Symmetry, Integrability and Geometry: Methods and Applications SIGMA **9** (2013), 011, 10 pages

On the n-Dimensional Porous Medium Diffusion Equation and Global Actions of the Symmetry Group

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Received September 10, 2012, in final form February 08, 2013; Published online February 12, 2013
http://dx.doi.org/10.3842/SIGMA.2013.011

Abstract. By restricting to a special class of smooth functions, the local action of the symmetry group is globalized. This special class of functions is constructed using parabolic induction.

Key words: globalization; porous medium equation; Lie group representation; Lorentz group; parabolic induction

2010 Mathematics Subject Classification: 22E70; 35Q35

1 Introduction

The theory of Lie groups finds its genesis when Sophus Lie sets up the task to develop an analogue of Galois theory for differential equations. The original prolongation algorithm of Sophus Lie provides a set of point symmetries of a differential equation (for a modern treatment, see [11]). It is well-known that these point symmetries generate a *local* Lie group action on the space of solutions of the differential equation and this action seldom globalizes. However, most of the modern results on Lie group theory apply to *global* Lie groups and *global* Lie group representations. As a result, many of the standard techniques of representation theory are not always applicable to differential equations. Therefore, it is an important problem to find a globalization of the local action of the symmetry group of a differential equation (see [3, 6, 7, 10, 15] and references therein). This is usually achieved by restricting to a special class of functions.

In this paper, we study the globalization problem for the n-dimensional porous medium equation

$$u_t = \Delta_n(u^m), \tag{1}$$

where Δ_n is the n-dimensional Laplacian and u is a function of $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. When $m > 1$, this equation models slow diffusion phenomena so this condition is often assumed (cf. [17]). However, for reasons that will become evident below, we will allow $m \in \mathbb{R} \setminus \{0, 1\}$. It is worth noting that the globalization problem for the special case $m = 1$ was solved in [15], where applications of the global action of the group can be found. We will pay particular attention to the case $m = \frac{n-2}{n+2}$, because for this special value of m the symmetry group is n-dimensions larger than in the generic case.

The goal of this paper is to describe a class of functions on which the action of the symmetry group globalizes and to describe the action of the group on these functions. This idea is quite profitable when applied to linear PDE's (see [6, 7, 15]). However, it can also be applied to nonlinear equations as well. For example, the globalization problem was studied for a family of nonlinear heat equations in [13] and for the nonlinear potential filtration equation in [14]. In both cases, the group action is globalized by using parabolic induction on solvable groups. In this article, the globalization is achieved by parabolic induction on a semisimple group. In [13, 14], the global actions of the group are not linear. In contrast, the globalization of the action of the symmetry group of the porous medium equation is linear.

The paper is organized as follows. In Section 2, we list the infinitesimal generators of the symmetry group of equation (1). The symmetry group for the one-dimensional case is calculated in [12] (see also [1, 5] and references therein). For the n-dimensional case, it was be found in [4] (the result is available in the handbook [1]). In either case, the symmetry group is realized as a subgroup of $G := \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(n+1,1)_0$, where $\mathrm{SO}(n+1,1)_0$ stands for the identity component of the generalized special orthogonal group with signature $(n+1,1)$.

In Section 3, we briefly study the structure of the group G. In Section 4, we use parabolic induction to construct a family of representations of G and by restriction, of the symmetry group of (1). We use these representations to define class a of smooth functions on which the action of the symmetry group globalizes (see equation (6)). In Section 5, we write explicitly the action of the 1-parameter subgroups generated by the infinitesimal generators of G. Finally, in Section 6, we realize well-known solutions of equation (1) as elements of the constructed representation of G and we point references for possible generalizations of this work.

2 Symmetry group

Using Lie's prolongation algorithm, we calculate the symmetry group of equation (1) and re-obtain the result of [4]. This yields two cases depending on the value of m. When $m \neq \frac{n-2}{n+2}$, the infinitesimal generators of the symmetry group are

$$X_1 = \partial_t, \qquad X_2 = \sum_{i=1}^{n} x_i\partial_i + \frac{2u}{m-1}\partial_u, \qquad X_3 = -t\partial_t + \frac{u}{m-1}\partial_u, \tag{2}$$

$$Y_i = \partial_i \qquad \text{for} \quad 1 \le i \le n, \tag{3}$$

$$Z_{i,j} = x_i\partial_j - x_j\partial_i \qquad \text{for} \quad 1 \le i < j \le n. \tag{4}$$

Let $\mathfrak{s}$ denote the parabolic subalgebra of upper triangular matrices in $\mathfrak{sl}_2(\mathbb{R})$ and let $\mathfrak{g} := \mathfrak{s} \oplus \mathfrak{so}(n+1,1)$. Then, the infinitesimal generators (2)–(4) span an algebra isomorphic to a parabolic subalgebra of $\mathfrak{g}$.

In addition to these generators, in the special case $m = \frac{n-2}{n+2}$ the symmetry group is extended by the one-parameter groups generated by the operators

$$W_i = \left(x_i^2 - \sum_{j\neq i} x_j^2 \right) \partial_i + \sum_{j\neq i} 2x_ix_j\partial_j + \frac{4x_iu}{m-1}\partial_u \tag{5}$$

for $1 \le i \le n$. The infinitesimal generators (2)–(5) span an algebra isomorphic to $\mathfrak{g}$. The isomorphism for the $\mathfrak{so}(n+1,1)$ part is explicitly defined in the following way

$$\begin{aligned}
&X_2 \mapsto -E_{n+1,n+2} - E_{n+2,n+1},\\
&\frac{1}{2}(W_i + Y_i) \mapsto E_{n+1,i} - E_{i,n+1} \qquad \text{for} \quad 1 \le i \le n,\\
&\frac{1}{2}(W_i - Y_i) \mapsto E_{n+2,i} + E_{i,n+2} \qquad \text{for} \quad 1 \le i \le n,\\
&Z_{i,j} \mapsto E_{i,j} - E_{j,i} \qquad \text{for} \quad 1 \le i < j \le n,
\end{aligned}$$

where $E_{k,l}$ is the $(n+2)\times(n+2)$ matrix with single non-zero entry in the kth row and lth column. The generators X_1 and X_3 generate the parabolic subalgebra of $\mathfrak{sl}_2(\mathbb{R})$ of upper triangular matrices.

3 The group

With an eye toward the construction of a family of induced representations of the group $G := \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(n+1,1)_0$, we will study its structure in this section. Since most of this material is standard, most details will be omitted (for a general treatment see [8, Chapter VI]).

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition under the standard Cartan involution. Then, $\mathfrak{k}$ is isomorphic to $\mathfrak{so}(2) \times \mathfrak{so}(n+1)$ with $\mathfrak{so}(2) \subset \mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(n+1)$ embedded in $\mathfrak{so}(n+1,1)$ in the upper left $(n+1) \times (n+1)$ block.

We consider the minimal parabolic subalgebras $\mathfrak{q}^{\pm}$ of $\mathfrak{g}$ and their respective Langlands decompositions $\mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^{\pm}$. The maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ is given by

$$\mathfrak{a} := \operatorname{span}\left\{ H_{v,y} := \left[\begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}, \left(\begin{array}{ccc|c} & & & 0 \\ & 0_{(n+1)\times(n+1)} & & \vdots \\ & & & 0 \\ & & & -y \\ \hline 0 & \cdots & 0 \quad -y & 0 \end{array}\right) \right] \;\middle|\; v, y \in \mathbb{R} \right\}.$$

The centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, that is denoted by $\mathfrak{m}$, is trivial in the $\mathfrak{sl}_2(\mathbb{R})$ component and it embeds as $\mathfrak{so}(n)$ in the upper left $n \times n$ block in the $\mathfrak{so}(n+1,1)$ component.

If $\nu_i^{\pm} := E_{n+1,i} - E_{i,n+1} \pm E_{n+2,i} \pm E_{i,n+2} \in \mathrm{Mat}_{(n+2)\times(n+2)}$ for $1 \le i \le n$, then the nilpotent subalgebras

$$\mathfrak{n}^+ = \operatorname{span}\left\{ \eta^+_{i,a,\sigma} := \left[\begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix}, a\nu_i^+ \right] \;\middle|\; a, \sigma \in \mathbb{R}, 1 \le i \le n \right\}$$

and

$$\mathfrak{n}^- = \operatorname{span}\left\{ \eta^-_{i,a,\sigma} := \left[\begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}, a\nu_i^- \right] \;\middle|\; a, \sigma \in \mathbb{R}, 1 \le i \le n \right\}.$$

Consequently, the minimal parabolic subalgebras of $\mathfrak{g}$ are defined as $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $\mathfrak{q}^- = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^-$.

At the level of the group, these subalgebras exponentiate to

$$A := \left\{ h_{a,y} := \left[\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \left(\begin{array}{c|cc} I_n & \multicolumn{2}{c}{0} \\ \hline 0 & \cosh y & \sinh y \\ & \sinh y & \cosh y \end{array}\right) \right] \;\middle|\; y \in \mathbb{R}, a > 0 \right\},$$

$$M := \left\{ m_{j,B} := \left[\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^j, \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I_2 \end{array}\right) \right] \;\middle|\; B \in \mathrm{SO}(n), j \in \mathbb{Z}_2 \right\},$$

$$N := \left\{ n_{t,x} := \left[\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \left(\begin{array}{c|cc} I_n & -x & x \\ \hline x^{\mathrm{T}} & 1 - \frac{1}{2}\|x\|^2 & \frac{1}{2}\|x\|^2 \\ x^{\mathrm{T}} & -\frac{1}{2}\|x\|^2 & 1 + \frac{1}{2}\|x\|^2 \end{array}\right) \right] \;\middle|\; (t,x) \in \mathbb{R}^{1,n} \right\},$$

and

$$N^- := \left\{ n^-_{t,x} := \left[\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \left(\begin{array}{c|cc} I_n & x & x \\ \hline -x^{\mathrm{T}} & 1 - \frac{1}{2}\|x\|^2 & -\frac{1}{2}\|x\|^2 \\ x^{\mathrm{T}} & \frac{1}{2}\|x\|^2 & 1 + \frac{1}{2}\|x\|^2 \end{array}\right) \right] \;\middle|\; (t,x) \in \mathbb{R}^{1,n} \right\}.$$

The minimal parabolic subgroups corresponding to $\mathfrak{q}$ and $\mathfrak{q}^-$ are $Q = MAN$ and $Q^- = MAN^-$ respectively.

4 Induced representations

In this section, we will construct a family of characters on Q^-. We will induce a representation of G from each of these characters and we will identify each of these induced representations with a subspace of $C^\infty(\mathbb{R}^{1,n})$. It is shown in [2] that G acts globally on these special classes of functions. Since the symmetry group of the porous medium equation can be embedded in G, the action of G restricts to a global action of the symmetry group of the porous medium equation.

We start by noticing that a general element $\phi_{b,c} \in \mathfrak{a}^*$ is a linear functional that acts by $\phi_{b,c}(H_{v,y}) = bv + cy$ for fixed constants $b, c \in \mathbb{C}$. By exponentiating $\phi_{b,c}$ we obtain a character on A. The resulting character, $\chi_{r,s} : A \to \mathbb{C}$, is determined by two continuous parameters $r, s \in \mathbb{C}$ and defined by

$$\chi_{r,s}(h_{a,y}) = a^r e^{sy}.$$

A character $\chi_p : M \to \mathbb{C}$ can be defined by

$$\chi_p(m_{j,B}) = (-1)^{jp},$$

for $p \in \mathbb{Z}_2$. We will consider the characters parametrized by two continuous parameters $r, s \in \mathbb{C}$ and a discrete parameter $p \in \mathbb{Z}_2$, $\chi_{p,r,s} : Q^- \to \mathbb{C}$ defined by

$$\chi_{p,r,s}(q^-) = \chi_p(m)\chi_{r,s}(a),$$

where $q^- = man^-$ and by requiring it to be trivial on $\mathfrak{n}^-$.

We consider the infinite dimensional induced representation

$$\mathrm{Ind}_{Q^-}^G(\chi_{p,r,s}) := \left\{ \varphi \in C^\infty(G) \;\middle|\; \varphi(gq^-) = \chi_{p,r,s}(q^-)^{-1}\varphi(g) \text{ for } g \in G \text{ and } q^- \in Q^- \right\},$$

with the G-action defined by left translation. This space is known as the induced picture.

The unipotent group N is isomorphic to $\mathbb{R}^{1,n}$ via $(t,x) \mapsto n_{t,x}$. Since NQ^- embeds in G as an open dense set, it is easy to see from the definition of $\mathrm{Ind}_{Q^-}^G(\chi_{p,r,s})$ that an element $\varphi \in \mathrm{Ind}_{Q^-}^G(\chi_{p,r,s})$ is completely determined by its value on N. Therefore, the restriction map between $\mathrm{Ind}_{Q^-}^G(\chi_{p,r,s})$ and the space

$$I'(p,r,s) := \left\{ f \in C^\infty(\mathbb{R}^{1,n}) \;\middle|\; f(t,x) = \varphi(n_{t,x}) \text{ for some } \varphi \in \mathrm{Ind}_{Q^-}^G(\chi_{p,r,s}) \right\} \tag{6}$$

is injective. By the definition of $I'(p,r,s)$, the restriction map $\varphi \to f$ is surjective, thus an isomorphism of vector spaces. A G-module structure can be given to $I'(p,r,s)$ so that the map $\varphi \to f$ is intertwining. Therefore, $\mathrm{Ind}_{Q^-}^G(\chi_{p,r,s})$ is isomorphic to $I'(p,r,s)$ as G-module. This space is known as the non-compact picture.

5 Actions on $I'(p,r,s)$

In this section, we will describe the actions of G and of $\mathfrak{g}$ on $I'(p,r,s)$. As a result, we will find special values of r and s that will determine the class of functions on which the action of the symmetry group of equation (1), globalizes.

5.1 Actions of $\mathfrak{sl}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$

In this section, when we consider the action of an element $g \in \mathrm{SL}(2,\mathbb{R})$, it should be understood as the action of its image under the natural inclusion map $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow G$. The analogue will be true for elements in the Lie algebra. If possible, we will avoid writing this image explicitly to avoid cumbersome notation.

Proposition 1. *An element* $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$ *acts on* $I'(p,r,s)$ *by*

$$\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}.f(t,x) = \operatorname{sgn}(a)^p|a|^r f\left(\frac{t-ab}{a^2}, x\right).$$

Let $\{H,E,F\}$ *be the standard basis for* $\mathfrak{sl}(2,\mathbb{R})$*. Then,* H *acts by the differential operator* $-2t\partial_t + ru\partial_u$ *and* E *acts by* ∂_t *on* $I'(p,r,s)$*.*

Proof. First, we write

$$\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix} = \begin{pmatrix} 1 & ba \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}$$

and we calculate each action of each element individually. Let $f \in I'(p,r,s)$. Then, using the definition of $I'(p,r,s)$ and the G action on it, we have

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}.f(t,x) = \begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}.\varphi(n_{t,x}) = \varphi\left(\left[\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix}, I_{n+2}\right] n_{t,x}\right).$$

Now we notice that

$$\left[\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix}, I_{n+2}\right] n_{t,x} = n_{a^{-2}t,x}\cdot\left[\begin{pmatrix} a^{-1} & 0 \\ 0 & a\end{pmatrix}, I_{n+2}\right].$$

Using the character $\chi_{p,r,s}$ and the definition of $\mathrm{Ind}_{Q^-}^G(\chi_{p,r,s})$, we obtain

$$\begin{aligned}\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}.f(t,x) &= \chi_{p,r,s}\left(\left[(\operatorname{sgn}(a)\begin{pmatrix} |a|^{-1} & 0 \\ 0 & |a|\end{pmatrix}, I_{n+2}\right]\right)^{-1}\varphi(n_{a^{-2}t,x}) \\ &= \operatorname{sgn}(a)^p|a|^r f(a^{-2}t,x).\end{aligned}$$

The calculation for the other element is similar, thus omitted. The result on the Lie algebra action follows by differentiation. ■

Remark 1. Setting $r = \frac{2}{m-1}$ we recover the actions of the elements X_1 and X_3 in the symmetry group. We did not calculate the action of F because the space of solutions to the porous medium equation is not invariant under the subalgebra generated by this element. So, for the goals of this paper, the action of this element is not relevant.

Remark 2. Notice that exponentiation of the infinitesimal generators in $\mathfrak{sl}(2,\mathbb{R})$ gives the action of $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)$ when $a > 0$. With the appropriate value of r this agrees with the action calculated in the previous proposition. However, the previous proposition gives us a way to extend the action when $a < 0$. This reflects the global nature of the action of G on $I'(p,r,s)$.

5.2 Actions of $\mathfrak{so}(n+1,1)$ and $\mathrm{SO}(n+1,1)_0$

As in the previous section, in this section when we consider the action of an element $g \in \mathrm{SO}(n+1,1)_0$, it should be understood as the action of its image under the natural inclusion map $\mathrm{SO}(n+1,1)_0 \hookrightarrow G$. The same convention will be used for the actions of the Lie algebra.

5.2.1 Actions of $\mathfrak{m}$ and M

Lemma 1. *Let $g_{i,j,\theta} = \exp_{\mathrm{SO}(n+1,1)}(\theta(E_{i,j} - E_{j,i}))$ for $1 \leq i < j \leq n$. Then*

$$[I_2, g_{i,j,-\theta}] \cdot n_{t,x} = n_{t,g_{i,j,-\theta}x} \cdot [I_2, g_{i,j,-\theta}].$$

Proof. This is a straightforward matrix calculation. ■

Proposition 2. *An element $g_{i,j,\theta}$ for $1 \leq i < j \leq n$ acts on a function $f \in I'(p,r,s)$ by*

$$g_{i,j,\theta}.f(t,x) = f(t, g_{i,j,-\theta}x).$$

The element $E_{i,j} - E_{j,i} \in \mathfrak{so}(n+1,1)$ for $1 \leq i < j \leq n$ acts on $I'(p,r,s)$ by the differential operator $x_i\partial_j - x_j\partial_i$.

Proof. By the definition of $I'(p,r,s)$ and by the action of G on $\mathrm{Ind}_{Q^-}^{G}(\chi_{p,r,s})$ we have

$$g_{i,j,\theta}.f(t,x) = g_{i,j,\theta}.\varphi(n_{t,x}) = \varphi([I_2, g_{i,j,-\theta}].n_{t,x}).$$

The result now follows from the lemma and the definition of $\chi_{p,r,s}$. The result for the Lie algebra follows by taking $\frac{d}{d\theta}|_{\theta=0}$. ■

5.2.2 Actions of $\mathfrak{n}$ and N

A straightforward matrix calculation shows that

$$n_{t',x_ie_i}n_{t,x} = n_{t-t',x-x_ie_i},$$

where e_i is the standard basis element of $\mathbb{R}^n$. We use this fact to prove the following proposition.

Proposition 3. *The action of $n_{t',x'}$ on $f \in I'(p,r,s)$ is given by*

$$n_{t',x'}.f(t,x) = f(t-t', x-x')$$

and ν_i^+ acts on $I'(p,r,s)$ by the differential operator ∂_i, for $1 \leq i \leq n$.

5.2.3 Actions of $\mathfrak{a}$ and A

Using the fact that $h_{1,\epsilon}^{-1}n_{t,x} = n_{t,e^{\epsilon}x}h_{1,\epsilon}^{-1}$, we prove the following proposition.

Proposition 4. *The action of $h_{1,\epsilon}$ on $f \in I'(p,r,s)$ is given by $h_{1,\epsilon}.f(t,x) = e^{s\epsilon}f(t, e^{\epsilon}x)$ and $H_{0,1}$ acts on $I'(p,r,s)$ by the differential operator $\sum_{i=1}^{n} x_i\partial_i + su\partial_u$.*

Remark 3. When $s = \frac{2}{m-1}$, this action corresponds to the action of the infinitesimal generator X_2 of the symmetry group of the porous medium equation (1).

5.2.4 Actions of $\mathfrak{n}^-$ and N^-

To describe this action, we need to introduce the maps $\delta_i : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\delta_i(x) = 1 - 2x_i + \|x\|^2.$$

We will also need the maps $\gamma_i : \mathbb{R}^{1,n} \to \mathbb{R}^n$ defined by

$$\gamma_i(\epsilon, x) = \delta_i(\epsilon x)^{-1}\big(x - \epsilon\|x\|^2e_i\big)$$

and the maps $\kappa_i : \mathbb{R}^{1,n} \to \mathbb{R}^n$ given by

$$\kappa_i(\epsilon, x) = \delta_i(\epsilon x)^{-1}\epsilon(\epsilon x - e_i).$$

Then

$$\exp\left(\epsilon\nu_i^-\right).f(t,x) = n^-_{0,\epsilon e_i}.\varphi\left(n_{t,x}\right) = \varphi\left(n^-_{0,-\epsilon e_i}.n_{t,x}\right).$$

Now, in order to write the action back in terms of f we need to decompose $n^-_{0,-\epsilon e_i}.n_{t,x}$ as a product of its $N \times MAN^-$ components.

Lemma 2. *For some $m \in M$*

$$n^-_{0,-\epsilon e_i}.n_{t,x} = n_{t,\gamma_i(\epsilon,x)} m h_{1,-\log(\delta_i(\epsilon x))} n^-_{0,\kappa_i(\epsilon,x)}.$$

Proof. Since $\mathrm{SO}(n+1,1)$ has real rank 1, the Weyl group generated by the restricted roots $W(G,A)$ has two elements. The non-trivial element $\omega \in W(G,A)$ acts on N by $\omega^{-1}N\omega = N^-$. Then, by the uniqueness of the Bruhat decomposition, it suffices to show that

$$\left(n_{t,\gamma_i(\epsilon,x)}\right)^{-1} n^-_{0,-\epsilon e_i} n_{t,x} \left(n^-_{0,\kappa_i(\epsilon,x)}\right)^{-1} h^{-1}_{1,-\log(\delta_i(\epsilon x))}$$

differs from the $(n+2)\times(n+2)$ identity matrix only in the upper left $n \times n$ block (i.e. it is in M). This is a straightforward but long matrix calculation. ■

Proposition 5. *Let $f \in I'(p,r,s)$. Then,*

$$n^-_{0,-\epsilon e_i} f(t,x) = \delta_i(\epsilon x)^{-s} f(t, \gamma_i(\epsilon, x))$$

and ν_i^- acts on $I'(p,r,s)$ as the differential operator

$$\left(x_i^2 - \sum_{j\neq i} x_j^2\right)\partial_i + \sum_{j\neq i} 2x_i x_j \partial_j + 2x_i s u \partial_u.$$

Proof. The first equation follows from the previous lemma and the second equation follows form differentiating the first. ■

We summarize the previous propositions in the following theorem.

Theorem 1. *The action of the group G on $I'\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$, for $p \in \mathbb{Z}_2$, gives a globalization of the action of the local symmetry group of equation (1).*

Proof. Follows from Propositions 1 to 5. ■

6 Applications

The problem of explicitly describing the solution space of the porous medium equation inside $I'\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$ is a complicated problem. A possible strategy is to look at the compact picture of $I'\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$ and analyze what conditions need to be satisfied by the functions in this space to correspond to a solution of the differential equation. For references in this direction see [6] and [7]. However, some solutions can be realized in $I'\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$ and for illustration purposes we will examine a few in this section.

6.1 Stationary solutions

Since we work in a smooth class, the simplest example to consider is provided by stationary solutions. In this case, the solutions are given as harmonic polynomials raised to the $1/m$th power.

Let $k : \mathbb{R}^n \to \mathbb{R}$ be a harmonic polynomial and define $f \in C^\infty(\mathbb{R}^{1,n})$ by $f(t,x) = k(x)^{1/m}$. It is well-known that f satisfies equation (1).

Since sections in $I\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$ are completely determined by their values on N and we know the values that φ must take, namely $\varphi(n_{t,x}) = f(t,x)$, we can extend φ to $NMAN^-$ using the character $\chi_{p,\frac{2}{m-1},\frac{2}{m-1}}$. This can be used to define φ on all of G via limits, because $NMAN^-$ sits as an open dense subset of G. To exemplify how this process works, let us choose the simplest example. Let $k : \mathbb{R} \to \mathbb{R}$ be given by $k(x) = x$. By explicitly decomposing the element

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right]$$

in G in its $N \times MAN^-$ components, we determine that

$$\begin{aligned} &\varphi\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right]\right) \\ &\qquad = \operatorname{sgn}(d)^p |d|^{\frac{2}{m-1}} \left(\frac{a_{21}+a_{31}}{1+a_{11}}\right)^{\frac{1}{m}} \left(\frac{-2(a_{21}+a_{31})}{(1+a_{11})(a_{12}-a_{13})}\right)^{\frac{2}{1-m}} \end{aligned}$$

for $a_{11} \neq -1$ and $a_{12} \neq a_{13}$. As expected, φ restricts to f on N, more specifically $\varphi(n_{t,x}) = f(t,x)$. To extend φ to the elements in G for which $a_{11} = -1$, we notice that in this case the conditions on the group elements force $a_{12} = \pm a_{13}$ and $a_{21} = \pm a_{31}$. If $a_{11} = -1$, $a_{12} = \pm a_{13} \neq 0$, and $a_{21} = -a_{31}$, we can use limits to determine the appropriate value for φ

$$\varphi\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} -1 & \pm a_{13} & a_{13} \\ -a_{31} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right]\right) = \operatorname{sgn}(d)^p |d|^{\frac{2}{m-1}} \left(\frac{\mp 1}{a_{31}}\right)^{\frac{1}{m}} \left(\frac{\mp 2}{a_{31}a_{13}}\right)^{\frac{2}{1-m}}.$$

The values of φ when $a_{21} = a_{31}$ and when $a_{12} = a_{13} = 0$ can be determined in a similar fashion. The resulting map can be shown to be smooth and it was constructed in such way that the condition $\varphi(gq^-) = \chi_{p,\frac{2}{m-1},\frac{2}{m-1}}(q^-)^{-1}\varphi(g)$ is satisfied. Therefore, the so defined φ belongs to $I\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$ and its image f belongs to $I'\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$.

This procedure can be repeated for higher dimensions. For example, if we let $k : \mathbb{R}^2 \to \mathbb{R}$ be a 2-dimensional harmonic polynomial, then the corresponding map φ is given by

$$\begin{aligned} &\varphi\left(\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}\right]\right) = \operatorname{sgn}(d)^p |d|^{\frac{2}{m-1}} (k(z_1,z_2))^{\frac{1}{m}} \\ &\qquad \times \big(a_{44}(1+z_1^2+z_2^2) - a_{34}\big(-1+z_1^2+z_2^2\big) - 2(a_{14}z_1 + a_{24}z_2)\big)^{\frac{2}{1-m}}, \end{aligned}$$

where

$$z_1 = \frac{(a_{11}+a_{22})(a_{31}+a_{41}) - (a_{12}-a_{21})(a_{32}+a_{42})}{(a_{12}-a_{21})^2 + (a_{11}+a_{22})^2}$$

and

$$z_2 = \frac{(a_{11}+a_{22})(a_{32}+a_{42})-(a_{12}-a_{21})(a_{31}+a_{41})}{(a_{12}-a_{21})^2+(a_{11}+a_{22})^2}$$

whenever $a_{12} \neq a_{21}$ or $a_{11} \neq -a_{22}$. In this case determining the smooth extension to the whole group is more involved, but it mimics the procedure used in the one-dimensional case. It is easy to see that the so defined map φ restricts to $k(x)^{1/m}$ on N as desired.

6.2 Other solutions

To stay within the smooth class of functions, some of the well-known solutions of the porous medium equation can only be considered for specific values of m. This is the case of the solutions obtained via separation of variables. It is well known that these solutions are of the form

$$u(t,x) = ((m-1)(t-t_0))^{-1/(m-1)}F(x),$$

where $\Delta F^m(x) + F(x) = 0$, see [16, Chapter 4]. When $m > 1$ these solutions have a non-removable singularity at $t = t_0$. However, for $m < 1$ these solutions satisfy the condition $u(t_0, x) = 0$. Solutions of this type would need to be extended to sections in $I\big(p, \frac{2}{m-1}, \frac{2}{m-1}\big)$ in a similar fashion as we extended the stationary solutions. This in itself can be a complicated problem.

For source-type solutions, the smooth category would need to be abandoned and other categories may be considered. However, the induced representation may no longer carry the structure of a global G-module. For more information in this direction see [2].

7 Compact picture

To study the structure of $\mathrm{Ind}_{Q^-}^G\left(\chi_{p,\frac{2}{m-1},\frac{2}{m-1}}\right)$ as a G-representation, it is useful to look at an isomorphic copy of it called the compact picture. To construct $I'\Big(p, \frac{2}{m-1}, \frac{2}{m-1}\Big)$, we used restriction to the non-compact subgroup $N \cong \mathbb{R}^{1,n}$. To construct the compact picture of $\mathrm{Ind}_{Q^-}^G\left(\chi_{p,\frac{2}{m-1},\frac{2}{m-1}}\right)$ we use restriction to the maximal compact subgroup $K \subset G$.

The Iwasawa decomposition of G is given by $G = KAN^-$. Since $AN^- \subset Q^-$, a map in the induced picture is completely determined by its restriction to K. The compact picture is defined as the image of this restriction, that is

$$I''\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right) = \left\{\zeta \in C^\infty(K) \,\middle|\, \exists \varphi \in \mathrm{Ind}_{Q^-}^G\left(\chi_{p,\frac{2}{m-1},\frac{2}{m-1}}\right) : \zeta(k) = \varphi(k)\ \forall\, k \in K\right\}$$

and it is isomorphic to $\mathrm{Ind}_{Q^-}^G\left(\chi_{p,\frac{2}{m-1},\frac{2}{m-1}}\right)$ as vector spaces. The space can be given the structure of a G-module so that the restriction map is intertwining. Consequently,

$$\mathrm{Ind}_{Q^-}^G\left(\chi_{p,\frac{2}{m-1},\frac{2}{m-1}}\right) \cong I'\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$$

and $I'\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right) \cong I''\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$ as G-modules. The space $I''\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$ contains an open dense subset given by

$$\left\{\zeta \in C^\infty(K) \,\middle|\, \zeta(k\mu) = \chi_{p,\frac{2}{m-1},\frac{2}{m-1}}(\mu)^{-1}\zeta(k) \text{ for } k \in K \text{ and } \mu \in M \cap K\right\}$$

(see [9, Chapter 2]). Since, $\mathrm{SO}(n+1)/\mathrm{SO}(n) \cong S^n$ we obtain

$$\left\{F \in C^\infty(S^1 \times S^n) \,\middle|\, F(\theta + j\pi, z) = (-1)^{jp}F(\theta, z) \text{ for } j \in \mathbb{Z}\right\}$$

is an open dense set in $I''\left(p, \frac{2}{m-1}, \frac{2}{m-1}\right)$. This gives a more concrete realization of the representation spaces and exhibits their infinite dimensionality.

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