Tilting Modules in Truncated Categories[*]

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Abstract. We begin the study of a tilting theory in certain truncated categories of modules $\mathcal{G}(\Gamma)$ for the current Lie algebra associated to a finite-dimensional complex simple Lie algebra, where $\Gamma = P^+ \times J$, J is an interval in $\mathbb{Z}$, and P^+ is the set of dominant integral weights of the simple Lie algebra. We use this to put a tilting theory on the category $\mathcal{G}(\Gamma')$ where $\Gamma' = P' \times J$, where $P' \subseteq P^+$ is saturated. Under certain natural conditions on Γ', we note that $\mathcal{G}(\Gamma')$ admits full tilting modules.

Key words: current algebra; tilting module; Serre subcategory

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1   Introduction

Associated to any finite-dimensional complex simple Lie algebra $\mathfrak{g}$ is its current algebra $\mathfrak{g}[t]$. The current algebra is just the Lie algebra of polynomial maps from $\mathbb{C} \to \mathfrak{g}$ and can be identified with the space $\mathfrak{g} \otimes \mathbb{C}[t]$ with the obvious commutator. The study of the representation theory of current algebras was largely motivated by its relationship to the representation theory of affine and quantum affine algebras associated to $\mathfrak{g}$. However, it is also now of independent interest since the current algebra has connections with problems arising in mathematical physics, for instance the $X = M$ conjectures, see [1, 17, 25]. Also, the current algebra, and many of its modules, admits a natural grading by the integers, and this grading gives rise to interesting combinatorics. For example, [22] relates certain graded characters to the Poincaré polynomials of quiver varieties.

Let P^+ be the set of dominant integral weights of $\mathfrak{g}$, $\Lambda = P^+ \times \mathbb{Z}$, and $\widehat{\mathcal{G}}$ the category of $\mathbb{Z}$-graded modules for $\mathfrak{g}[t]$ with the restriction that the graded pieces are finite-dimensional. Also, let $\mathcal{G}$ be the full subcategory of $\widehat{\mathcal{G}}$ consisting of modules whose grades are bounded above. Then Λ indexes the simple modules in $\widehat{\mathcal{G}}$. In this paper we are interested in studying Serre subcategories $\widehat{\mathcal{G}}(\Gamma)$ where $\Gamma \subset \Lambda$ is of the form $P' \times J$ where $J \subset \mathbb{Z}$ is a (possibly infinite) interval and $P' \subset P^+$ is closed with respect to a natural partial order. In particular, we study the tilting theories in these categories. This generalized the work of [3], where Γ was taken to be all of Λ.

The category $\widehat{\mathcal{G}}(\Gamma)$ contains the projective cover and injective envelope of its simple objects. Given a partial order on the set Γ, we can define the standard and costandard objects, as in [18]. The majority of the paper is concerned with a particular order, in which case the standard objects $\Delta(\lambda, r)(\Gamma)$ are quotients of the finite-dimensional local Weyl modules, and the costandard objects $\nabla(\lambda, r)(\Gamma)$ are submodules of (appropriately defined) duals of the infinite-dimensional global Weyl modules. We recall (see, for example, [23]) that a module T is called

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tilting if T admits a filtration by standard modules and a filtration by costandard modules. In our case, both sets of objects have been extensively studied (see [12, 19, 20, 24] for the local Weyl modules, and [6, 15], for the global Weyl modules). Both families of modules live in a subcategory $\mathcal{G}_{\text{bdd}}(\Gamma)$ consisting of objects whose weights are in a finite union of cones (as in $\mathcal{O}$) and whose grades are bounded above. The main goal of this paper is to construct another family of modules indexed by Γ and which are in $\mathcal{G}_{\text{bdd}}(\Gamma)$. These modules are denoted by $T(\lambda, r)(\Gamma)$, and admit an infinite filtration whose quotients are of the form $\Delta(\mu, s)(\Gamma)$, for $(\mu, s) \in \Gamma$. They also satisfy the homological property that $\text{Ext}^1_{\mathcal{G}}(\Delta(\mu, s)(\Gamma), T(\lambda, r)(\Gamma)) = 0$ for all $(\mu, s) \in \Gamma$. We use the following theorem to prove that this homological property is equivalent to having a $\nabla(\Gamma)$-filtration, proving that the $T(\lambda, r)(\Gamma)$ are tilting. The theorem was proved in [4, 2], and [10] for $\mathfrak{sl}_2[t]$, $\mathfrak{sl}_{n+1}[t]$ and general $\mathfrak{g}[t]$ respectively.

Theorem 1.1. *Let $P(\lambda, r)$ denote the projective cover of the simple module $V(\lambda, r)$. Then $P(\lambda, r)$ admits a filtration by global Weyl modules, and we have an equality of filtration multiplicities $[P(\lambda, r) : W(\mu, s)] = [\Delta(\mu, r) : V(\lambda, s)]$, where $\Delta(\mu, r)$ is the local Weyl module.*

The following is the main result of this paper.

Theorem 1.2.

1. *Given $(\lambda, r) \in \Gamma$, there exists an indecomposable module $T(\lambda, r)(\Gamma) \in \text{Ob}\,\mathcal{G}_{\text{bdd}}(\Gamma)$ which admits a $\Delta(\Gamma)$-filtration and a $\nabla(\Gamma)$-filtration. Further,*

$$T(\lambda, r)(\Gamma)[s]_\lambda = 0 \quad \text{if} \quad s > r, \quad T(\lambda, r)(\Gamma)[r]_\lambda = 1, \quad \text{wt}\,T(\lambda, r)(\Gamma) \subset \text{conv}\,W\lambda,$$

   and $T(\lambda, r)(\Gamma) \cong T(\mu, s)(\Gamma)$ if and only if $(\lambda, r) = (\mu, s)$.

2. *Moreover, any indecomposable tilting module in $\mathcal{G}_{\text{bdd}}(\Gamma)$ is isomorphic to $T(\lambda, r)(\Gamma)$ for some $(\lambda, r) \in \Gamma$, and any tilting module in $\mathcal{G}_{\text{bdd}}(\Gamma)$ is isomorphic to a direct sum of indecomposable tilting modules.*

The majority of the paper is devoted to the case where $\Gamma = P^+ \times J$. It is easy to see from the construction that the module $T(\lambda, r)(\Gamma)$ has its weights bounded above by λ. It follows that if we let $P' \subset P^+$ be saturated (downwardly closed with respect to the normal partial order on weights), and set $\Gamma' = P' \times J$, then $T(\lambda, r)(\Gamma') = T(\lambda, r)(\Gamma)$.

We use the convention that $\Delta(\lambda, r)(\Lambda)$ is simply written $\Delta(\lambda, r)$, and similarly for other objects. Keeping $\Gamma = P^+ \times J$, there is a natural functor taking $M \in \text{Ob}\,\mathcal{G}$ to $M^\Gamma \in \text{Ob}\,\mathcal{G}(\Gamma)$. For $(\lambda, r) \in \Gamma$ this functor preserves many objects, and in particular we have $\Delta(\lambda, r)^\Gamma = \Delta(\lambda, r)(\Gamma)$ and $\nabla(\lambda, r)^\Gamma = \nabla(\lambda, r)(\Gamma)$. So it is natural to ask if $T(\lambda, r)^\Gamma = T(\lambda, r)(\Gamma)$. The answer is "no", and is a result of the following phenomena: for $(\mu, s) \notin \Gamma$, the module $\nabla(\mu, s)^\Gamma$ is not in general zero, and does not correspond to any simple module. Hence $\nabla(\mu, s)^\Gamma$ can not be considered costandard. So the modules $T(\lambda, r)(\Gamma)$ must be studied independently.

Another purpose of this paper is the following. In [3], the tilting modules $T(\lambda, r)$ are constructed for all $(\lambda, r) \in \Lambda$. It is normal to then consider the module $T = \bigoplus_{(\lambda, r) \in \Lambda} T(\lambda, r)$, the algebra $A = \text{End}\,T$, and use several functors to find equivalences of categories. However, it is not hard to see that if T is defined in this way, then T fails to have finite-dimensional graded components, and hence $T \notin \text{Ob}\,\mathcal{G}$. One of the purposes of this paper is to find Serre subcategories with index sets Γ such that $T(\Gamma) = \bigoplus_{(\lambda, r) \in \Gamma} T(\lambda, r)(\Gamma) \in \text{Ob}\,\mathcal{G}(\Gamma)$. It is not hard to see that (except for the degenerate case where $\Gamma = \{0\} \times J$) a necessary and sufficient pair of conditions on Γ is that P' be finite and J have an upper bound. It is natural to study the algebra $\text{End}\,T(\Gamma)$ in the case that $T(\Gamma) \in \mathcal{G}(\Gamma)$, and this will be pursued elsewhere. We also note that in the case that Γ is finite then $\text{End}\,T(\Gamma)$ is a finite-dimensional associative algebra.

We end the paper by considering other partial orders which can be used on $\Gamma \subset \Lambda$. In particular, we consider partial orders induced by the so-called covering relations. One tends to

get trivial tilting theories in these cases (one of the standard-costandard modules is simple, and the other is projective or injective), but the partial orders are natural for other reasons, and we include their study for completeness. One of the reasons to study these other subcategories is that one can obtain directed categories as in [7] (in the sense of [16]).

The paper is organized as follows. In Section 2 we establish notation and recall some basic results on the finite-dimensional representations of a finite-dimensional simple Lie algebra. In Section 3 we introduce several important categories of modules for the current algebra. We also introduce some important objects, including the local and global Weyl modules. In Section 4 we state the main results of the paper and establish some homological results. Section 5 is devoted to constructing the modules $T(\lambda, r)(\Gamma)$ and establishing their properties. Finally, in Section 6, we consider the tilting theories which arise when considering partial orders on Λ which are induced by covering relations.

We also provide for the reader's convenience a brief index of the notation which is used repeatedly in this paper.

2 Preliminaries

2.1 Simple Lie algebras and current algebras

Throughout the paper $\mathfrak{g}$ denotes a finite-dimensional complex simple Lie algebra and $\mathfrak{h}$ a fixed Cartan subalgebra of $\mathfrak{g}$. Set $I = \{1, \ldots, \dim \mathfrak{h}\}$ and let $\{\alpha_i : i \in I\} \subset \mathfrak{h}^*$ be a set of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $R \subset \mathfrak{h}^*$ (respectively, R^+, P^+, Q^+) be the corresponding set of roots (respectively, positive roots, dominant integral weights, the $\mathbb{Z}_+$-span of R^+) and let $\theta \in R^+$ be the highest root. Given $\lambda, \mu \in \mathfrak{h}^*$, we say that $\lambda \geq \mu$ iff $\lambda - \mu \in Q^+$. Let $W \subset \mathrm{Aut}(\mathfrak{h}^*)$ be the Weyl group of $\mathfrak{g}$ and $w_\circ$ be the longest element of W. For $\alpha \in R$ denote by $\mathfrak{g}_\alpha$ the corresponding root space. The subspaces $\mathfrak{n}^\pm = \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\pm\alpha}$, are Lie subalgebras of $\mathfrak{g}$. Fix a Chevalley basis $\{x_\alpha^\pm, h_i \,|\, \alpha \in R^+, i \in I\}$ of $\mathfrak{g}$ and for $\alpha \in R^+$ set $h_\alpha = [x_\alpha, x_{-\alpha}]$. Note that $h_{\alpha_i} = h_i$, $i \in I$. For $i \in I$, let $\omega_i \in P^+$ be defined by $\omega_i(h_j) = \delta_{ij}$ for all $j \in I$.

Given any Lie algebra $\mathfrak{a}$ let $\mathfrak{a}[t] = \mathfrak{a} \otimes \mathbb{C}[t]$ be the current algebra of $\mathfrak{a}$. Let $\mathfrak{a}[t]_+$ be the Lie ideal $\mathfrak{a} \otimes t\mathbb{C}[t]$. Both $\mathfrak{a}[t]$ and $\mathfrak{a}[t]_+$ are $\mathbb{Z}_+$-graded Lie algebras with the grading given by powers of t. Let $\mathbf{U}(\mathfrak{a})$ denote the universal enveloping algebra of $\mathfrak{a}$. Then $\mathbf{U}(\mathfrak{a}[t])$ has a natural $\mathbb{Z}_+$-grading as an associative algebra and we let $\mathbf{U}(\mathfrak{a}[t])[k]$ be the kth-graded piece. These graded pieces are $\mathfrak{a}$-modules under left and right multiplication by elements of $\mathfrak{a}$ and, hence, also under the adjoint action of $\mathfrak{a}$. If $\dim \mathfrak{a} < \infty$, then $\mathbf{U}(\mathfrak{a}[t])[k]$ is a free module for $\mathfrak{a}$ (via left or right multiplication) of finite rank.

The algebra $\mathbf{U}(\mathfrak{a})$ is a Hopf algebra, the comultiplication being given by extending the assignment $x \to x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{a}$ to an algebra homomorphism of $\mathbf{U}(\mathfrak{a})$. In the case of $\mathbf{U}(\mathfrak{a}[t])$ the comultiplication is a map of graded algebras. We shall repeatedly use the fact that, if $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$ (notice that this is the case for the Lie algebra $\mathfrak{g}$), then $\mathbf{U}(\mathfrak{a}[t])$ is generated as a graded algebra by $\mathfrak{a}$ and $\mathfrak{a} \otimes t$ without further comment.

2.2 Finite-dimensional modules

Let $\mathcal{F}(\mathfrak{g})$ be the category of finite-dimensional $\mathfrak{g}$-modules with the morphisms being maps of $\mathfrak{g}$-modules. In particular, we may write $\mathrm{Hom}_\mathfrak{g}$ for $\mathrm{Hom}_{\mathcal{F}(\mathfrak{g})}$. The set P^+ parametrizes the isomorphism classes of simple objects in $\mathcal{F}(\mathfrak{g})$. For $\lambda \in P^+$, let $V(\lambda)$ be the simple module in the corresponding isomorphism class which is generated by an element $v_\lambda \in V(\lambda)$ satisfying the defining relations:

$$\mathfrak{n}^+ v_\lambda = 0, \qquad h v_\lambda = \lambda(h) v_\lambda, \qquad (x_{\alpha_i}^-)^{\lambda(h_i)+1} v_\lambda = 0,$$

for all $h \in \mathfrak{h}$, $i \in I$. The module $V(-w_\circ \lambda)$ is the $\mathfrak{g}$-dual of $V(\lambda)$. If $V \in \mathcal{F}(\mathfrak{g})$, write $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V : hv = \lambda(h)v, \forall h \in \mathfrak{h}\}$. Set $\mathrm{wt}(V) = \{\lambda \in \mathfrak{h}^* : V_\lambda \neq 0\}$. Finally, recall also that the category $\mathcal{F}(\mathfrak{g})$ is semi-simple, i.e. any object in $\mathcal{F}(\mathfrak{g})$ is isomorphic to a direct sum of the modules $V(\lambda)$, $\lambda \in P^+$. We shall use the following standard results without further mentioning.

Lemma 2.1.

1. *Let $\lambda \in P^+$. Then $\mathrm{wt}(V(\lambda)) \subset \lambda - Q^+$.*

2. *Let $V \in \mathcal{F}(\mathfrak{g})$. Then $w\,\mathrm{wt}(V) \subset \mathrm{wt}(V)$ and $\dim V_\lambda = V_{w\lambda}$ for all $w \in W$.*

3. *Let $V \in \mathcal{F}(\mathfrak{g})$. Then $\dim \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), V) = \dim\{v \in V_\lambda : \mathfrak{n}^+ v = 0\}$.*

3 The main category and its subcategories

In this section we define the categories which are our objects of study and present several properties and functors between them. We also introduce several families of modules which will be relevant. Most of these have been studied elsewhere.

3.1 The main category

Let $\widehat{\mathcal{G}}$ be the category whose objects are $\mathbb{Z}$-graded $\mathfrak{g}[t]$-modules with finite-dimensional graded pieces and where the morphisms are 0-graded maps of $\mathfrak{g}[t]$-modules. More precisely, if $V \in \mathrm{Ob}\,\widehat{\mathcal{G}}$ then

$$V = \bigoplus_{r \in \mathbb{Z}} V[r],$$

where $V[r]$ is a finite-dimensional subspace of V such that $(xt^k)V[r] \subset V[r+k]$ for all $x \in \mathfrak{g}$ and $r, k \in \mathbb{Z}$. In particular, $V[r] \in \mathrm{Ob}\,\mathcal{F}(\mathfrak{g})$ and, if $V, W \in \mathrm{Ob}\,\widehat{\mathcal{G}}$, then

$$\mathrm{Hom}_{\widehat{\mathcal{G}}}(V, W) = \{f \in \mathrm{Hom}_{\mathfrak{g}[t]}(V, W) : f(V[r]) \subset W[r], r \in \mathbb{Z}\}.$$

For $M \in \widehat{\mathcal{G}}$, we define its graded character to be $\mathrm{ch}_{\mathrm{gr}} M = \sum_{r \in \mathbb{Z}} \mathrm{ch}\, M[r]u^r$.

Define a covariant functor $\mathrm{ev} : \mathcal{F}(\mathfrak{g}) \to \widehat{\mathcal{G}}$ by the requirements:

$$\mathrm{ev}(V)[0] = V \qquad \text{and} \qquad \mathrm{ev}(V)[r] = 0, \qquad \text{for all} \quad 0 \neq r \in \mathbb{Z},$$

with $\mathfrak{g}[t]$-action given by

$$\left(xt^k\right)v = \delta_{k,0}xv, \qquad \text{for all} \quad x \in \mathfrak{g}, \quad k \in \mathbb{Z}_+, \quad v \in V,$$

and

$$\mathrm{Hom}_{\widehat{\mathcal{G}}}(\mathrm{ev}(V), \mathrm{ev}(W)) = \mathrm{Hom}_{\mathfrak{g}}(V, W).$$

Further, for $s \in \mathbb{Z}$ let τ_s be the grading shift given by

$$(\tau_s V)[k] = V[k - s], \qquad \text{for all} \quad k \in \mathbb{Z}, \quad V \in \mathrm{Ob}\,\widehat{\mathcal{G}}.$$

Clearly $\tau_s(V) \in \mathrm{Ob}\,\widehat{\mathcal{G}}$. In particular, for $\lambda \in P^+$ and $r \in \mathbb{Z}$, set $V(\lambda, r) := \tau_r(\mathrm{ev}(V(\lambda)))$ and $v_{\lambda, r} := \tau_r(v_\lambda)$.

Proposition 3.1. *The isomorphism classes of simple objects in $\widehat{\mathcal{G}}$ are parametrized by pairs* (λ, r) *and we have*

$$\operatorname{Hom}_{\widehat{\mathcal{G}}}(V(\lambda, r), V(\mu, s)) = \begin{cases} 0, & \text{if} \quad (\lambda, r) \neq (\mu, s), \\ \mathbb{C}, & \text{if} \quad (\lambda, r) = (\mu, s). \end{cases}$$

Moreover, if $V \in \operatorname{Ob}\widehat{\mathcal{G}}$ is such that $V = V[n]$ for some $n \in \mathbb{Z}$, then V is semi-simple.

For $M \in \operatorname{Ob}\widehat{\mathcal{G}}$ we define its (graded) dual to be the module $M^* \in \operatorname{Ob}\widehat{\mathcal{G}}$, where

$$M^* = \bigoplus_{r \in \mathbb{Z}} M^*[-r] \qquad \text{and} \qquad M^*[-r] = M[r]^*$$

and equipped with the usual action where

$$(xt^r)m^*(V) = -m^*(xt^r.v).$$

We note that $M^{**} \cong M$ and that $\operatorname{ch}_{\mathrm{gr}} M^* = \sum_{r \in \mathbb{Z}} \operatorname{ch}(M[r]^*)u^{-r}$.

From now on we set $\Lambda = P^+ \times \mathbb{Z}$ and we equip Λ with the dictionary partial order $\leq$, i.e.

$$(\mu, r) \leq (\lambda, s) \qquad \Leftrightarrow \qquad \text{either} \quad \mu < \lambda \quad \text{or} \quad \mu = \lambda \quad \text{and} \quad r \leq s.$$

3.2   Some bounded subcategories of the main category

Let $\mathcal{G}_{\leq s}$ be the full subcategory of $\widehat{\mathcal{G}}$ whose objects V satisfy $V[r] = 0$, for all $r > s$, and let $\mathcal{G}$ be the full subcategory of $\widehat{\mathcal{G}}$ consisting of $V \in \operatorname{Ob}\widehat{\mathcal{G}}$ such that $V \in \operatorname{Ob}\mathcal{G}_{\leq s}$ for some $s \in \mathbb{Z}$. It follows from the definition that $\mathcal{G}_{\leq s}$ is a full subcategory of $\mathcal{G}_{\leq r}$ for all $s < r \in \mathbb{Z}$. Further, let $\mathcal{G}_{\mathrm{bdd}}$ be the full subcategory of $\mathcal{G}$ consisting of objects M satisfying the following condition: there exists $\mu_1, \ldots, \mu_s \in P^+$ (depending on M) such that

$$\operatorname{wt} M \subset \bigcup_{\ell=1}^{s} \operatorname{conv} W\mu_\ell.$$

Given $s \in \mathbb{Z}$ and $V \in \operatorname{Ob}\widehat{\mathcal{G}}$, define

$$V_{>s} = \bigoplus_{r > s} V[r] \qquad \text{and} \qquad V_{\leq s} = V/V_{>s}.$$

Then $V_{\leq s} \in \operatorname{Ob}\mathcal{G}_{\leq s}$. Furthermore, if $f \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(V, W)$, then $V_{>s}$ is contained in the kernel of the canonical morphism $\bar{f} : V \to W_{\leq s}$ and hence we have a natural morphism $f_{\leq s} \in \operatorname{Hom}_{\widehat{\mathcal{G}}_{\leq s}}(V_{\leq s}, W_{\leq s})$.

Lemma 3.2. *The assignments $V \mapsto V_{\leq r}$ for all $V \in \operatorname{Ob}\widehat{\mathcal{G}}$ and $f \mapsto f_{\leq r}$ for all $f \in \operatorname{Hom}_{\widehat{\mathcal{G}}}(V, W)$, $V, W \in \operatorname{Ob}\widehat{\mathcal{G}}$ define a full, exact and essentially surjective functor from $\widehat{\mathcal{G}}$ to $\mathcal{G}_{\leq r}$.*

Given $V \in \operatorname{Ob}\mathcal{G}$ we denote by $[V : V(\lambda, r)]$ the multiplicity of $V(\lambda, r)$ in a composition series for V. Furthermore, given $W \in \operatorname{Ob}\widehat{\mathcal{G}}$, we set $[W : V(\lambda, r)] := [W_{\leq r} : V(\lambda, r)]$. Observe that $[V : V(\lambda, r)]$ equals the $\mathfrak{g}$-module multiplicity of $V(\lambda)$ in $V[r]$. For any $V \in \operatorname{Ob}\widehat{\mathcal{G}}$, define

$$\Lambda(V) = \{(\lambda, r) \in \Lambda : [V : V(\lambda, r)] \neq 0\}.$$

3.3 Projective and injective objects in the main category

Given $(\lambda, r) \in \Lambda$, set

$$P(\lambda, r) = \mathbf{U}(\mathfrak{g}[t]) \otimes_{\mathbf{U}(\mathfrak{g})} V(\lambda, r) \qquad \text{and} \qquad I(\lambda, r) \cong P(-w_\circ\lambda, -r)^*.$$

Clearly, $P(\lambda, r)$ is an infinite-dimensional $\mathbb{Z}$-graded $\mathfrak{g}[t]$-module. Using the PBW theorem we have an isomorphism of graded vector spaces $\mathbf{U}(\mathfrak{g}[t]) \cong \mathbf{U}(\mathfrak{g}[t]_+) \otimes \mathbf{U}(\mathfrak{g})$, and hence we get $P(\lambda, r)[k] = \mathbf{U}(\mathfrak{g}[t]_+)[k-r] \otimes V(\lambda, r)$, where we understand that $\mathbf{U}(\mathfrak{g}[t]_+)[k-r] = 0$ if $k < r$. This shows that $P(\lambda, r) \in \mathrm{Ob}\,\widehat{\mathcal{G}}$ and also that $P(\lambda, r)[r] = 1 \otimes V(\lambda, r)$. Set $p_{\lambda, r} = 1 \otimes v_{\lambda, r}$.

Proposition 3.3. *Let $(\lambda, r) \in \Lambda$, and $s \geq r$.*

1. *$P(\lambda, r)$ is generated as a $\mathfrak{g}[t]$-module by $p_{\lambda, r}$ with defining relations*

   $$(\mathfrak{n}^+)p_{\lambda, r} = 0, \qquad h p_{\lambda, r} = \lambda(h)p_{\lambda, r}, \qquad (x_{\alpha_i}^-)^{\lambda(h_i)+1}p_{\lambda, r} = 0,$$

   for all $h \in \mathfrak{h}$, $i \in I$. Hence, $P(\lambda, r)$ is the projective cover in the category $\widehat{\mathcal{G}}$ of its unique simple quotient $V(\lambda, r)$.

2. *The modules $P(\lambda, r)_{\leq s}$ are projective in $\mathcal{G}_{\leq s}$.*

3. *Let $V \in \mathrm{Ob}\,\widehat{\mathcal{G}}$. Then $\dim\mathrm{Hom}_{\widehat{\mathcal{G}}}(P(\lambda, r), V) = [V : V(\lambda, r)]$.*

4. *Any injective object of $\mathcal{G}$ is also injective in $\widehat{\mathcal{G}}$.*

5. *Let $(\lambda, r) \in \Lambda$. The object $I(\lambda, r)$ is the injective envelope of $V(\lambda, r)$ in $\mathcal{G}$ or $\widehat{\mathcal{G}}$.*

3.4 Local and global Weyl modules

The next two families of modules in $\mathcal{G}_{\mathrm{bdd}}$ we need are the local and global Weyl modules which were originally defined in [15].

For the purposes of this paper, we shall denote the local Weyl modules by $\Delta(\lambda, r)$, $(\lambda, r) \in P^+ \times \mathbb{Z}$. Thus, $\Delta(\lambda, r)$ is generated as a $\mathfrak{g}[t]$-module by an element $w_{\lambda, r}$ with relations:

$$\mathfrak{n}^+[t]w_{\lambda, r} = 0, \qquad (x_i^-)^{\lambda(h_i)+1}w_{\lambda, r} = 0, \qquad (h \otimes t^s)w_{\lambda, r} = \delta_{s,0}\lambda(h)w_{\lambda, r},$$

here $i \in I$, $h \in \mathfrak{h}$ and $s \in \mathbb{Z}_+$.

Next, let $W(\lambda, r)$ be the global Weyl modules, which is $\mathfrak{g}[t]$-module generated as a $\mathfrak{g}[t]$-module by an element $w_{\lambda, r}$ with relations:

$$\mathfrak{n}^+[t]w_{\lambda, r} = 0, \qquad (x_i^-)^{\lambda(h_i)+1}w_{\lambda, r} = 0, \qquad h w_{\lambda, r} = \lambda(h)w_{\lambda, r},$$

where $i \in I$ and $h \in \mathfrak{h}$. Clearly the module $\Delta(\lambda, r)$ is a quotient of $W(\lambda, r)$ and moreover $V(\lambda, r)$ is the unique irreducible quotient of $W(\lambda, r)$. It is known (see [6] or [15]) that $W(0, r) \cong \mathbb{C}$ and that, if $\lambda \neq 0$, the modules $W(\lambda, r)$ are infinite-dimensional and satisfy $\mathrm{wt}\,W(\lambda, r) \subset \mathrm{conv}\,W\lambda$ and $W(\lambda, r)[s] \neq 0$ iff $s \geq r$, from which we see that $W(\lambda, r) \notin \mathrm{Ob}\,\mathcal{G}$. It follows that, if we set

$$\nabla(\lambda, r) = W(-w_\circ\lambda, -r)^*,$$

then $\nabla(\lambda, r) \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}$ and $\mathrm{soc}\,\nabla(\lambda, r) \cong V(\lambda, r)$.

We note that $\Delta(\lambda, r)$ (resp. $\nabla(\lambda, r)$) is the maximal quotient of $P(\lambda, r)$ (resp. submodule of $I(\lambda, r)$) satisfying

$$[\Delta(\lambda, r) : V(\mu, s)] \neq 0 \implies (\mu, s) \leq (\lambda, r),$$
$$\left(\text{resp.} \quad [\nabla(\lambda, r) : V(\mu, s)] \neq 0 \implies (\mu, s) \leq (\lambda, r)\right).$$

Hence these are the standard (resp. costandard) modules in $\mathcal{G}$ associated to (λ, r).

3.5 Truncated subcategories

In this section, we recall the definition of certain Serre subcategories of $\widehat{\mathcal{G}}$.

Given $\Gamma \subset \Lambda$, let $\widehat{\mathcal{G}}(\Gamma)$ be the full subcategory of $\widehat{\mathcal{G}}$ consisting of all M such that

$$M \in \mathrm{Ob}\,\widehat{\mathcal{G}}, \qquad [M : V(\lambda, r)] \neq 0 \implies (\lambda, r) \in \Gamma.$$

The subcategories $\mathcal{G}(\Gamma)$ and $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ are defined in the obvious way. Observe that if $(\lambda, r) \in \Gamma$, then $V(\lambda, r) \in \widehat{\mathcal{G}}(\Gamma)$, and we have the following trivial result.

Lemma 3.4. *The isomorphism classes of simple objects of $\widehat{\mathcal{G}}(\Gamma)$ are indexed by Γ.*

Remark 3.5. Let $\mathcal{C}$ be one of the categories $\mathcal{G}_{\leq s}$, $\mathcal{G}$, $\mathcal{G}_{\mathrm{bdd}}$, $\mathcal{G}(\Gamma)$, $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$, which are full subcategories of $\widehat{\mathcal{G}}$. Then, we have $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, N) = \mathrm{Ext}^1_{\mathcal{C}}(M, N)$ for all $M, N \in \mathcal{C}$.

3.6 A specific truncation

We now focus on Γ of the form $\Gamma = P^+ \times J$, where J is an interval in $\mathbb{Z}$ with one of the forms $(-\infty, n]$, $[m, n]$, $[m, \infty)$ or $\mathbb{Z}$, where $n, m \in \mathbb{Z}$. We set $a = \inf J$ and $b = \sup J$. Throughout this section, we assume that $(\lambda, r) \in \Gamma$.

Let $P(\lambda, r)(\Gamma)$ be the maximal quotient of $P(\lambda, r)$ which is an object of $\mathcal{G}(\Gamma)$ and let $I(\lambda, r)(\Gamma)$ be the maximal submodule of $I(\lambda, r)$ which is an object of $\mathcal{G}(\Gamma)$. These are the indecomposable projective and injective modules associated to the simple module $V(\lambda, r) \in \mathcal{G}(\Gamma)$.

For an object $M \in \mathcal{G}$, let M^Γ be the subquotient

$$M^\Gamma = \frac{M_{\geq a}}{M_{>b}},$$

where $M_{\geq a} = \bigoplus_{r \geq a} M[r]$, and we understand $M_{\geq a} = M$ if $a = -\infty$ and $M_{>b} = 0$ if $b = \infty$.

Remark 3.6.

1. If $M = \bigoplus_{s \geq p} M[s]$ for some $p \geq a$, then $M^\Gamma = \frac{M}{M_{>b}}$.
2. If $M = \bigoplus_{s < p} M[s]$ for some $p \leq b$, then $M^\Gamma = M_{\geq a}$.

Clearly $M^\Gamma \in \mathcal{G}(\Gamma)$, and because morphisms are graded, this assignment defines a functor from $\mathcal{G}$ to $\mathcal{G}(\Gamma)$. It follows from Lemma 3.2 that Γ is exact.

If we define another subset $\Gamma' = P^+ \times \{-J\}$, then it follows from the definition of the graded duality that if $M \in \mathrm{Ob}\,\mathcal{G}(\Gamma)$ then $M^* \in \mathrm{Ob}\,\mathcal{G}(\Gamma')$.

Lemma 3.7. *The module $P(\lambda, r)(\Gamma) = P(\lambda, r)^\Gamma$ and $I(\lambda, r)(\Gamma) = I(\lambda, r)^\Gamma$.*

We set

$$\Delta(\lambda, r)(\Gamma) := \Delta(\lambda, r)^\Gamma, \qquad W(\lambda, r)(\Gamma) := W(\lambda, r)^\Gamma \qquad \text{and} \qquad \nabla(\lambda, r)(\Gamma) := \nabla(\lambda, r)^\Gamma.$$

In light of the above remark, we can see that

$$\Delta(\lambda, r)(\Gamma) = \frac{\Delta(\lambda, r)}{\bigoplus_{s>b} \Delta(\lambda, r)[s]}, \qquad W(\lambda, r)(\Gamma) = \frac{W(\lambda, r)}{\bigoplus_{s>b} W(\lambda, r)[s]} \qquad \text{and}$$
$$\nabla(\lambda, r)(\Gamma) = \nabla(\lambda, r)_{\geq a}.$$

Note that, with respect to the partial order $\leq$, for each $(\lambda, r) \in \Gamma$ we have $\Delta(\lambda, r)(\Gamma)$ the maximal quotient of $P(\lambda, r)(\Gamma)$ such that

$$[\Delta(\lambda, r)(\Gamma) : V(\mu, s)] \neq 0 \implies (\mu, s) \leq (\lambda, r).$$

Similarly, we see that $\nabla(\lambda, r)(\Gamma)$ is the maximal submodule of $I(\lambda, r)(\Gamma)$ satisfying

$$[\nabla(\lambda, r)(\Gamma) : V(\mu, s)] \neq 0 \implies (\mu, s) \leq (\lambda, r).$$

These modules $\Delta(\lambda, r)(\Gamma)$ and $\nabla(\lambda, r)(\Gamma)$ are called, respectively, standard and co-standard modules associated to $(\lambda, r) \in \Gamma$.

The following proposition summarizes the properties of $\Delta(\lambda, r)(\Gamma)$ which are necessary for this paper. They can easily be derived from the properties of the functor Γ.

Proposition 3.8.

1. *The module $\Delta(\lambda, r)(\Gamma)$ is generated as a $\mathfrak{g}[t]$-module by an element $w_{\lambda, r}$ with relations:*

$$\mathfrak{n}^+[t]w_{\lambda, r} = 0, \qquad (x_i^-)^{\lambda(h_i)+1}w_{\lambda, r} = 0,$$
$$(h \otimes t^s)w_{\lambda, r} = \delta_{s,0}\lambda(h)w_{\lambda, r}, \qquad \boldsymbol{U}(\mathfrak{g}[t])[p]w_{\lambda, r} = 0, \qquad if \quad p > b - r,$$

   for all $i \in I$, $h \in \mathfrak{h}$ and $s \in \mathbb{Z}_+$, where if $b = \infty$, then the final relation is empty relation.

2. *The module $\Delta(\lambda, r)(\Gamma)$ is indecomposable and finite-dimensional and, hence, an object of $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$.*

3. $\dim \Delta(\lambda, r)(\Gamma)_\lambda = \dim \Delta(\lambda, r)(\Gamma)[r]_\lambda = 1$.

4. $\mathrm{wt}\, \Delta(\lambda, r)(\Gamma) \subset \mathrm{conv}\, W\lambda$.

5. *The module $V(\lambda, r)$ is the unique irreducible quotient of $\Delta(\lambda, r)(\Gamma)$.*

6. $\{\mathrm{ch}_{\mathrm{gr}}\, \Delta(\lambda, r)(\Gamma) : (\lambda, r) \in \Gamma\}$ *is a linearly independent subset of $\mathbb{Z}[P][u, u^{-1}]$.*

3.7 The truncated global Weyl modules

Here we collect the results on $W(\lambda, r)(\Gamma)$ which we will need for this paper.

Proposition 3.9.

1. *The module $W(\lambda, r)(\Gamma)$ is generated as a $\mathfrak{g}[t]$-module by an element $w_{\lambda, r}$ with relations:*

$$\mathfrak{n}^+[t]w_{\lambda, r} = 0, \qquad (x_i^-)^{\lambda(h_i)+1}w_{\lambda, r} = 0, \qquad hw_{\lambda, r} = \lambda(h)w_{\lambda, r},$$
$$\boldsymbol{U}(\mathfrak{g}[t])[p]w_{\lambda, r} = 0, \qquad if \quad p > b - r,$$

   where, if $b = \infty$, then the final relation is empty relation. Here $i \in I$ and $h \in \mathfrak{h}$.

2. *The module $W(\lambda, r)(\Gamma)$ is indecomposable and an object of $\widehat{\mathcal{G}}(\Gamma)$.*

3. $\dim W(\lambda, r)(\Gamma)[r]_\lambda = 1$ *and* $\dim W(\lambda, r)(\Gamma)_\lambda[s] \neq 0$ *if and only if $r \leq s \leq b$.*

4. $\mathrm{wt}\, W(\lambda, r)(\Gamma) \subset \mathrm{conv}\, W\lambda$.

5. *The module $V(\lambda, r)$ is the unique irreducible quotient of $W(\lambda, r)(\Gamma)$.*

6. $\{\mathrm{ch}_{\mathrm{gr}}\, W(\lambda, r)(\Gamma) : (\lambda, r) \in \Gamma\}$ *is a linearly independent subset of $\mathbb{Z}[P][u, u^{-1}]$.*

3.8 The costandard modules

The following proposition summarizes the main results on $\nabla(\lambda, r)(\Gamma)$ that are needed for this paper. All but the final result can be found by considering the properties of the functor Γ and the paper [3].

Proposition 3.10.

1. *The module $\nabla(\lambda, r)(\Gamma)$ is an indecomposable object of $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$.*

2. $\dim \nabla(\lambda, r)(\Gamma)[r]_\lambda = 1$ *and* $\dim \nabla(\lambda, r)(\Gamma)[s]_\lambda \neq 0 \Leftrightarrow a \leq s \leq r$.

3. $\mathrm{wt}\, \nabla(\lambda, r)(\Gamma) \subset \mathrm{conv}\, W\lambda$.

4. *Any submodule of* $\nabla(\lambda, r)(\Gamma)$ *contains* $\nabla(\lambda, r)(\Gamma)[r]_\lambda$ *and the socle of* $\nabla(\lambda, r)(\Gamma)$ *is the simple module* $V(\lambda, r)$.

5. $\{\mathrm{ch}_{\mathrm{gr}}\, \nabla(\lambda, r)(\Gamma) : (\lambda, r) \in \Gamma\}$ *is a linearly independent subset of* $\mathbb{Z}[P][u, u^{-1}]]$.

6. *Let* $\Gamma' = P^+ \times \{-J\}$ *and* $(\lambda, r) \in \Gamma$. *Then* $\nabla(\lambda, r)(\Gamma) \cong W(-\omega_0\lambda, -r)(\Gamma')^*$.

Proof. We prove the final item. As a vector space we have

$$W(-\omega_0\lambda, -r)(\Gamma') \cong \bigoplus_{s=-r}^{-a} W(-\omega_0\lambda, -r)[s].$$

Since $W(-\omega_0\lambda, -r)(\Gamma')$ is a quotient of $W(-\omega_0\lambda, -r)$, its dual must be a submodule of $\nabla(\lambda, r)$. By the definition of the graded dual, we see that, as a vector space,

$$W(-\omega_0\lambda, -r)(\Gamma')^* \cong \bigoplus_{s=a}^{r} \nabla(\lambda, r)[s].$$

Hence, as vector spaces, we see that $\nabla(\lambda, r)(\Gamma) \cong W(-\omega_0\lambda, -r)(\Gamma')^*$. Now, the fact that $W(-\omega_0\lambda, -r)(\Gamma')^*$ is a submodule completes the proof. ∎

4   The main theorem and some homological results

Definition 4.1. We say that $M \in \mathrm{Ob}\,\mathcal{G}(\Gamma)$ admits a $\Delta(\Gamma)$ (resp. $\nabla(\Gamma)$)-filtration if there exists an increasing family of submodules $0 \subset M_0 \subset M_1 \subset \cdots$ with $M = \bigcup_k M_k$, such that

$$M_k/M_{k-1} \cong \bigoplus_{(\lambda, r) \in \Gamma} \Delta(\lambda, r)(\Gamma)^{m_k(\lambda, r)} \qquad \left(\text{resp.}\quad M_k/M_{k-1} \cong \bigoplus_{(\lambda, r) \in \Gamma} \nabla(\lambda, r)(\Gamma)^{m_k(\lambda, r)}\right)$$

for some choice of $m_k(\lambda, r) \in \mathbb{Z}_+$. We do not require $\sum_{(\lambda, r)} m_k(\lambda, r) < \infty$. If $M_k = M$ for some $k \geq 0$, then we say that M admits a finite $\Delta(\Gamma)$ (resp. $\nabla(\Gamma)$)-filtration. Because our modules have finite-dimensional graded components, we can conclude that the multiplicity of a fixed $\Delta(\lambda, r)(\Gamma)$ (resp. $\nabla(\lambda, r)(\Gamma)$) in a $\Delta(\Gamma)$-filtration (resp. $\nabla(\Gamma)$-filtration) must be finite, and we denote this multiplicity by $[M : \Delta(\lambda, r)(\Gamma)]$ (resp. $[M : \nabla(\lambda, r)(\Gamma)]$). Finally, we say that $M \in \mathrm{Ob}\,\mathcal{G}(\Gamma)$ is tilting if M has both a $\Delta(\Gamma)$ and a $\nabla(\Gamma)$-filtration.

The main goal of this paper is to understand tilting modules in $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$. (The case where $J = \mathbb{Z}$ was studied in [3].) In the case of algebraic groups (see [18, 23]) a crucial necessary result is to give a cohomological characterization of modules admitting a $\nabla(\Gamma)$-filtration. The analogous result in our situation is to prove the following statement:

An object M of $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ admits a $\nabla(\Gamma)$-filtration if and only if $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}((\Delta(\lambda, r)(\Gamma), M) = 0$ for all $(\lambda, r) \in \Gamma$.

It is not hard to see that the forward implication is true. The converse statement however requires one to prove that any object of $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ can be embedded in a module which admits a $\nabla(\Gamma)$-filtration. This in turn requires Theorem 5.8. Summarizing, the first main result that we shall prove in this paper is:

Proposition 4.2. *Let* $M \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$. *Then the following are equivalent:*

1. *The module M admits a $\nabla(\Gamma)$-filtration.*

2. *M satisfies $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), M) = 0$ for all $(\lambda, r) \in \Gamma$.*

The second main result that we shall prove in this paper is the following:

Theorem 4.3.

1. *Given $(\lambda, r) \in \Gamma$, there exists an indecomposable module $T(\lambda, r)(\Gamma) \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ which admits a $\Delta(\Gamma)$-filtration and a $\nabla(\Gamma)$-filtration. Further,*

$$T(\lambda, r)(\Gamma)[s]_\lambda = 0 \qquad if \quad s > r, \quad T(\lambda, r)(\Gamma)[r]_\lambda = 1, \quad \mathrm{wt}\,T(\lambda, r)(\Gamma) \subset \mathrm{conv}\,W\lambda,$$

   and $T(\lambda, r)(\Gamma) \cong T(\mu, s)(\Gamma)$ if and only if $(\lambda, r) = (\mu, s)$.

2. *Moreover, any indecomposable tilting module in $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ is isomorphic to $T(\lambda, r)(\Gamma)$ for some $(\lambda, r) \in \Gamma$, and any tilting module in $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ is isomorphic to a direct sum of indecomposable tilting modules.*

5   Proof of Proposition 4.2

5.1   Initial homological results

We begin by proving the implication (1) $\implies$ (2) from Proposition 4.2. In order to do this, we first establish some homological properties of the standard and costandard modules which will be used throughout the paper.

Proposition 5.1. *Let $\lambda, \mu \in P^+$. Then we have the following*

1. $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(W(\lambda, r)(\Gamma), W(\mu, s)(\Gamma)) = 0 = \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\nabla(\mu, s)(\Gamma), \nabla(\lambda, r)(\Gamma))$ *for all $s, r \in \mathbb{Z}$ if $\lambda \not< \mu$.*

2. $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), \nabla(\mu, s)(\Gamma)) = 0.$

3. *If $\lambda \not\leq \mu$ then $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), \Delta(\mu, s)(\Gamma)) = 0$ for all $s, r \in \mathbb{Z}$.*

4. *If $s \geq r$, then $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, s)(\Gamma), \Delta(\lambda, r)(\Gamma)) = 0.$*

Proof. For part (1), suppose that we have a sequence $0 \to W(\mu, s)(\Gamma) \to X \to W(\lambda, r)(\Gamma) \to 0$, and let $x \in X_\lambda[r]$ be a pre-image of $w_{\lambda, r}$. It is clear from the hypothesis on μ that $\mathfrak{n}^+[t].x = 0$. If $b < \infty$ and $p > b - r$, then $\mathbf{U}(\mathfrak{g}[t])[p].x = 0$ by grade considerations. Note that, since $\dim X[r] < \infty$, we have $\dim \mathbf{U}(\mathfrak{g})x < \infty$. It follows from the finite-dimensional representation theory that $(x_i^-)^{\lambda(h_i)+1}x = 0$, for all $i \in I$, and so the sequence splits. The proof for $\nabla(\Gamma)$ is similar and omitted.

For part (2), suppose we have a sequence $0 \to \nabla(\mu, s)(\Gamma) \to X \to \Delta(\lambda, r)(\Gamma) \to 0$ and $\mu \not\geq \lambda$. Then $\dim X_\lambda[r] = 1$, and if $x \in X_\lambda[r]$ is a pre-image of $w_{\lambda, r}$ then x satisfies the defining relations of $w_{\lambda, r}$ and the sequence splits. If $\mu \geq \lambda$, then by taking duals we get a sequence $0 \to \Delta(\lambda, r)(\Gamma)^* \to Y \to W(-\omega_0\mu, -s)(\Gamma') \to 0$. Again, if $y \in Y_{-\omega_0\mu}[-s]$ is a pre-image of $w_{-\omega_0\mu, -s}$, we see that y satisfies the defining relations of $w_{-\omega_0\mu, -s} \in W(-\omega_0\mu, -s)(\Gamma')$, and the sequence splits.

The proofs for parts (3) and (4) are similar to that for part (1) and are omitted. ∎

The proof of the following lemma is standard (see, for example, [3]).

Lemma 5.2. *Suppose that $M \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ admits a (possibly infinite) $\nabla(\Gamma)$-filtration. Then M admits a finite $\nabla(\Gamma)$-filtration*

$$0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M \qquad with \quad M_s/M_{s-1} \cong \bigoplus_{r \in \mathbb{Z}} \nabla(\lambda_s, r)(\Gamma)^{[M:\nabla(\lambda_s, r)(\Gamma)]},$$

where $\lambda_i > \lambda_j$ implies $i > j$. In particular if μ is maximal such that $M_\mu \neq 0$, then there exists $s \in \mathbb{Z}$ and a surjective map $M \to \nabla(\mu, s)(\Gamma)$ such that the kernel has a $\nabla(\Gamma)$-filtration.

We can now prove the implication (1) $\implies$ (2) from Proposition 4.2.

Corollary 5.3. *If $M \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ admits a $\nabla(\Gamma)$-filtration, then $\mathrm{Ext}^1_{\mathcal{G}(\Gamma)}(\Delta(\lambda, r)(\Gamma), M) = 0$, for all $(\lambda, r) \in \Gamma$.*

Proof. Let $0 \subset M_1 \subset M_2 \subset \cdots \subset M_k = M$ be a finite $\nabla(\Gamma)$-filtration as in Lemma 5.2. Then, M_s/M_{s-1} is a (possibly infinite) direct sum of $\nabla(\lambda_s, r)(\Gamma)$, and

$$\mathrm{Ext}^1_{\mathcal{G}(\Gamma)}\big(\Delta(\lambda, r)(\Gamma), M_s/M_{s-1}\big) \cong \mathrm{Ext}^1_{\mathcal{G}(\Gamma)}\left(\Delta(\lambda, r)(\Gamma), \bigoplus_{r \in \mathbb{Z}} \nabla(\lambda_s, r)(\Gamma)^{m_s(\lambda_s, r)}\right)$$

$$\cong \prod \mathrm{Ext}^1_{\mathcal{G}(\Gamma)}(\Delta(\lambda, r)(\Gamma), \nabla(\lambda_s, r)(\Gamma)) = 0,$$

by Proposition 5.1 (2). The result follows by induction on k, the length of the filtration. ∎

5.2 Towards understanding extensions between the standard and costandard modules

Proposition 5.4. *Suppose that $N \in \mathrm{Ob}\,\mathcal{G}(\Gamma)$ is such that $\mathrm{Ext}^1_{\mathcal{G}(\Gamma)}(\Delta(\lambda, r)(\Gamma), N) = 0$ for all $(\lambda, r) \in \Gamma$. If $M \in \mathrm{Ob}\,\mathcal{G}(\Gamma)$ has a $\Delta(\Gamma)$-filtration then $\mathrm{Ext}^1_{\mathcal{G}(\Gamma)}(M, N) = 0$.*

Proof. Consider a short exact sequence $0 \to N \to U \to M \to 0$. Suppose that $M_k \subset M_{k+1}$ is a part of the $\Delta(\Gamma)$-filtration of M and assume that

$$M_{k+1}/M_k \cong \bigoplus_{(\mu, s) \in \Lambda} \Delta(\mu, s)(\Gamma)^{m_k(\mu, s)}.$$

By assumption we have $\mathrm{Ext}^1_{\mathcal{G}(\Gamma)}(M_{k+1}/M_k, N) = 0$. Let $U_k \subset U$ be the pre-image of M_k, which contains N because $0 \in M_k$. Note that $U_{k+1}/U_k \cong M_{k+1}/M_k$. Now, consider the short exact sequence $0 \to N \to U_k \to M_k \to 0$. This sequence defines an element of $\mathrm{Ext}^1_{\mathcal{G}(\Gamma)}(M_k, N)$. Since M_k has a finite $\Delta(\Gamma)$-filtration it follows that $\mathrm{Ext}^1_{\mathcal{G}(\Gamma)}(M_k, N) = 0$. Hence the sequence splits and we have a retraction $\varphi_k : U_k \to N$. We want to prove that $\varphi_{k+1} : U_{k+1} \to N$ can be chosen to extend φ_k. For this, applying $\mathrm{Hom}_{\mathcal{G}(\Gamma)}(-, N)$ to $0 \to U_k \to U_{k+1} \to U_{k+1}/U_k \to 0$, we get $\mathrm{Hom}_{\mathcal{G}(\Gamma)}(U_{k+1}, N) \to \mathrm{Hom}_{\mathcal{G}(\Gamma)}(U_k, N) \to 0$, which shows that we can choose φ_{k+1} to lift φ_k. Now defining $\varphi : U \to N$ by $\varphi(u) = \varphi_k(u)$, for all $u \in U_k$, we have the desired splitting of the original short exact sequence. ∎

Together with Proposition 5.1 and taking $N = \nabla(\lambda, r)(\Gamma)$ in the proposition above, we now have:

Corollary 5.5. *Suppose $M \in \mathrm{Ob}\,\mathcal{G}(\Gamma)$ admits a $\Delta(\Gamma)$-filtration. Then, $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, \nabla(\lambda, r)(\Gamma)) = 0$, for all $(\lambda, r) \in \Gamma$.*

5.3 A natural embedding

In this section we show that every $M \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ embeds into an injective module $I(M) \in \mathrm{Ob}\,\mathcal{G}(\Gamma)$. Let $\mathrm{soc}\, M \subset M$ be the maximal semi-simple submodule of M.

Lemma 5.6. *Let $M \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$.*

1. *If $M \neq 0$, then $\mathrm{soc}\, M \neq 0$.*
2. *Suppose $\mathrm{soc}\, M = \bigoplus V(\lambda, r)^{m_{\lambda, r}}$. Then $M \hookrightarrow \bigoplus I(\lambda, r)(\Gamma)^{m_{\lambda, r}}$.*

Proof. For the first part, let $M \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}$, suppose $M \neq 0$, and let $s \in \mathbb{Z}$ be minimal such that $M \in \mathrm{Ob}\,\mathcal{G}_{\leq s}$. Then $M[s] \neq 0$ and $M[s] \subset \mathrm{soc}\,M$.

For the second, let $\mathrm{soc}\,M = \bigoplus_{(\lambda, r) \in \Lambda} V(\lambda, r)^{m_{\lambda, r}}$ from which we get a natural injection $\mathrm{soc}\,M \xhookrightarrow{\iota} \bigoplus I(\lambda, r)^{m_{\lambda, r}}$. By injectivity, we have a morphism $M \xrightarrow{f} \bigoplus I(\lambda, r)^{m_{\lambda, r}}$, through which ι factors. In fact, we can show that f is an injection. If not, $\mathrm{soc}\,\ker f \neq 0$. On the other hand, $\mathrm{soc}\,\ker f \subset \mathrm{soc}\,M$, and f is injective on $\mathrm{soc}\,M$. If we assume that $M \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$, then it is easy to conclude that $\mathrm{im}\,f \subset \bigoplus I(\lambda, r)(\Gamma)^{m_{\lambda, r}}$, completing the proof. ∎

5.4   o-canonical filtration

In this section we shall establish a finite filtration on modules $M \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ where the successive quotients embed into direct sums of $\nabla(\Gamma)$. We then use the filtration to establish lower and upper bounds on the graded character of M. We use the character bounds to prove Proposition 4.2.

Now fix an ordering of $P^+ = \{\lambda_0, \lambda_1, \dots\}$ such that $\lambda_r > \lambda_s$ implies that $r > s$. For $M \in \mathrm{Ob}\,\mathcal{G}(\Gamma)$ we set $M_s \subset M$ as the maximal submodule whose weights lie in $\{\mathrm{conv}\,W\lambda_r \mid r \leq s\}$. Evidently $M_{s-1} \subset M_s$. We call this the o-canonical filtration, because it depends on the order. This is a finite filtration for $M \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ and we set $k(M)$ to be minimal such that $M = M_{k(M)}$. Clearly

$$M_{s-1} \subset M_s, \qquad M = \bigcup_{s=0}^{k(M)} M_s, \qquad \text{and}$$

$$\mathrm{Hom}_{\mathcal{G}}(V(\lambda, r), M_s/M_{s-1}) \neq 0 \implies \lambda = \lambda_s. \tag{5.1}$$

It follows from Lemma 5.6 and (5.1) that the quotient M_s/M_{s-1} embeds into a module of the form $\bigoplus I(\lambda_s, r)(\Gamma)^{m_{s,r}}$, where $m_{s,r} = \dim \mathrm{Hom}_{\mathcal{G}}(V(\lambda_s, r), M_s/M_{s-1})$. Since the weights of M_s/M_{s-1} are bounded above by λ_s, they embed into the maximal submodule of this direct sum, whose weights are bounded above by λ_s. Hence we have M_s/M_{s-1} embedding into a direct sum of modules of the form $\nabla(\lambda_s, r)$, with $r \in J$. This gives,

$$\mathrm{ch}_{\mathrm{gr}}\,M = \sum_{s \geq 0} \mathrm{ch}_{\mathrm{gr}}\,M_s/M_{s-1} \leq \sum_{s \geq 0} \sum_{r \in J} \dim \mathrm{Hom}_{\mathcal{G}}(V(\lambda_s, r), M_s/M_{s-1})\,\mathrm{ch}_{\mathrm{gr}}\,\nabla(\lambda_s, r)(\Gamma),$$

i.e.,

$$[M : V(\mu, \ell)] \leq \sum_{s \geq 0} \sum_{r \in J} \dim \mathrm{Hom}_{\mathcal{G}}(V(\lambda_s, r), M_s/M_{s-1})[\nabla(\lambda_s, r)(\Gamma) : V(\mu, \ell)],$$

for all $(\mu, \ell) \in \Lambda$. We claim that this is equivalent to

$$\mathrm{ch}_{\mathrm{gr}}\,M = \sum \mathrm{ch}_{\mathrm{gr}}\,M_s/M_{s-1} \leq \sum_{s \geq 0} \sum_{r \in J} \dim \mathrm{Hom}_{\mathcal{G}}(\Delta(\lambda_s, r)(\Gamma), M)\,\mathrm{ch}_{\mathrm{gr}}\,\nabla(\lambda_s, r)(\Gamma).$$

The claim follows from Lemma 5.7 below, and [3, § 3.5], which states that

$$\mathrm{Hom}_{\mathcal{G}}(\Delta(\lambda_s, r), M) \cong \mathrm{Hom}_{\mathcal{G}}(V(\lambda_s, r), M_s/M_{s-1}).$$

Lemma 5.7. *Let $M \in \mathrm{Ob}\,\mathcal{G}(\Gamma)$ and $(\lambda, r) \in \Gamma$. We have*

$$\mathrm{Hom}_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), M) \cong \mathrm{Hom}_{\widehat{\mathcal{G}}}(\Delta(\lambda, r), M)$$

and

$$\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), M) \cong \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r), M).$$

Proof. As $M[s] = 0$ for all $s > b$, we must have $\text{Hom}_{\widehat{\mathcal{G}}}(\bigoplus_{s>b} \Delta(\lambda, r)[s], M) = 0$. Similarly, if $0 \to M \to X \to \bigoplus_{s>b} \Delta(\lambda, r)[s] \to 0$ is exact, by using again that $M[s] = 0$ for $s > b$, we have $\dim X[n] = \dim(\bigoplus_{s>b} \Delta(\lambda, r)[s])[n]$ for any $n > b$ and so we have an injective map $\iota : \bigoplus_{s>b} \Delta(\lambda, r)[s] \to X$ which splits the sequence. Thus, $\text{Ext}^1_{\widehat{\mathcal{G}}}(\bigoplus_{s>b} \Delta(\lambda, r)[s], M) = 0$. Now the other statements are easily deduced. ∎

Therefore, we get

$$\text{ch}_{\text{gr}} M \le \sum_{s \ge 0} \sum_{r \in J} \dim \text{Hom}_{\mathcal{G}(\Gamma)}(\Delta(\lambda_s, r)(\Gamma), M) \, \text{ch}_{\text{gr}} \nabla(\lambda_s, r)(\Gamma) \tag{5.2}$$

and the equality holds if, and only if, the o-canonical filtration is $\nabla(\Gamma)$-filtration.

5.5 A homological characterization of costandard modules

Note that even if $N \notin \text{Ob}\,\mathcal{G}_{\text{bdd}}$, we can still define the submodules N_s, and by definition $N_s \in \text{Ob}\,\mathcal{G}_{\text{bdd}}$. The following result, or more precisely a dual statement about projective modules and global Weyl filtrations, was proved for $\mathfrak{g} = \mathfrak{sl}_2$ in [4], for $\mathfrak{g} = \mathfrak{sl}_{n+1}$ in [2] and for general $\mathfrak{g}$ in [10]. In particular, we note that the argument in [2, Section 5.5] works in general.

Theorem 5.8. *For all $(\lambda, r) \in \Lambda$ and for all $p \in \mathbb{Z}_+$ the o-canonical filtration on $I(\lambda, r)_p$ is a ∇-filtration. Moreover, for all (μ, s) we have $[I(\lambda, r)_p : \nabla(\mu, s)] = [\Delta(\mu, r) : V(\lambda, s)]$.*

We combine this with equation (5.2) and the linear independence of the graded characters of the $\nabla(\lambda, r)(\Gamma)$ to see that $[I(\lambda, r)(\Gamma) : \nabla(\mu, s)] = \dim \text{Hom}_{\mathcal{G}(\Gamma)}(\Delta(\lambda_s, r), I(\lambda, r)_p)$.

As a consequence of the theorem and the exactness of the functor Γ, we conclude that $I(\lambda, r)_p(\Gamma)$ has a $\nabla(\Gamma)$-filtration. It is easy to see that $I(\lambda, r)_p(\Gamma) \in \mathcal{G}_{\text{bdd}}(\Gamma)$.

For $M \in \mathcal{G}_{\text{bdd}}(\Gamma)$, let p be minimal such that $M_p = M$. Then it is clear that we can refine the embedding from Lemma 5.6 to $M \hookrightarrow \bigoplus I(\lambda, r)_p(\Gamma)$. We can conclude the following:

Corollary 5.9. *For all $M \in \text{Ob}\,\mathcal{G}_{\text{bdd}}(\Gamma)$, we have $M \subset I(M) \in \text{Ob}\,\mathcal{G}_{\text{bdd}}(\Gamma)$, where the o-canonical filtration of $I(M)$ is a $\nabla(\Gamma)$-filtration.*

We now complete the proof of Proposition 4.2 following the argument in [3]. Let $M \in \mathcal{G}_{\text{bdd}}(\Gamma)$ and assume that $\text{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), M) = 0$. Let $I(M) \in \mathcal{G}_{\text{bdd}}(\Gamma)$ be as in Corollary 5.9, and consider the short exact sequence $0 \to M \to I(M) \to Q \to 0$. where $Q \in \text{Ob}\,\mathcal{G}_{\text{bdd}}(\Gamma)$. The assumption on the module M implies that, if we apply the functor $\text{Hom}_{\mathcal{G}(\Gamma)}(\Delta(\lambda, r)(\Gamma), -)$, we get the short exact sequence

$$0 \to \text{Hom}_{\mathcal{G}(\Gamma)}(\Delta(\lambda, r)(\Gamma), M) \to \text{Hom}_{\mathcal{G}(\Gamma)}(\Delta(\lambda, r)(\Gamma), I(M))$$
$$\to \text{Hom}_{\mathcal{G}(\Gamma)}(\Delta(\lambda, r)(\Gamma), Q) \to 0. \tag{5.3}$$

Since the o-canonical filtration of $I(M)$ is a $\nabla(\Gamma)$-filtration, we can conclude that (5.2) is an equality for $I(M)$. We get that

$$\text{ch}_{\text{gr}} M = \text{ch}_{\text{gr}} I(M) - \text{ch}_{\text{gr}} Q \ge \sum_{\substack{r \in J \\ s \ge 0}} (\dim \text{Hom}_{\mathcal{G}(\Gamma)}(\Delta(\lambda_s, r)(\Gamma), I(M))$$

$$- \dim \text{Hom}_{\mathcal{G}(\Gamma)}(\Delta(\lambda_s, r)(\Gamma), Q)) \, \text{ch}_{\text{gr}} \nabla(\lambda_s, r)$$

$$= \sum_{\substack{r \in J \\ s \ge 0}} \dim \text{Hom}_{\mathcal{G}(\Gamma)}(\Delta(\lambda_s, r)(\Gamma), M) \, \text{ch}_{\text{gr}} \nabla(\lambda_s, r),$$

where the final equality is from the exactness of (5.3). We now get that the character bound in (5.2) is an equality for M, and, hence, that M has a $\nabla(\Gamma)$-filtration.

Finally, we can prove the following.

Proposition 5.10. *The following are equivalent for a module $M \in \operatorname{Ob} \mathcal{G}_{\mathrm{bdd}}(\Gamma)$*

1. *For all $(\lambda, r) \in \Gamma$, we have $\operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), M) = 0$.*

2. *M admits a $\nabla(\Gamma)$-filtration.*

3. *The o-canonical filtration on M is a $\nabla(\Gamma)$-filtration.*

Proof. The equivalence of (1) and (2) is precisely the statement of Proposition 4.2. Clearly (3) implies (2), so it is enough to show that (1) implies (3). Assuming (1), we have shown that the character bound in (5.2) is an equality, which is true if and only if the o-canonical filtration is a $\nabla(\Gamma)$-filtration. ∎

5.6 Extensions between simple modules

Our final result before constructing the tilting modules $T(\lambda, r)(\Gamma)$ shows that the space of extensions between standard modules is always finite-dimensional. The proof is analogous to the proof in [3].

Proposition 5.11. *For all $(\lambda, r), (\mu, s) \in \Gamma$ we have $\dim \operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), \Delta(\mu, s)(\Gamma)) < \infty$.*

Proof. Consider the short exact sequence $0 \to K \to P(\lambda, r) \to \Delta(\lambda, r)(\Gamma) \to 0$ and apply the functor $Hom_{\widehat{\mathcal{G}}}(-, \Delta(\mu, s)(\Gamma))$. Since $P(\lambda, r)$ is projective, we see that the result follows if $\dim \operatorname{Hom}_{\widehat{\mathcal{G}}}(K, \Delta(\mu, s)(\Gamma)) < \infty$. Let ℓ be such that $\Delta(\mu, s)(\Gamma)[p] = 0$ for all $p > \ell$. Then $\operatorname{Hom}_{\widehat{\mathcal{G}}}(K_{>\ell}, \Delta(\mu, s)(\Gamma)) = 0$, and, hence, we have an injection $\operatorname{Hom}_{\widehat{\mathcal{G}}}(K, \Delta(\mu, s)(\Gamma)) \hookrightarrow \operatorname{Hom}_{\widehat{\mathcal{G}}}(\frac{K}{K_{>\ell}}, \Delta(\mu, s)(\Gamma))$. The proposition follows because $\frac{K}{K_{>\ell}}$ is finite-dimensional. ∎

6 Construction of tilting modules

6.1 Defining a subset which can be appropriately enumerated

In this section we construct a family of indecomposable modules in the category $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$, denoted by $\{T(\lambda, r)(\Gamma) : (\lambda, r) \in \Gamma\}$, each of which admits a $\Delta(\Gamma)$-filtration and satisfies

$$\operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\mu, s)(\Gamma), T(\lambda, r)(\Gamma)) = 0, \qquad (\mu, s) \in \Gamma.$$

It follows that the modules $T(\lambda, r)(\Gamma)$ are tilting and we prove that any tilting module in $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ is a direct sum of copies of $T(\lambda, r)(\Gamma)$, $(\lambda, r) \in \Gamma$. The construction is a generalization of the one from [3], and the ideas are similar to the ones given in [23]. One of the first difficulties we encounter when trying to construct $T(\lambda, r)(\Gamma)$, using the algorithm given in [23], is to find a suitable subset (depending on (λ, r)) of Γ which can be appropriately enumerated. Hence we assume the following result, whose proof we postpone to Section 6.6.

Proposition 6.1. *Fix $(\lambda, r) \in \Gamma$ and assume that under the enumeration we have $\lambda = \lambda_k$. Then there exists a subset $\mathcal{S}(\lambda, r) \subset \Gamma$ such that*

1. *$(\lambda, r) \in \mathcal{S}(\lambda, r)$.*

2. *there exists r_i for each $i \leq k$ such that $r_i \geq r$, $r_k = r$, and*

$$\mathcal{S}(\lambda, r) = \{(\lambda_i, s) \,|\, i \leq k, s \leq r_i\}.$$

3. *$\operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\mu', s')(\Gamma), \Delta(\mu, s)(\Gamma)) = 0$ for all $(\mu, s) \in \mathcal{S}(\lambda, r)$ and $(\mu', s') \notin \mathcal{S}(\lambda, r)$.*

Furthermore, there exists an injection $\eta : \mathcal{S}(\lambda, r) \to \mathbb{Z}_{\geq 0}$ such that for $(\mu_i, p_i) = \eta^{-1}(i)$ we have $\operatorname{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\mu_i, p_i)(\Gamma), \Delta(\mu_j, p_j)(\Gamma)) = 0$ if $i < j$, and $\Delta(\mu_j, p_j)(\Gamma)_{\mu_i}[p_i] = 0$ for $i < j$.

Without loss of generality we may assume that $\eta(\lambda, r) = 0$ and the image of η is an interval. We need the following elementary result.

Lemma 6.2. *Suppose that $M, N \in \mathrm{Ob}\,\widehat{\mathcal{G}}$ are such that*

$$0 < \dim \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, N) < \infty \qquad and \qquad \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, M) = 0.$$

Then, there exists $U \in \mathrm{Ob}\,\widehat{\mathcal{G}}$, $d \in \mathbb{Z}_+$ and a non-split short exact sequence

$$0 \to N \to U \to M^d \to 0$$

so that $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, U) = 0$.

Proof. The proof follows from induction on $\dim \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, N)$. The base case is obvious. For the inductive step, chose any non-split sequence $0 \to N \to U' \to M \to 0$. Apply the functor $\mathrm{Hom}_{\widehat{\mathcal{G}}}(M, -)$ to the sequence, and note that the image of map $1_M \in \mathrm{Hom}_{\widehat{\mathcal{G}}}(M, M)$ is in the kernel of the surjection from $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, N) \to \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, U')$. It follows that $\dim \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, U') < \dim \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, N)$. By the induction hypothesis, we now have a module U and a non-split sequence $0 \to U' \to U \to M^{d-1} \to 0$. Now, considering the sequence $0 \to U'/N \to U/N \to M^{d-1} \to 0$, and again using that $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(M, M) = 0$, we get a non-split sequence $0 \to N \to U \to M^d \to 0$. ∎

6.2   Constructing tilting modules

We now use η to construct an infinite family of finite-dimensional modules M_i, whose direct limit will be $T(\lambda, r)(\Gamma)$. We note that the construction, at this point, will seem to be dependent on the ordering of P^+ we have chosen, and on the set $\mathcal{S}(\lambda, r)$ and η. We prove independence at the end of this section.

Set $M_0 = \Delta(\mu_0, p_0)(\Gamma)$. If $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\mu_1, p_1)(\Gamma), \Delta(\mu_0, p_0)(\Gamma)) = 0$, then set $M_1 = M_0$. If not, then since $\dim(\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\mu_1, p_1)(\Gamma), \Delta(\mu_0, p_0)(\Gamma))) < \infty$ by Proposition 5.11, Lemma 6.2 gives us an object $M_1' \in \mathrm{Ob}\,\widehat{\mathcal{G}}(\Gamma)$ and a non-split short exact sequence

$$0 \to M_0 \to M_1' \to \Delta(\mu_1, p_1)(\Gamma)^{d_1'} \to 0$$

with $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\mu_1, p_1)(\Gamma), M_1') = 0$.

Let $M_1 \subseteq M_1'$ be an indecomposable summand containing $(M_1')_{\mu_0}[p_0]$. By Proposition 6.1 we see that $(M_1')_{\mu_0}[p_0] = (M_0)_{\mu_0}[p_0]$. Then we have $M_0 \overset{\iota_0}{\hookrightarrow} M_1$. Now, suppose that $M_1 \neq M_1'$. Then, since M_1' is generated by $(M_1')_{\mu_0}[p_0]$ and $(M_1')_{\mu_1}[p_1]$, we must have $(M_1)_{\mu_1}[p_1] \neq (M_1')_{\mu_1}[p_1]$. We must have $\dim(M_1')_{\mu_1}[p_1] - \dim(M_1)_{\mu_1}[p_1]$ linearly independent vectors in $(M_1')_{\mu_1}[p_1]$ which do not have a pre-image in M_0, and each one must then generate a copy of $\Delta(\mu_1, p_1)(\Gamma)$. So, $M_1' = M_1 \oplus \Delta(\mu_1, p_1)(\Gamma)^d$ for some d and we obtain the sequence

$$0 \to M_0 \to M_1 \to \Delta(\mu_1, p_1)(\Gamma)^{d_1} \to 0. \tag{6.1}$$

By applying the $\mathrm{Hom}(\Delta(\mu, p)(\Gamma), -)$ in the sequence (6.1), we get

$$\cdots \to \mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu, p)(\Gamma), M_0) \to \mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu, p)(\Gamma), M_1)$$
$$\to \mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu, p)(\Gamma), \Delta(\mu_1, p_1)(\Gamma)^{d_1}) \to \cdots.$$

If $(\mu, p) \notin \mathcal{S}(\lambda, r)$ or $(\mu, p) = (\mu_0, p_0)$ by Proposition 6.1 we get

$$\mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu, p)(\Gamma), M_0) = 0, \qquad \mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu, p)(\Gamma), \Delta(\mu_1, p_1)(\Gamma)) = 0$$

and we see that the middle term is also trivial, i.e.

$$\mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu,p)(\Gamma), M_1) = 0 \qquad \text{for} \quad (\mu,p) \notin \mathcal{S}(\lambda,r),$$

$$\text{and} \qquad \mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu_0,p_0)(\Gamma), M_1) = 0.$$

The fact that $\mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu_1,p_1)(\Gamma), M_1) = 0$ follows from the fact that M_1 is a summand of M_1' and that $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\mu_1,p_1)(\Gamma), M_1') = 0$.

We use Lemma 6.2 again, with $N = M_1$ and $M = \Delta(\mu_2,p_2)(\Gamma)$, and we get

$$0 \to M_1 \to M_2' \to \Delta(\mu_2,p_2)(\Gamma)^{d_2'} \to 0,$$

a summand $M_2 \subseteq M_2'$ containing $(M_2')_{\mu_i}[p_i]$, $i = 0,1$, such that

$$M_1 \overset{\iota_1}{\hookrightarrow} M_2, \qquad \mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu_i,p_i)(\Gamma), M_2) = 0 \qquad \text{for} \quad i = 0,1,2,$$

$$\text{and} \qquad \mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu,p)(\Gamma), M_2) = 0 \qquad \text{for} \quad (\mu,p) \notin \mathcal{S}(\lambda,r).$$

Repeating this procedure, and using Lemma 6.2 and the properties of η, we have the following proposition. The condition on weights is a consequence that $\mathrm{wt}\,\Delta(\lambda_i,p) \subset \mathrm{conv}\,W\lambda_k$ for all $i \leq k$.

Proposition 6.3. *There exists a family $\{M_s\}$, $s \in \mathbb{Z}_{\geq 0}$, of indecomposable finite-dimensional modules and injective morphisms $\iota_s : M_s \to M_{s+1}$ of objects of $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ which have the following properties:*

1. $M_0 = \Delta(\lambda_k,r)(\Gamma) = \Delta(\mu_0,p_0)(\Gamma)$, *and for $s \geq 1$,*

$$M_s/\iota_{s-1}(M_{s-1}) \cong \Delta(\mu_s,p_s)(\Gamma)^{d_s}, \qquad d_s \in \mathbb{Z}_+,$$
$$\dim M_s[r]_{\lambda_k} = 1, \qquad \mathrm{wt}\,M_s \subset \mathrm{conv}\,W\lambda_k.$$

2. *The spaces $M_s[p] = 0$, for all $s \geq 0$, $p \gg \max\{r_i\}$.*

3. *For all $0 \leq \ell \leq s$ we have $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\mu_\ell,p_\ell)(\Gamma), M_s) = 0$, and, for all $(\mu,p) \notin \mathcal{S}(\lambda,r)$, we have $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\mu,p)(\Gamma), M_s) = 0$.*

4. *M_s is generated as a $\mathfrak{g}[t]$-module by the spaces $\{M_s[p_\ell]_{\mu_\ell} : \ell \leq s\}$. Moreover, if we let*

$$\iota_{r,s} = \iota_{s-1} \cdots \iota_r : \; M_r \to M_s, \quad r < s, \qquad \iota_{r,r} = \mathrm{id},$$

then $M_s[p_\ell]_{\mu_\ell} = \iota_{\ell,s}(M_\ell[p_\ell]_{\mu_\ell})$, $s \geq \ell$.

6.3   Defining the tilting modules

Let $T(\lambda_k,r)(\Gamma) = T(\lambda,r)(\Gamma)$ be the direct limit of $\{M_s, \iota_{r,s} \,|\, r, s \in \mathbb{Z}_+, r \leq s\}$. We have an injection $M_s \hookrightarrow T(\lambda,r)(\Gamma)$, and, letting $\widetilde{M}_s$ the image of M_s, we have $\widetilde{M}_s \subset \widetilde{M}_{s+1}$, $T(\lambda,r)(\Gamma) = \bigcup \widetilde{M}_s$, and $\frac{\widetilde{M}_s}{\widetilde{M}_{s-1}} \cong \frac{M_s}{M_{s-1}}$. In particular we see that $T(\lambda,r)(\Gamma)$ has $\Delta(\Gamma)$-filtration. We identify M_s with $\widetilde{M}_s$.

The argument that $T(\lambda,r)(\Gamma)$ is indecomposable is identical to that from [3], which we include for completeness. We begin with an easy observation:

$$T(\lambda,r)(\Gamma)[p_\ell]_{\mu_\ell} = M_\ell[p_\ell]_{\mu_\ell}, \qquad M_s = \sum_{\ell \leq s} \mathbf{U}(\mathfrak{g}[t])T(\lambda_k,r)[p_\ell]_{\mu_\ell}. \tag{6.2}$$

To prove that $T(\lambda, r)(\Gamma)$ is indecomposable, suppose that $T(\lambda, r)(\Gamma) = U_1 \oplus U_2$. Since $\dim T(\lambda, r)(\Gamma)[r]_\lambda = 1$, we may assume without loss of generality that $T(\lambda, r)(\Gamma)[r]_\lambda \subset U_1$ and hence $M_0 \subset U_1$. Assume that we have proved by induction that $M_{s-1} \subset U_1$. Since M_s is generated as a $\mathfrak{g}[t]$-module by the spaces $\{M_s[p_\ell]_{\mu_\ell} : \ell \leq s\}$, it suffices to prove that $M_s[p_s]_{\mu_s} \subset U_1$. By (6.2), we have $U_i[p_s]_{\mu_s} \subset M_s$ and hence

$$M_s = (M_{s-1} + \mathbf{U}(\mathfrak{g}[t])U_1[p_s]_{\mu_s}) \oplus \mathbf{U}(\mathfrak{g}[t])U_2[p_s]_{\mu_s}.$$

Since M_s is indecomposable by construction, it follows that $U_2[p_s]_{\mu_s} = 0$ and $M_s \subset U_1$ which completes the inductive step.

Proposition 6.4. *Let $(\lambda, r) \in \Gamma$.*

1. *Then there exists an indecomposable module $T(\lambda, r)(\Gamma) \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ which admits a filtration by finite-dimensional modules $M_s = \sum\limits_{\ell \leq s} \boldsymbol{U}(\mathfrak{g}[t])T(\lambda, r)(\Gamma)[p_\ell]_{\mu_\ell}$, $s \geq 0$, such that $M_0 \cong \Delta(\lambda, r)(\Gamma)$ and the successive quotients are isomorphic to a finite-direct sum of $\Delta(\mu, s)(\Gamma)$, $(\mu, s) \in \mathcal{S}(\lambda, r)$.*

2. *We have $\mathrm{wt}\, T(\lambda, r)(\Gamma) \subset \mathrm{conv}\, W\lambda$, $\dim T(\lambda, r)(\Gamma)[r]_\lambda = 1$.*

3. *For all $(\mu, s) \in \Gamma$, we have $\mathrm{Ext}^1_{\widehat{\mathcal{G}}(\Gamma)}(\Delta(\mu, s)(\Gamma)), T(\lambda, r)(\Gamma)) = 0$.*

Proof. Part (1) and (2) are proved in the proceeding discussion. The proof for part (3) is identical to that found in [3]. ∎

6.4 Initial properties of tilting modules

The next result is an analog of Fitting's lemma for the infinite-dimensional modules $T(\lambda, r)(\Gamma)$.

Lemma 6.5. *Let $\psi : T(\lambda, r)(\Gamma) \to T(\lambda, r)(\Gamma)$ be any morphism of objects of $\widehat{\mathcal{G}}$. Then $\psi(M_s) \subset M_s$ for all $s \geq 0$ and ψ is either an isomorphism or locally nilpotent, i.e., given $m \in M$, there exists $\ell \geq 0$ (depending on m) such that $\psi^\ell(m) = 0$.*

Proof. Since ψ preserves both weight spaces and graded components it follows that $\psi(M_s) \subset M_s$ for all $s \geq 0$. Moreover, since M_s is indecomposable and finite–dimensional it follows from Fitting's lemma that the restriction of ψ to M_s, $s \geq 0$ is either nilpotent or an isomorphism. If all the restrictions are isomorphisms then since $T(\lambda, r)(\Gamma)$ is the union of M_s, $s \geq 0$, it follows that ψ is an isomorphism. On the other hand, if the restriction of ψ to some M_s is nilpotent, then the restriction of ψ to all M_ℓ, $\ell \geq 0$ is nilpotent which proves that ψ is locally nilpotent. ∎

In the rest of the section we shall complete the proof of the main theorem by showing that any indecomposable tilting module is isomorphic to some $T(\lambda, r)(\Gamma)$ and that any tilting module in $\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ is isomorphic to a direct sum of indecomposable tilting modules. Let $T \in \mathcal{G}_{\mathrm{bdd}}(\Gamma)$ be a fixed tilting module. Then we have

$$\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(T, \nabla(\lambda, r)(\Gamma)) = 0 = \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), T), \qquad (\lambda, r) \in \Gamma, \qquad (6.3)$$

where the first equality is due to Corollary 5.5.

Lemma 6.6. *Suppose that T_1 is any summand of T. Then, T_1 admits a $\nabla(\Gamma)$-filtration and $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(T_1, \nabla(\lambda, r)(\Gamma)) = 0$, for all $(\lambda, r) \in \Gamma$.*

Proof. Since Ext^1 commutes with finite direct sums, for all $(\lambda, r) \in \Gamma$ we have

$$\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(T_1, \nabla(\lambda, r)(\Gamma)) = 0, \qquad \text{and} \qquad \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), T_1) = 0.$$

By Proposition 4.2, the second equality implies that T_1 has a $\nabla(\Gamma)$-filtration and the proof of the lemma is complete. ∎

6.5   Completing the proof of the main theorem

The preceding lemma illustrates one of the difficulties we face in our situation. Namely, we cannot directly conclude that T_1 has a $\Delta(\Gamma)$-filtration from the vanishing Ext-condition by using the dual of Proposition 4.2. However, we have the following, whose proof is given in [3].

Proposition 6.7. *Suppose that $N \in \mathcal{G}_{\mathrm{bdd}}(\Gamma)$ has a $\nabla(\Gamma)$-filtration and satisfies*

$$\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(N, \nabla(\lambda, r)(\Gamma)) = 0, \qquad \text{for all} \quad (\lambda, r) \in \Gamma.$$

Let (μ, s) be such that $N \to \nabla(\mu, s)(\Gamma) \to 0$. Then $T(\mu, s)(\Gamma)$ is a summand of N.

The following is immediate. Note that this also shows that our construction of the indecomposable tilting modules is independent of the choice of enumeration of P^+, the set $\mathcal{S}(\lambda, r)$ and η.

Corollary 6.8. *Any indecomposable tilting module is isomorphic to $T(\lambda, r)(\Gamma)$ for some $(\lambda, r) \in \Gamma$. Further if $T \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$ is tilting there exists $(\lambda, r) \in \Gamma$ such that $T(\lambda, r)(\Gamma)$ is isomorphic to a direct summand of T.*

Proof. Since T and $T(\lambda, r)(\Gamma)$ are tilting they satisfy (6.3) and the corollary follows.   ∎

We can now prove the following theorem.

Theorem 6.9. *Let $T \in \mathrm{Ob}\,\mathcal{G}_{\mathrm{bdd}}(\Gamma)$. The following are equivalent.*

1. *T is tilting.*
2. *$\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), T) = 0 = \mathrm{Ext}^1_{\widehat{\mathcal{G}}}(T, \nabla(\lambda, r)(\Gamma)), (\lambda, r) \in \Gamma$.*
3. *T is isomorphic to a direct sum of objects $T(\mu, s)(\Gamma), (\mu, s) \in \Gamma$.*

Proof. The implication (1) $\implies$ (2) is given by Corollary 5.5 and Proposition 4.2, while the fact that (3) implies (1) is clear. We complete the proof by showing that (2) implies (3).

By Proposition 5.10, that $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), T) = 0$ implies that T admits a $\nabla(\Gamma)$-filtration. By Lemma 5.2, we can assume that the filtration is finite, and if $\lambda_k = \lambda$ is maximal such that $T_\lambda \neq 0$, then $T \to \nabla(\lambda, r)(\Gamma) \to 0$ for r. Indeed, if we choose integers $r_1 \geq r_2 \geq \cdots$ such that

$$[T : \nabla(\lambda, s)(\Gamma)] \neq 0 \qquad \text{iff} \quad s = r_i \quad \text{for some} \quad i,$$

then Lemma 5.2 says that we may assume that $T \to \nabla(\lambda, r_1)(\Gamma) \to 0$. By Proposition 6.7 we have $T \cong T(\lambda, r_1)(\Gamma) \oplus T_1$ and we see that T_1 has a $\nabla(\Gamma)$-filtration. The same argument implies that T_1 maps onto $\nabla(\lambda, r_2)(\Gamma)$, and hence $T_1 \cong T(\lambda, r_2)(\Gamma) \oplus T_2$.

Continuing, we find that for $j \geq 1$, there exists a summand T_j of T with

$$T = T_j \bigoplus_{s=1}^{j} T(\lambda, r_s)(\Gamma).$$

Let $\pi_j : T \to \oplus_{s=1}^{j} T(\lambda, r_s)(\Gamma)$ be the canonical projections. Because T has finite-dimensional graded components, and the r_i are decreasing, it follows that for $m \in T$ there exists an integer $k(m)$ such that $\pi_j(m) = \pi_{k(m)}(m)$ for all $k(m) \leq j$. Hence, we have a surjection

$$\pi : \ T \to \bigoplus_{j \geq 1} T(\lambda, r_j)(\Gamma) \to 0 \qquad \text{and} \qquad \ker \pi = \bigcap T_j,$$

where $\pi(m) := \pi_{k(m)}(m)$. In particular, we have $T = (\bigoplus T(\lambda, r_i)(\Gamma)) \oplus \ker \pi$, where $(\ker \pi)_\lambda = 0$, $\ker \pi$ admits a $\nabla(\Gamma)$-filtration and $\mathrm{Ext}^1_{\widehat{\mathcal{G}}}(\ker \pi, \nabla(\mu, r)(\Gamma)) = 0$, for all $(\mu, r) \in \Gamma$. It follows that we may apply to $\ker \pi$ the same arguments we used on T. The result follows by induction on k.   ∎

6.6   Proof of Proposition 6.1

We construct here the set $\mathcal{S}(\lambda, r)$ and the enumeration η.

The set $\mathcal{S}(\lambda, r)$. Recall that $\Gamma = P^+ \times J$, and that $a = \inf J$ and $b = \sup J$. Using the enumeration of P^+, let $\lambda = \lambda_k$ and define integers $r_k \leq r_{k-1} \leq \cdots \leq r_0$ recursively by setting $r_k = r$ and

$$r_s = \max\{r \,|\, \Delta(\lambda_{s+1}, r_{s+1})(\Gamma)[r] \neq 0\}.$$

Note that because $\Delta(\lambda_i, r_i)(\Gamma)[p] = 0$ for any $p > r_{i-1}$, and $r_i \leq r_j$ for all $j < i$, we have $\Delta(\lambda_i, r_i)(\Gamma)[p] = 0$ if $p > r_j$ for any $j < i$. Then, it follows that

$$\Delta(\lambda_i, s)(\Gamma)[p] = 0 \qquad \text{for any} \quad s \leq r_i < p. \tag{6.4}$$

We set $\mathcal{S}(\lambda, r) = \{(\lambda_i, s) | i \leq k, s \leq r_i\}$, and note that it satisfies conditions (1) and (2) of Proposition 6.1 by construction.

We now verify condition (3). Let $(\mu, s) \in \mathcal{S}(\lambda, r)$ and $(\mu', s') \notin \mathcal{S}(\lambda, r)$. There are two possibilities for (μ', s'): either $\mu' = \lambda_i$ for $i > k$, or $\mu' = \lambda_i$ for some $0 \leq i \leq k$ and $s' > r_i$. The first case is covered by Proposition 5.1.3, which tells us that if $\lambda \not\leq \mu$ then $\operatorname{Ext}^1_{\hat{\mathcal{G}}}(\Delta(\lambda, r)(\Gamma), \Delta(\mu, s)(\Gamma)) = 0$ for all $s, r \in \mathbb{Z}$. For the second case, again using Proposition 5.1.3, it is enough to prove that $\operatorname{Ext}^1_{\hat{\mathcal{G}}}(\Delta(\lambda_i, s')(\Gamma), \Delta(\lambda_j, s)(\Gamma)) = 0$ for $\lambda_i \leq \lambda_j$, $s' > r_i$, and $s \leq r_j$. By the total order (cf. Section 5.4), it follows that $i \leq j$, and, hence, $s \leq r_j \leq r_i < s'$. By Proposition 5.1.4, we can in fact assume that $i < j$. Consider a short exact sequence $0 \to \Delta(\lambda_j, s)(\Gamma) \to M \to \Delta(\lambda_i, s')(\Gamma) \to 0$. From (6.4) we have $\Delta(\lambda_j, s)(\Gamma)[s'] = 0$, and it follows that the sequence splits, as we desired.

The enumeration η. It remains to define the enumeration η. The case where $J = \mathbb{Z}$ is done in [3], and the case where J is a finite or of the form $[a, \infty)$ (in which case $\mathcal{S}(\lambda, r)$ is in fact finite), we use the enumeration defined by the following rules

1) $\eta(\lambda_i, s) < \eta(\lambda_j, s')$ if $i > j$,

2) $\eta(\lambda_i, s) < \eta(\lambda_i, s - 1)$.

Suppose that $i < j$ and let $(\mu_i, p_i) = \eta^{-1}(i)$ and $(\mu_j, p_j) = \eta^{-1}(j)$. If $\mu_i = \mu_j$, then rule (2) implies that $p_j < p_i$, and Proposition 5.1 says that $\operatorname{Ext}^1_{\hat{\mathcal{G}}}(\Delta(\mu_i, p_i)(\Gamma), \Delta\mu_j, p_j)(\Gamma)) = 0$. Otherwise, we have $\mu_j \not\leq \mu_i$, and the result again follows by Proposition 5.1.

We are left with the case where $J = (-\infty, b]$. In this case η will in fact be a bijection. Note that it is enough to define a bijective, set theoretic inverse η^{-1}.

We recursively define another set of integers $\{r'_i\}$ by setting $r'_k = r_k$ and letting $r'_i = \max\{r | \Delta(\lambda_{i+1}, r'_{i+1})[r] \neq 0\}$. It is easy to see that $r_i \leq r'_i$. If $r_i < r'_i$, then we must have $r'_i > b$, which implies that $r_i = b$. This implies that $r_j = b$ for all $j < i$. We note that $\Delta(\lambda_j, b)(\Gamma) = V(\lambda_j, b)$. Set $a_s := r'_s - r'_{s+1}$.

Lemma 6.10. *We have* $\operatorname{Ext}^1_{\hat{\mathcal{G}}}(\Delta(\lambda_i, c)(\Gamma), \Delta(\lambda_s, d)(\Gamma)) = 0$ *if* $c - d \geq a_{s-1} + 1$ *and* $i < s$.

Proof. We first prove that under these conditions $\operatorname{Ext}^1_{\hat{\mathcal{G}}}(\Delta(\lambda_i, c), \Delta(\lambda_s, d)) = 0$. Note that we can shift by $-d + r'_s$, and so we examine $\operatorname{Ext}^1_{\hat{\mathcal{G}}}(\Delta(\lambda_i, c - d + r'_s), \Delta(\lambda_s, r'_s))$. According to our hypothesis, we have $c - d + r'_s \geq r'_{s-1} + 1$. It follows by the definition of r'_{s-1} that $\Delta(\lambda_s, r'_s)[p] = 0$ if $p \geq c - d + r'_s$. If we examine a sequence $0 \to \Delta(\lambda_s, r'_s) \to M \to \Delta(\lambda_i, c - d + r'_s) \to 0$ and let $m \in M_{\lambda_i}[c - d + r'_s]$ be a pre-image of w_{λ_i}, it is clear that m satisfies the defining relations of w_{λ_i}. Therefore, the sequence splits. Finally, observe that $\Delta(\lambda_s, d)(\Gamma)[c] = 0$, again by the definition of the r'_i and noting that $\Delta(\lambda_s, d)(\Gamma)$ is a quotient of $\Delta(\lambda_s, d)$. The same argument as above also shows that $\operatorname{Ext}^1_{\hat{\mathcal{G}}}(\Delta(\lambda_i, c)(\Gamma), \Delta(\lambda_s, d)(\Gamma)) = 0$. ∎

We define $\eta^{-1} : \mathbb{Z}_{\geq 0} \to \mathcal{S}(\lambda, r)$ in the following way.

Set $\eta^{-1}(0) = (\lambda_k, r_k)$. If η^{-1} is defined on $\{0, \ldots, m-1\}$ and $\eta^{-1}(m-1) = (\lambda_i, p_i)$, we define $\eta^{-1}(m)$ as follows. Suppose that $i > 0$, and $(\lambda_{i-1}, p_i + a_{i-1}) \in \mathcal{S}(\lambda, r)$, then we set $\eta^{-1}(m) = (\lambda_{i-1}, p_i + a_{i-1})$. Otherwise, we let $\eta^{-1}(m) = (\lambda_k, p_m - 1)$, where p_m is the minimal integer such that (λ_k, p) has a pre-image under η^{-1}.

To prove that $\mathrm{Ext}^1_{\hat{\mathcal{G}}}(\Delta(\mu_i, p_i)(\Gamma), \Delta(\mu_j, p_j)(\Gamma)) = 0$ if $i < j$, we assume that $\mu_i \leq \mu_j$. If $\mu_i = \mu_j$ then this follows from Proposition 5.1. So we assume that $\mu_i < \mu_j$, and lets say that $\mu_j = \lambda_\ell$. In this case we must have $p_i - p_j > a_{\ell-1}$ and so the result follows by Lemma 6.10.

7   Some different considerations on truncated categories

Throughout this section we discuss some "trivial" tilting theories for the category $\mathcal{G}(\Gamma)$ by considering different type of orders on the set Λ. These categories, equipped with the orders described below, have already appeared in the literature (see [5, 7, 8, 9, 11] and references therein).

7.1   Truncated categories with the covering relation

Consider a strict partial order on Λ in the following way. Given $(\lambda, r), (\mu, s) \in \Lambda$, say that (μ, s) covers (λ, r) if and only if $s = r + 1$ and $\mu - \lambda \in R \sqcup \{0\}$. It follows immediately that for any $(\mu, s) \in \Lambda$ the set of $(\lambda, r) \in \Lambda$ such that (μ, s) covers (λ, r) is finite. Let $\preccurlyeq$ be the unique partial order on Λ generated by this cover relation.

One of the main inspirations to consider this relation comes from the following proposition

Proposition 7.1 ([7, Proposition 2.5]). *For* $(\lambda, r), (\mu, s) \in \Lambda$*, we have*

$$\mathrm{Ext}^1_{\mathcal{G}}(V(\lambda, r), V(\mu, s)) = \begin{cases} 0, & \text{if } s \neq r+1, \\ \mathrm{Hom}_{\mathfrak{g}}(V(\lambda), \mathfrak{g} \otimes V(\mu)), & \text{if } s = r+1. \end{cases}$$

In other words, $\mathrm{Ext}^1_{\mathcal{G}}(V(\lambda, r), V(\mu, s)) = 0$ *unless* (μ, s) *covers* (λ, r)*.*

Given $\Gamma \subset \Lambda$, set

$$V_\Gamma^+ = \{v \in V[s]_\mu : \mathfrak{n}^+ v = 0, \ (\mu, s) \in \Gamma\}, \qquad V_\Gamma = \mathbf{U}(\mathfrak{g}) V_\Gamma^+, \qquad V^\Gamma = V / V_{\Lambda \setminus \Gamma}.$$

Proposition 7.2 ([7, Propositions 2.1, 2.4, and 2.7]). *Let* Γ *be finite and convex and assume that* $(\lambda, r), (\mu, s) \in \Gamma$*.*

1. *The object* $I(\lambda, r)_\Gamma$ *is the injective envelope of* $V(\lambda, r)$ *in* $\mathcal{G}[\Gamma]$ *while* $P(\lambda, r)^\Gamma$ *is the projective cover of* $V(\lambda, r)$ *in* $\mathcal{G}[\Gamma]$*. In particular,* $\mathcal{G}[\Gamma]$ *has enough projectives.*

2. $[P(\lambda, r)^\Gamma : V(\mu, s)] = [P(\lambda, r) : V(\mu, s)] = [I(\mu, s) : V(\lambda, r)] = [I(\mu, s)_\Gamma : V(\lambda, r)]$.

3. $\mathrm{Hom}_{\mathcal{G}}(P(\mu, s), P(\lambda, r)) \cong \mathrm{Hom}_{\mathcal{G}[\Gamma]}(P(\mu, s)^\Gamma, P(\lambda, r)^\Gamma)$.

4. *Let* $K(\lambda, r)$ *be the kernel of the canonical projection* $P(\lambda, r) \twoheadrightarrow V(\lambda, r)$*, and let* $(\mu, s) \in \Lambda$*. Then* $[K(\lambda, r) : V(\mu, s)] \neq 0$ *only if* $(\lambda, r) \prec (\mu, s)$*.*

5. *Let* $(\mu, s) \in \Lambda$*. Then* $[I(\lambda, r)/V(\lambda, r) : V(\mu, s)] \neq 0$ *only if* $(\mu, s) \prec (\lambda, r)$*.*

Following Section 3.6 but using the partial order $\preceq$ defined above, for each $(\lambda, r) \in \Gamma$ we denote by $\Delta(\lambda, r)(\Gamma)$ the maximal quotient of $P(\lambda, r)$ such that

$$[\Delta(\lambda, r)(\Gamma) : V(\mu, s)] \neq 0 \implies (\mu, s) \preceq (\lambda, r).$$

Similarly, we denoted by $\nabla(\lambda, r)(\Gamma)$ the maximal submodule of $I(\lambda, r)$ satisfying

$$[\nabla(\lambda, r)(\Gamma) : V(\mu, s)] \neq 0 \implies (\mu, s) \preceq (\lambda, r).$$

The modules $\Delta(\lambda, r)(\Gamma)$ and $\nabla(\lambda, r)(\Gamma)$ are called, respectively, the standard and co-standard modules associated to (λ, r). Further, any module in $\widehat{\mathcal{G}}$ with a $\Delta(\Gamma)$-filtration and a $\nabla(\Gamma)$-filtration is called tilting.

Proposition 7.3. *Let $\Gamma \subseteq \Lambda$ finite and convex.*

1. *For all $(\lambda, r) \in \Gamma$, there exists indecomposable tilting module $T(\lambda, r)(\Gamma)$ and $T(\lambda, r)(\Gamma) = I(\lambda, r)$.*

2. *For all indecomposable tilting module T, we have $T \cong T(\lambda, r)(\Gamma)$ for some $(\lambda, r) \in \Gamma$.*

3. *Every tilting module is isomorphic to a direct sum of indecomposable tilting modules.*

Before proving this proposition, we have some remarks to make:

Remark 7.4.

1. It follows from Proposition 3.3.3 and Proposition 7.2, parts (2) and (5), that the costandard modules in $\widehat{\mathcal{G}}$ associated to (λ, r) is $I(\lambda, r)$ and similarly it follows from Proposition 7.2, parts (2) and (4), that the standard module in $\widehat{\mathcal{G}}$ associated to (λ, r) is the simple module $V(\lambda, r)$.

2. For any $M \in \mathcal{G}$ let $k(M)$ the such that $M \in \mathcal{G}_{\leq k(M)}$. Thus M admits a filtration $\{M_i\}$ where $M_i = \bigoplus_{j=0}^{i} M[k(M) - j]$ which can be refined into a Jordan-Holder series since each quotient M_{i+1}/M_i is a finite-dimensional $\mathfrak{g}$-module.

Proof of Proposition 7.3. Part (1) follows from the Remark 7.4 since we have $T(\lambda, r)(\Gamma) := I(\lambda, r)$. Part (2) and (3) are direct consequences of the injectivity of $I(\lambda, r)$. ∎

7.2   Truncated categories related to restricted Kirillov–Reshetikhin modules

One of the goals of [5, 9] was to study the modules $P(\lambda, r)^{\Gamma}$ (and their multigraded version) under certain very specific conditions on Γ. In these papers it was shown that the modules $P(\lambda, r)^{\Gamma}$ are giving in terms of generators and relations which allows us to regard these modules as specializations of the famous Kirillov–Reshetikhin modules (in the sense of [13, 14]). These papers develop a general theory over a $\mathbb{Z}_+$-graded Lie algebra $\mathfrak{a} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}[i]$ where $\mathfrak{g}_0$ is a finite-dimensional complex simple Lie algebra and its non-zero graded components $\mathfrak{g}[i]$, $i > 0$, are finite-dimensional $\mathfrak{g}_0$- modules. By focusing in these algebras with $\mathfrak{g}[i] = 0$ for $i > 1$, we have $\mathfrak{a} \cong \mathfrak{g} \ltimes V$, where V is a $\mathfrak{g}$-module, and in this context a very particular tilting theory can be described as follows.

Assume that $\mathrm{wt}(V) \neq \{0\}$ and fix a subset $\Psi \subseteq \mathrm{wt}(V)$ satisfying

$$\sum_{\nu \in \Psi} m_\nu \nu = \sum_{\mu \in \mathrm{wt}(V)} n_\mu \mu (m_\nu, n_\mu \in \mathbb{Z}_+) \implies \sum_{\nu \in \Psi} m_\nu \leq \sum_{\mu \in \mathrm{wt}(V)} n_\mu$$

and

$$\sum_{\nu \in \Psi} m_\nu = \sum_{\mu \in \mathrm{wt}(V)} n_\mu \quad \text{only if} \quad n_\mu = 0 \quad \text{for all} \quad \mu \notin \Psi.$$

Remark 7.5. It is proved in [21] that such subsets are precisely those lying on a proper face of the convex polytope determined by $\mathrm{wt}(V)$.

Consider the reflexive and transitive binary relation on P given by

$$\mu \leq_\Psi \lambda \qquad \text{if} \quad \lambda - \mu \in \mathbb{Z}_+ \Psi,$$

where $\mathbb{Z}_+ \Psi$ is the $\mathbb{Z}_+$-span of Ψ. Set also

$$d_\Psi(\mu, \lambda) = \min \left\{ \sum_{\nu \in \Psi} m_\nu : \lambda - \mu = \sum_{\nu \in \Psi} m_\nu \nu, \, m_\nu \in \mathbb{Z}_+ \, \forall \, \nu \in \Psi \right\}.$$

By [11, Proposition 5.2], $\leq_\Psi$ is in fact a partial order on P and

$$d_\Psi(\nu, \mu) + d_\Psi(\mu, \lambda) = d_\Psi(\nu, \lambda) \qquad \text{whenever} \quad \nu \leq_\Psi \mu \leq_\Psi \lambda.$$

Moreover, it induces a refinement of the partial order $\preccurlyeq$ on Λ by setting

$$(\lambda, r) \preccurlyeq_\Psi (\mu, s) \qquad \text{if} \quad \lambda \leq_\Psi \mu, \quad s - r \in \mathbb{Z}_+, \quad \text{and} \quad d_\Psi(\lambda, \mu) = s - r.$$

Finally, if $\Gamma \subseteq \Lambda$ is finite and convex with respect to $\preccurlyeq_\Psi$ and there exists $(\lambda, r) \in \Lambda$ such that $(\lambda, r) \preccurlyeq_\Psi (\mu, s)$ for all $(\mu, s) \in \Gamma$, it was shown [11, Lemma 5.5] that

$$\operatorname{Hom}_{\mathcal{G}[\Gamma]}(P(\mu, s)^\Gamma, P(\nu, t)^\Gamma) \neq 0 \qquad \text{only if} \quad (\nu, t) \preccurlyeq_\Psi (\mu, s) \tag{7.1}$$

and

$$\operatorname{Ext}^j_{\mathcal{G}[\Gamma]}(V(\mu, s), V(\nu, t)) \neq 0 \qquad \text{only if} \quad (\nu, t) \preccurlyeq_\Psi (\mu, s) \quad \text{and} \quad j = d_\Psi(\nu, \mu).$$

In particular, it follows from Proposition 3.3.3, Proposition 7.2, parts (2) and (3), and (7.1) that

$$[P(\lambda, r) : V(\mu, s)] \neq 0 \implies (\lambda, r) \preccurlyeq_\Psi (\mu, s)$$

and

$$[I(\lambda, r) : V(\mu, s)] \neq 0 \implies (\mu, s) \preccurlyeq_\Psi (\lambda, r).$$

We conclude that the standard modules in $\mathcal{G}(\Gamma)$ are the simple modules and the costandard modules are the injective hulls of the simple modules and, hence, $T(\lambda, r)(\Gamma) = I(\lambda, r)_\Gamma$.

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Acknowledgements

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