ and $Q(x) < 0$ for $x < z'_{n+1}$. Since $z'_n > 0$ and $Q(0) > 0$, this implies $z'_{n+1} < 0 < z'_n$. Therefore

$$|Z_Q \cap (-\infty, 0)| = 1 = |Z_P \cap (-\infty, 0)| + 1, \qquad |Z_Q \cap (0, \infty)| = n = |Z_P \cap (0, \infty)|\,.$$

This ends our discussion of the case $P(0) > 0$ and $Q(0) > 0$. The remaining cases are proven similarly. ■

Taneda [6] proved that for $n \in \mathbb{N}$:

- if $n \equiv 1 \pmod 3$, then $\frac{Q_n}{z} \in \mathbb{Z}[z^3]$;
- if $n \not\equiv 1 \pmod 3$, then $Q_n \in \mathbb{Z}[z^3]$.

Hence $Q_n(0) = 0$ if $n \equiv 1 \pmod 3$. By Theorem 4, for every $n \geq 1$, Q_{n-1} and Q_n do not have a common root. Therefore $Q_n(0) = 0$ if and only if $n \equiv 1 \pmod 3$.

Let us denote the coefficient of the lowest degree term in Q_n by x_n. That is, we define $x_n := Q_n(0)$ if $n \not\equiv 1 \pmod 3$, and $x_n := Q'_n(0)$ if $n \equiv 1 \pmod 3$. In [5] we derived the following recursion for the x_n:

$$x_0 = 1, \qquad x_1 = 1$$

and

$$x_{n+1}x_{n-1} = \begin{cases} (2n+1)x_n^2 & \text{if } n \equiv 0 \pmod 3, \\ 4x_n^2 & \text{if } n \equiv 1 \pmod 3, \\ -(2n+1)x_n^2 & \text{if } n \equiv 2 \pmod 3. \end{cases} \tag{10}$$

We remark that the above recursion can be used to determine the x_n explicitly, a direct formula for x_n is given by Kaneko and Ochiai [4].

Lemma 3. *For every $n \in \mathbb{N}$,*

$$\operatorname{sgn}(Q_n(0)) = \begin{cases} -1 & \text{if } n \equiv 3,5,6,8 \pmod{12}, \\ 0 & \text{if } n \equiv 1,4,7,10 \pmod{12}, \\ 1 & \text{if } n \equiv 0,2,9,11 \pmod{12}, \end{cases}$$

where sgn denotes the sign function on $\mathbb{R}$.

Proof. By induction using recursion (10), we have

$$\operatorname{sgn}(x_n) = \begin{cases} -1 & \text{if } n \equiv 3,5,7,6,8,10 \pmod{12}, \\ 1 & \text{if } n \equiv 0,1,2,4,9,11 \pmod{12}. \end{cases}$$

The lemma follows from this and the fact that $Q_n(0) = 0$ if and only if $n \equiv 1 \pmod 3$. ■

We apply Lemma 2 to the Yablonskii–Vorob'ev polynomials to prove Theorem 3.

Proof of Theorem 3. Let $n \geq 2$, then by Proposition 2, Theorem 4 and Theorem 5, $P := Q_{n-1}$ and $Q := Q_{n+1}$ are monic polynomials with only simple roots such that the real roots interlace. Furthermore by Theorem 2, both P and Q have a real root and

$$\min(Z_P) > \min(Z_Q), \qquad \max(Z_P) < \max(Z_Q).$$

So we can apply Lemma 2 together with Lemma 3 and obtain:

$$|Z_{n+1}\cap(-\infty,0)| = |Z_{n-1}\cap(-\infty,0)| + \begin{cases} 0 & \text{if } n\equiv 0,3 \pmod 6, \\ 1 & \text{if } n\equiv 1,2,4,5 \pmod 6, \end{cases}$$

$$|Z_{n+1}\cap(0,\infty)| = |Z_{n-1}\cap(0,\infty)| + \begin{cases} 0 & \text{if } n\equiv 0,1 \pmod 3, \\ 1 & \text{if } n\equiv 2 \pmod 3. \end{cases}$$

Observe that $Z_0=\varnothing$, $Z_1=\{0\}$ and $Z_2=\left\{-\sqrt[3]{4}\right\}$. The theorem is obtained by applying the above recursive formulas inductively. ■

Let us discuss an example. By Theorem 1, the unique rational solution of $P_{\rm II}(\alpha)$ for the parametervalue $\alpha := 21$ is given by

$$w_{21} = \frac{Q'_{20}}{Q_{20}} - \frac{Q'_{21}}{Q_{21}}.$$

By Theorem 4, Q_{20} and Q_{21} do not have common roots and the roots of Q_{20} and Q_{21} are simple. Hence the poles of w_{21} are precisely the roots of Q_{20} and Q_{21}, the roots of Q_{20} are poles of w_{21} with residue 1 and the roots of Q_{21} are poles of w_{21} with residue -1.

By Theorem 2, Q_{20} has 10 real roots and by Theorem 3, 7 of them are negative and 3 of them are positive. Similarly Q_{21} has 11 real roots, 7 of them are negative and 4 of them are positive.

Therefore w_{21} has 21 real poles, 10 with residue 1 and 11 with residue -1. More precisely w_{21} has 7 positive real poles, 3 with residue 1 and 4 with residue -1 and w_{21} has 14 negative real poles, 7 with residue 1 and 7 with residue -1.

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