

Generalised Chern–Simons Theory and G_2 -Instantons over Associative Fibrations

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Abstract. Adjusting conventional Chern–Simons theory to G_2 -manifolds, one describes G_2 -instantons on bundles over a certain class of 7-dimensional flat tori which fiber non-trivially over T^4 , by a pullback argument. Moreover, if $c_2 \neq 0$, any (generic) deformation of the G_2 -structure away from such a fibred structure causes all instantons to vanish. A brief investigation in the general context of (conformally compatible) associative fibrations $f : Y^7 \rightarrow X^4$ relates G_2 -instantons on pullback bundles $f^*E \rightarrow Y$ and self-dual connections on the bundle $E \rightarrow X$ over the base, a fact which may be of independent interest.

Key words: Chern–Simons; Yang–Mills; G_2 -manifolds; associative fibrations

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1 Introduction

This article fits in the context of gauge theory in higher dimensions, following the seminal works of S. Donaldson & R. Thomas, G. Tian and others [4, 16]. The common thread to such generalisations is the presence of a closed differential form on the base manifold Y , inducing an analogous notion of anti-self-dual connections, or *instantons*, on bundles over Y . In the case at hand, G_2 -manifolds are 7-dimensional Riemannian manifolds with holonomy in the Lie group G_2 , which implies the existence of precisely such a structure. This allows one to make sense of G_2 -instantons as the energy-minimising gauge classes of connections, solutions to the corresponding Yang–Mills equation.

Heuristically, G_2 -instantons are somewhat analogous to flat connections in dimension 3. Given a bundle over a compact 3-manifold, with space of connections \mathcal{A} and gauge group \mathcal{G} , the *Chern–Simons functional* is a multi-valued real function on the quotient $\mathcal{B} = \mathcal{A}/\mathcal{G}$, with integer periods, whose critical points are precisely the flat connections [3, § 2.5]. Similar theories can be formulated in higher dimensions in the presence of a suitable closed differential form [4, 15]; e.g. on a G_2 -manifold (Y, φ) , the coassociative 4-form $*\varphi$ allows for the definition of a functional of Chern–Simons type¹. Its ‘gradient’, the Chern–Simons 1-form, vanishes precisely at the G_2 -instantons, hence it detects the solutions to the Yang–Mills equation. These gauge-theoretic preliminaries are covered in Section 2.

On the other hand, one may understand G_2 -manifolds as a particular case of the rich theory of calibrated geometries [6], for which the G_2 -structure φ is a calibration 3-form. Then a 3-dimensional submanifold P is said to be *associative* if it is calibrated by φ , i.e., if $\varphi|_P = d\text{Vol}|_P$. The deformation theory of associative submanifolds is known to be obstructed [9], so their occurrence in families, e.g. fibering over a 4-manifold, is nongeneric and somewhat exotic. Nonetheless, we may consider theoretically, at first, the existence of instantons over associative

¹In fact only the condition $d*\varphi = 0$ is required, so the discussion extends to cases in which the G_2 -structure φ is not necessarily torsion-free.

fibrations $f : Y^7 \rightarrow X^4$. Given a bundle $E \rightarrow X$, a connection \mathbf{A} on its pullback \mathbf{E} is locally of the form

$$\mathbf{A} \stackrel{\text{loc}}{=} A_t(x) + \sum_{i=1}^3 \sigma_i(x, t) dt^i,$$

where $\{A_t\}$ is a family of connections on E parametrised by the associative fibers $P_x := f^{-1}(x)$ and $\sigma_i \in \Omega^0(Y, f^*\mathfrak{g}_E)$. In Section 3.1 I prove the following relation between G_2 -instantons and families of self-dual connections over the base:

Theorem 1. *Let $f : Y \rightarrow X$ define an associative fibration and $\mathbf{E} \rightarrow Y$ be the pullback from an indecomposable vector bundle $E \rightarrow X$.*

(i) *If a connection \mathbf{A} on \mathbf{E} is a G_2 -instanton, then $\{A_t\}$ is a family of self-dual connections on E , satisfying*

$$\frac{\partial A_t}{\partial t^i} = d_{A_t} \sigma_i, \quad i = 1, 2, 3.$$

(ii) *If, moreover, the family $A_t \equiv A_{t_0}$ is constant, then $\mathbf{A} = f^* A_{t_0}$ is a pullback.*

NB: We denote henceforth by \mathcal{M}_+^4 the moduli space of SD connections on the base and by \mathcal{M}_φ^7 the moduli space of G_2 -instantons relative to G_2 -structure φ .

Finally, over the remaining of Section 3, these ideas are applied to a concrete example of certain T^3 -fibrations over T^4 , topologically equivalent to the 7-torus, which I will call G_2 -torus fibrations [11]. Deforming the metric (i.e. the lattice) on T^4 induces a change on the fibration map and hence on the G_2 -structure, and one can use Chern–Simons formalism to see how this affects the moduli of G_2 -instantons:

Theorem 2. *Let $f : \mathbb{T} \rightarrow T^4$ be a G_2 -torus fibration, $\mathbf{E} \rightarrow \mathbb{T}$ be the pullback of an indecomposable vector bundle $E \rightarrow T^4$ and φ denote the G_2 -structure of \mathbb{T} ; then*

(i) *every SD connection on E lifts to a G_2 -instanton on \mathbf{E} , i.e.,*

$$f^* \mathcal{M}_+^4 \subset \mathcal{M}_\varphi^7;$$

(ii) *if, moreover, $c_2(E) \neq 0$, then any perturbation $\varphi + \phi$ away from the class of fibred structures causes the moduli space of G_2 -instantons to vanish, i.e.,*

$$\mathcal{M}_{\varphi+\phi}^7 = \emptyset.$$

The construction of G_2 -instantons is a recent and active research area. Indeed Theorem 2 yields nontrivial, albeit nongeneric, examples of G_2 -instanton moduli, whenever a complex vector bundle $E \rightarrow T^4$ admits SD connections. The interested reader will find other examples in works of Walpuski, Clarke and the author [2, 11, 12, 13, 17]. In the high-energy physics community, solutions to a very similar problem in the context of G_2 -structures with torsion have been found eg. for cylinders over nearly-Kähler homogeneous spaces [5] and more generally for cones over nontrivial manifolds admitting real Killing spinors [7].

Finally, a paper just published by Wang [18] makes significant progress towards a Donaldson theory over higher-dimensional foliations, which seems to encompass our G_2 -torus fibration as a special, codimension 4 tight foliation, whose leaf space is the smooth 4-manifold X . It is inspiring to speculate whether an invariant of the corresponding foliated moduli space can be explicitly computed for some suitable bundle $\mathbf{E} \rightarrow \mathbb{T}$, or indeed if that space coincides with our definition of \mathcal{M}^7 .

2 Gauge theory over G_2 -manifolds

I will concisely recall the essentials of gauge theory on G_2 -manifolds, while referring the interested reader to a more detailed exposition in [12].

Let Y be an oriented smooth 7-manifold; a G_2 -structure is a smooth 3-form $\varphi \in \Omega^3(Y)$ such that, at every point $p \in Y$, one has $\varphi_p = f_p^*(\varphi_0)$ for some frame $f_p : T_p Y \rightarrow \mathbb{R}^7$ and (adopting the conventions of [14])

$$\varphi_0 = e^{567} + \omega_1 \wedge e^5 + \omega_2 \wedge e^6 + \omega_3 \wedge e^7 \quad (1)$$

with

$$\omega_1 = e^{12} - e^{34}, \quad \omega_2 = e^{13} - e^{42} \quad \text{and} \quad \omega_3 = e^{14} - e^{23}.$$

Moreover, φ determines a Riemannian metric $g(\varphi)$ induced by the pointwise inner-product

$$\langle u, v \rangle e^{1\dots 7} := \frac{1}{6} (u \lrcorner \varphi_0) \wedge (v \lrcorner \varphi_0) \wedge \varphi_0, \quad (2)$$

under which $*_\varphi \varphi$ is given pointwise by

$$*_\varphi \varphi_0 = e^{1234} - \omega_1 \wedge e^{67} - \omega_2 \wedge e^{75} - \omega_3 \wedge e^{56}. \quad (3)$$

Such a pair (Y, φ) is a G_2 -manifold if $d\varphi = 0$ and $d*_\varphi \varphi = 0$.

2.1 The G_2 -instanton equation

The G_2 -structure allows for a 7-dimensional analogue of conventional Yang–Mills theory, yielding a notion of (anti-)self-duality for 2-forms. Under the usual identification between 2-forms and matrices, we have $\mathfrak{g}_2 \subset \mathfrak{so}(7) \simeq \Lambda^2$, so we denote $\Lambda_{14}^2 := \mathfrak{g}_2$ and Λ_7^2 its orthogonal complement in Λ^2 :

$$\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{14}^2. \quad (4)$$

It is easy to check that $\Lambda_7^2 = \langle e_1 \lrcorner \varphi_0, \dots, e_7 \lrcorner \varphi_0 \rangle$, hence the orthogonal projection onto Λ_7^2 in (4) is given by

$$\begin{aligned} L_{*\varphi_0} : \Lambda^2 &\rightarrow \Lambda^6, \\ \eta &\mapsto \eta \wedge *_\varphi \varphi_0 \end{aligned}$$

in the sense that [1, p. 541]

$$L_{*\varphi_0}|_{\Lambda_7^2} : \Lambda_7^2 \xrightarrow{\sim} \Lambda^6 \quad \text{and} \quad L_{*\varphi_0}|_{\Lambda_{14}^2} = 0.$$

Furthermore, since (4) splits Λ^2 into irreducible representations of G_2 , a little inspection on generators reveals that $(\Lambda^2)_{14}^2$ is respectively the ${}_{+1}^{-2}$ -eigenspace of the G_2 -equivariant linear map

$$\begin{aligned} T_{\varphi_0} : \Lambda^2 &\rightarrow \Lambda^2, \\ \eta &\mapsto T_{\varphi_0} \eta := *(\eta \wedge \varphi_0). \end{aligned}$$

Consider now a vector bundle $E \rightarrow Y$ over a compact G_2 -manifold (Y, φ) ; the curvature $F := F_A$ of some connection A decomposes according to the splitting (4):

$$F_A = F_7 \oplus F_{14}, \quad F_i \in \Omega_i^2(\text{End } E), \quad i = 7, 14.$$

The L^2 -norm of F_A is the *Yang–Mills functional*:

$$\text{YM}(A) := \|F_A\|^2 = \|F_7\|^2 + \|F_{14}\|^2. \quad (5)$$

It is well-known that the values of $\text{YM}(A)$ can be related to a certain characteristic class of the bundle E , given (up to choice of orientation) by

$$\kappa(E) := - \int_Y \text{tr}(F_A^2) \wedge \varphi.$$

Using the property $d\varphi = 0$, a standard argument of Chern–Weil theory [10] shows that the de Rham class $[\text{tr}(F_A^2) \wedge \varphi]$ is independent of A , thus the integral is indeed a topological invariant. The eigenspace decomposition of T_φ implies (up to a sign)

$$\kappa(E) = -2\|F_7\|^2 + \|F_{14}\|^2,$$

and combining with (5) we get

$$\text{YM}(A) = -\frac{1}{2}\kappa(E) + \frac{3}{2}\|F_{14}\|^2 = \kappa(E) + 3\|F_7\|^2.$$

Hence $\text{YM}(A)$ attains its absolute minimum at a connection whose curvature lies either in Λ_7^2 or in Λ_{14}^2 . Moreover, since $\text{YM} \geq 0$, the sign of $\kappa(E)$ obstructs the existence of one type or the other, so we fix $\kappa(E) \geq 0$ and define G_2 -instantons as connections with $F \in \Lambda_{14}^2$, i.e., such that $\text{YM}(A) = \kappa(E)$. These are precisely the solutions of the G_2 -instanton equation:

$$F_A \wedge *\varphi = 0 \quad (6a)$$

or, equivalently,

$$F_A - *(F_A \wedge \varphi) = 0. \quad (6b)$$

If instead $\kappa(E) \leq 0$, we may still reverse orientation and consider $F \in \Lambda_{14}^2$, but then the above eigenvalues and energy bounds must be adjusted accordingly, which amounts to a change of the $(-)$ sign in (6b).

2.2 Definition of the Chern–Simons functional ϑ

Gauge theory in higher dimensions can be formulated in terms of the geometric structure of manifolds with exceptional holonomy [4]. In particular, instantons can be characterised as critical points of a Chern–Simons functional, hence zeroes of its gradient 1-form [3]. The explicit case of G_2 -manifolds, which we now describe, was first examined in the author’s thesis [11].

Let $E \rightarrow Y$ be a vector bundle; the space \mathcal{A} is an affine space modelled on $\Omega^1(\mathfrak{g}_E)$ so, fixing a reference connection $A_0 \in \mathcal{A}$,

$$\mathcal{A} = A_0 + \Omega^1(\mathfrak{g}_E)$$

and, accordingly, vectors at $A \in \mathcal{A}$ are 1-forms $a, b, \dots \in T_A \mathcal{A} \simeq \Omega^1(\mathfrak{g}_E)$ and vector fields are maps $\alpha, \beta, \dots : \mathcal{A} \rightarrow \Omega^1(\mathfrak{g}_E)$. In this notation we define the *Chern–Simons functional* by

$$\vartheta(A) := \frac{1}{2} \int_Y \text{tr} \left(d_{A_0} a \wedge a + \frac{2}{3} a \wedge a \wedge a \right) \wedge *\varphi,$$

fixing $\vartheta(A_0) = 0$. This function is obtained by integration of the *Chern–Simons 1-form*

$$\rho(\beta)_A = \rho_A(\beta_A) := \int_Y \text{tr}(F_A \wedge \beta_A) \wedge *\varphi. \quad (7)$$

We find ϑ explicitly by integrating ρ over paths $A(t) = A_0 + ta$, from A_0 to any $A = A_0 + a$:

$$\begin{aligned}\vartheta(A) - \vartheta(A_0) &= \int_0^1 \rho_{A(t)}(\dot{A}(t)) dt = \int_0^1 \left(\int_Y \operatorname{tr} \left((F_{A_0} + td_{A_0}a + t^2a \wedge a) \wedge a \right) \wedge * \varphi \right) dt \\ &= \frac{1}{2} \int_Y \operatorname{tr} \left(d_{A_0}a \wedge a + \frac{2}{3}a \wedge a \wedge a \right) \wedge * \varphi + K,\end{aligned}$$

where $K = K(A_0, a)$ is a constant and vanishes if A_0 is an instanton.

The co-closedness condition $d * \varphi = 0$ implies that the 1-form (7) is closed, so the procedure doesn't depend on the path $A(t)$. Indeed, given tangent vectors $a, b \in \Omega^1(\mathfrak{g}_E)$ at A , the leading term in the expansion of ρ ,

$$\rho_{A+a}(b) - \rho_A(b) = \int_Y \operatorname{tr}(d_A a \wedge b) \wedge * \varphi + O(|b|^2),$$

is symmetric by Stokes' theorem:

$$\int_Y \operatorname{tr}(d_A a \wedge b - a \wedge d_A b) \wedge * \varphi = \int_Y d(\operatorname{tr}(b \wedge a) \wedge * \varphi) = 0.$$

We conclude that

$$\rho_{A+a}(b) - \rho_A(b) = \rho_{A+b}(a) - \rho_A(a) + O(|b|^2)$$

and, comparing reciprocal Lie derivatives on parallel vector fields $\alpha \equiv a$, $\beta \equiv b$ near a point A , we have:

$$\begin{aligned}d\rho(\alpha, \beta)_A &= (\mathcal{L}_b \rho)_A(a) - (\mathcal{L}_a \rho)_A(b) = \lim_{h \rightarrow 0} \frac{1}{h} \{ \rho_{A+hb}(a) - \rho_A(a) - (\rho_{A+ha}(b) - \rho_A(b)) \} \\ &= \lim_{h \rightarrow 0} \frac{1}{h^2} \underbrace{ \{ (\rho_{A+hb}(ha) - \rho_A(ha)) - (\rho_{A+ha}(hb) - \rho_A(hb)) \} }_{O(|h|^3)} = 0.\end{aligned}$$

Since \mathcal{A} is contractible, by the Poincaré lemma ρ is the derivative of some function ϑ . Again by Stokes, ρ vanishes along \mathcal{G} -orbits $\operatorname{im}(d_A) \simeq T_A\{\mathcal{G}.A\}$. Thus ρ descends to the quotient \mathcal{B} and so does ϑ , locally.

2.3 Periodicity of ϑ

Consider the gauge action of $g \in \mathcal{G}$ and some path $\{A(t)\}_{t \in [0,1]} \subset \mathcal{A}$ connecting an instanton A to $g.A$. The natural projection $p_1 : Y \times [0, 1] \rightarrow Y$ induces a bundle

$$\begin{array}{ccc} \mathbf{E}_g & \xrightarrow{\tilde{p}_1} & E \\ \downarrow & & \downarrow \\ Y \times [0, 1] & \xrightarrow{p_1} & Y \end{array}$$

and, using g to identify the fibres $(\mathbf{E}_g)_0 \xrightarrow{g} (\mathbf{E}_g)_1$, one may think of \mathbf{E}_g as a bundle over $Y \times S^1$. Moreover, in some local trivialisation, the path $A(t) = A_i(t)dx^i$ gives a connection $\mathbf{A} = \mathbf{A}_0 dt + \mathbf{A}_i dx^i$ on \mathbf{E}_g :

$$(\mathbf{A}_0)_{(t,p)} = 0, \quad (\mathbf{A}_i)_{(t,p)} = A_i(t)_p.$$

The corresponding curvature 2-form is $F_{\mathbf{A}} = (F_{\mathbf{A}})_{i0} dx^i \wedge dt + (F_{\mathbf{A}})_{jk} dx^j \wedge dx^k$, where

$$(F_{\mathbf{A}})_{i0} = \dot{A}_i(t), \quad (F_{\mathbf{A}})_{jk} = (F_A)_{jk}.$$

The periods of ϑ are then of the form

$$\begin{aligned} \vartheta(g.A) - \vartheta(A) &= \int_0^1 \rho_{A(t)}(\dot{A}(t)) dt = \int_{Y \times [0,1]} \text{tr}(F_{A(t)} \wedge \dot{A}_i(t) dx^i) \wedge dt \wedge * \varphi \\ &= \int_{Y \times S^1} \text{tr} F_{\mathbf{A}} \wedge F_{\mathbf{A}} \wedge * \varphi = \frac{1}{8\pi^2} \langle c_2(\mathbf{E}_g) \smile [* \varphi], Y \times S^1 \rangle. \end{aligned}$$

The Künneth formula for $Y \times S^1$ gives

$$H^4(Y \times S^1, \mathbb{R}) = H^4(Y, \mathbb{R}) \oplus H^3(Y, \mathbb{R}) \otimes \underbrace{H^1(S^1, \mathbb{R})}_{\mathbb{Z}}$$

and obviously $H^4(Y) \smile [* \varphi] = 0$ so, denoting by $c'_2(\mathbf{E}_g)$ the component lying in $H^3(Y)$ and by $S_g := [\frac{1}{8\pi^2} c'_2(\mathbf{E}_g)]^{PD}$ its normalised Poincaré dual, we are left with

$$\vartheta(g.A) - \vartheta(A) = \langle [* \varphi], S_g \rangle.$$

Consequently, the periods of ϑ lie in the set

$$\left\{ \int_{S_g} * \varphi \mid S_g \in H_4(Y, \mathbb{R}) \right\}.$$

That may seem odd at first, because $* \varphi$ is not, in general, an integral class and so the set of periods is *dense*. However, as long as our interest remains in the study of the moduli space $\mathcal{M} = \text{Crit}(\rho)$ of G_2 -instantons, there is not much to worry, for the gradient $\rho = d\vartheta$ is unambiguously defined on \mathcal{B} .

3 Instantons over G_2 -torus fibrations

Instances of G_2 -manifolds fibred by associative submanifolds in the literature are relatively scarce, not least because their deformation theory is zero-index elliptic [9] and therefore any new examples will be somewhat exotic. A few trivial cases include the products $T^7 = T^4 \times T^3$ and $K3 \times T^3$ and also $CY^3 \times S^1$ given a family of curves in the Calabi–Yau [8, § 10.8]. The example I will propose is unique in the sense that the total space is not a Riemannian product.

3.1 Instantons over associative fibrations

We consider pullback bundles over smooth associative fibrations, and relate G_2 -instantons to their gauge theory over the base; in particular we do not address the possibility of singular fibres.

Definition 1. A G_2 -manifold (Y^7, φ) is called an *associative fibration* over a compact oriented Riemannian four-manifold (X^4, η) if it is the total space of a Riemannian submersion $f : Y \rightarrow X$ such that each fibre $P_x := f^{-1}(x) \subset Y$ is a smooth associative submanifold.

Since each fibre P_x is 3-dimensional and orientable, its tangent bundle is differentiably trivial and we may choose global coordinates $t = (t^1, t^2, t^3)$ induced respectively by a global coframe $\{e_5, e_6, e_7\} := \{dt^1, dt^2, dt^3\}$. Thus near each $y \in P_x$ we may complete the triplet into a local orthogonal coframe $\{e_1, \dots, e_7\}$ of T^*Y such that φ_y has the form (1), and the point y is unambiguously described by $(x, t(y))$.

Lemma 1. *Let $f : Y \rightarrow X$ define an associative fibration and $\mathbf{E} \rightarrow Y$ be the pullback from a vector bundle $E \rightarrow X$; then a connection A on E is self-dual if, and only if, f^*A is a G_2 -instanton on \mathbf{E} .*

Proof. Let $F := (F_{f^*A})_y$ be the curvature 2-form at $y \in P_x$; then

$$*\varphi(F \wedge \varphi) \stackrel{\text{loc}}{=} *\varphi[F \wedge (\varphi|_{P_x} + \omega_1 \wedge e^5 + \omega_2 \wedge e^6 + \omega_3 \wedge e^7)] = *\eta F + *\varphi[O(F^-) \wedge f^*d\text{Vol}_\eta],$$

where $O(F^-) := (F_{34} - F_{12})e^5 + (F_{42} - F_{13})e^6 + (F_{23} - F_{14})e^7$ vanishes precisely when A is self-dual, i.e., when $F = *\eta F$ satisfies the G₂-instanton equation (6b). ■

We are now in position to prove Theorem 1. Let us examine the general form of a G₂-instanton on \mathbf{E} . An arbitrary connection \mathbf{A} on \mathbf{E} is locally of the form

$$\mathbf{A}(y) \stackrel{\text{loc}}{=} A_t(x) + \sum_{i=1}^3 \sigma_i(x, t) dt^i,$$

where $\{A_t\}_{t \in t(P_x)}$ is a family of connections on E and $\sigma_i \in \Omega^0(Y, f^*\mathfrak{g}_E)$. The curvature of \mathbf{A} is

$$F_{\mathbf{A}} = F_{A_t} + \sum_{i=1}^3 \left(d_{A_t} \sigma_i - \frac{\partial A_t}{\partial t^i} \right) \wedge dt^i + F_\sigma$$

with

$$F_\sigma := \sum_{i,j=1}^3 \left(\frac{\partial \sigma_i}{\partial t^j} - \frac{\partial \sigma_j}{\partial t^i} + \frac{1}{2} [\sigma_i, \sigma_j] \right) dt^i \wedge dt^j.$$

Replacing $F_{\mathbf{A}}$ into the G₂-instanton equation (6a) and using the expression (3) of $*\varphi$ in the natural frame $\{e_1, \dots, e_7\}$, we have

$$\left(F_{A_t} + \sum_{i=1}^3 \left(d_{A_t} \sigma_i - \frac{\partial A_t}{\partial t^i} \right) \wedge e^{4+i} + F_\sigma \right) \wedge (e^{1234} - \omega_1 \wedge e^{67} - \omega_2 \wedge e^{75} - \omega_3 \wedge e^{56}) = 0.$$

Using the following elementary properties

$$\begin{aligned} F_{A_t} \wedge e^{1234} &= 0, & F_{A_t} \wedge \omega_1 \wedge e^{67} &= [(F_{A_t})_{34} - (F_{A_t})_{12}] (*e^5), \\ F_{A_t} \wedge \omega_2 \wedge e^{75} &= [(F_{A_t})_{42} - (F_{A_t})_{13}] (*e^6), & F_{A_t} \wedge \omega_3 \wedge e^{56} &= [(F_{A_t})_{23} - (F_{A_t})_{14}] (*e^7), \\ F_\sigma \wedge e^{4+i} \wedge e^{4+j} &= 0, & F_\sigma \wedge e^{1234} &= (F_\sigma)_{23} (*e^5) + (F_\sigma)_{31} (*e^6) + (F_\sigma)_{12} (*e^7), \end{aligned}$$

and the fact that each $d_{A_t} \sigma_i$ and $\frac{\partial A_t}{\partial t^i}$ are locally 1-forms on the base, hence their wedge product with $e^{1234} = d\text{Vol}_\eta$ vanishes, the equation simplifies to

$$\sum_{i=1}^3 \left(d_{A_t} \sigma_i - \frac{\partial A_t}{\partial t^i} \right) \wedge \omega_i = 0 \quad \text{and} \quad F_{A_t}^- - Q(F_\sigma) = 0,$$

where Q is the linear map on 2-forms defined by

$$Q(dt^i \wedge dt^j) = Q(e^{4+i} \wedge e^{4+j}) := \sum_{k=1}^3 \epsilon^{ijk} \omega_k.$$

On the other hand, if $\mathbf{A} = A_t + \sum \sigma_i$ is a G₂-instanton, then it minimises the Yang–Mills functional (5). This implies

$$\sum \left\| d_{A_t} \sigma_i - \frac{\partial A_t}{\partial t^i} \right\|^2 + \|F_\sigma\|^2 = 0,$$

since otherwise the pullback component A_t alone would violate the minimum energy:

$$\text{YM}(A_t) = \|F_{A_t}\|^2 < \|F_{\mathbf{A}}\|^2 = \text{YM}(\mathbf{A}).$$

In particular $F_\sigma \equiv 0$ and so every A_t must be SD. Finally, if the family $A_t \equiv A_{t_0}$ is constant, then $d_{A_{t_0}}\sigma_i = 0$ implies $\sigma \equiv 0$, since by assumption \mathbf{E} is indecomposable and therefore does not admit nontrivial parallel sections. This concludes the proof of Theorem 1.

Remark 1. If \mathcal{M}_+^4 is discrete, then by continuity the family $\{A_t\}$ is contained in a gauge orbit; if the family is constant, then \mathbf{A} is a pullback.

3.2 G_2 -torus fibrations

A 7-torus $T^7 = \mathbb{R}^7/\Lambda$ naturally inherits the G_2 -structure φ from \mathbb{R}^7 . Recall from Section 2.2 that a connection A on some bundle over T^7 is a G_2 -instanton if and only if it is a zero of the Chern–Simons 1-form (7):

$$\rho_A(b) = \int_{T^7} \text{tr}(F_A \wedge b) \wedge *\varphi. \quad (8)$$

One asks what is the behaviour of the moduli space of G_2 -instantons under perturbations $\varphi \rightarrow \varphi + \phi$ of the G_2 -structure. More precisely, given suitable assumptions, one asks whether $(\varphi + \phi)$ -instantons exist at all once we deform the lattice. As a working example, we consider the following class of flat T^3 -fibred 7-tori:

Definition 2. A G_2 -torus fibration structure is a triplet (η, L, α) in which:

- η is a metric on \mathbb{R}^4 ;
- L is a lattice on the subspace $\Lambda_+^2(\mathbb{R}^4, \eta)$ of η -self-dual 2-forms;
- $\alpha : \mathbb{R}^4 \rightarrow \Lambda_+^2(\mathbb{R}^4, \eta)$ is a linear map.

Given the above data, set $V \doteq \mathbb{R}^4 \oplus \Lambda_+^2$ and form the torus $\mathbb{T} = V/\tilde{L}$, with the lattice

$$\tilde{L} \doteq \{(\mu, \nu + \alpha\mu) \mid \mu \in \mathbb{Z}^4, \nu \in L\} \subset V.$$

Then \mathbb{T} inherits from V the G_2 -structure φ which makes the generators of \tilde{L} orthonormal with respect to the induced inner-product (2). It is straightforward to check that \mathbb{T} is an associative fibration as in Definition 1: denoting by e^5, e^6, e^7 the $(\nu + \alpha\mu)$ -orthonormal basis of the fibre Λ_+^2 , the flat G_2 -structure (1) simplifies to $\varphi|_{\Lambda_+^2} = e^{567} = d\text{Vol}_\varphi|_{\Lambda_+^2}$; moreover the lattice \tilde{L} on every tangent subspace normal to the fibre is just the lattice μ from the base, so the corresponding metrics are the same. Although \mathbb{T} fibres over the 4-torus \mathbb{R}^4/μ , the induced metric $g(\varphi)$ is *not*, in general, a Riemannian product.

Suppose the moduli space \mathcal{M}_+^4 of self-dual connections on $E \rightarrow T^4$ is nonempty; then we have trivial solutions to the G_2 -instanton equation on the pullback $\mathbf{E} \rightarrow \mathbb{T}$ simply by lifting \mathcal{M}_+^4 as in Lemma 1, which proves the first part of Theorem 2:

Corollary 1. *If A is a self-dual connection on $E \rightarrow T^4$, then its pullback f^*A by the fibration map $f : \mathbb{T} \rightarrow T^4$ is a G_2 -instanton on \mathbf{E} .*

For future reference, I denote the set of such φ -instantons obtained by lifts from \mathcal{M}_+^4 by

$$\widetilde{\mathcal{M}}_+^4 := f^*\mathcal{M}_+^4 \subset \mathcal{B}^7. \quad (9)$$

We know from 4-dimensional gauge theory that SD connections on a complex vector bundle $E \rightarrow T^4$ correspond to stable holomorphic structures on E , thus in such cases we have examples of G_2 -instantons on bundles over \mathbb{T} .

3.3 Deformations of \mathbb{T}

Working on a bundle $\mathbf{E} \rightarrow \mathbb{T}$ with compact structure group over a fixed G_2 -torus fibration, let us ponder in generality about the behaviour of instantons under a deformation of the G_2 -structure:

$$\varphi \rightarrow \varphi + \phi, \quad *_\varphi\varphi \rightarrow *_\varphi\varphi + \xi_\phi, \quad \xi_\phi := *_{\varphi+\phi}(\varphi + \phi) - *_\varphi\varphi \in \Omega^4(\mathbb{T}).$$

An arbitrary deformation ϕ does not in general preserve the fibred structure of \mathbb{T} :

Proposition 1. *A deformation $\xi_\phi \in \Lambda^4(\mathbb{T})$ of the coassociative 4-form $*_\varphi\varphi$ has four orthogonal components, with the following significance:*

$$\Lambda^4(\mathbb{R}^4 \oplus \Lambda_+^2) = \underbrace{\Lambda^4(\mathbb{R}^4)}_{\text{(I)}} \oplus \underbrace{\Lambda^3(\mathbb{R}^4) \otimes \Lambda^1(\Lambda_+^2)}_{\text{(II)}} \oplus \underbrace{\Lambda^2(\mathbb{R}^4) \otimes \Lambda^2(\Lambda_+^2)}_{\text{(III)}} \oplus \underbrace{\Lambda^1(\mathbb{R}^4) \otimes \Lambda^3(\Lambda_+^2)}_{\text{(IV)}},$$

(I) corresponds to a rescaling of the metric η on \mathbb{R}^4 ;

(II) redefines the map α ;

(III) splits as $\text{Hom}(\Lambda_+^2, \Lambda_+^2) \oplus \text{Hom}(\Lambda_-^2, \Lambda_+^2)$, where the first factor modifies the lattice L and the second one affects the conformal class of η ;

(IV) parametrises deformations transverse to the fibred structures.

Proof. Let us examine the four cases.

(I) If $\xi_\phi \in \Lambda^4(\mathbb{R}^4) \simeq \mathbb{R}$, then it must be a multiple of $*\varphi|_{\mathbb{R}^4} = e^{1234} = d\text{Vol}_\eta$.

(II) Since $\Lambda^3(\mathbb{R}^4) \otimes \Lambda^1(\Lambda_+^2) \simeq \mathbb{R}^4 \otimes (\Lambda_+^2)^* \simeq \text{Hom}(\mathbb{R}^4, \Lambda_+^2)$, such deformations are precisely linear maps $\mathbb{R}^4 \rightarrow \Lambda_+^2$.

(III) Clearly $\Lambda^2(\mathbb{R}^4) = \Lambda_+^2 \oplus \Lambda_-^2$ and $\Lambda^2(\Lambda_+^2) \simeq (\Lambda_+^2)^*$, so the product decomposes as

$$(\Lambda_+^2 \otimes (\Lambda_+^2)^*) \oplus (\Lambda_-^2 \otimes (\Lambda_+^2)^*) \simeq \text{Hom}(\Lambda_+^2, \Lambda_+^2) \oplus \text{Hom}(\Lambda_-^2, \Lambda_+^2).$$

Now, on one hand, acting with an endomorphism on Λ_+^2 is equivalent to redefining the triplet $\{e^5, e^6, e^7\}$, hence the lattice $L \subset \Lambda_+^2$. On the other hand, since the orthogonal split $\Lambda^2 = \Lambda_-^2 \oplus \Lambda_+^2$ is conformally invariant, a map $\Lambda_-^2 \rightarrow \Lambda_+^2$ redefines the orthogonal complement of Λ_-^2 and hence the conformal class.

(IV) Since $\Lambda^3(\Lambda_+^2) \simeq \mathbb{R}$, this component is just $\Lambda^1(\mathbb{R}^4)$, which is irreducible in the sense that \mathbb{T} has no distinguished subspaces in \mathbb{R}^4 . Then either every 7-torus is a G_2 -fibration, which is obviously false, or these are precisely the deformations away from said structures. \blacksquare

We will now describe what happens to the zeroes of (8) under the corresponding perturbation of the Chern–Simons 1-form:

$$\rho \rightarrow \rho_\phi := \rho + r_\phi, \quad (r_\phi)_A(b) = \int_{\mathbb{T}} \text{tr}(F_A \wedge b) \wedge \xi_\phi.$$

Clearly a φ -instanton A is also a $(\varphi + \phi)$ -instanton if and only if $(r_\phi)_A \equiv 0$. There is little reason, however, to expect such a coincidence; as we will see, the topology of the bundle may constrain the existence of instantons under certain – indeed most – deformations.

Denoting henceforth by \mathcal{A} the space of connections over the 7-manifold \mathbb{T} , let us briefly digress into the translation action of some vector $v \in \mathbb{T}$ on some $A \in \mathcal{A}$. The first order variation is given by the bundle-valued 1-form

$$(\beta_v)_A := v \lrcorner F_A,$$

which we interpret as a vector in $T_A\mathcal{A}$. Notice first that in the direction β_v the value of the Chern–Simons 1-form is independent of the base-point:

Lemma 2. *The function $\rho(\beta_v) : \mathcal{A} \rightarrow \mathbb{R}$ is constant.*

Proof. The computation is straightforward:

$$\begin{aligned} \rho(\beta_v)_{A+ha} &= \int_{\mathbb{T}} \operatorname{tr} F_{A+ha} \wedge v \lrcorner F_{A+ha} \wedge * \varphi = -\frac{1}{2} \int_{\mathbb{T}} \operatorname{tr} F_{A+ha} \wedge F_{A+ha} \wedge (v \lrcorner * \varphi) \\ &= -\frac{1}{2} \int_{\mathbb{T}} (\operatorname{tr} F_A \wedge F_A + d\chi) \wedge (v \lrcorner * \varphi) = -\frac{1}{2} \int_{\mathbb{T}} \operatorname{tr} F_A \wedge F_A \wedge (v \lrcorner * \varphi) = \rho(\beta_v)_A, \end{aligned}$$

where $d\chi$ is the exact differential given by Chern–Weil theory and we use Stokes’ theorem and Cartan’s identity $d(v \lrcorner * \varphi) = \mathcal{L}_v(*\varphi) = 0$, since φ is constant on the flat torus. ■

Similarly, evaluating r_ϕ on β_v gives

$$r_\phi(\beta_v)_A = \int_{\mathbb{T}} \operatorname{tr}(F_A \wedge (\beta_v)_A) \wedge \xi_\phi = -\frac{1}{2} \int_{\mathbb{T}} \operatorname{tr}(F_A \wedge F_A) \wedge (v \lrcorner \xi_\phi) = \langle c_2(E), S_\phi(v) \rangle,$$

where $S_\phi(v) \doteq -\frac{1}{2}[v \lrcorner \xi_\phi]^{PD}$, and this depends only on the topology of E , not on the point A .

Remark 2. Hence we may interpret ϕ as defining a linear functional

$$\begin{aligned} N_\phi : \mathbb{R}^7 &\rightarrow \mathbb{R}, \\ v &\mapsto \langle c_2(E), S_\phi(v) \rangle, \end{aligned}$$

such that $N_\phi \neq 0$ implies no φ -instanton is still a $(\varphi + \phi)$ -instanton. This is, however, a rather weak obstruction, since the map $\phi \mapsto N_\phi$ has kernel of dimension at least 28 and thus, in principle, leaves plenty of possibilities for instantons of perturbed G_2 -structures.

Now consider specifically a translation vector on the base $v \in T^4$. Notice that for deformations ϕ of types (I), (II) or (III) the contraction of ξ_ϕ with such v gives $S_\phi(v) = 0$, so ϕ only effectively contributes to the function $\rho(\beta_v)$ when $\xi_\phi \in \Lambda^1(\mathbb{R}^4)$, which means the perturbed torus is no longer a fibred structure (Proposition 1). Moreover, either the bundle E is flat and β_v vanishes identically, or $c_2(E) \neq 0$ and the following holds:

Lemma 3. *If $c_2(E) \neq 0$ and ϕ is of type (IV), then there exists $v \in T^4$ such that $r_\phi(\beta_v)$ is a non-zero constant.*

Proof. Denoting T^3 the typical fibre of f (and setting $\operatorname{Vol}(T^3) = 1$), we may assume

$$\xi_\phi = -2\varepsilon \wedge d \operatorname{Vol}_{T^3}$$

for some $0 \neq \varepsilon \in \Lambda^1(T^4)$. One can always choose $v \in T^4$ such that $\varepsilon(v) \neq 0$, and consider $(\beta_v)_A = v \lrcorner F_A$. Then

$$r_\phi(\beta_v)_A = -2 \int_{\mathbb{T}} \operatorname{tr}(F_A \wedge v \lrcorner F_A) \wedge \varepsilon \wedge d \operatorname{Vol}_{T^3} = -2 \int_{T^4} \operatorname{tr}(F_A \wedge v \lrcorner F_A) \wedge \varepsilon = \varepsilon(v) \cdot c_2(E),$$

which is nonzero by assumption. ■

So far we know from Corollary 1 that the set \mathcal{M}_+^4 of self-dual connections (modulo gauge) over T^4 lifts to instantons (cf. (9)) of the original G_2 -structure φ (i.e. to zeroes of ρ). However, for bundles with non-trivial c_2 , this generic case degenerates precisely under deformations of type (IV):

Proposition 2. *Let $\mathbf{E} \rightarrow (\mathbb{T}, \varphi)$ be the pullback of a stable $SU(n)$ -bundle E over T^4 with $c_2(E) \neq 0$; then E admits no $(\varphi + \phi)$ -instantons, for any perturbation ϕ away from a fibred structure (i.e. of type (IV) in Proposition 1).*

Proof. Fix a lifted φ -instanton $A \in \widetilde{\mathcal{M}}_+^4$; for any $A + ha \in \mathcal{A}$, Lemma 2 gives $\rho_{A+ha}(\beta_v) \equiv \rho_A(\beta_v) = 0$. Taking $v \in T^4$ as in Lemma 3 we have

$$\rho_\phi(\beta_v)_{A+ha} = r_\phi(\beta_v)_{A+ha} + \rho(\beta_v)_{A+ha} = \underbrace{\varepsilon(v) \cdot c_2(E)}_{\neq 0} + \underbrace{\rho(\beta_v)_A}_0,$$

hence $A + ha$ is not a $(\varphi + \phi)$ -instanton. ■

Combining Corollary 1 and Proposition 2 we obtain Theorem 2.

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