Symmetry, Integrability and Geometry: Methods and Applications SIGMA 11 (2015), 018, 13 pages

# Irreducible Generic Gelfand–Tsetlin Modules of $\mathfrak{gl}(n)$*

*Vyacheslav FUTORNY* †, *Dimitar GRANTCHAROV* ‡ *and Luis Enrique RAMIREZ* †

† *Instituto de Matemática e Estatística, Universidade de São Paulo, São Paulo SP, Brasil*
E-mail: *futorny@ime.usp.br*, *luiser@ime.usp.br*

‡ *Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019, USA*
E-mail: *grandim@uta.edu*

Received October 01, 2014, in final form February 24, 2015; Published online February 28, 2015
http://dx.doi.org/10.3842/SIGMA.2015.018

**Abstract.** We provide a classification and explicit bases of tableaux of all irreducible generic Gelfand–Tsetlin modules for the Lie algebra $\mathfrak{gl}(n)$.

*Key words:* Gelfand–Tsetlin modules; Gelfand–Tsetlin basis; tableaux realization

*2010 Mathematics Subject Classification:* 17B67

## 1 Introduction

Let $\mathfrak{g}$ be a complex finite-dimensional semisimple Lie algebra. The category of weight modules of $\mathfrak{g}$ is interesting on its own on the one hand, and it contains some fundamental subcategories like the category $\mathcal{O}$, categories of parabolically induced modules, Harish-Chandra modules on the other. A weight $\mathfrak{g}$-module is a module which is a direct sum of simple $\mathfrak{h}$-modules, where $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$. The classification of the simple weight modules is a very hard problem which is solved only for $\mathfrak{g} = \mathfrak{sl}(2)$. However, the classification of the simple objects is known for various subcategories of weight modules, including those with finite weight multiplicities [5, 17].

The classification of the simple weight $\mathfrak{sl}(2)$-modules involves two parameters that correspond to eigenvalues of the generators of a maximal commutative subalgebra of $U(\mathfrak{sl}(2))$, the *Gelfand–Tsetlin subalgebra*. Such subalgebra can be defined for any $\mathfrak{sl}(n)$ and has a joint spectrum on every finite-dimensional module. This observation leads naturally to the definition of a *Gelfand–Tsetlin module*: a module that is the direct sum of its common generalized eigenspaces with respect to the Gelfand–Tsetlin subalgebra $\Gamma$. Such modules were introduced in [2, 3, 4]. Note that an irreducible Gelfand–Tsetlin modules does not need to be $\Gamma$-diagonalizable [6].

Gelfand–Tsetlin subalgebras and modules appear in various contexts. Such subalgebras were considered in [22] in connection with subalgebras of maximal Gelfand–Kirillov dimension in the universal enveloping algebra of a simple Lie algebra. Furthermore, Gelfand–Tsetlin subalgebras are related to: general hypergeometric functions on the complex Lie group $\mathrm{GL}(n)$ [13, 14]; solutions of the Euler equation [22]; and problems in classical mechanics in general [15, 16].

One natural question is to attempt the classification of all irreducible Gelfand–Tsetlin modules of $\mathfrak{sl}(n)$. An explicit construction of all irreducible Gelfand–Tsetlin modules for the case $n = 3$ was recently obtained in [10]. Various partial results for $\mathfrak{sl}(3)$ were previously obtained in [1, 6, 7, 8, 9].

A *generic Gelfand–Tsetlin module* is a module spanned by tableaux with noninteger differences of entries in each row (see Definition 5.1). The present paper provides a classification of all irreducible generic Gelfand–Tsetlin modules of $\mathfrak{sl}(n)$ extending the result in [21] for $n = 3$.

*This paper is a contribution to the Special Issue on New Directions in Lie Theory. The full collection is available at http://www.emis.de/journals/SIGMA/LieTheory2014.html

For simplicity we work with $\mathfrak{gl}(n)$ instead of $\mathfrak{sl}(n)$. We also obtain an explicit construction of all irreducible generic modules providing a Gelfand–Tsetlin type basis.

The organization of the paper is as follows. In Section 3 we introduce some basic definitions and preparatory results on Gelfand–Tsetlin modules. In Section 4 we list the Gelfand–Tsetlin formulas and use them to recall the classical result of Gelfand and Tsetlin for finite-dimensional $\mathfrak{gl}(n)$-modules. In Section 5 we introduce the notion of generic Gelfand–Tsetlin module and recall the classification of irreducible generic Gelfand–Tsetlin modules of $\mathfrak{gl}(3)$. The main theorem in the paper, the classification of irreducible generic Gelfand–Tsetlin $\mathfrak{gl}(n)$-modules, is included in Section 6. In the last section we compute the number of irreducible Gelfand–Tsetlin modules in the so-called generic blocks.

## 2 Notation and conventions

Throughout the paper we fix an integer $n \geq 2$. The ground field will be $\mathbb{C}$. For $a \in \mathbb{Z}$, we write $\mathbb{Z}_{\geq a}$ for the set of all integers $m$ such that $m \geq a$. Similarly, we define $\mathbb{Z}_{<a}$, etc. By $\mathfrak{gl}(n)$ we denote the general linear Lie algebra consisting of all $n \times n$ complex matrices, and by $\{E_{i,j} \,|\, 1 \leq i, j \leq n\}$, the standard basis of $\mathfrak{gl}(n)$ of elementary matrices. We fix the standard Cartan subalgebra $\mathfrak{h}$, the standard triangular decomposition and the corresponding basis of simple roots of $\mathfrak{gl}(n)$. The weights of $\mathfrak{gl}(n)$ will be written as $n$-tuples $(\lambda_1, \dots, \lambda_n)$.

For a Lie algebra $\mathfrak{a}$ by $U(\mathfrak{a})$ we denote the universal enveloping algebra of $\mathfrak{a}$. Throughout the paper $U = U(\mathfrak{gl}(n))$. For a commutative ring $R$, by $\operatorname{Specm} R$ we denote the set of maximal ideals of $R$.

We will write the vectors in $\mathbb{C}^{\frac{n(n+1)}{2}}$ in the following form:

$$L = (l_{ij}) = (l_{n1}, \dots, l_{nn} | \dots | l_{21}, l_{22} | l_{11}).$$

For $1 \leq j \leq i \leq n$, $\delta^{ij} \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ is defined by $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{k\ell}$ are zero.

For $i > 0$ by $S_i$ we denote the $i$th symmetric group. Throughout the paper we set $G := S_n \times \dots \times S_1$.

## 3 Gelfand–Tsetlin modules

Recall that $U = U(\mathfrak{gl}(n))$. Let for $m \leqslant n$, $\mathfrak{gl}_m$ be the Lie subalgebra of $\mathfrak{gl}(n)$ spanned by $\{E_{ij} \,|\, i, j = 1, \dots, m\}$. We have the following chain

$$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \dots \subset \mathfrak{gl}_n.$$

It induces the chain $U_1 \subset U_2 \subset \dots \subset U_n$ for the universal enveloping algebras $U_m = U(\mathfrak{gl}_m)$, $1 \leq m \leq n$. Let $Z_m$ be the center of $U_m$. The subalgebra of $U$ generated by $\{Z_m \,|\, m = 1, \dots, n\}$ will be called the (*standard*) *Gelfand–Tsetlin subalgebra* of $U$ and will be denoted by $\Gamma$ [2].

**Definition 3.1.** A finitely generated $U$-module $M$ is called a *Gelfand–Tsetlin module* (*with respect to* $\Gamma$) if

$$M = \bigoplus_{\mathsf{m} \in \operatorname{Specm} \Gamma} M(\mathsf{m}),$$

where $M(\mathsf{m}) = \{v \in M \,|\, \mathsf{m}^k v = 0 \text{ for some } k \geq 0\}$.

For each $\mathbf{m} \in \operatorname{Specm} \Gamma$ we have associated a character $\chi_{\mathbf{m}} : \Gamma \to \Gamma/\mathbf{m} \sim \mathbb{C}$. In the same way, for each non-zero character $\chi : \Gamma \to \mathbb{C}$ we have that $Ker(\chi)$ is a maximal ideal of $\Gamma$. So, we have a natural identification between characters of $\Gamma$ and elements of $\operatorname{Specm}\Gamma$. Using characters we can define Gelfand–Tsetlin modules. A $U$-module $M$ is called *Gelfand–Tsetlin module* (with respect to $\Gamma$) if

$$M = \bigoplus_{\chi\in\Gamma^*} M(\chi),$$

where $M(\chi) = \{v \in M : \forall\, g \in \Gamma, \exists\, k \in \mathbb{Z}_{>0} \text{ such that } (g - \chi(g))^k v = 0\}$. The *Gelfand–Tsetlin support* of $M$ is the set $\operatorname{Supp}_{\mathrm{GT}}(M) := \{\chi \in \Gamma^* : M(\chi) \neq 0\}$.

**Lemma 3.2.** *Any submodule of a Gelfand–Tsetlin module over $\mathfrak{gl}(n)$ is a Gelfand–Tsetlin module.*

**Proof.** The proof is standard, but for a sake of completeness, we provide the important details. Let $M$ be a Gelfand–Tsetlin $\mathfrak{gl}(n)$-module and $N$ any submodule of $M$. We will prove that, if $\{\chi_1, \dots, \chi_k\}$ is a set of distinct Gelfand–Tsetlin characters in $\operatorname{Supp}_{\mathrm{GT}}(M)$ such that $\sum_{i=1}^{k} v_i \in N$ with $v_i \in M(\chi_i)$, then $v_i \in N$ for all $i = 1, \dots, k$.

Without loss of generality we assume that $k = 2$. Since $\chi_1 \neq \chi_2$, there exist $g \in \Gamma$ and $r \leq s$ in $\mathbb{Z}_{\geq 0}$ such that $\chi_1(g) \neq \chi_2(g)$, $(g - \chi_1(g))^r(v_1) = 0$ and $(g - \chi_2(g))^s(v_2) = 0$. Let $a := \chi_1(g)$ and $b := \chi_2(g)$, Then, if $w = v_1 + v_2$ we have $(g - b)^s w = (g - b)^s v_1 \in N$. Let $y := (g - b)^s v_1$. We have that $y \in N$ on one hand and

$$y = ((g - a) + (a - b))^s v_1 = \sum_{k=0}^{r-1} \binom{s}{k} (a - b)^{s-k} (g - a)^k v_1 \in N$$

on the other. As $\binom{s}{k}(a - b)^{s-k} \neq 0$ for any $k$, using that $(g - a)^{r-1} y \in N$, we obtain $(g - a)^{r-1} v_1 \in N$. Reasoning in the same way, from $(g - a)^{r-i} y \in N$, and $(g - a)^{r-1} v_1, \dots, (g - a)^{r-i+1} v_1 \in N$ we obtain $x^{r-i} v_1 \in N$. Hence $v_1 \in N$ and consequently, $v_2 \in N$. ■

One can choose the following generators of $\Gamma$: $\{c_{mk} \,|\, 1 \leq k \leq m \leq n\}$, where

$$c_{mk} = \sum_{(i_1,\dots,i_k)\in\{1,\dots,m\}^k} E_{i_1 i_2} E_{i_2 i_3} \cdots E_{i_k i_1}. \tag{3.1}$$

Let $\Lambda$ be the polynomial algebra in the variables $\{\lambda_{ij} \,|\, 1 \leqslant j \leqslant i \leqslant n\}$. The action of the symmetric group $S_i$ on $\{\lambda_{ij} |\; 1 \leqslant j \leqslant i\}$ induces the action of $G = S_n \times \cdots \times S_1$ on $\Lambda$. There is a natural embedding $\imath : \Gamma \longrightarrow \Lambda$ given by $\imath(c_{mk}) = \gamma_{mk}(\lambda)$, where

$$\gamma_{mk}(\lambda) = \sum_{i=1}^{m} (\lambda_{mi} + m - 1)^k \prod_{j\neq i} \left(1 - \frac{1}{\lambda_{mi} - \lambda_{mj}}\right). \tag{3.2}$$

Hence, $\Gamma$ can be identified with $G$-invariant polynomials in $\Lambda$.

**Remark 3.3.** In what follows, we will identify the set $\operatorname{Specm}\Lambda$ of maximal ideals of $\Lambda$ with the set $\mathbb{C}^{\frac{n(n+1)}{2}}$. Then we have a surjective map $\pi : \operatorname{Specm}\Lambda \to \operatorname{Specm}\Gamma$. Moreover, since $\Lambda$ is integral over $\Gamma$, there are finitely many maximal ideals of $\Lambda$ that map to a fixed maximal ideal of $\Gamma$. The different maximal ideals of $\Lambda$ are obtained from each other under permutations in the group $G$.

If $\pi(\ell) = \mathsf{m}$ for some $\ell \in \operatorname{Specm}\Lambda$, then we write $\ell = \ell_{\mathsf{m}}$ and say that $\ell_{\mathsf{m}}$ is *lying over* $\mathsf{m}$.

# 4 Finite-dimensional modules of $\mathfrak{gl}(n)$

In this section we recall a classical result of Gelfand and Tsetlin which provides an explicit basis for every irreducible finite-dimensional $\mathfrak{gl}(n)$-module.

**Definition 4.1.** For a vector $L = (l_{ij})$ in $\mathbb{C}^{\frac{n(n+1)}{2}}$, by $T(L)$ we will denote the following array with entries $\{l_{ij} : 1 \leq j \leq i \leq n\}$

$$\begin{array}{ccccccc} \boxed{l_{n1}} & \boxed{l_{n2}} & & \cdots & & \boxed{l_{n,n-1}} & \boxed{l_{nn}} \\ & \boxed{l_{n-1,1}} & & \cdots & & \boxed{l_{n-1,n-1}} & \\ & \cdots & & \cdots & & \cdots & \\ & & \boxed{l_{21}} & & \boxed{l_{22}} & & \\ & & & \boxed{l_{11}} & & & \end{array}$$

Such an array will be called a *Gelfand–Tsetlin tableau* of height $n$. A Gelfand–Tsetlin tableau of height $n$ is called *standard* if $l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{>0}$ for all $1 \leq i \leq k \leq n-1$.

Note that, for sake of convenience, the second condition above is slightly different from the original condition in [12].

**Theorem 4.2** ([12])**.** *Let $L(\lambda)$ be the finite-dimensional irreducible module over $\mathfrak{gl}(n)$ of highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then there exist a basis of $L(\lambda)$ consisting of all standard tableaux $T(L) = T(l_{ij})$ with fixed top row $l_{nj} = \lambda_j - j + 1$. Moreover, the action of the generators of $\mathfrak{gl}(n)$ on $L(\lambda)$ is given by the Gelfand–Tsetlin formulas:*

$$E_{k,k+1}(T(L)) = -\sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k+1} (l_{ki} - l_{k+1,j})}{\prod_{j\neq i}^{k} (l_{ki} - l_{kj})} \right) T\big(L + \delta^{ki}\big),$$

$$E_{k+1,k}(T(L)) = \sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k-1} (l_{ki} - l_{k-1,j})}{\prod_{j\neq i}^{k} (l_{ki} - l_{kj})} \right) T\big(L - \delta^{ki}\big),$$

$$E_{kk}(T(L)) = \left( k - 1 + \sum_{i=1}^{k} l_{ki} - \sum_{i=1}^{k-1} l_{k-1,i} \right) T(L), \tag{4.1}$$

*if the new tableau $T(L \pm \delta^{ki})$ is not standard, then the corresponding summand of $E_{k,k+1}(T(L))$ or $E_{k+1,k}(T(L))$ is zero by definition. Furthermore, for $s \leq r$,*

$$c_{rs}(T(L)) = \gamma_{rs}(l) T(L), \tag{4.2}$$

*where $\{c_{rs}\}$ are the generators of $\Gamma$ defined in (3.1) and $\gamma_{rs}$ are defined in (3.2) (see [23]).*

The formulas above are called *Gelfand–Tsetlin formulas* for $\mathfrak{gl}(n)$. These formulas were extended to the case of $U_q(\mathfrak{gl}(n))$ in [19].

# 5 Generic Gelfand–Tsetlin modules of $\mathfrak{gl}(n)$

Theorem 4.2 gives an explicit realization of any irreducible finite-dimensional $\mathfrak{gl}(n)$-module. Using the Gelfand–Tsetlin formulas, Drozd, Futorny and Ovsienko defined the class of infinite-dimensional generic modules for $\mathfrak{gl}(n)$ in [2].

**Definition 5.1.** A Gelfand–Tsetlin tableau $T(L)$ $\big($equivalently, $L \in \mathbb{C}^{\frac{n(n+1)}{2}}\big)$ is called *generic* if $l_{ki} - l_{kj} \notin \mathbb{Z}$ for all $1 \leq i \neq j \leq k \leq n-1$. A character $\chi$ and $\mathsf{n} = \operatorname{Ker}\chi$ are called *generic* if $\ell_{\mathsf{n}}$ is generic for one choice (hence for all choices) of $\ell_{\mathsf{n}}$ lying over $\mathsf{n}$. A Gelfand–Tsetlin module $M$ will be called a *generic Gelfand–Tsetlin module* if every $\mathsf{n}$ in $\operatorname{Supp}_{\mathrm{GT}}(M)$ is generic.

**Theorem 5.2** ([2, Section 2.3] and [18, Theorem 2])**.** *Let $T(L) = T(l_{ij})$ be a generic Gelfand–Tsetlin tableau of height $n$. Denote by $\mathcal{B}(T(L))$ the set of all Gelfand–Tsetlin tableaux $T(R) = T(r_{ij})$ satisfying $r_{nj} = l_{nj}$, $r_{ij} - l_{ij} \in \mathbb{Z}$ for $1 \leq j \leq i \leq n-1$.*

*(i) The vector space $V(T(L)) = \operatorname{span}\mathcal{B}(T(L))$ has a structure of a $\mathfrak{gl}(n)$-module with action of the generators of $\mathfrak{gl}(n)$ given by the Gelfand–Tsetlin formulas (4.1).*

*(ii) The action of the generators of $\Gamma$ on the basis elements of $V(T(L))$ is given by (4.2).*

*(iii) The $\mathfrak{gl}(n)$-module $V(T(L))$ is a Gelfand–Tsetlin module all of whose Gelfand–Tsetlin multiplicities are 1.*

**Remark 5.3.** The basis of the module in the previous theorem is

$$\mathcal{B}(T(L)) = \big\{T(L+z) : z \in \mathbb{Z}^{\frac{n(n+1)}{2}} \text{ and } z_{n1} = \cdots = z_{nn} = 0\big\}.$$

By a slight abuse of notation we will identify elements in $\mathbb{Z}^{\frac{n(n-1)}{2}}$ with elements $z \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ such that $z_{n1} = \cdots = z_{nn} = 0$. This will allow us to write $T(L+z)$ for $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$.

**Remark 5.4.** In what follows, we will apply Lemma 3.2 and use that the elements of $\Gamma$ separate the tableaux in the submodules of $V(T(L))$ in the following sense. Let $N$ be a $\mathfrak{gl}(n)$-submodule of $V(T(L))$, $g \in \mathfrak{gl}(n)$, and $T(R)$ be a tableau in $N$. Then, if $g \cdot T(R) = \sum_i c_i T(R_i)$ for some distinct tableaux $T(R_i)$ in $\mathcal{B}(T(L))$ and nonzero $c_i \in \mathbb{C}$, we have $T(R_i) \in N$ for all $i$.

**Theorem 5.5.** *If $\mathsf{n} \in \operatorname{Specm}\Gamma$ is generic, then there exists a unique irreducible Gelfand–Tsetlin module $N$ such that $N(\mathsf{n}) \neq 0$.*

**Proof.** Let $X_{\mathsf{n}} = U/U\mathsf{n}$. We know that $X_{\mathsf{n}} = U/U\mathsf{n}$ is a Gelfand–Tsetlin module. Furthermore, any irreducible Gelfand–Tsetlin module $M$ with $M(\mathsf{n}) \neq 0$ is a homomorphic image of $X_{\mathsf{n}}$, and $X_{\mathsf{n}}(\mathsf{n})$ maps onto $M(\mathsf{n})$. Since both spaces $X_{\mathsf{n}}(\mathsf{n})$ and $M(\mathsf{n})$ are $\Gamma$-modules then the projection $X_{\mathsf{n}}(\mathsf{n}) \to M(\mathsf{n})$ is a homomorphism of $\Gamma$-modules (see also [11, Corollary 5.3]). Taking into account that $\dim X_{\mathsf{n}}(\mathsf{n}) \leq 1$, we conclude that $X_{\mathsf{n}}$ has a unique maximal submodule (which does not intersect $X_{\mathsf{n}}(\mathsf{n})$) and hence there exist a unique irreducible module $N$ with $N(\mathsf{n}) \neq 0$. ■

**Definition 5.6.** If $T(R)$ is a generic tableau and $\mathsf{r} \in \operatorname{Specm}\Gamma$ corresponds to $R$ then, the unique module $N$ such that $N(\mathsf{r}) \neq 0$ is called the *irreducible Gelfand–Tsetlin module containing* $T(R)$, or simply, the *irreducible module containing* $T(R)$.

Our goal is to describe explicitly the irreducible Gelfand–Tsetlin module containing $T(R)$ for every generic tableau $T(R)$. Below we recall how this is achieved in the case $n = 3$ in [20]. One should note that the methods used in [20] involve direct computations based on a case-by-case consideration, while in the present paper we provide an invariant proof. Also, we reformulate the result in [20] in terms of $T(L+z)$.

For any tableau $T(R) \in \{T(L+z) : z \in \mathbb{Z}^3\}$ and any $1 < p \leq 3$, $1 \leq s \leq p$, and $1 \leq u \leq p-1$, define

$$\Omega^+(T(R)) := \{(p,s,u) : r_{p,s} - r_{p-1,u} \in \mathbb{Z}_{\geq 0}\}.$$

**Theorem 5.7** ([20])**.** *If $T(L)$ is a generic Gelfand–Tsetlin tableau of height 3, then the following is a basis for the irreducible $\mathfrak{gl}(3)$-module containing $T(L)$:*

$$\mathcal{I}(T(L)) := \big\{T(L+z) : z \in \mathbb{Z}^3 \text{ and } \Omega^+(T(L)) = \Omega^+(T(L+z))\big\}.$$

*The action of $\mathfrak{gl}(3)$ on this irreducible module is given by the Gelfand–Tsetlin formulas.*

**Example 5.8.** Consider $a, b, c \in \mathbb{C}$ such that $\{a-b, a-c, b-c\} \bigcap \mathbb{Z} = \varnothing$, $L = (a,b,c|a,b+1|a)$ and

$$T(L)= \begin{array}{ccccc} a & & b & & c \\ & a & & b+1 & \\ & & a & & \end{array}$$

then $\Omega^+(T(L)) = \{(3,1,1),(2,1,1)\}$. So, by Theorem 5.7, the irreducible module containing $T(L)$ has basis

$$\mathcal{I}(T(L)) = \big\{T(L+(m,n,k)) : (m,n,k) \in \mathbb{Z}^3,\ m \leq 0,\ k \leq m,\ \text{and } n > -1\big\}.$$

# 6 Classification of irreducible generic Gelfand–Tsetlin $\mathfrak{gl}(n)$-modules

In this section we prove the main result in the paper, i.e. the generalization of Theorem 5.7 for $\mathfrak{gl}(n)$. For convenience we introduce and recall some notation.

**Notation 6.1.** Let $T(L) = T(l_{ij})$ be a fixed tableau of height $n$.

(i) $\mathcal{B}(T(L)) := \big\{T(L+z) : z \in \mathbb{Z}^{\frac{n(n-1)}{2}}\big\}$.

(ii) $V(T(L)) := \operatorname{span} \mathcal{B}(T(L))$.

(iii) For any $T(R) = T(r_{ij}) \in \mathcal{B}(T(L))$ and for any $1 < p \leq n$, $1 \leq s \leq p$ and $1 \leq u \leq p-1$ we define:

  (a) $\omega_{p,s,u}(T(R)) := r_{p,s} - r_{p-1,u}$;
  (b) $\Omega(T(R)) := \{(p,s,u) : \omega_{p,s,u}(T(R)) \in \mathbb{Z}\}$;
  (c) $\Omega^+(T(R)) := \{(p,s,u) : \omega_{p,s,u}(T(R)) \in \mathbb{Z}_{\geq 0}\}$;
  (d) $\mathcal{N}(T(R)) := \{T(Q) \in \mathcal{B}(T(L)) : \Omega^+(T(R)) \subseteq \Omega^+(T(Q))\}$;
  (e) $W(T(R)) := \operatorname{span} \mathcal{N}(T(R))$;
  (f) $U \cdot T(R)$: the $\mathfrak{gl}(n)$-submodule of $V(T(L))$ generated by $T(R)$.

## 6.1 Basis for the module generated by a single tableau

In order to find an explicit basis of every irreducible generic module, we first find a basis of $U \cdot T(R)$ for any tableau $T(R)$ in $\mathcal{B}(T(L))$.

**Proposition 6.2.** *For any $T(R) \in \mathcal{B}(T(L))$, the Gelfand–Tsetlin formulas endow $W(T(R))$ with a $\mathfrak{gl}(n)$-module structure.*

**Proof.** It is enough to prove $U \cdot T(Q) \subseteq W(T(R))$ for any $T(Q) = T(q_{ij}) \in \mathcal{N}(T(R))$. We will show $g \cdot T(Q)$ is in $W(T(R))$ for every (standard) generator $g$ of $\mathfrak{gl}(n)$.

Suppose $g = E_{k,k+1}$ for some $1 \leq k \leq n-1$. By the Gelfand–Tsetlin formulas, we have

$$E_{k,k+1}(T(Q)) = -\sum_{i=1}^{k} \left( \frac{\prod_{j=1}^{k+1}(q_{ki} - q_{k+1,j})}{\prod_{j\neq i}^{k}(q_{ki} - q_{kj})} \right) T\big(Q + \delta^{ki}\big).$$

If $E_{k,k+1}(T(Q)) \notin W(T(R))$, then there exist $k$ and $i$ such that $T(Q) \in \mathcal{N}(T(R))$ but $T(Q+\delta^{ki}) \notin \mathcal{N}(T(R))$. That implies

$$\Omega^+(T(R)) \subseteq \Omega^+(T(Q)) \text{ and } \Omega^+(T(R)) \nsubseteq \Omega^+\big(T\big(Q + \delta^{ki}\big)\big).$$

Hence, there exists $(p, s, u) \in \Omega^+(T(R))$ such that $\omega_{p,s,u}(T(Q)) \in \mathbb{Z}_{\geq 0}$ and $\omega_{p,s,u}(T(Q + \delta^{ki})) \notin \mathbb{Z}_{\geq 0}$. The latter holds only in two cases:

$$(p, s, u) \in \{(k, i, u), (k + 1, s, i) : 1 \leq u \leq k - 1;\ 1 \leq s \leq k + 1\}.$$

Note that if neither of these two cases hold, we have $\omega_{p,s,u}(T(Q + \delta^{ki})) = \omega_{p,s,u}(T(Q))$. We consider now each of the two cases separately.

(i) Suppose $(p, s, u) = (k, i, u)$. Then $\omega_{k,i,u}(T(Q)) = q_{ki} - q_{k-1,u} \in \mathbb{Z}_{\geq 0}$ and $\omega_{k,i,u}(T(Q+\delta^{ki})) = (q_{ki} + 1) - q_{k-1,u} \notin \mathbb{Z}_{\geq 0}$, which is impossible.

(ii) Suppose $(p, s, u) = (k + 1, s, i)$. Then

$$\omega_{k+1,s,i}(T(Q)) = q_{k+1,s} - q_{ki} \in \mathbb{Z}_{\geq 0}$$

and

$$\omega_{k+1,s,i}(T(Q + \delta^{ki})) = q_{k+1,s} - (q_{ki} + 1) \notin \mathbb{Z}_{\geq 0}.$$

Hence $q_{k+1,s} - q_{k,i} = 0$ and then the coefficient of $T(Q + \delta^{ki})$ in the decomposition of $E_{k,k+1}(T(Q))$ is $-\frac{\prod_{j=1}^{k+1}(q_{ki}-q_{k+1,j})}{\prod_{j\neq i}^{k}(q_{ki}-q_{kj})} = 0$.

Therefore, the tableaux that appear with nonzero coefficients in $E_{k,k+1}(T(Q))$ are elements of $N(T(R))$. Hence, $E_{k,k+1}(T(Q)) \in W(T(R))$. The proof that $E_{k+1,k}(T(Q)) \in W(T(R))$ is analogous to the one of $E_{k,k+1}(T(Q)) \in W(T(R))$. The case $g = E_{kk}$ is trivial because $E_{kk}$ acts as a multiplication by a scalar on $T(Q)$ and $T(Q) \in \mathcal{N}(T(R)) \subseteq W(T(R))$. ■

Given any tableau $T(R)$, there are three modules containing $T(R)$: $V(T(L))$, $W(T(R))$ and $U \cdot T(R)$. We will show that $W(T(R)) = U \cdot T(R)$. For this we need the following lemmas.

**Lemma 6.3.** *Let $T(L)$ be a generic tableau. If $0 \neq z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ is such that $\Omega^+(T(L)) \subseteq \Omega^+(T(L + z))$ then, there exist $i$, $j$ such that $z_{ij} \neq 0$ and*

$$\Omega^+(T(L)) \subseteq \Omega^+\big(T\big(L + z_{ij}\delta^{ij}\big)\big) \subseteq \Omega^+(T(L + z)). \tag{6.1}$$

**Proof.** We will use the following definition in the proof of the lemma.

**Definition 6.4.** Given a generic tableau $T(R) \in \mathcal{B}(T(L))$, a *chain in* $T(R)$ *of length* $\ell$ *starting in row* $d$ is a subset of the entries of $T(R)$, $C = \{r_{d-i,s^{(d-i)}}\}_{i=0,\dots,\ell}$, where $1 \leq s^{(d-i)} \leq d-i$ are such that $r_{d-i,s^{(d-i)}} - r_{d-i-1,s^{(d-i-1)}} \in \mathbb{Z}$ for any $i = 0, \dots, \ell-1$ (i.e. $\{(d-i, s^{(d-i)}, s^{(d-i-1)})\}_{i=0,\dots,\ell} \subseteq \Omega(T(R))$). The chain is called maximal if

(i) $(d+1, i, s^{(d)}) \notin \Omega(T(R))$ for any $1 \leq i \leq d+1$,

(ii) $(d-\ell, s^{(d-\ell)}, j) \notin \Omega(T(R))$ for any $1 \leq j \leq d-\ell-1$.

For every $T(R)$ in $\mathcal{B}(T(L))$ we have that $\Omega^+(T(R)) = \bigsqcup_{1\leq c\leq n} \Omega^+_c(T(R))$, where $\Omega^+_c(T(R)) := \{(p,s,u) \in \Omega^+(T(R)) : p = c\}$. In particular, (6.1) holds if and only if

$$\Omega^+_c(T(L)) \subseteq \Omega^+_c\big(T\big(L + z_{ij}\delta^{ij}\big)\big) \subseteq \Omega^+_c(T(L+z)) \tag{6.2}$$

for any $1 \leq c \leq n$. For $c \notin \{i, i+1\}$ we have $\Omega^+_c(T(L)) = \Omega^+_c(T(L + z_{ij}\delta^{ij}))$. So, in order to verify (6.2), it is enough to consider the cases $c = i, i+1$.

Lets consider $k$, $l$ such that $z_{kl} \neq 0$. Set for convenience $Q := L + z$. There exists a maximal chain $C$ in $T(Q)$ of length $\ell$, starting in row $d$ such that $q_{kl} \in C$. Suppose that $C = \{q_{[i]}\}_{i=0,\dots,\ell}$ where $[i] := (d-i, s^{(d-i)})$. If $\ell = 0$, then $C = \{q_{kl}\}$ and (6.1) is obvious for $z_{ij} = z_{kl}$.

Let $a$ and $b$ be the minimum and maximum of $\{i : z_{[i]} \neq 0\}$, respectively. We have

$$\begin{aligned} &\Omega^+_{d-a+1}\big(T\big(L + z_{[a]}\delta^{[a]}\big)\big) = \Omega^+_{d-a+1}(T(L+z)),\\ &\Omega^+_{d-b}\big(T\big(L + z_{[b]}\delta^{[b]}\big)\big) = \Omega^+_{d-b}(T(L+z)). \end{aligned} \tag{6.3}$$

Therefore (6.2) holds for the pairs $c = d-a+1$, $z_{ij} = z_{[a]}$ and $c = d-b$, $z_{ij} = z_{[b]}$, respectively. Now, let $a \leq m \leq b$ and consider the 4 cases depending on what the signs of $z_{[a]}$ and $z_{[a+1]}$ are.

(i) $z_{[m]} > 0$ and $z_{[m+1]} \leq 0$. In this case (6.2) holds for $c = d-m$ and $z_{ij} = z_{[m]}$. In particular, if $z_{[a]} > 0$ and $z_{[a+1]} \leq 0$, using the first equation in (6.3), we conclude that (6.1) holds for $z_{ij} = z_{[a]}$.

(ii) $z_{[m]} < 0$ and $z_{[m-1]} \geq 0$. In this case (6.2) holds for $c = d-m+1$ and $z_{ij} = z_{[m-1]}$. In particular, if $z_{[b]} < 0$ and $z_{[b-1]} \geq 0$, using the second equation in (6.3) we conclude that (6.1) holds for $z_{ij} = z_{[b]}$.

(iii) $z_{[m]} > 0$ and $z_{[m+1]} > 0$. In this case (6.2) holds for $c = d-m$ and

$$z_{ij} = \begin{cases} z_{[m]} & \text{if } \ l_{[m]} - l_{[m+1]} \in \mathbb{Z}_{\geq 0},\\ z_{[m+1]} & \text{if } \ l_{[m+1]} - l_{[m]} \in \mathbb{Z}_{>0}. \end{cases}$$

(iv) $z_{[m]} < 0$ and $z_{[m-1]} < 0$. In this case (6.2) holds for $c = d-m+1$ and

$$z_{ij} = \begin{cases} z_{[m]} & \text{if } \ l_{[m-1]} - l_{[m]} \in \mathbb{Z}_{\geq 0},\\ z_{[m-1]} & \text{if } \ l_{[m]} - l_{[m-1]} \in \mathbb{Z}_{>0}. \end{cases}$$

Now combining (i)–(iv) we reduce the proof to the following two cases:

(a) $z_{[a]} > 0, z_{[a+1]} > 0, \dots, z_{[b]} > 0$ and for any $t = 1, \dots, b-a$, (6.2) holds for $c = d-a+t+1$ and $z_{ij} = z_{[a+l]}$. In particular, (6.2) holds for $c = d-b+1$ and $z_{ij} = z_{[b]}$. So, by the second equation in (6.3) we have that (6.1) holds for $z_{ij} = z_{[b]}$.

(b) $z_{[b]} < 0, z_{[b-1]} < 0, \dots, z_{[a]} < 0$ and for any $t = 1, \dots, b-a$, (6.2) holds for $c = d-(b-t)$ and $z_{ij} = z_{[b-t]}$. In particular, (6.2) holds for $c = d-a$ and $z_{ij} = z_{[a]}$. So, by the first equation in (6.3) we have that (6.1) holds for $z_{ij} = z_{[a]}$. ■

**Definition 6.5.** Given $T(Q)$ and $T(R)$ in $\mathcal{B}(T(L))$, we write $T(R) \preceq_{(1)} T(Q)$ if there exist $g \in \mathfrak{gl}(n)$ such that $T(Q)$ appears with nonzero coefficient in the decomposition of $g \cdot T(R)$ into a linear combination of tableaux. For any $p \geq 1$ we write $T(R) \preceq_{(p)} T(Q)$ if there exist tableaux $T(L^{(1)})$,..., $T(L^{(p)})$, such that

$$T(R) = T\big(L^{(0)}\big) \preceq_{(1)} T\big(L^{(1)}\big) \preceq_{(1)} \cdots \preceq_{(1)} T\big(L^{(p)}\big) = T(Q).$$

As an immediate consequence of the definition of $\preceq_{(p)}$ we have the following.

**Lemma 6.6.** *If $T(Q)$, $T(Q^{(0)})$, $T(Q^{(1)})$ and $T(Q^{(2)})$ are tableaux in $\mathcal{B}(T(L))$ then:*

*(i)* $T(Q^{(0)}) \preceq_{(p)} T(Q^{(1)})$ *and* $T(Q^{(1)}) \preceq_{(q)} T(Q^{(2)})$ *imply* $T(Q^{(0)}) \preceq_{(p+q)} T(Q^{(2)})$*;*

*(ii)* $T(Q) \preceq_{(1)} T(Q)$.

**Corollary 6.7.** *If $T(R), T(Q) \in \mathcal{B}(T(L))$ are generic Gelfand–Tsetlin tableaux such that $T(R) \preceq_{(p)} T(Q)$ for some $p \in \mathbb{Z}_{\geq 0}$, then $T(Q) \in U \cdot T(R)$.*

**Proof.** By Lemma 5.4 and the definition of the relation $\preceq_{(1)}$, we first verify that $T(R) \preceq_{(1)} T(Q)$ implies $T(Q) \in U \cdot T(R)$. Now, using Lemma 6.6(i), if $T(R) \preceq_{(p)} T(Q)$ for some $p$ then $T(Q) \in U \cdot T(R)$. ■

The next theorem provides a convenient basis for the submodule of $V(T(L))$ generated by a fixed tableau. Recall the definition of $\mathcal{N}(T(R))$ in Notation 6.1(iii)(d).

**Theorem 6.8.** *For any tableau $T(R) \in \mathcal{B}(T(L))$, $U \cdot T(R) = W(T(R))$. In particular, $\mathcal{N}(T(R))$ forms a basis of $U \cdot T(R)$, and the action of $\mathfrak{gl}(n)$ on $U \cdot T(R)$ is given by the Gelfand–Tsetlin formulas.*

**Proof.** By Proposition 6.2, $U \cdot T(R) \subseteq W(T(R))$. To prove that $W(T(R)) \subseteq U \cdot T(R)$ we will show that $T(Q) \in U \cdot T(R)$ for any $T(Q) \in \mathcal{N}(T(R))$. By Corollary 6.7, it is enough to prove that $T(R) \preceq_{(p)} T(Q)$ for some positive integer $p$.

Suppose that $T(Q) = T(R + z) \in \mathcal{N}(T(R))$ for some $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$. Let $t$ be the number of non-zero components of $z$. We will prove that $T(R) \preceq_{(p)} T(Q)$ using induction on $t$.

Let us first consider the case $t = 1$ (the case $t = 0$ is trivial, since then $T(Q) = T(R)$) and $z_{ij} > 0$. We will first prove that $T(R + l\delta^{ij}) \preceq_{(1)} T(R + (l + 1)\delta^{ij})$ for any $0 \leq l \leq z_{ij} - 1$. This will imply

$$T(R) \preceq_{(1)} T\big(R + \delta^{ij}\big) \preceq_{(1)} T\big(R + 2\delta^{ij}\big) \preceq_{(1)} \cdots \preceq_{(1)} T\big(R + z_{ij}\delta^{ij}\big) = T(Q),$$

and then $T(R) \preceq_{(z_{ij})} T(Q)$. To prove that $T(R + l\delta^{ij}) \preceq_{(1)} T(R + (l + 1)\delta^{ij})$ we show that the coefficient of $T(R + (l + 1)\delta^{ij})$ in the decomposition of $E_{i,i+1}(T(R + l\delta^{ij}))$ is not zero. In fact, by the Gelfand–Tsetlin formulas, that coefficient is

$$a_l := -\frac{\prod_{k=1}^{i+1} (r_{ij} - r_{i+1,k} + l)}{\prod_{k \neq j}^{i} (r_{ij} - r_{ik} + l)}.$$

Assume that $a_l = 0$. Then $r_{ij} - r_{i+1,k} + l = 0$ for some $k$, which implies $\omega_{i+1,k,j}(T(R)) = r_{i+1,k} - r_{ij} = l \in \mathbb{Z}_{\geq 0}$. But, since $T(Q) \in \mathcal{N}(T(R))$, we have

$$l - z_{ij} = r_{i+1,k} - r_{ij} - z_{ij} = \omega_{i+1,k,j}(T(Q)) \in \mathbb{Z}_{\geq 0}.$$

Therefore we have $0 \leq l \leq z_{ij} - 1$ and $z_{ij} \leq l$, which is a contradiction. Hence, $T(R) \preceq_{(z_{ij})} T(Q)$.

Let now $t = 1$ and $z_{ij} < 0$. Using the same arguments as in the case $z_{ij} > 0$, we prove that $T(R) \preceq_{(-z_{ij})} T(Q)$ using $|z_{ij}|$ applications of $E_{i+1,i}$. This completes the proof for $t = 1$.

Assume now that for any $w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ with at most $t$ nonzero components, and such that $\Omega^+(T(R)) \subseteq \Omega^+(T(R+w))$, we have $T(R) \preceq_{(p)} T(R+w)$ for some $p$. Let us consider $z$ with $t+1$ nonzero components. Since $\Omega^+(T(R)) \subseteq \Omega^+(T(R+z))$, by Lemma 6.3, there exist $i$, $j$ such that

$$\Omega^+(T(R)) \subseteq \Omega^+\big(T\big(R + z_{ij}\delta^{ij}\big)\big) \subseteq \Omega^+(T(R+z)).$$

Using the induction hypothesis for the pairs of tableaux $(T(R), T(R+z_{ij}\delta^{ij}))$ and $(T(R+z_{ij}\delta^{ij}), T(R+z))$, there exist $p, q \in \mathbb{Z}_{\geq 0}$ such that

$$T(R) \preceq_{(p)} T\big(R + z_{ij}\delta^{ij}\big) \qquad \text{and} \qquad T\big(R + z_{ij}\delta^{ij}\big) \preceq_{(q)} T(R+z).$$

Thus, by Lemma 6.6(i), $T(R) \preceq_{(p+q)} T(R+z)$. ■

**Proposition 6.9.** *Let $T(R)$ and $T(Q)$ be in $\mathcal{B}(T(L))$. Then $U \cdot T(R) = U \cdot T(Q)$ if and only if $\Omega^+(T(Q)) = \Omega^+(T(R))$.*

**Proof.** Using Theorem 6.8 and the definitions of $W(T(R))$, $W(T(Q))$, $\Omega^+(T(R))$, and $\Omega^+(T(Q))$, we can prove a stronger statement: $U \cdot T(R) \subseteq U \cdot T(Q)$ if and only if $\Omega^+(T(Q)) \subseteq \Omega^+(T(R))$. ■

**Corollary 6.10.** *$U \cdot T(R) = V(T(L))$ whenever $\Omega^+(T(R)) = \varnothing$.*

**Definition 6.11.** We will write $T(Q) \sim_{\Omega^+} T(R)$ if $\Omega^+(T(R)) = \Omega^+(T(Q))$.

**Proposition 6.12.** *Every submodule of $V(T(L))$ is finitely generated.*

**Proof.** Let $N$ be any submodule of $V(T(L))$ and $\Phi$ the set of all tableaux $T(R)$ in $N$ such that $\Omega^+(T(P)) \subseteq \Omega^+(T(R))$ implies $\Omega^+(T(P)) = \Omega^+(T(R))$. By Theorem 6.8, $N = \sum\limits_{T(R)\in\Phi} U \cdot T(R)$ and by Proposition 6.9, we can write $N = \bigoplus_{T(R)\in\tilde{\Phi}} U \cdot T(R)$, where $\tilde{\Phi}$ is a set of distinct representatives of $\Phi / \sim_{\Omega^+}$ (hence $\Omega^+(T(R)) \neq \Omega^+(T(Q))$ for any $T(R)$, $T(Q)$ in $\tilde{\Phi}$). Now, since $\Omega(T(L))$ is a finite set, then $\tilde{\Phi}$ is finite. ■

### 6.2 Basis for irreducible modules containing a given tableau

By Theorem 6.8, the module generated by a tableau $T(R)$ has basis $\mathcal{N}(T(R))$. For the purpose of the next theorem let us introduce the following equivalence on $\mathbb{C}^{\frac{n(n+1)}{2}}$.

**Definition 6.13.** We write $z \sim w$ for $z, w \in \mathbb{C}^{\frac{n(n+1)}{2}}$ if and only if one of the two cases hold.

(i) $z - w \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ and $z \sim_{\Omega^+} w$.

(ii) $z \in Gw$.

Now we are ready to formulate and prove the main theorem in the paper.

**Theorem 6.14.** *The irreducible module containing $T(R)$ has a basis of tableaux*

$$\mathcal{I}(T(R)) = \big\{T(Q) \in \mathcal{B}(T(R)) : \Omega^+(T(Q)) = \Omega^+(T(R))\big\}.$$

*The action of $\mathfrak{gl}(n)$ on this irreducible module is given by the Gelfand–Tsetlin formulas (4.1). Therefore the set of irreducible generic Gelfand–Tsetlin modules is in one-to-one correspondence with $\mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}} / \sim$, where $\mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}}$ stands for the set of generic vectors in $\mathbb{C}^{\frac{n(n+1)}{2}}$.*

**Proof.** For each tableau $T(R)$, we have an explicit construction of the module containing $T(R)$ (recall Definition 5.6):

$$M(T(R)) := U \cdot T(R) / \left( \sum U \cdot T(Q) \right),$$

where the sum is taken over tableaux $T(Q)$ such that $T(Q) \in U \cdot T(R)$ and $U \cdot T(Q)$ is a proper submodule of $U \cdot T(R)$.

The module $M(T(R))$ is simple. Indeed, this follows from the fact that for any nonzero tableau $T(S)$ in $M(T(R))$ we have $U \cdot T(S) = U \cdot T(R)$ and, hence, $T(S)$ generates $M(T(R))$.

By Theorem 6.8 and Proposition 6.9, a basis for a proper submodule $U \cdot T(Q)$ of $U \cdot T(R)$ is $\{T(S) : \Omega^+(T(R)) \subsetneq \Omega^+(T(Q)) \subseteq \Omega^+(T(S))\}$ so, a basis for the module $\sum U \cdot T(Q)$ is $\{T(S) : \Omega^+(T(R)) \subsetneq \Omega^+(T(S))\}$. Therefore, $\mathcal{I}(T(R))$ is a basis for $M(T(R))$.

To show that $\mathbb{C}_{\text{gen}}^{\frac{n(n+1)}{2}} / \sim$ parameterizes the set of all irreducible generic Gelfand–Tsetlin modules we use Theorem 5.5 and the fact that $\ell, \ell' \in \operatorname{Specm} \Lambda$ lie over the same $\mathsf{m}$ in $\operatorname{Specm} \Gamma$ if and only if $\ell \in G\ell'$ (see Remark 3.3). ■

# 7 Number of irreducible modules in generic blocks

**Definition 7.1.** For any generic tableau $T(L)$, the *block associated with* $T(L)$ is the set of all Gelfand–Tsetlin $\mathfrak{gl}(n)$-modules with Gelfand–Tsetlin support contained in $\operatorname{Supp}_{\mathrm{GT}}(V(T(L)))$.

Theorem 6.14 describe explicit bases of the irreducible modules in the block associated with $V(T(L))$. In this section we will use this description to compute the number of nonisomorphic irreducible modules in this block.

**Definition 7.2.** For any $T(R) = T(r_{ij}) \in \mathcal{B}(T(L))$, $1 < p \leq n$ and $1 \leq u \leq p-1$, define $d_{pu}(T(R))$ to be the number of distinct elements in

$$\{r_{ps} : (p, s, u) \in \Omega(T(R))\}.$$

**Remark 7.3.** For any generic tableau $T(R) = T(r_{ij}) \in \mathcal{B}(T(L))$ of height $n$ we have:

(i) $d_{pu}(T(L)) = d_{pu}(T(R))$ for any $1 < p \leq n$, $1 \leq u \leq p-1$;

(ii) if $p \neq n$, then $d_{pu}(T(R)) \leq 1$ for any $1 \leq u \leq p-1$.

**Example 7.4.** Suppose $a, b, c \in \mathbb{C}$ are such that $\{a-b, a-c, b-c\} \cap \mathbb{Z} = \varnothing$. If $R = (a, a-1, b|a, b|c)$, then

$$T(R) := \begin{array}{ccccc} \boxed{a} & & \boxed{a-1} & & \boxed{b} \\ & \boxed{a} & & \boxed{b} & \\ & & \boxed{c} & & \end{array}$$

$d_{31}(T(R)) = 2$, $d_{32}(T(R)) = 1$, $d_{21}(T(R)) = 0$ and $d_{22}(T(R)) = 0$.

**Remark 7.5.** For each tableau $T(R)$ we have an one-to-one correspondence between the set $\{0, 1, \ldots, d_{pu}(T(L))\}$ and the subset $\{0, i_1, \ldots, i_{d_{pu}(T(L))}\}$ of $\{0, 1, \ldots, p\}$ defined as follows: $i_1 = 1$ and $i_k = \min\{x : r_{px} \notin \{r_{pi_1}, \ldots, r_{pi_{k-1}}\}\}$.

**Theorem 7.6.** *For any generic tableau $T(L)$, the number of irreducible modules in the block associated with $T(L)$ is*

$$\prod_{1\leq u\leq p-1<n} (d_{pu}(T(L))+1).$$

*In particular, $V(T(L))$ is irreducible if and only if $d_{pu}(T(L))=0$ for any $p$ and $u$, or equivalently, if and only if $\Omega(T(L))=\varnothing$.*

**Proof.** By Theorem 6.14, the irreducible modules are in one-to-one correspondence with the subsets of $\Omega(T(L))$ of the form $\Omega^+(T(L+z))$. For any $T(R)\in\mathcal{B}(T(L))$, we can decompose $\Omega(T(R))$ into a disjoint union $\Omega(T(R))=\bigsqcup_{p,u}\Omega_{pu}(T(R))$, where

$$\Omega_{p,u}(T(R))=\{(p,1,u),(p,2,u),\dots,(p,p,u)\}\cap\Omega(T(R)).$$

Now, if $\Omega^+_{p,u}(T(R)):=\Omega_{p,u}\cap\Omega^+(T(R))$, one can write $\Omega^+(T(R))=\bigsqcup_{p,u}\Omega^+_{pu}(T(R))$. For $p$, $u$ fixed, let us denote by $s_{p,u}$ the number of different subsets of the form $\Omega^+_{p,u}(T(R))$. So, the number of different subsets of the form $\Omega^+(T(R))$ is $\prod\limits_{p,u} s_{p,u}$.

Let $\{T(R^{(i)})\}_{i=1}^{s_{pu}}$ be a set of tableaux such that $\{\Omega^+_{p,u}(T(R^{(i)}))\}_{i=1}^{s_{pu}}$ is the set of all distinct sets of the form $\Omega^+_{p,u}(T(R))$. We have a one-to-one correspondence between $\{T(R^{(i)})\}_{i=1}^{s_{pu}}$ and the set $\{0,i_1,\dots,i_{d_{pu}(T(L))}\}$ constructed as in Remark 7.5. More explicitly, this correspondence is defined my the map:

$$T(R^{(i)})\to\begin{cases}\min\{j:(p,j,u)\in\Omega^+(T(R^{(i)}))\}, & \text{if } \ \Omega^+_{pu}(T(R^{(i)}))\neq\varnothing,\\ 0, & \text{if } \ \Omega^+_{pu}(T(R^{(i)}))=\varnothing.\end{cases}$$

Therefore, $s_{pu}=d_{pu}(T(L))+1$. ■

### Acknowledgements

V.F. gratefully acknowledges the hospitality and excellent working conditions at the CRM, University of Montreal, where part of this work was completed. V.F. is supported in part by the CNPq grant (301320/2013-6) and by the Fapesp grant (2014/09310-5). D.G. is supported in part by the Fapesp grant (2011/21621-8) and by the NSA grant H98230-13-1-0245. L.E.R. is supported by the Fapesp grant (2012/23450-9).

## References

[1] Britten D.J., Lemire F.W., Futorny V.M., Simple $A_2$-modules with a finite-dimensional weight space, *Comm. Algebra* **23** (1995), 467–510.

[2] Drozd Yu.A., Futorny V.M., Ovsienko S.A., Harish-Chandra subalgebras and Gel'fand–Zetlin modules, in Finite-Dimensional Algebras and Related Topics (Ottawa, ON, 1992), *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, Vol. 424, Kluwer Acad. Publ., Dordrecht, 1994, 79–93.

[3] Drozd Yu.A., Ovsienko S.A., Futorny V.M., Irreducible weighted sl(3)-modules, *Funct. Anal. Appl.* **23** (1989), 217–218.

[4] Drozd Y.A., Ovsienko S.A., Futorny V.M., Gel'fand–Zetlin modules over Lie algebra SL(3), in Proceedings of the International Conference on Algebra, Part 2 (Novosibirsk, 1989), *Contemp. Math.*, Vol. 131, Amer. Math. Soc., Providence, RI, 1992, 23–29.

[5] Fernando S.L., Lie algebra modules with finite-dimensional weight spaces. I, *Trans. Amer. Math. Soc.* **322** (1990), 757–781.

[6] Futorny V.M., A generalization of Verma modules, and irreducible representations of the Lie algebra sl(3), *Ukrain. Math. J.* **38** (1986), 422–427.

[7] Futorny V.M., Weight representations of semi-simple finite-dimensional Lie algebras, Ph.D. Thesis, Kiev University, 1986.

[8] Futorny V.M., Irreducible sl(3)-modules with infinite-dimensional weight subspaces, *Ukrain. Math. J.* **41** (1989), 856–859.

[9] Futorny V.M., Weighted sl(3)-modules generated by semiprimitive elements, *Ukrain. Math. J.* **43** (1991), 250–254.

[10] Futorny V.M., Grantcharov D., Ramirez L.E., Classification of irreducible Gelfand–Tsetlin modules for $\mathfrak{sl}(3)$, in preparation.

[11] Futorny V.M., Ovsienko S.A., Fibers of characters in Gel'fand–Tsetlin categories, *Trans. Amer. Math. Soc.* **366** (2014), 4173–4208, math.RT/0610071.

[12] Gel'fand I.M., Tsetlin M.L., Finite-dimensional representations of the group of unimodular matrices, *Dokl. Akad. Nauk USSR* **71** (1950), 825–828.

[13] Graev M.I., Infinite-dimensional representations of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ related to complex analogs of the Gelfand–Tsetlin patterns and general hypergeometric functions on the Lie group $\mathrm{GL}(n,\mathbb{C})$, *Acta Appl. Math.* **81** (2004), 93–120.

[14] Graev M.I., A continuous analogue of Gelfand–Tsetlin schemes and a realization of the principal series of irreducible unitary representations of the group $\mathrm{GL}(n,\mathbb{C})$ in the space of functions on the variety of these schemes, *Dokl. Math.* **75** (2007), 31–35.

[15] Kostant B., Wallach N., Gelfand–Zeitlin theory from the perspective of classical mechanics. I, in Studies in Lie Theory, *Progr. Math.*, Vol. 243, Birkhäuser Boston, Boston, MA, 2006, 319–364, math.SG/0408342.

[16] Kostant B., Wallach N., Gelfand–Zeitlin theory from the perspective of classical mechanics. II, in The Unity of Mathematics, *Progr. Math.*, Vol. 244, Birkhäuser Boston, Boston, MA, 2006, 387–420, math.SG/0501387.

[17] Mathieu O., Classification of irreducible weight modules, *Ann. Inst. Fourier (Grenoble)* **50** (2000), 537–592.

[18] Mazorchuk V., On categories of Gelfand-Zetlin modules, in Noncommutative Structures in Mathematics and Physics (Kiev, 2000), *NATO Sci. Ser. II Math. Phys. Chem.*, Vol. 22, Kluwer Acad. Publ., Dordrecht, 2001, 299–307.

[19] Nazarov M., Tarasov V., Yangians and Gelfand–Zetlin bases, *Publ. Res. Inst. Math. Sci.* **30** (1994), 459–478, hep-th/9302102.

[20] Ramirez L.E., Combinatorics of irreducible Gelfand–Tsetlin $\mathfrak{sl}(3)$-modules, *Algebra Discrete Math.* **14** (2012), 276–296.

[21] Ramirez L.E., Classificação dos $\mathfrak{sl}(3)$-módulos de Gelfand–Tsetlin irredutíveis, Ph.D. Thesis, Universidade de São Paulo, 2013.

[22] Vinberg E.B., Some commutative subalgebras of a universal enveloping algebra, *Math. USSR Izvestiya* **36** (1991), 1–22.

[23] Zhelobenko D.P., Compact Lie groups and their representations, *Translations of Mathematical Monographs*, Vol. 40, Amer. Math. Soc., Providence, R.I., 1973.