The $(n,1)$-Reduced DKP Hierarchy, the String Equation and W Constraints*

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Abstract. The total descendent potential of a simple singularity satisfies the Kac–Wakimoto principal hierarchy. Bakalov and Milanov showed recently that it is also a highest weight vector for the corresponding W-algebra. This was used by Liu, Yang and Zhang to prove its uniqueness. We construct this principal hierarchy of type D in a different way, viz. as a reduction of some DKP hierarchy. This gives a Lax type and a Grassmannian formulation of this hierarchy. We show in particular that the string equation induces a large part of the W constraints of Bakalov and Milanov. These constraints are not only given on the tau function, but also in terms of the Lax and Orlov–Schulman operators.

Key words: affine Kac–Moody algebra; loop group orbit; Kac–Wakimoto hierarchy; isotropic Grassmannian; total descendent potential; W constraints

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1 Introduction

Givental, Milanov, Frenkel, and Wu, showed a in a series of publications [6, 8, 9, 25] that the total descendant potential of an A, D or E type singularity satisfies the Kac–Wakimoto hierarchy [17]. Recently Bakalov and Milanov showed in [2] that this potential is also a highest weight vector for the corresponding W-algebra. For type A Fukuma, Kawai and Nakayama [7] showed that these W constraints can be obtained completely from the string equation. This was used by Kac and Schwarz [14] to show that this A_n potential is a unique $(n+1)$-reduced KP tau function, if one assumes that it corresponds to a point in the big cell of the Sato Grassmannian.

Uniqueness for type D and E singularities, together with the A case as well, was recently shown by Liu, Yang and Zhang in [20]. They use the results of [2] and the twisted vertex algebra construction to obtain this result. Both constructions use the Kac–Wakimoto principal hierarchy construction of [17].

In this paper we obtain the principal realization of the basic module of type $D_n^{(1)}$ as a certain reduction of a representation of D_∞. The reduction of the corresponding DKP-type (or sometimes also called 2-component BKP) hierarchy gives Hirota bilinear equations for the corresponding tau functions. This gives an equivalent but slightly different formulation of Kac–Wakimoto D_n principal hierarchy [17]. The total descendent potential of a D_n type singularity satisfies these equations. This approach has 3 advantages: (1) there is a Lax type formulation for this hierarchy; (2) there is a Grassmannian formulation for this reduced hierarchy; (3) one can show that the string equation generates part of the W-algebra constraints. This makes it

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possible to describe – at least part of – the W-algebra constraints in terms of pseudo-differential operators and in terms of the corresponding Grassmannian.

This approach, viz. obtaining the principal hierarchy of type D as a reduction of the 2-component BKP hierarchy, which describes the D_∞-group orbit of the highest weight vector, was also considered by Liu, Wu and Zhang in [19]. They even obtain Lax equations. However, their Lax equations are formulated differently than the ones in this paper. They use certain (scalar) pseudo-differential operators of the second type, where we need not only the basic representation of type D, but also the other level one module. As such we obtain a pair of tau functions τ_0 and τ_1, which are related. The equations on both tau functions provide (2×2)-matrix pseudo-differential operators, with which we can formulate a slightly different, but probably equivalent, Lax equation. However, in both approaches the equations on the tau-function τ_0 is the same. Wu [26] used the approach of [19] to study the Virasoro-constraints, he showed that they can be obtained from the string equation. Using the (2×2)-matrix pseudo-differential approach of this paper, we recover Wu's result and even more, the string equation not only produces the Virasoro constraints but even produces a large part of the Bakalov–Milanov [2] W constraints, but not all.

2 The $(n,1)$-reduced DKP hierarchy

2.1 The principal hierarchy for $D^{(1)}_{n+1}$

The principal hierarchy of the affine Lie algebra $D^{(1)}_{n+1}$ can be described in many different ways [11, 17]. Here we take the approach of ten Kroode and the author [21] and describe this hierarchy as a reduction of the 2-component BKP hierarchy, i.e., we introduce two neutral or twisted fermionic fields and obtain a representation of the Lie algebra of d_∞. We define an equation which describes the corresponding D_∞ group orbit of the highest weight vector. Following Jimbo and Miwa [10] we use a certain reduction procedure, which reduces the group to a smaller group, viz., to the group corresponding to $D^{(1)}_{n+1}$ in its principal realization and thus obtain a larger set of equations for elements in the group orbit.

Remark 2.1. It is important to note the following. The Kac–Wakimoto principal hierarchy of type $D^{(1)}_{n+1}$ characterizes the group orbit of the highest weight vector of type $D^{(1)}_{n+1}$ in the principal realization (see [17, Theorem 0.1] or [11]). Jimbo and Miwa show in [10] that elements of this group orbit satisfy this $D^{(1)}_{n+1}$ reduction of this DKP or 2 component BKP hierarchy. Since the total descendent potential of a D_{n+1} singularity satisfies the Kac–Wakimoto hierarchy it is an element in this $D^{(1)}_{n+1}$ group orbit and hence also satisfies this Jimbo–Miwa $D^{(1)}_{n+1}$ principal reduction or $(n,1)$-reduced DKP hierarchy.

Let n be a positive integer, consider the following Clifford algebra $\mathrm{Cl}(\mathbb{C}^\infty)$ on the vector space $\mathbb{C}^\infty$ with basis $\phi^1_{\frac{i}{2n}}$, $\phi^2_{\frac{i}{2}}$, with $i\in\mathbb{Z}$ and symmetric bilinear form

$$\left(\phi^1_{\frac{i}{2n}},\phi^1_{\frac{j}{2n}}\right)=\left(\phi^2_{\frac{i}{2}},\phi^2_{\frac{j}{2}}\right)=(-)^i\delta_{i,-j},\qquad \left(\phi^1_{\frac{i}{2n}},\phi^2_{\frac{j}{2}}\right)=0.$$

The Clifford algebra has the usual commutation relations:

$$\phi^1_{\frac{i}{2n}}\phi^1_{\frac{j}{2n}}+\phi^1_{\frac{j}{2n}}\phi^1_{\frac{i}{2n}}=(-)^i\delta_{i,-j}=\phi^2_{\frac{i}{2}}\phi^2_{\frac{j}{2}}+\phi^2_{\frac{j}{2}}\phi^2_{\frac{i}{2}},\qquad \phi^1_{\frac{i}{2n}}\phi^2_{\frac{j}{2}}+\phi^2_{\frac{j}{2}}\phi^1_{\frac{i}{2n}}=0.$$

We define its corresponding Spin module V with vacuum vector $|0\rangle$ as follows (cf. [21]):

$$\phi^1_{\frac{i}{2n}}|0\rangle=\phi^2_{\frac{i}{2}}|0\rangle=0,\quad i>0,\qquad \left(\phi^1_0+i\phi^2_0\right)|0\rangle=0.$$

The normal ordered elements $:\phi_i^a\phi_j^b:$ form the infinite Lie algebra of type d_∞, where the central elements acts as 1, see [21] for more details. The best way to describe the affine Lie algebra $D_{n+1}^{(1)}$ is to introduce, following [1], $\omega = e^{\frac{\pi i}{n}}$ the $2n$-th root of 1, and write

$$\varphi(z) = \sum_{m\in\frac{1}{2n}\mathbb{Z}} \varphi_{(m)} z^{-m-1}, \qquad \text{then} \qquad \varphi(e^{2\pi i k}z) = \sum_{m\in\frac{1}{2n}\mathbb{Z}} \omega^{-2kmn}\varphi_{(m)} z^{-m-1}.$$

The fields corresponding to the elements in the Clifford algebra are

$$\phi^1(z) = \sum_{i\in\mathbb{Z}} \phi^1_{\frac{i}{2n}} z^{\frac{-n-i}{2n}}, \qquad \phi^2(z) = \sum_{i\in\mathbb{Z}} \phi^2_{\frac{i}{2}} z^{\frac{-1-i}{2}}.$$

Then the commutation relations can be described as follows in term of the anti-commutator $\{\,,\}$

$$\big\{\phi^1(z), \phi^1\big(e^{2\pi i n}w\big)\big\} = (-)^n \sum_{j=0}^{2n-1} \delta_j(z-w), \qquad \big\{\phi^2(z), \phi^2\big(e^{2\pi i}w\big)\big\} = -\sum_{j=0}^{1} \delta_{jn}(z-w),$$
$$\big\{\phi^1(z), \phi^2(w)\big\} = 0,$$

where $\delta_j(z-w)$ is the $2n$-twisted delta function, e.g. [1]:

$$\delta_j(z-w) = z^{\frac{j}{2n}} w^{\frac{-j}{2n}} \delta(z-w) = \sum_{k\in\frac{j}{2n}+\mathbb{Z}} z^k w^{-k-1}.$$

Then, see [21], the modes of the fields

$$:\phi^a(e^{2\pi i k}z)\phi^b\big(e^{2\pi i \ell}z\big):, \qquad 1\le a,b\le 2, \qquad 0\le k,\ell\le 2n-1,$$

together with 1 span the affine Lie algebra of type $D_{n+1}^{(1)}$ in its principal realization. The spin module V splits in the direct sum of two irreducible components when restricted to $D_{n+1}^{(1)}$. The irreducible components $V = V_0$ and V_1 correspond to the $\mathbb{Z}_2$ gradation given by

$$\deg|0\rangle = 0, \qquad \deg\phi_k^{\pm a} = 1.$$

The highest weight vector of V_0 is $|0\rangle$, the highest weight vector of V_1 is

$$|1\rangle = \frac{1}{\sqrt{2}}\big(\phi_0^1 - i\phi_0^2\big)|0\rangle.$$

Here V_0 is the basic representation, V_1 is an other level 1 module. Both modules are isomorphic.

2.2 The DKP hierarchy and its principal reduction

The DKP hierarchy is the following equation on $\mathfrak{T}\in V_0$:

$$\mathrm{Res}_z\left((-)^n\phi^1(z)\mathfrak{T}\otimes\phi^1\big(e^{2\pi i n}z\big)\mathfrak{T} - \phi^2(z)\mathfrak{T}\otimes\phi^2\big(e^{2\pi i}z\big)\mathfrak{T}\right) = 0.$$

This equation describes an element in the D_∞-group orbit of $|0\rangle$.

If one restricts the action on $|0\rangle$ to the loop group of type $D_{n+1}^{(1)}$, the orbit is smaller and is given by more equations. The principal reduction, of [3, 10] induces the following. If $\mathfrak{T}\in V_0$ is in this loop group orbit of $|0\rangle$, it satisfies the $(n,1)$-reduced DKP hierarchy for all integers $p\ge 0$

$$\mathrm{Res}_z\, z^p\left((-)^n\phi^1(z)\mathfrak{T}\otimes\phi^1\big(e^{2\pi i n}z\big)\mathfrak{T} - \phi^2(z)\mathfrak{T}\otimes\phi^2\big(e^{2\pi i}z\big)\mathfrak{T}\right) = 0. \tag{2.1}$$

However, for us it will be more convenient not only to use the action on $|0\rangle$ but also on $|1\rangle$ and write $\mathfrak{T}_a$ for the action of the loop group on $|a\rangle$, where $a=0,1$. One thus obtains

$$\mathrm{Res}_z\, z^p\left((-)^n\phi^1(z)\mathfrak{T}_a\otimes\phi^1\big(e^{2\pi i n}z\big)\mathfrak{T}_b - \phi^2(z)\mathfrak{T}_a\otimes\phi^2\big(e^{2\pi i}z\big)\mathfrak{T}_b\right) = \delta_{a+b,1}\delta_{p0}\mathfrak{T}_b\otimes\mathfrak{T}_a \tag{2.2}$$

for all integers $p\ge 0$, here $a,b=0,1$.

2.3 A Grassmannian description

We follow the description of [15]. The Clifford algebra $\mathrm{Cl}(\mathbb{C}^\infty)$ has a natural $\mathbb{Z}_2$-gradation $\mathrm{Cl}(\mathbb{C}^\infty) = \mathrm{Cl}_0(\mathbb{C}^\infty) \oplus \mathrm{Cl}_1(\mathbb{C}^\infty)$, where $\mathrm{Cl}_0(\mathbb{C}^\infty)_0$ consists of products of an even number of elements from $\mathbb{C}^\infty$. Let $\mathrm{Spin}(\mathbb{C}^\infty)$ denote the multipicative group of invertible elements in $a \in \mathrm{Cl}_0(\mathbb{C}^\infty)$ such that $a\mathbb{C}^\infty a^{-1} = \mathbb{C}^\infty$. There exists a homomorphism $T : \mathrm{Spin}(\mathbb{C}^\infty) \to D_\infty$ such that $T(g)(v) = gvg^{-1}$. Thus $T(g)$ is orthogonal, i.e., $(T(g)(v), T(g)(w)) = (v, w)$, in fact it is an element in $\mathrm{SO}(\mathbb{C}^\infty)$. Let $a = 0, 1$, then

$$\mathrm{Ann}(g|a\rangle) = \{v \in \mathbb{C}^\infty | vg|a\rangle = 0\} = \{gvg^{-1} \in \mathbb{C}^\infty | v|a\rangle = 0\} = T(g)\big(\mathrm{Ann}(|a\rangle)\big).$$

Since

$$\mathrm{Ann}(|a\rangle) = \mathbb{C}\frac{\phi_0^1 + (-)^a i\phi_0^2}{\sqrt{2}} \oplus \bigoplus_{i>0} \mathbb{C}\phi^1_{\frac{i}{2n}} \oplus \mathbb{C}\phi^2_{\frac{i}{2}}, \tag{2.3}$$

it is easy to verify that $\mathrm{Ann}(|a\rangle)$ for $a = 0, 1$ is a maximal isotropic subspace of $\mathbb{C}^\infty$ and hence $\mathrm{Ann}(g|a\rangle)$ for $a = 0, 1$ and $g \in \mathrm{Spin}(\mathbb{C}^\infty)$ is also maximal isotropic. Hence an element in the D_∞ group orbit of the vacuum vector produces two unique maximal isotropic subspaces. We can say even more, the modified DKP hierarchy, i.e. equation (2.2) with $p = 0$ and $\{a, b\} = \{0, 1\}$, has the following geometric interpretation, see also [15] for more information,

$$\dim\left(\mathrm{Ann}(g|a\rangle) - \mathrm{Ann}(g|b\rangle)\right) = 1, \qquad 0 \le a \ne b \le 1.$$

Note that this follows immediately from (2.3). Let e_1 and e_2 be the orthonormal basis of $\mathbb{C}^2$ we identify

$$\phi^1_{\frac{i}{2n}} = t^{\frac{i}{2n}} e_1, \qquad \phi^2_{\frac{i}{2}} = t^{\frac{i}{2}} e_2, \tag{2.4}$$

where we assume that the bilinear form does not change, i.e.,

$$\left(t^{\frac{i}{2n}} e_1, t^{\frac{j}{2n}} e_1\right) = (-)^i \delta_{i,-j}, \qquad \left(t^{\frac{i}{2}} e_2, t^{\frac{j}{2}} e_2\right) = (-)^i \delta_{i,-j}, \qquad \left(t^{\frac{i}{2n}} e_1, t^{\frac{j}{2}} e_2\right) = 0.$$

We think of $t = e^{i\theta}$ as the loop parameter. Now if g corresponds to an element in $D^{(1)}_{n+1}$, then $\mathrm{Ann}(g|a\rangle)$ satisfies

$$t\,\mathrm{Ann}(g|a\rangle) \subset \mathrm{Ann}(g|a\rangle), \qquad a = 0, 1.$$

2.4 A bosonization procedure

In general there are many different bosonizations for the same level one $D^{(1)}_{n+1}$ module (see [13] and [21]). Kac and Peterson [13] showed that for every conjugacy class of the Weyl group of type D_{n+1} there is a different realization. The principal realization first obtained in [12] is the realization which is connected to a Coxeter element in the Weyl group (all Coxeter elements form one conjugacy class). As such the bosonization procedure for this principal realization is unique and well known, see, e.g., [21]. Here we do not take the usual one, but the one which is related to the D_{n+1} singularities as in the paper of Bakalov and Milanov [2]. This means that we introduce a parameter $\sqrt{\hbar}$ and that we choose the realization of the Heisenberg algebra slightly different from the usual one.

The bosonization of this principal hierarchy consists of identifying V with the space $F = \mathbb{C}[\theta, q^a_k; a = 1, 2, \ldots, n+1, k = 0, 1, \ldots]$. Here θ is a Grassmann variable satisfying $\theta^2 = 0$. Let σ be the isomorphism that maps V into F, we take $\sigma(V_0) = F_0 = \mathbb{C}[q^a_k; a = 1, 2, \ldots, n+1, k =$

$0,1,\dots]$ and $\sigma(V_1) = F_1 = \theta\mathbb{C}[q_k^a; a = 1,2\dots n+1, k = 0,1,\dots]$. The Heisenberg algebra, α_k^a is defined by

$$\alpha^1(z) = \sum_{i\in\frac{1}{2n}+\frac{1}{n}\mathbb{Z}} \alpha_i^1 z^{-i-1} := \frac{(-1)^n}{2\sqrt{n}} : \phi^1(z)\phi^1\big(e^{2\pi i n}z\big) :,$$
$$\alpha^2(z) = \sum_{i\in\frac{1}{2}+\mathbb{Z}} \alpha_i^2 z^{-i-1} := \frac{-1}{2} : \phi^2(z)\phi^2\big(e^{2\pi i}z\big) : .$$

Then

$$\big[\alpha_k^a, \alpha_\ell^b\big] = k\delta_{ab}\delta_{k,-\ell} \qquad \text{and} \qquad \big[\alpha_k^1, \phi^1(z)\big] = \frac{z^k}{\sqrt{n}}\phi^1(z), \qquad \big[\alpha_k^2, \phi^2(z)\big] = z^k\phi^1(z).$$

Remark 2.2. Note that in the notation of [2], $n = N-1$,

$$\alpha^1(z) = Y(\sqrt{n}v_1, z) = \sqrt{n}Y(v_1,z), \qquad \alpha^2(z) = Y(v_{n+1}, z)$$

and

$$\phi^1(z) = \frac{1}{\sqrt{2n}}Y\big(e^{v_1}, z\big), \qquad \phi^2(z) = \frac{1}{\sqrt{2}}Y\big(e^{v_{n+1}}, z\big). \tag{2.5}$$

Here the v_i form an orthonormal basis of the Cartan subalgebra of the Lie algebra of type D_{n+1}. Elements e^{v_i} are elements in the group algebra of the root lattice of type B_{n+1}, which has as basis the elements v_i. This construction is related to an automorphism ρ, which is a lift of a Coxeter element in the Weyl group and which gives the Kac–Peterson twisted realization [13], see also [21] for more details. ρ acts on the $v_1, v_2, \dots, v_n, v_{n+1}$ as follows

$$v_1 \mapsto v_2 \mapsto \cdots \mapsto v_n \mapsto -v_1, \qquad v_{n+1} \mapsto -v_{n+1},$$

then (see [2] or [1])

$$Y(v_j, z) = Y\big(\rho^{j-1}(v_1), z\big) = Y\big(v_1, e^{2(j-1)\pi i}z\big),$$
$$Y\big(e^{v_j}, z\big) = Y\big(e^{v_1}, e^{2(j-1)\pi i}z\big), \qquad 1 < j \le n.$$

The factors $\frac{1}{\sqrt{2n}}$ and $\frac{1}{\sqrt{2}}$ in (2.5) follow from the fact that $B_{v_1,-v_1} = 4n$ and $B_{v_{n+1},-v_{n+1}} = 4$ (see [2, p. 853] for the definition of these constants).

Let σ be the isomorphism which sends V to F, such that $\sigma(|0\rangle) = 1$ and $\sigma(|1\rangle) = \theta$,

$$\sigma\alpha^1_{-\frac{2j-1}{2n}-k}\sigma^{-1} = \frac{\hbar^{-\frac{1}{2}}q_k^j}{((2j-1)/(2n))_k}, \qquad \sigma\alpha^1_{\frac{2j-1}{2n}+k}\sigma^{-1} = ((2j-1)/(2n))_{k+1}\,\hbar^{\frac{1}{2}}\frac{\partial}{\partial q_k^j}, \tag{2.6}$$

$$\sigma\alpha^2_{-\frac{1}{2}-k}\sigma^{-1} = \frac{\hbar^{-\frac{1}{2}}q_k^{n+1}}{(1/2)_k}, \qquad \sigma\alpha^2_{\frac{1}{2}+k}\sigma^{-1} = (1/2)_{k+1}\,\hbar^{\frac{1}{2}}\frac{\partial}{\partial q_k^{n+1}}, \tag{2.7}$$

for $k = 0,1,2,\dots$ and $1 \le j \le n$, where $(x)_k = x(x+1)\cdots(x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}$ is the (raising) Pochhammer symbol (N.B. $(x)_0 = 1$). To describe $\sigma\phi^a(z)\sigma^{-1}1$, we introduce two extra operators θ and $\frac{\partial}{\partial\theta}$, then

$$\sigma\phi^1(z)\sigma^{-1} = \frac{\big(\theta + \frac{\partial}{\partial\theta}\big)}{\sqrt{2}}z^{-\frac{1}{2}}\Gamma^1\big(q, z^{\frac{1}{2n}}\big), \qquad \sigma\phi^2(z)\sigma^{-1} = i\frac{\big(\theta - \frac{\partial}{\partial\theta}\big)}{\sqrt{2}}z^{-\frac{1}{2}}\Gamma^2\big(q, z^{\frac{1}{2}}\big),$$

where

$$\Gamma^1\big(q,z^{\frac{1}{2n}}\big)=\Gamma^1_+\big(q,z^{\frac{1}{2n}}\big)\Gamma^1_-\big(q,z^{\frac{1}{2n}}\big),\qquad \Gamma^2\big(q,z^{\frac{1}{2}}\big)=\Gamma^2_+\big(q,z^{\frac{1}{2}}\big)\Gamma^2_-\big(q,z^{\frac{1}{2}}\big) \tag{2.8}$$

with

$$\Gamma^1_+\big(q,z^{\frac{1}{2n}}\big)=\exp\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\sum_{k=0}^{\infty}\frac{\hbar^{-\frac{1}{2}}q_k^j}{((2j-1)/(2n))_{k+1}}z^{\frac{2j-1}{2n}+k}\right), \tag{2.9}$$

$$\Gamma^1_-\big(q,z^{\frac{1}{2n}}\big)=\exp\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}\sum_{k=0}^{\infty}-((2j-1)/(2n))_k\,\hbar^{\frac{1}{2}}\frac{\partial}{\partial q_k^j}z^{-\frac{2j-1}{2n}-k}\right), \tag{2.10}$$

$$\Gamma^2_+\big(q,z^{\frac{1}{2}}\big)=\exp\left(\sum_{k=0}^{\infty}\frac{\hbar^{-\frac{1}{2}}q_k^{n+1}}{(1/2)_{k+1}}z^{\frac{1}{2}+k}\right), \tag{2.11}$$

$$\Gamma^2_-\big(q,z^{\frac{1}{2}}\big)=\exp\left(\sum_{k=0}^{\infty}-(1/2)_k\,\hbar^{\frac{1}{2}}\frac{\partial}{\partial q_k^{n+1}}z^{-\frac{1}{2}-k}\right). \tag{2.12}$$

Now let $\sigma(\mathfrak{T}_0)=\tau_0$ and $\sigma(\mathfrak{T}_1)=\tau_1\theta$ Using (2.9)–(2.12) we can rewrite the equation (2.2) and thus obtain a family of Hirota bilinear equations on τ_a, here $p\geq 0$:

$$\begin{aligned}&\mathrm{Res}_\lambda\Big(\lambda^{2np-1}\Gamma^1(q,\lambda)\tau_a\otimes\Gamma^1(q,-\lambda)\tau_b-(-)^{a+b}\lambda^{2p-1}\Gamma^2(q,\lambda)\tau_a\otimes\Gamma^2(q,-\lambda)\tau_b\Big)\\&\quad=2\delta_{a+b,1}\delta_{p0}\tau_b\otimes\tau_a.\end{aligned} \tag{2.13}$$

From now on we will often omit σ.

Using Remark 2.1, we obtain that the total descendent potential of a D_{n+1} singularity satisfies (2.13).

3 Sato–Wilson and Lax equations

3.1 Pseudo-differential operator approach

We want to reformulate (2.13) in terms of pseudo-differential operators. For this we introduce an extra variable x by replacing q_0^1 and q_0^{n+1} by $q_0^1+\frac{\hbar^{\frac{1}{2}}}{2n}x$ and $q_0^{n+1}+\frac{\hbar^{\frac{1}{2}}}{2}x$ and write ∂ for ∂_x. Then both τ_a and $\Gamma^b(q,\lambda)\tau_a$ for $b=1,2$, defined in (2.8), will depend on x. We keep the dependence in τ_a but remove it in the second term by writing $\Gamma^b(x,q,\lambda)\tau_a=\Gamma^b(q,\lambda)\tau_a e^{x\lambda}$. Next we rewrite (2.2):

$$\mathrm{Res}_\lambda\Big(W(\lambda)\mathrm{diag}\big(\lambda^{2np-1},\lambda^{2p-1}\big)\otimes W(-\lambda)^T\Big)=\delta_{p0}V\otimes V^T,$$

where

$$W(\lambda)=\begin{pmatrix}\Gamma^1(q,\lambda)\tau_0 & i\Gamma^2(q,\lambda)\tau_0\\ i\Gamma^1(q,\lambda)\tau_1 & \Gamma^2(q,\lambda)\tau_1\end{pmatrix}e^{x\lambda},\qquad V=\begin{pmatrix}\tau_1 & i\tau_1\\ i\tau_0 & \tau_0\end{pmatrix}. \tag{3.1}$$

Divide the first row of W and V by τ_1 and the second by τ_0, one thus obtains

$$\mathrm{Res}_\lambda\Big(P(\lambda)\mathrm{diag}\big(\lambda^{2np-1},\lambda^{2p-1}\big)E(\lambda)e^{x\lambda}\otimes e^{-x\lambda}E(-\lambda)^TP(-\lambda)^TJ\Big)=\delta_{p0}I, \tag{3.2}$$

where

$$P(\lambda)=\frac{1}{\sqrt{2}}\begin{pmatrix}\dfrac{\Gamma^1_-(q,\lambda)\tau_0}{\tau_1} & i\dfrac{\Gamma^2_-(q,\lambda)\tau_0}{\tau_1}\\ i\dfrac{\Gamma^1_-(q,\lambda)\tau_1}{\tau_0} & \dfrac{\Gamma^2_-(q,\lambda)\tau_1}{\tau_0}\end{pmatrix}$$

and

$$E(\lambda)=\begin{pmatrix}\Gamma_+^1(q,\lambda) & 0\\ 0 & \Gamma_+^2(q,\lambda)\end{pmatrix},\qquad J=\begin{pmatrix}0 & -i\\ -i & 0\end{pmatrix}.$$

Then using the fundamental Lemma of [16], equation (3.2) leads to:

$$\big(P(\partial)\mathrm{diag}\big(\partial^{2np-1},\partial^{2p-1}\big)P^*(\partial)J\big)_-=\delta_{p0}\partial^{-1}I.$$

Taking $p=0$ one deduces that

$$P^{-1}\partial^{-1}=\partial^{-1}P^*J \tag{3.3}$$

and for $p>0$ that

$$\big(P\mathrm{diag}\big(\partial^{2np},\partial^{2p}\big)P^{-1}\big)_{\le 0}=0.$$

Now differentiate (3.2) for $p=0$ to some q_k^j and apply the fundamental lemma then one gets the following Sato–Wilson equations:

$$\frac{\partial P}{\partial q_k^j}P^{-1}=-\big(B_k^j\big)_{\le 0},$$

where

$$B_k^j=\begin{cases}\dfrac{1}{\sqrt{n}}\dfrac{\hbar^{-\frac{1}{2}}}{(2j-1)/(2n))_{k+1}}PE_{11}\partial^{2j-1+2kn}P^{-1} & \text{if}\quad j\le n,\\[2ex] \dfrac{\hbar^{-\frac{1}{2}}}{(1/2)_{k+1}}PE_{22}\partial^{1+2k}P^{-1} & \text{if}\quad j=n+1.\end{cases}$$

Now introduce the operators

$$L=P\partial P^{-1},\qquad C_a=PE_{aa}P^{-1}.$$

Then clearly

$$[L,C_a]=0,\qquad C_aC_b=\delta_{ab}C_a,\qquad C_1+C_2=I,\qquad \big(L^{2np}C_1+L^{2p}C_2\big)_{\le 0}=0 \tag{3.4}$$

and one has the following Lax equations:

$$\frac{\partial L}{\partial q_k^j}=\big[\big(B_k^j\big)_{>0},L\big],\qquad \frac{\partial C_a}{\partial q_k^j}=\big[\big(B_k^j\big)_{>0},C_a\big].$$

Note that in the important Drinfeld–Sokolov paper [5], in the case of the Coxeter element in the Weyl group of type D, also 2×2 pseudo-differential operators appear. The principal realization of the basic representation is definitely related to this Drinfeld–Sokolov hierarchy, see, e.g., [4]. However, a direct relation between our 2×2 operators and the ones appearing in [5] is unclear.

3.2 The Orlov–Schulman and S operator

Introduce the Orlov–Schulman operator

$$M=PExE^{-1}P^{-1}=PRP^{-1},$$

where

$$R = xI + 2\hbar^{-\frac{1}{2}} \sum_{k=0}^{\infty} \left(\sqrt{n} E_{11} \sum_{j=1}^{n} \frac{q_k^j}{((2j-1)/(2n))_k} \partial^{2nk+2j-2} + E_{22} \frac{q_k^{n+1}}{(1/2)_k} \partial^{2k} \right).$$

Then $[L, M] = I$ and the wave function $W(\lambda)$ satisfies

$$LW(\lambda) = \lambda W(\lambda), \qquad C_i W(\lambda) = W(\lambda) E_{ii}, \qquad MW(\lambda) = \frac{\partial W(\lambda)}{\partial \lambda}.$$

Moreover,

$$M = \frac{\partial P}{\partial \partial} P^{-1} + 2\hbar^{-\frac{1}{2}} \sum_{k=0}^{\infty} \left(\sqrt{n} \sum_{j=1}^{n} \frac{q_k^j}{((2j-1)/(2n))_k} L^{2nk+2j-2} C_1 + \frac{q_k^{n+1}}{(1/2)_k} L^{2k} C_2 \right).$$

We introduce the operator

$$S = \left(\frac{1}{2n} M L^{1-2n} C_1 + \frac{1}{2} M L^{-1} C_2 \right)_{\leq 0} P,$$

which will play a crucial role in the deduction of the W constraints. S is explicitly given by

$$\begin{aligned}
S &= \frac{1}{2n} \frac{\partial P}{\partial \partial} \partial^{1-2n} E_{11} + \frac{1}{2} \frac{\partial P}{\partial \partial} \partial^{-1} E_{22} \\
&\quad + \sum_{k=0}^{\infty} \left(\frac{1}{\sqrt{n\hbar}} \sum_{j=1}^{n} \frac{q_k^j}{((2j-1)/(2n))_k} L^{2n(k-1)+2j-1} C_1 + \frac{1}{\sqrt{\hbar}} \frac{q_k^{n+1}}{(1/2)_k} L^{2k-1} C_2 \right)_{\leq 0} P \\
&= \frac{1}{2n} \frac{\partial P}{\partial \partial} \partial^{1-2n} E_{11} + \frac{1}{2} \frac{\partial P}{\partial \partial} \partial^{-1} E_{22} + \frac{1}{\sqrt{n\hbar}} \sum_{j=1}^{n} q_0^j P \partial^{2j-2n-1} E_{11} + \frac{1}{\sqrt{\hbar}} q_0^{n+1} P \partial^{-1} E_{22} \\
&\quad + \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{n\hbar}} \sum_{j=1}^{n} \frac{q_k^j}{((2j-1)/(2n))_k} L^{2n(k-1)+2j-1} C_1 + \frac{1}{\sqrt{\hbar}} \frac{q_k^{n+1}}{(1/2)_k} L^{2k-1} C_2 \right)_{\leq 0} P \\
&= \frac{1}{2n} \frac{\partial P}{\partial \partial} \partial^{1-2n} E_{11} + \frac{1}{2} \frac{\partial P}{\partial \partial} \partial^{-1} E_{22} + \frac{1}{\sqrt{n\hbar}} \sum_{j=1}^{n} q_0^j P \partial^{2j-2n-1} E_{11} + \frac{1}{\sqrt{\hbar}} q_0^{n+1} P \partial^{-1} E_{22} \\
&\quad - \sum_{j=1}^{n+1} \sum_{k=0}^{\infty} q_{k+1}^j \frac{\partial P}{\partial q_k^j}.
\end{aligned} \tag{3.5}$$

4 The string equation and W constraints

4.1 The principal Virasoro algebra

The principal realization of the basic representation of type $D_{n+1}^{(1)}$ has a natural Virasoro algebra with central charge $n+1$. It is given by (see, e.g., [21])

$$\begin{aligned}
L_k &= \sum_{j \in \mathbb{Z}} (-)^j \frac{j}{4n} : \phi^1_{\frac{-j}{2n}} \phi^1_{\frac{j}{2n}+k} : + (-)^j \frac{j}{4} : \phi^2_{\frac{-j}{2}} \phi^2_{\frac{j}{2}+k} : + \delta_{k,0} \left(\frac{n+1}{16n} + \frac{n^2-1}{24n} \right) \\
&= \sum_{j \in \mathbb{Z}} \frac{1}{2} : \alpha^1_{-\frac{1}{2n}-\frac{j}{n}} \alpha^1_{\frac{1}{2n}+\frac{j}{n}+k} : + \frac{1}{2} : \alpha^2_{-\frac{1}{2}-j} \alpha^2_{\frac{1}{2}+j+k} : + \delta_{k,0} \left(\frac{n+1}{16n} + \frac{n^2-1}{24n} \right),
\end{aligned} \tag{4.1}$$

or in terms of the field

$$L(z)=\sum_{k\in\mathbb{Z}}L_k z^{-k-2}=\frac{1}{2}w^{-\frac{1}{2}}\frac{\partial}{\partial w}w^{\frac{1}{2}}$$
$$\times\left((-)^n:\phi^1(w)\phi^1\left(e^{2\pi i n}z\right):-:\phi^2(w)\phi^2\left(e^{2\pi i}z\right):\right)\big|_{w=z}+\left(\frac{n+1}{16n}+\frac{n^2-1}{24n}\right)z^{-2}.$$

Using (2.6) we can express L_k in terms of the "times" q_k^j, in particular L_{-1} is equal to

$$\sigma L_{-1}\sigma^{-1}=\frac{1}{2\hbar}\left(q_0^{n+1}\right)^2+\frac{1}{2\hbar}\sum_{j=1}^{n}q_0^j q_0^{n+1-j}+\sum_{\ell=1}^{n+1}\sum_{k=0}^{\infty}q_{k+1}^{\ell}\frac{\partial}{\partial q_k^{\ell}}. \tag{4.2}$$

Let $\tau\in V_0$, the string equation is the following equation on τ

$$L_{-1}\tau=\frac{\partial\tau}{\partial q_0^1}. \tag{4.3}$$

However, following, e.g., [7], we remove the right-hand side of (4.3) by introducing the shift $q_1^1\mapsto q_1^1-1$. This reduces the string equation to

$$L_{-1}\tau=0. \tag{4.4}$$

However, this would introduce in the vertex operator $\Gamma_+^1(q,\lambda)$ of (2.9) some extra part

$$e^{-\frac{(2n)^2\hbar^{-\frac{1}{2}}}{\sqrt{n}}\frac{\lambda^{2n+1}}{2n+1}},$$

which fortunately cancels in (2.13). Therefor we will assume that the string equation is of the form (4.4) and that the hierarchy is given by (2.13), where the operators (2.9) do not have this extra term. We will show that if τ is in the $D_{n+1}^{(1)}$ group orbit of the vacuum vector, hence satisfies (2.1), and τ satisfies the string equation (4.4), i.e., that τ is annihilated by L_{-1}, that this induces the annihilation of other elements in the $W_{D_{n+1}}$ W-algebra. We will follow the approach of [24] (see also [23]). For this we use the following. If $\tau=\tau_0=g|0\rangle$ satisfies the string equation, then also its companion $\tau_1=g|1\rangle$, satisfies the string equations. This is because $\sigma L_{-1}\sigma^{-1}$ commutes with the operator $\theta+\frac{\partial}{\partial\theta}$ which intertwines F_0 with F_1.

4.2 A consequence of the string equation

Assume that the string equation (4.4) $L_{-1}\tau_a=0$ holds for both $a=0,1$. Then clearly also

$$\frac{\Gamma_-^c(\lambda)\left(L_{-1}\tau_a\right)}{\tau_b}-\frac{L_{-1}\tau_b}{(\tau_b)^2}\Gamma_-^c(\lambda)(\tau_a)=0. \tag{4.5}$$

Denote by $\tau_d^c=\Gamma_-^c(\lambda)(\tau_d)$, then (4.5) is equivalent to

$$\frac{\tau_b\Gamma_-^c(\lambda)\left(L_{-1}\right)\tau_a^c-\tau_a^c L_{-1}\tau_b}{(\tau_b)^2}=0. \tag{4.6}$$

Now,

$$\Gamma_-^c(\lambda)\left(L_{-1}\right)=\frac{1}{2\hbar}\Gamma_-^c(\lambda)\left(\left(q_0^{n+1}\right)^2+\sum_{j=1}^{n}q_0^j q_0^{n+1-j}\right)+\sum_{\ell=1}^{n+1}\sum_{k=0}^{\infty}\Gamma_-^c(\lambda)\left(q_{k+1}^{\ell}\right)\frac{\partial}{\partial q_k^{\ell}},$$

hence (4.6) turns into

$$\frac{1}{2\hbar}\frac{\tau_a^c}{\tau_b}\left(\Gamma_-^c(\lambda)-1\right)\left(\left(q_0^{n+1}\right)^2+\sum_{j=1}^{n}q_0^j q_0^{n+1-j}\right)$$
$$+\sum_{\ell=1}^{n+1}\sum_{k=0}^{\infty}\left(\frac{\Gamma_-^c(\lambda)\left(q_{k+1}^\ell\right)}{\tau_b}\frac{\partial\tau_a^c}{\partial q_k^\ell}-\frac{\tau_a^c}{(\tau_b)^2}q_{k+1}^\ell\frac{\partial\tau_b}{\partial q_k^\ell}\right)=0.$$

We rewrite this as

$$\sum_{\ell=1}^{n+1}\sum_{k=0}^{\infty}q_{k+1}^\ell\frac{\partial\frac{\tau_a^c}{\tau_b}}{\partial q_k^\ell}+R_{abc}=0, \tag{4.7}$$

where

$$R_{ab1}=\frac{\tau_a^1}{2\tau_b}\lambda^{-2n}-\frac{1}{\sqrt{n\hbar}}\frac{\tau_a^1}{\tau_b}\sum_{j=1}^{n}\lambda^{1-2j}q_0^{n+1-j}$$
$$-\frac{\sqrt{\hbar}}{\sqrt{n}\tau_b}\sum_{\ell=1}^{n}\sum_{k=0}^{\infty}((2\ell-1)/2n)_{k+1}\lambda^{1-2nk-2n-2\ell}\frac{\partial\tau_a^1}{\partial q_k^\ell}$$

and

$$R_{ab2}=\frac{\tau_a^2}{2\tau_b}\lambda^{-2}-\frac{1}{\sqrt{\hbar}}\frac{\tau_a^2}{\tau_b}\lambda^{-1}q_0^{n+1}-\frac{\sqrt{\hbar}}{\tau_b}\sum_{k=0}^{\infty}(1/2)_{k+1}\lambda^{-2k-3}\frac{\partial\tau_a^2}{\partial q_k^{n+1}}.$$

We will now prove the following

Proposition 4.1. *The string equation* (4.4) *induces*

$$\left(\left(\frac{1}{2n}ML^{1-2n}-\frac{1}{2}L^{-2n}\right)C_1+\left(\frac{1}{2}ML^{-1}-\frac{1}{2}L^{-2}\right)C_2\right)_{\leq 0}=0. \tag{4.8}$$

Proof. To prove this we first observe that (4.8) is equivalent to

$$\left(\frac{1}{2}P\partial^{-2n}E_{11}+\frac{1}{2}P\partial^{-2}E_{22}\right)_{\leq 0}-S=0, \tag{4.9}$$

where S is given by (3.5). We calculate the various parts of this formula:

$$\frac{1}{2}P_{a1}\lambda^{-2n}-\frac{1}{\sqrt{n\hbar}}\sum_{j=1}^{n}q_0^jP_{a1}\lambda^{2j-2n-1}=\frac{i^{a-1}\tau_{a-1}^1}{2\sqrt{2}\tau_{2-a}}\lambda^{-2n}-\frac{i^{a-1}\tau_{a-1}^1}{\sqrt{2n\hbar}\tau_{2-a}}\sum_{j=1}^{n}\lambda^{2j-2n-1}q_0^j,$$
$$\frac{1}{2}P_{a2}\lambda^{-2}-\frac{1}{\sqrt{\hbar}}q_0^{n+1}P_{a2}\lambda^{-}=\frac{i(-i)^{a-1}\tau_{a-1}^1}{2\sqrt{2}\tau_{2-a}}\lambda^{-2}-\frac{i(-i)^{a-1}\tau_{a-1}^1}{\sqrt{2\hbar}\tau_{2-a}}\lambda^{-1}.$$

Now

$$\frac{1}{2n}\frac{\partial P_{a1}(\lambda)}{\partial\lambda}\lambda^{1-2n}=\frac{i^{a-1}\hbar^{\frac{1}{2}}}{\sqrt{2n}\tau_{2-a}}\frac{\partial\tau_{a-1}^1}{\partial\lambda}\lambda^{1-2n}$$
$$=\frac{i^{a-1}\hbar^{\frac{1}{2}}}{\sqrt{2n}\tau_{2-a}}\sum_{\ell=1}^{n}\sum_{k=0}^{\infty}((2\ell-1)/2n)_{k+1}\lambda^{1-2n(k+1)-2\ell}\frac{\partial\tau_{a-1}^1}{\partial q_k^\ell}$$

and

$$\frac{1}{2}\frac{\partial P_{a2}(\lambda)}{\partial\lambda}\lambda^{-1}=\frac{i(-i)^{a-1}\hbar^{\frac{1}{2}}}{\sqrt{2}\tau_{2-a}}\frac{\partial\tau^1_{a-1}}{\partial\lambda}\lambda^{-1}=\frac{i(-i)^{a-1}\hbar^{\frac{1}{2}}}{\sqrt{2}\tau_{2-a}}\sum_{k=0}^{\infty}(1/2)_{k+1}\lambda^{-2k-3}\frac{\partial\tau^1_{a-1}}{\partial q_k^{n+1}}.$$

Substituting these formulas into (4.9) one obtains up to a multiplicative scalar

$$\begin{aligned}&\sum_{\ell=1}^{n+1}\sum_{k=0}^{\infty}q^{\ell}_{k+1}\frac{\partial\tau^1_{a-1}/\tau_{2-a}}{\partial q^{\ell}_k}+\frac{\tau^1_{a-1}}{2\tau_{2-a}}\lambda^{-2n}-\frac{\tau^1_{a-1}}{\sqrt{n\hbar}\tau_{2-a}}\sum_{j=1}^{n}\lambda^{2j-2n-1}q^j_0\\&\quad-\frac{\hbar^{\frac{1}{2}}}{\sqrt{n}\tau_{2-a}}\sum_{\ell=1}^{n}\sum_{k=0}^{\infty}((2\ell-1)/2n)_{k+1}\lambda^{1-2n(k+1)-2\ell}\frac{\partial\tau^1_{a-1}}{\partial q^{\ell}_k}=0\end{aligned}$$

and

$$\begin{aligned}&\sum_{\ell=1}^{n+1}\sum_{k=0}^{\infty}q^{\ell}_{k+1}\frac{\partial\tau^1_{a-1}/\tau_{2-a}}{\partial q^{\ell}_k}+\frac{\tau^1_{a-1}}{2\tau_{2-a}}\lambda^{-2}-\frac{\tau^1_{a-1}}{\sqrt{\hbar}\tau_{2-a}}\lambda^{-1}q_0^{n+1}\\&\quad-\frac{\hbar^{\frac{1}{2}}}{\tau_{2-a}}\sum_{k=0}^{\infty}(1/2)_{k+1}\lambda^{-2k+3}\frac{\partial\tau^1_{a-1}}{\partial q_k^{n+1}}=0,\end{aligned}$$

which is exactly equation (4.7). ■

A consequence of (3.4) and Proposition 4.1:

Proposition 4.2. *Let τ satisfy the string equation, then for all $p,q\geq 0$, except $p=q=0$, the following equation holds:*

$$\left(\left(\frac{1}{2n}ML^{1-2n}-\frac{1}{2}L^{-2n}\right)^q L^{2np}C_1+\left(\frac{1}{2}ML^{-1}-\frac{1}{2}L^{-2}\right)^q L^{2p}C_2\right)_{\leq 0}=0. \tag{4.10}$$

We rewrite the formula (4.10), using (3.3):

$$\left(\left(\frac{1}{2n}PR\partial^{1-2n}-\frac{1}{2}\partial^{-2n}\right)^q\partial^{2np-1}E_{11}P^*J+\left(\frac{1}{2}R\partial^{-1}-\frac{1}{2}\partial^{-2}\right)^q\partial^{2p-1}E_{22}P^*J\right)_{-}=0.$$

Now using again the fundamental Lemma of [16] this gives

$$\begin{aligned}&\mathrm{Res}_\lambda\left(\lambda^{2np-1}\left(\frac{1}{2n}\lambda^{1-2n}\partial_\lambda-\frac{1}{2}\lambda^{-2n}\right)^q W(\lambda)E_{11}\right. &(4.11)\\&\left.+\lambda^{2p-1}\left(\frac{1}{2}\lambda^{-1}\partial_\lambda-\frac{1}{2}\lambda^{-2}\right)^q W(\lambda)E_{22}\right)\otimes W(-\lambda)^T=0. &(4.12)\end{aligned}$$

Now let $\lambda^k=z$, then $\partial_z=\frac{1}{k}\lambda^{1-k}\partial_\lambda$ and $\frac{1}{k}\lambda^{1-k}\partial_\lambda-\frac{1}{2}\lambda^{-k}=z^{\frac{1}{2}}\partial_z z^{-\frac{1}{2}}$, then (4.12) is equivalent to

$$\begin{aligned}&\mathrm{Res}_z\left(z^p\partial_z^q\left(z^{-\frac{1}{2}}\Gamma^1\big(q,z^{\frac{1}{2n}}\big)\tau_a\right)\otimes z^{-\frac{1}{2}}\Gamma^1\big(q,-z^{\frac{1}{2n}}\big)\tau_b\right.\\&\left.\quad-(-)^{a+b}z^p\partial_z^q\left(z^{-\frac{1}{2}}\Gamma^2\big(q,z^{\frac{1}{2}}\big)\tau_a\right)\otimes z^{-\frac{1}{2}}\Gamma^2\big(q,-z^{\frac{1}{2}}\big)\tau_b\right)=0.\end{aligned}$$

And this formula induces

$$\begin{aligned}&\mathrm{Res}_z\left((-)^n z^p\partial_z^q\big(\phi^1(z)\big)\mathfrak{T}_a\otimes\phi^1\big(e^{2\pi in}z\big)\mathfrak{T}_b-z^p\partial_z^q\big(\phi^2(z)\big)\mathfrak{T}_a\otimes\phi^2\big(e^{2\pi i}z\big)\mathfrak{T}_b\right)=0,\\&\mathrm{Res}_z\left((-)^n z^p\phi^1\big(e^{2\pi in}z\big)\mathfrak{T}_a\otimes\partial_z^q\big(\phi^1(z)\big)\mathfrak{T}_b-z^p\phi^2\big(e^{2\pi i}z\big)\mathfrak{T}_a\otimes\partial_z^q\big(\phi^2(z)\big)\mathfrak{T}_b\right)=0. &(4.13)\end{aligned}$$

4.3 Some useful formulas

We have

$$\begin{aligned}\big[:\phi^1(y)\phi^1\big(e^{2\pi in}w\big):,\phi^1(z)\big] &= \phi^1(y)\big\{\phi^1\big(e^{2\pi in}w\big),\phi^1(z)\big\} - \big\{\phi^1(y),\phi^1(z)\big\}\phi^1\big(e^{2\pi in}w\big)\\ &= (-)^n\sum_{j=0}^{2n-1}\delta_j(z-w)\phi^1(y) - \delta_j\big(z-e^{2\pi in}y\big)\phi^1\big(e^{2\pi in}w\big),\end{aligned}$$

and similarly

$$\big[:\phi^2(y)\phi^2\big(e^{2\pi i}w\big):,\phi^2(z)\big] = -\sum_{j=0}^{1}\delta_{jn}(z-w)\phi^2(y) - \delta_{jn}\big(z-e^{2\pi i}y\big)\phi^2\big(e^{2\pi i}w\big).$$

We calculate the action of

$$X(y,w)\otimes 1 = \big((-)^n:\phi^1(y)\phi^1\big(e^{2\pi in}w\big):-:\phi^2(y)\phi^2\big(e^{2\pi i}w\big):\big)\otimes 1$$

on the bilinear identity (2.2), using the above formulas one obtains

$$\begin{aligned}&\delta_{a+b,1}\delta_{p0}X(y,w)\mathfrak{T}_b\otimes\mathfrak{T}_a\\ &\quad= \mathrm{Res}_z\, z^p\Bigg\{(-)^n\Bigg(\sum_{j=0}^{2n-1}\delta_j(z-w)\phi^1(y) - \delta_j\big(z-e^{2\pi in}y\big)\phi^1\big(e^{2\pi in}w\big)\Bigg)\mathfrak{T}_a\otimes\phi^1\big(e^{2\pi in}z\big)\mathfrak{T}_b\\ &\qquad-\Bigg(\sum_{j=0}^{1}\delta_{jn}(z-w)\phi^2(y) - \delta_{jn}\big(z-e^{2\pi i}y\big)\phi^2\big(e^{2\pi i}w\big)\Bigg)\mathfrak{T}_a\otimes\phi^1\big(e^{2\pi in}z\big)\mathfrak{T}_b\\ &\qquad+(-)^n\phi^1(z)X(y,w)\mathfrak{T}_a\otimes\phi^1(e^{2\pi in}z)\mathfrak{T}_b - \phi^2(z)X(y,w)\mathfrak{T}_a\otimes\phi^2(e^{2\pi i}z)\mathfrak{T}_b\Bigg\}.\end{aligned}$$

Thus

$$\begin{aligned}&\delta_{a+b,1}\delta_{p0}X(y,w)\mathfrak{T}_b\otimes\mathfrak{T}_a\\ &\quad-\mathrm{Res}_z\Big(z^p\big((-)^n\phi^1(z)X(y,w)\mathfrak{T}_a\otimes\phi^1\big(e^{2\pi in}z\big)\mathfrak{T}_b - \phi^2(z)X(y,w)\mathfrak{T}_a\otimes\phi^2\big(e^{2\pi i}z\big)\mathfrak{T}_b\big)\Big)\\ &= w^p\big((-)^n\phi^1(y)\mathfrak{T}_a\otimes\phi^1\big(e^{2\pi in}w\big)\mathfrak{T}_b - \phi^2(y)\mathfrak{T}_a\otimes\phi^2\big(e^{2\pi i}w\big)\mathfrak{T}_b\big)\\ &\quad-y^p\big((-)^n\phi^1\big(e^{2\pi in}w\big)\mathfrak{T}_a\otimes\phi^1(y)\mathfrak{T}_b - \phi^2\big(e^{2\pi i}w\big)\mathfrak{T}_a\otimes\phi^2(y)\mathfrak{T}_b\big).\end{aligned}\tag{4.14}$$

4.4 W constraints

Now, let $X_{k\ell} = \mathrm{Res}_w\, w^k\partial_y^\ell X(y,w)|_{y=w}$, then putting $p=0$ in formula (4.14) and using (4.13), one deduces

$$\mathrm{Res}_z\Big((-)^n\phi^1(z)X_{pq}\mathfrak{T}_a\otimes\phi^1\big(e^{2\pi in}z\big)\mathfrak{T}_b - \phi^2(z)X_{pq}\mathfrak{T}_a\otimes\phi^2\big(e^{2\pi i}z\big)\mathfrak{T}_b\Big) = \delta_{a+b,1}X_{pq}\mathfrak{T}_b\otimes\mathfrak{T}_a.$$

Thus

$$\begin{aligned}&\mathrm{Res}_\lambda\,\lambda^{-1}\Big(\Gamma^1(q,\lambda)X_{pq}\tau_a\otimes\Gamma^1(q,-\lambda)\tau_b - (-)^{a+b}\Gamma^2(q,\lambda)X_{pq}\tau_a\otimes\Gamma^2(q,-\lambda)\tau_b\Big)\\ &\quad= 2\delta_{a+b,1}X_{pq}\tau_b\otimes\tau_a.\end{aligned}\tag{4.15}$$

Note that here we abuse the notation, we write X_{pq} for $\sigma X_{pq}\sigma^{-1}$. Consider this as equation in two sets of variables x, q and x', q'. Let $a\neq b$ and set $x=x'$ and $q=q'$, This gives

$$\frac{X_{pq}\tau_a}{\tau_a} = \frac{X_{pq}\tau_b}{\tau_b}.\tag{4.16}$$

Now divide $\Gamma^c(q,\lambda)\,(X_{pq}\tau_a)$ by τ_b, then

$$\frac{\Gamma^c(q,\lambda)\,(X_{pq}\tau_a)}{\tau_b} = \Gamma^c_+(q,\lambda)\frac{\Gamma^c_-(q,\lambda)\tau_a}{\tau_b}\Gamma^c_-(q,\lambda)\left(\frac{X_{pq}\tau_a}{\tau_a}\right).$$

Now using (4.16), we rewrite (4.15) in the matrix version

$$\mathrm{Res}_\lambda\left(\lambda^{-1}\sum_{c=1}^{2}\Gamma^c_-(q,\lambda)\left(\frac{X_{pq}\tau_a}{\tau_a}\right)P(\lambda)E_{cc}E(\lambda)e^{x\lambda}\otimes e^{-x\lambda}E(-\lambda)^TP(-\lambda)^TJ\right) = \frac{X_{pq}\tau_a}{\tau_a}I.$$

Let

$$\Gamma^c_-(q,\lambda)\left(\frac{X_{pq}\tau_a}{\tau_a}\right) = \sum_{k=0}^{\infty}S^c_k(x,q)\lambda^{-k},$$

then

$$\mathrm{Res}_\lambda\left(\sum_{c=1}^{2}\sum_{k=0}^{\infty}S^c_kP(\lambda)E_{cc}\lambda^{-k-1}E(\lambda)e^{x\lambda}\otimes e^{-x\lambda}E(-\lambda)^TP(-\lambda)^TJ\right) = \frac{X_{pq}\tau_a}{\tau_a}I.$$

This gives

$$\sum_{k=1}^{\infty}S^c_kP(\partial)E_{cc}\partial^{-k}P(\partial)^{-1} = 0.$$

Now multiplying with $P(\partial)\partial^{\ell-1}$ from the right and taking the residue, one deduces that

$$S^c_\ell(x,q) = 0 \qquad \text{for} \quad \ell = 1,2,\ldots,$$

hence,

$$\left(\Gamma^c_-(q,\lambda) - 1\right)\left(\frac{X_{pq}\tau_a}{\tau_a}\right) = 0,$$

from which we conclude that

$$\frac{X_{pq}\tau_a}{\tau_a} = \text{const.}$$

In order to calculate these constants, we determine $[X_{01}, X_{pq}]$ and $[X_{11}, X_{0q}]$. The action of both operators on τ give zero. Now write $X_{pq} = X^1_{pq} + X^2_{pq}$, then

$$X^a_{pq} = \sum_{k>(q-p)n}(-)^k b_{pq}(k)\phi^a_{-\frac{k}{2n}}\phi^a_{\frac{k}{2n}+p-q},$$

where

$$b_{pq}(k) = \left(\frac{k}{2n}+\frac{1}{2}-q\right)_q - \left(-\frac{k}{2n}+\frac{1}{2}-p\right)_q.$$

From now on we assume $n = 1$ if $a = 2$, in particular

$$X^a_{01} = \sum_{k>n}(-)^k a(k)\phi^a_{-\frac{k}{2n}}\phi^a_{\frac{k}{2n}-1}, \qquad \text{where} \qquad a(k) = \frac{k}{n} - 1.$$

Then

$$[X^a_{01}, X^b_{pq}] = \delta_{ab} \sum_{j>n,k>(q-p)n} (-)^k a(j) b_{pq}(k) \Big(\delta_{j-2n,k} \phi^a_{-\frac{j}{2n}} \phi^a_{\frac{j}{2n}+p-q-1} + \delta_{j,-k+2(q-p+1)n} \phi^a_{-\frac{k}{2n}} \phi^a_{\frac{k}{2n}+p-q-1} - \delta_{j,k} \phi^a_{\frac{j}{2n}-1} \phi^a_{-\frac{j}{2n}+p-q} - \delta_{j,k+2(p-q)n} \phi^a_{-\frac{k}{2n}} \phi^a_{\frac{k}{2n}+p-q-1} \Big). \tag{4.17}$$

Now, if $p-q \neq 1$ the right hand side is normally ordered and we obtain

$$[X^a_{01}, X^b_{pq}] = \delta_{ab} \sum_{j>(q-p+1)n} (-)^j \big(a(j) b_{pq}(j-2n) - a(j+2(p-q)n) b_{pq}(j) \big) \phi^a_{-\frac{j}{2n}} \phi^a_{\frac{j}{2n}+p-q-1}.$$

It is straightforward to check that

$$a(j) b_{pq}(j-2n) - a(j+2(p-q)n) b_{pq}(j) = -2p b_{p-1,q}(j),$$

thus

$$\big[X^a_{01}, X^b_{pq}\big] = -2\delta_{ab} p X^b_{p-1,q}, \qquad \text{if} \quad p-q \neq 1. \tag{4.18}$$

If $p-q=1$ we have to normal order the right hand side of (4.17). Note that in that case, the second and third term of the right hand side of (4.17) are equal to 0 and the first term is normally ordered, the last one not. This gives

$$\big[X^a_{01}, X^b_{q+1,q}\big] = -2\delta_{ab}(q+1) X^b_{q,q} - 2c^b_{q+1},$$

where

$$\begin{aligned} c^b_{q+1} &= \left(\frac{1}{2} a(2n) b_{q+1,q}(0) + \sum_{-n<k<0} a(k+2n) b_{q+1,q}(k) \right) \\ &= \sum_{j=1-n}^{n} \left(\frac{j}{2n} - q \right)_{q+1} + \left(-\frac{j}{2n} - q \right)_{q+1}. \end{aligned}$$

Clearly, if $p=0$ the right hand side of (4.18) is equal to 0. For that case, one calculates

$$[X^a_{11}, X_{0q}] = 2q X_{0q},$$

so finally we obtain the following result. Note that $X_{p,0}=0$ and let $c_q = c^1_q + c^2_q$, then

Theorem 4.3. *For all $p \geq 0$ and $q > 0$, one has the following W constraints:*

$$\left(X_{pq} + \frac{\delta_{p,q}}{2q+2} c_{q+1} \right) \tau_a = 0, \qquad \textit{for both} \quad a = 0, 1,$$

where

$$c_q = \sum_{j=1-n}^{n} \left(\frac{j}{2n} - q \right)_q + \left(-\frac{j}{2n} - q \right)_q + \sum_{j=0}^{1} \left(\frac{j}{2} - q \right)_q + \left(-\frac{j}{2} - q \right)_q.$$

It is straightforward to check that for $|y|>|z|$

$$\begin{aligned}X^a(y,z)&=(-)^n:\phi^a(y)\phi^a(e^{2\pi in}z):\\&=\frac{1}{2}(yz)^{-\frac{1}{2}}\frac{y^{\frac{1}{2n}}+z^{\frac{1}{2n}}}{y^{\frac{1}{2n}}-z^{\frac{1}{2n}}}\left(\Gamma_+^a\big(q,y^{\frac{1}{2n}}\big)\Gamma_+^a\big(q,-z^{\frac{1}{2n}}\big)\Gamma_-^a\big(q,y^{\frac{1}{2n}}\big)\Gamma_-^a\big(q,-z^{\frac{1}{2n}}\big)-1\right)\end{aligned}$$

and

$$X^a_{pq}=\frac{(-1)^n}{q+1}\operatorname{Res}_z z^p\partial_y^{q+1}(y-z):\phi^a(y)\phi^a\big(e^{2\pi in}z\big):\Big|_{y=z}. \tag{4.19}$$

Now

$$\partial_y^k\left((y-z)(yz)^{-\frac{1}{2}}\frac{y^{\frac{1}{2n}}+z^{\frac{1}{2n}}}{y^{\frac{1}{2n}}-z^{\frac{1}{2n}}}\right)\Big|_{y=z}=c_k^a z^{-k}.$$

Let

$$\Gamma^a(y,z)=\Gamma_+^a\big(q,y^{\frac{1}{2n}}\big)\Gamma_+^a\big(q,-z^{\frac{1}{2n}}\big)\Gamma_-^a\big(q,y^{\frac{1}{2n}}\big)\Gamma_-^a\big(q,-z^{\frac{1}{2n}}\big),$$

then

$$X^a_{pq}=\operatorname{Res}_z\frac{1}{2q+2}\sum_{k=0}^{q+1}\binom{q+1}{k}c_k^a z^{p-k}\partial_y^{q-k+1}\left(\Gamma^a(y,z)-1\right)\Big|_{y=z}$$

and

$$X^a_{pq}+\frac{\delta_{p,q}}{2q+2}c^a_{q+1}=\operatorname{Res}_z\frac{1}{2q+2}\sum_{k=0}^{q+1}\binom{q+1}{k}c_k^a z^{p-k}\partial_y^{q-k+1}\left(\Gamma^a(y,z)\right)\Big|_{y=z}.$$

We now want to obtain one formula in which we combine all our W constraints. For this, we first write the generating series of the c_k^a:

$$\sum_{k=0}^{\infty}\frac{c_k^a}{k!}z^k=\sum_{k=0}^{\infty}\left(\sum_{j=1-n}^{n}\binom{\frac{j}{2n}}{k}+\binom{-\frac{j}{2n}}{k}\right)z^k=\sum_{j=1-n}^{n}\left((1+z)^{\frac{j}{2n}}+(1+z)^{-\frac{j}{2n}}\right).$$

Next we calculate for $|u|>|z|>|w|$,

$$\begin{aligned}&\sum_{p,q=0}^{\infty}\frac{X^a_{pq}}{q!}u^{-p-1}w^{q+1}\\&\quad=\frac{1}{2}\sum_{p,q=0}^{\infty}\operatorname{Res}_z\frac{u^{-p-1}w^{q+1}}{(q+1)!}\sum_{k=0}^{q+1}\binom{q+1}{k}c_k^a z^{p-k}\left(\partial_y\right)^{q-k+1}\left(\Gamma^a(y,z)-1\right)\Big|_{y=z}\\&\quad=\frac{1}{2}\operatorname{Res}_z\frac{1}{u-z}\sum_{q=0}^{\infty}\sum_{k=0}^{q+1}\frac{c_k^a}{k!}\left(\frac{w}{z}\right)^k\frac{(w\partial_y)^{q-k+1}}{(q-k+1)!}\left(\Gamma^a(y,z)-1\right)\Big|_{y=z}\\&\quad=\frac{1}{2}\operatorname{Res}_z\frac{1}{u-z}\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}\frac{c_k^a}{k!}\left(\frac{w}{z}\right)^k\frac{(w\partial_y)^{\ell}}{\ell!}\left(\Gamma^a(y,z)-1\right)\Big|_{y=z}\\&\quad=\frac{1}{2}\operatorname{Res}_z\frac{1}{u-z}\sum_{j=1-n}^{n}\left(\left(1+\frac{w}{z}\right)^{\frac{j}{2n}}+\left(1+\frac{w}{z}\right)^{-\frac{j}{2n}}\right)\left(\Gamma^a(z+w,z)-1\right).\end{aligned}$$

Note that

$$\operatorname{Res}_z \frac{1}{u-z} \sum_{j=1-n}^{n} \left(\left(1+\frac{w}{z}\right)^{\frac{j}{2n}} + \left(1+\frac{w}{z}\right)^{-\frac{j}{2n}} \right)$$
$$= \sum_{j=1-n}^{n} \left(\left(1+\frac{w}{u}\right)^{\frac{j}{2n}} + \left(1+\frac{w}{u}\right)^{-\frac{j}{2n}} \right) - c_0^a.$$

Thus we have

Theorem 4.4. *For $|u| > |z| > |w|$, one has the following W constraints:*

$$\operatorname{Res}_z \frac{1}{u-z} \left(\sum_{j=1-n}^{n} \left(\left(1+\frac{w}{z}\right)^{\frac{j}{2n}} + \left(1+\frac{w}{z}\right)^{-\frac{j}{2n}} \right) \Gamma^1(z+w,z) \right.$$
$$\left. + \sum_{j=0}^{1} \left(\left(1+\frac{w}{z}\right)^{\frac{j}{2}} + \left(1+\frac{w}{z}\right)^{-\frac{j}{2}} \right) \Gamma^2(z+w,z) \right) \tau_a = 0.$$

We can express this in a different manner. Define

$$q_1^a[z] = \partial_y \Gamma^a(y,z)\big|_{y=z} \qquad \text{and} \qquad q_r^a[z] = \partial_z^{r-1} q_1^a[z],$$

then

$$q_r^1[z] = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \sum_{k=0}^{\infty} \left(\frac{\hbar^{-\frac{1}{2}} q_k^j}{((2j-1)/(2n))_{k+1-r}} z^{\frac{2j-1}{2n}+k-r} \right.$$
$$\left. - ((2j-1)/(2n))_{k+r}\, \hbar^{\frac{1}{2}} \frac{\partial}{\partial q_k^j} z^{-\frac{2j-1}{2n}-k-r} \right),$$
$$q_r^2[z] = \sum_{k=0}^{\infty} \left(\frac{\hbar^{-\frac{1}{2}} q_k^{n+1}}{(1/2)_{k+1-r}} z^{\frac{1}{2}+k-r} - (1/2)_{k+r}\, \hbar^{\frac{1}{2}} \frac{\partial}{\partial q_k^{n+1}} z^{-\frac{1}{2}-k-r} \right),$$

here we use the convention that for $m > 0$

$$\frac{1}{(a)_{-m}} = \frac{\Gamma(a)}{\Gamma(a-m)} = (a-m)_m.$$

Thus

$$\frac{X_{pq}^a}{q!} + \frac{\delta_{pq}}{2} \frac{c_{q+1}}{(q+1)!} = \frac{1}{2} \operatorname{Res}_z \sum_{\ell=0}^{q+1} z^{p-\ell} \frac{c_\ell}{\ell!} : S_{q-\ell+1}\left(\frac{q_r^a[z]}{r!} \right) :,$$

where the $S_\ell(x)$ are the elementary Schur functions defined by

$$\sum_{\ell=0}^{\infty} S_\ell(x) = \exp\left(\sum_{k=1}^{\infty} x_k z^k \right).$$

Thus we have the following consequence of Theorem 4.4:

Corollary 4.5. *For $|u| > |z| > |w|$,*

$$\operatorname{Res}_z \frac{1}{u-z} \left(\sum_{j=1-n}^{n} \left(\left(1+\frac{w}{z}\right)^{\frac{j}{2n}} + \left(1+\frac{w}{z}\right)^{-\frac{j}{2n}} \right) : e^{\sum_{r=1}^{\infty} \frac{q_r^1[z] w^r}{r!}} : \right.$$
$$\left. + \sum_{j=0}^{1} \left(\left(1+\frac{w}{z}\right)^{\frac{j}{2}} + \left(1+\frac{w}{z}\right)^{-\frac{j}{2}} \right) : e^{\sum_{r=1}^{\infty} \frac{q_r^2[z] w^r}{r!}} : \right) \tau_a = 0.$$

A similar result is described in [2, Section 3.5].

5 A comparison with the results of Bakalov and Milanov [2]

Unfortunately we do not obtain all the W constraints of Bakalov and Milanov [2] from the string equation. Kac, Wang and Yan gave a description in [18] of the corresponding W algebra. As is mentioned in [2, Example 2.5], this W algebra is generated by the elements (cf. Remark 2.2)

$$\begin{aligned}\nu^d &:= \sum_{i=1}^{n+1} e^{v_i}{}_{(-d)}e^{-v_i} + e^{-v_i}{}_{(-d)}e^{v_i} \\ &= \sum_{i=1}^{2n} e^{\rho^i(v_1)}{}_{(-d)}e^{\rho^{n+i}(v_1)} + \sum_{i=1}^{2} e^{\rho^i(v_{n+1})}{}_{(-d)}e^{\rho^{i+1}(v_{n+1})}, \qquad d>0,\end{aligned}$$

and the element

$$\pi^{n+1} := v_{1(-1)}v_{2(-1)}\cdots{}_{(-1)}\, v_{n(-1)}v_{n+1}.$$

Our constraints come from the elements ν^d, the constraints related to the element π^{n+1} cannot be obtained from the string equation.

Since the total descendent potential is a highest weight vector of this W algebra, this means (Theorem 1.1 of [2]) that it is annihilated by all coefficients of the fractional powers of z, where the power is ≤ -1, of all $Y(\nu^d,z)$ and $Y(\pi^{n+1},z)$. Now

$$\begin{aligned}Y\big(\nu^d,z\big) &= \frac{1}{(d+1)!}\partial_y^{d+1}(y-z) \\ &\quad\times\left(\sum_{j=1}^{2n} Y\big(e^{\rho^j(v_1)},y\big)Y\big(e^{\rho^{j+n}(v_1)},z\big) + \sum_{j=1}^{2} Y\big(e^{\rho^j(v_{n+1})},y\big)Y\big(e^{\rho^{j+1}(v_{n+1})},z\big)\right)\Bigg|_{y=z}.\end{aligned}$$

Using Remark 2.2 we obtain that

$$\begin{aligned}Y\big(\nu^d,z\big) &= \frac{1}{(d+1)!}\partial_y^{d+1}(y-z) \\ &\quad\times\left(\frac{(-)^n}{2n}\sum_{j=1}^{2n}\phi^1\big(e^{2j\pi i}y\big)\phi^1\big(e^{2(j+n)\pi i}z\big) - \frac{1}{2}\sum_{j=1}^{2}\phi^2\big(e^{2j\pi i}y\big)\phi^2\big(e^{2(j+1)\pi i}z\big)\right)\Bigg|_{y=z}. \qquad (5.1)\end{aligned}$$

Using the fact that $1+\omega+\omega^2+\cdots+\omega^{k-1}=0$ for $\omega\neq 1$ a k-th root of 1, one obtains that all non-integer powers of z do not appear in (5.1). Hence,

$$Y\big(\nu^d,z\big) = \operatorname{Res}_w \delta(z-w)\frac{1}{(d+1)!}\partial_y^{d+1}(y-w)\left((-)^n\phi^1(y)\phi^1\big(e^{2n\pi i}w\big) - \phi^2(y)\phi^2\big(e^{2\pi i}w\big)\right)\Big|_{y=w}.$$

Using (4.19), we see that the total descendent potential gets annihilated by

$$X_{pq} + \frac{\delta_{p,d}}{2q+2}c_{d+1}, \qquad p\geq 0, \qquad d=1,2,\ldots,$$

which are exactly the constraints appearing in Theorem 4.3.

6 The string equation on the Grassmannian

Using the (4.1)-formulation of L_{-1} in terms of the elements ϕ_i^a, one can show that

$$\left[L_{-1},\phi^1_{\frac{k}{2n}}\right] = \left(\frac{1}{2}-\frac{k}{2n}\right)\phi^1_{\frac{k}{2n}-1}, \qquad \left[L_{-1},\phi^2_{\frac{k}{2}}\right] = \left(\frac{1}{2}-\frac{k}{2}\right)\phi^1_{\frac{k}{2}-1}.$$

Then using the identification (2.4) we obtain

$$\left[L_{-1}, t^{\frac{k}{2n}}e_1\right] = -\left(t^{\frac{1}{2}}\frac{d}{dt}t^{-\frac{1}{2}}\right)(t^{\frac{k}{2n}})e_1, \qquad \left[L_{-1}, t^{\frac{k}{2}}e_2\right] = -\left(t^{\frac{1}{2}}\frac{d}{dt}t^{-\frac{1}{2}}\right)(t^{\frac{k}{2}})e_2.$$

Now applying the dilaton shift $q_1^1 \mapsto q_1^1 + 1$, then $\sigma L_1 \sigma^{-1}$ changes according to the description (4.2) to

$$\sigma L_1 \sigma^{-1} + \frac{\partial}{\partial q_0^1},$$

and by (2.6) one finds that L_{-1} changes into $L_{-1} + 2n\hbar^{-\frac{1}{2}}\alpha^1_{\frac{1}{2n}}$. Since

$$\left[\alpha^1_{\frac{1}{2n}}, \phi^a(z)\right] = \frac{\delta_{a1}}{\sqrt{n}} z^{\frac{1}{2n}}\phi^1(z),$$

we obtain

Proposition 6.1. *Let W be the point of the Grassmannian which corresponds to the τ-function that satisfies the string equation, then W satisfies*

$$tW \subset W \qquad \textit{and} \qquad \left(-\frac{d}{dt} + \frac{1}{2}t^{-1} + 2\sqrt{n}\hbar^{-\frac{1}{2}}t^{\frac{1}{2n}}E_{11}\right)W \subset W.$$

Note that the total descendent potential of a D_{n+1} type singularity is tau function, that satisfies this condition. Vakulenko, used a similar approach in [22]. He showed that the tau function is unique. However, his action on the Grassmannian seems somewhat strange.

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