Dynamics on Wild Character Varieties*

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Abstract. In the present paper, we will first present briefly a general research program about the study of the "natural dynamics" on character varieties and wild character varieties. Afterwards, we will illustrate this program in the context of the Painlevé differential equations $P_{\rm VI}$ and $P_{\rm V}$.

Key words: character varieties; wild fundamental groupoid; Painlevé equations

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To Juan J. Morales-Ruiz, for his 60^{th} *birthday.*

1 A sketch of a program

We begin with the sketch of a *work in progress* of the authors in collaboration with Julio Rebelo, based on (or related to) some results due to several people, mainly: Ph. Boalch [3, 5, 6, 7, 8], S. Cantat, F. Loray [10], B. Dubrovin [13, 14], M.A. Inaba, K. Iwasaki [16], M. Jimbo [17], B. Malgrange [21, 22, 23, 24, 25], M. Mazzoco, T. Miwa, M. van der Put, M.-H. Saito [29], K. Ueno [28], E. Witten [31], ..., and the Kyoto school around T. Kawai and Y. Takei [18, 19].

In the present state it is mainly a *PROGRAM.*

We would like to understand:

1. The *dynamics* and the *wild dynamics*[1] of equations of *isomonodromic deformations* and of *wild isomonodromic deformations* using the (*generalized*) *Riemann–Hilbert correspondances* and the corresponding (wild) dynamics on the (*wild*) *character varieties.* The notion of wild character variety was introduced by Boalch. The braid group action on character varieties for the Painlevé equations has been first defined by Dubrovin and Mazzocco for special parameters in [14], and by Iwasaki [16] in the general case.
2. The *confluence phenomena* for the equations of (wild) isomonodromic deformations and the corresponding confluence phenomena for the (wild) dynamics.

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[1] That is, roughly speaking, the ordinary dynamics coming from the nonlinear monodromy "plus" the dynamics coming from "nonlinear Stokes phenomena".

Our (long term!) aim is to built a *general theory*, testing it at each step on the case of the *Painlevé equations* (which is already far to be trivial).

Our initial motivation was to compute the *Malgrange groupoids* of the six Painlevé equations. Our conjecture is that it is *the biggest possible* (that is the groupoid of transformations conserving the area) in the generic cases (it is known for P_{I}: see Casale in [11], for P_{VI}: Cantat–Loray in [10], and for special parameters of P_{II} and P_{III}: Casale and Weil [12]). Using a result of Casale [11], it is possible to reprove, in "Painlevé style", the *irreducibility* of Painlevé equations (initially proved by the japanese school: Nishioka, Umemura, Okamoto, Noumi, ...). Our approach is to try to define in each case, using the Riemann–Hilbert map, a *wild dynamics* on the Okamoto variety of initial conditions and to prove that this dynamics is "chaotic", forcing Malgrange groupoid to be big (up to the conjecture that the wild dynamics is "into" the Malgrange groupoid).

The classical character varieties are moduli spaces of monodromy data of regular-singular connections, that is spaces of representations of the fundamental group of a punctured (or not) Riemann surface. Atiyah-Bott and Goldman prove that they admit holomorphic Poisson structure. This fact has been extended to wild character varieties by Boalch (see [4, 7]).

The wild character varieties generalize the classical (or *tame*) character varieties. They are moduli spaces of generalized monodromy data of meromorphic connections. In the irregular case it is necessary to add "Stokes data" to the classical monodromy. Then the wild character varieties are spaces of representations of a wild fundamental *groupoid*.

In the global irregular case it is *necessary* to use groupoids. They are explicitely used in [4], and implicitely used in [17, 31]. In the local irregular case it is sufficient to use a group, the Ramis wild fundamental group [26, 30].

Therefore in order to understand the confluence process of a classical representation of the fundamental group towards a representation of the wild groupoid, it is better to replace the classical fundamental group by a groupoid. This is a posteriori clear in the computations of [27] in the hypergeometric case. We plan to return to such problems in future papers.

We will show below, with the example of P_{VI}, that even in the classical case it is better to use fundamental groupoids than fundamental groups to study character varieties and their *natural dynamics*. This is in a line strongly suggested by Alexander Grothendieck.

Ceci est lié notamment au fait que les gens s'obstinent encore, en calculant avec des groupes fondamentaux, à fixer un seul point base, plutôt que d'en choisir astucieusement tout un paquet qui soit invariant par les symétries de la situation, lesquelles sont donc perdues en route. Dans certaines situations (comme des théorèmes de descente à la Van Kampen pour les groupes fondamentaux) il est bien plus élégant, voire indispensable pour y comprendre quelque chose, de travailler avec des groupoïdes fondamentaux par rapport à un paquet de points base convenable, et il en est certainement ainsi pour la tour de Teichmüller (cf. [15]).

... people are accustomed to work with fundamental groups and generators and relations for these and stick to it, even in contexts when this is wholly inadequate, namely when you get a clear description by generators and relations only when working simultaneously with a whole bunch of base-points chosen with care – or equivalently working in the algebraic context of groupoids, rather than groups. Choosing paths for connecting the base points natural to the situation to one among them, and reducing the groupoid to a single group, will then hopelessly destroy the structure and inner symmetries of the situation, and result in a mess of generators and relations no one dares to write down, because everyone feels they won't be of any use whatever, and just confuse the picture rather than clarify it. I have known such perplexity myself a long time ago, namely in Van Kampen type situations, whose only understandable formulation is in terms of (*amalgamated sums of*) *groupoids* (Alexandre Grothendieck, quoted by Ronald Brown[2]).

We recall that a groupoid is a small category in which every morphism is an isomorphism (for basic definitions and details cf. [9]).

[2] http://pages.bangor.ac.uk/~mas010/pstacks.htm.

Example 1.1. Let Y be a (topological) manifold. The fundamental groupoid $\pi_1(Y)$ of Y is the groupoid whose objects are the elements y of Y, and whose morphisms are the paths between elements of Y up to homotopy.

We have the following generalization.

Definition 1.2. Let Y be a (topological) manifold and $S \subset Y$. The fundamental groupoid $\pi_1(Y, S)$ of the pair (Y, S) is the groupoid whose objects are the elements s of S in Y, and whose morphisms are the paths between elements of S up to homotopy.

We have $\pi_1(Y) = \pi_1(Y, Y)$ and when S is reduced to a point a, $\pi_1(Y, \{a\})$ is the classical fundamental group of Y based at a: $\pi_1(Y, \{a\}) = \pi_1(Y, a)$.

2 Character varieties in the Painlevé context: the regular singular case revisited

We focus now on the "Painlevé context": let $\mathcal{E}$ be the set of linear rank 2 connections on the trivial bundle over $\mathbb{P}_1(\mathbb{C})$, with coefficients in $\mathfrak{sl}_2(\mathbb{C})$ and such that its singular locus contains at most 4 singular points. In this section, we first consider the "classical" case with 4 *regular* singular points, in order to be more familiar with the groupoid point of view which is essential to deal with the irregular cases. Furthermore it turns out that this point of view is yet useful to obtain the dynamics in the regular singular case. In this section, Δ is a linear differential system which represents the connection ∇.

2.1 The fundamental groupoid

We consider the "extended" singular locus S of Δ: S is the set of pairs $s = (p, d)$ where p is a singular point of Δ and d is a ray based in p. Therefore s is also a point on the divisor D_p of the real blowing up E_p at p. Let X be the manifold obtained by the real blowing up of each singular point. We denote by $\gamma_{s,s}$ a loop from s to s in X with positive orientation, homotopic to the exceptional divisor:

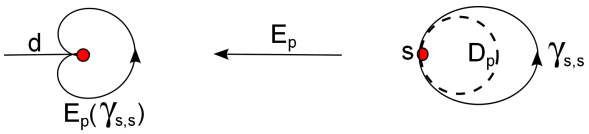

Figure 1. The real blowing up at p.

Definition 2.1. The fundamental groupoid $\pi_1(X, S)$ is the groupoid whose objects are the elements s of S in X, and whose morphisms are the paths between elements of S up to homotopy. The subgroupoid $\pi_1^{\text{loc}}(X, S)$ is the groupoid with same objects, whose morphisms are generated only by the loops $\gamma_{i,i}$ homotopic to each exceptional divisor at p_i.

We denote:

- $\text{Aut}(\pi_1(X, S))$ the group of the automorphisms of the groupoid, and $\text{Aut}_0(\pi_1(X, S))$ the subgroup of the "pure" automorphisms, which fix each object.
- $\text{Inn}_0(\pi_1(X, S))$ the normal subgroup of $\text{Aut}_0(\pi_1(X, S))$ of the inner automorphisms. An inner automorphism is defined by a collection of loops α_i at each object, by setting:

$$h_{\{\alpha_i\}} : \ \gamma_{i,j} \mapsto \alpha_i \gamma_{i,j} \alpha_j^{-1}.$$

– $\mathrm{Out}_0(\pi_1(X,S)) := \mathrm{Aut}_0(\pi_1(X,S))/\mathrm{Inn}_0(\pi_1(X,S))$.

– $\mathrm{Out}_0^*(\pi_1(X,S))$:= the subgroup of $\mathrm{Out}_0(\pi_1(X,S))$ whose elements fix each local morphism $\gamma_{i,i}$ up to conjugation.

We obtain the following presentation of $\pi_1(X,S)$:

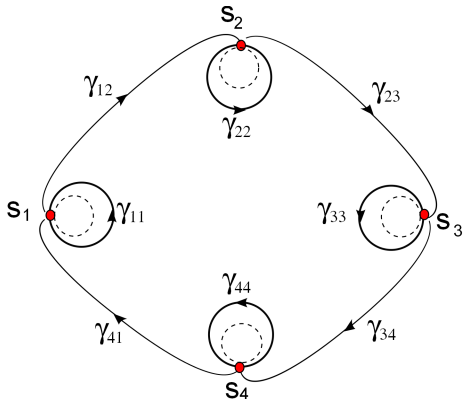

Figure 2. The groupoid $\pi_1(X,S)$ (classical case).

The exceptional divisors are circles in dotted lines. The morphisms are generated by 8 paths $\gamma_{i,i}$ and $\gamma_{i,i+1}$, $i=1,\dots,4$, (indexation modulo 4). On $\mathbb{P}_1(\mathbb{C})$, we have two relations, an exterior one and an interior one, namely,

$$r_{\mathrm{ext}}:\ \gamma_{1,2}\gamma_{2,3}\gamma_{3,4}\gamma_{4,1} = \star_1 \quad \text{(the trivial loop based in } s_1),$$
$$r_{\mathrm{int}}:\ \gamma_{1,1}\gamma_{1,2}\gamma_{2,2}\gamma_{2,3}\gamma_{3,3}\gamma_{3,4}\gamma_{4,4}\gamma_{4,1} = \star_1.$$

The local fundamental subgroupoid is generated by the loops $\gamma_{i,i}$, and is a disjoint union of four monogeneous groups.

Representations of the groupoid $\boldsymbol{\pi_1(X,S)}$. A representation of $\pi_1(X,S)$ in a group G is a morphism of groupoids ρ from $\pi_1(X,S)$ into G. The group G is here a groupoid with only one object, whose morphisms are the elements of G. Therefore ρ is characterized by its action on the morphisms of $\pi_1(X,S)$.

Analytic representations of the groupoid $\boldsymbol{\pi_1(X,S)}$ in $\boldsymbol{G}$ induced by a connection $\boldsymbol{\nabla}$ in $\boldsymbol{\mathcal{E}}$.

– For each object $s=(p,d)$, we consider a fundamental system of holomorphic solutions X_s in a neighborhood of s in X, i.e., in a small sector at p around the direction d, admitting an asymptotic expansion at p.

– At each morphism $\gamma_{i,j}$ joining s_i to s_j, corresponds a connection matrix $M_{i,j}$ between the fundamental systems of solutions X_i and X_j chosen at s_i and s_j defined by

$$X_j = \widetilde{X_i}^{\gamma_{i,j}} M_{i,j}, \tag{2.1}$$

where $\widetilde{X_i}^{\gamma_{i,j}}$ is the analytic continuation of X_i along $\gamma_{i,j}$. With this notation

$$\rho(\gamma_{i,j}\gamma_{j,k}) = \rho(\gamma_{i,j})\rho(\gamma_{j,k}).$$

2.2 The character variety

Definition 2.2. Let $\rho\colon \pi_1(X,S)\to G$ and $\rho'\colon \pi_1(X,S)\to G$ be two analytic representations of $\pi_1(X,S)$. Let $\rho(\gamma_{i,j}) = M_{i,j}$, and $\rho'(\gamma_{i,j}) = M'_{i,j}$, where $\gamma_{i,j}$ is a morphism from s_i to s_j. The

two representations ρ and ρ' are equivalent if and only if for each object s_i, there exists N_i in G such that

$$M'_{i,j} = N_i M_{i,j} N_j^{-1}.$$

Therefore, if we change the choice of the fundamental system attached to each object s_i, we obtain a new equivalent representation. All the representations induced by Δ are equivalent. Furthermore, if Δ and Δ' are gauge equivalent, their representations are equivalent. The class $[\rho]$ only depends on ∇.

Definition 2.3. Let $\mathcal{R}(S)$ (resp. $\mathcal{R}(S)^{\mathrm{loc}}$) be the space of the analytic representations of $\pi_1(X,S)$ (resp. of $\pi_1^{\mathrm{loc}}(X,S)$) induced by some Δ in $\mathcal{E}$, and $\sim$ the above equivalence relation on $\mathcal{R}(S)$.

- The character variety $\chi(S)$ is the quotient $\mathcal{R}(S)/\!\sim$.
- In the same way, the local character variety is $\chi(S)^{\mathrm{loc}} = \mathcal{R}(S)^{\mathrm{loc}}/\!\sim$. The morphism π from $\chi(S)$ to $\chi(S)^{\mathrm{loc}}$ is induced by restriction of the representations to $\pi_1^{\mathrm{loc}}(X,S)$.

Normalized representations. We construct a "good" representative of $[\rho]$ in $\chi(S)$ by using the following process:

– We choose freely a fundamental system of solutions X_1 in s_1.

– In s_2, we choose X_2 to be the analytic continuation of X_1 along the path $\gamma_{1,2}$, then we choose X_3 by analytic continuation of X_2 along $\gamma_{2,3}$, and finally X_4 by analytic continuation along $\gamma_{3,4}$. With these choices, we have

$$\rho(\gamma_{1,2}) = \rho(\gamma_{2,3}) = \rho(\gamma_{3,4}) = I,$$

and from the exterior relation, we obtain $\rho(\gamma_{1,4}) = I$. The representation ρ is now characterized by 4 matrices $M_{i,i} = \rho(\gamma_{i,i})$. From the interior relation, we have

$$M_{1,1}M_{2,2}M_{3,3}M_{4,4} = I.$$

A change in the initial choice of X_1 will give rise to 4 matrices related to the previous ones by a common conjugation. Finally, we have characterized $[\rho]$ in $\chi(S)$ by the data of 3 matrices $M_{i,i}$, $i = 1,2,3$ up to a common conjugation, as in the usual presentation with only one base point. Nevertheless, this groupoid point of view is more convenient, first for computing the isomonodromic dynamics, but also to get an extension to the irregular cases.

2.3 The character variety in trace coordinates

We will now describe the affine algebraic structure of $\chi(S)$ thanks to the following lemma:

Lemma 2.4 (Fricke lemma [20])**.** *Given* 3 *matrices* M_1, M_2 *and* M_3 *in* $\mathrm{SL}(2,\mathbb{C})$ *we denote*

$$a_i = \mathrm{tr}(M_i), \qquad x_{i,j} = \mathrm{tr}(M_i M_j), \qquad x_{i,j,k} = \mathrm{tr}(M_i M_j M_k),$$

where the indices are 2 *by* 2 *distincts. Since the trace map is invariant under cyclic permutations, we have* 3 *coordinates* a_i, 3 *coordinates* $x_{i,j}$ *and* 2 *coordinates* $x_{i,j,k}$. *We have the following relations*

$$\begin{aligned} x_{1,2,3} + x_{1,3,2} &= a_1x_{2,3} + a_2x_{3,1} + a_3x_{1,2} - a_1a_2a_3 := P, \\ x_{1,2,3} \times x_{1,3,2} &= a_1^2 + a_2^2 + a_3^2 + x_{1,2}^2 + x_{2,3}^2 + x_{3,1}^2 + x_{1,2}x_{2,3}x_{3,1} \\ &\quad - a_1a_2x_{1,2} - a_2a_3x_{2,3} - a_3a_1x_{3,1} - 4 := Q. \end{aligned}$$

Therefore, $x_{1,2,3}$ *and* $x_{1,3,2}$ *are the two solutions of the equation* $X^2 - PX + Q = 0$.

Let $M_4 = (M_1M_2M_3)^{-1}$ and $a_4 = \mathrm{tr}(M_4)$. We have $a_4 = x_{1,2,3}$. From the Fricke lemma, we obtain the following relation in $\mathbb{C}^4 \times \mathbb{C}^3$

$$F\colon\ a_4^2 - Pa_4 + Q = 0.$$

We call it the "Fricke hypersurface". It is a quartic in $\mathbb{C}^7$ endowed with the coordinates $(a_1, a_2, a_3, a_4, x_{1,2}, x_{2,3}, x_{3,1})$ and, with respect to the last 3 coordinates, a family of cubics F_a indexed by $a = (a_1, a_2, a_3, a_4)$.

By applying this lemma to the matrices $M_{i,i} = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$ for $i = 1, 2, 3$ of a normalized representation, the trace coordinates $(a_1, a_2, a_3, a_4, x_{1,2}, x_{2,3}, x_{3,1})$ define a morphism

$$T\colon\ \chi(S) \to F.$$

We consider the open set $\chi(S)^*$ defined by the following conditions:

(i) each matrix $M_{i,i}$ is semi-simple;

(ii) one of them (say $M_{1,1}$) is different from $\pm I$;

(iii) the two others satisfy $\beta_2\gamma_2\beta_3\gamma_3 \neq 0$.

We set $\chi(S)^{*\mathrm{loc}} = \pi^*\chi(S)^*$.

Proposition 2.5. *The morphism T is an isomorphism from $\chi(S)^*$ onto $F^* := (a_1 \neq \pm 2)$. The restriction of T on $\chi(S)^{*\mathrm{loc}}$ is an isomorphism onto $\mathbb{C}\backslash\{\pm 2\} \times \mathbb{C}^3$ and we have $T\pi = p_1T$, where p_1 is the first projection $\mathbb{C}^4 \times \mathbb{C}^3 \to \mathbb{C}^4$.*

Proof. By using a conjugation, we may suppose that

$$M_{1,1} = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_1^{-1} \end{pmatrix} \quad \text{with} \quad \alpha_1 \neq \pm 1, \qquad M_{i,i} = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \quad \text{for} \quad i = 2, 3.$$

This writing is not still unique: we may use a conjugation by a diagonal matrix D. Since the center do not act, we may suppose that $\det(D) = 1$, i.e.,

$$D = D_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

Lemma 2.6. *Under the conditions (i), (ii) and (iii) defining $\chi(S)^*$, two triples (M_1, M_2, M_3) and (M_1', M_2', M_3') are in the same orbit for the action of the group $\{D_\alpha,\ \alpha \in \mathbb{C}^*\}$ if and only if*

$$\alpha_1 = \alpha_1', \qquad \alpha_2 = \alpha_2', \qquad \delta_2 = \delta_2', \qquad \alpha_3 = \alpha_3', \qquad \delta_3 = \delta_3',$$
$$\beta_2\gamma_3 = \beta_2'\gamma_3', \qquad \gamma_2\beta_3 = \gamma_2'\beta_3'.$$

Proof. Since

$$D_\alpha M_{i,i} D_\alpha^{-1} = \begin{pmatrix} \alpha_i & \alpha^2\beta_i \\ \alpha^{-2}\gamma_i & \delta_i \end{pmatrix},$$

the condition is necessary. Suppose now that this condition holds for two triples. Since $\beta_2\gamma_3 \neq 0 \neq \beta_2'\gamma_3'$ we can choose α such that $\alpha^2 = \frac{\beta_2'}{\beta_2} = \frac{\gamma_3}{\gamma_3'}$. We also have $\beta_2\gamma_2 = 1 - \alpha_2\delta_2 = 1 - \alpha_2'\delta_2' = \beta_2'\gamma_2'$. Therefore $\alpha^{-2} = \frac{\gamma_2'}{\gamma_2} = \frac{\beta_3}{\beta_3'}$, which proves that $D_\alpha M_{i,i} D_\alpha^{-1} = M_{i,i}'$. ∎

Now we have to solve in SL_2 the system

$$\alpha_1 + \alpha_1^{-1} = a_1, \tag{2.2a}$$
$$\alpha_2 + \delta_2 = a_2, \tag{2.2b}$$
$$\alpha_3 + \delta_3 = a_3, \tag{2.2c}$$
$$\alpha_1\alpha_2 + \alpha_1^{-1}\delta_2 = x_{1,2}, \tag{2.2d}$$
$$\alpha_1\alpha_3 + \alpha_1^{-1}\delta_3 = x_{1,3}, \tag{2.2e}$$
$$\alpha_2\alpha_3 + \beta_2\gamma_3 + \gamma_2\beta_3 + \delta_2\delta_3 = x_{2,3}, \tag{2.2f}$$
$$\alpha_1\alpha_2\alpha_3 + \alpha_1\beta_2\gamma_3 + \alpha_1^{-1}\gamma_2\beta_3 + \alpha_1^{-1}\delta_2\delta_3 = x_{1,2,3}. \tag{2.2g}$$

We choose one of the two solutions of equation (2.2a). Since $\alpha_1 - \alpha_1^{-1} \neq 0$, we obtain from equations (2.2b), (2.2c), (2.2d) and (2.2e) a unique solution for α_2, δ_2, α_3, δ_3. Equations (2.2f), (2.2g) define a linear system in the 2 variables $(\beta_2\gamma_3, \gamma_2\beta_3)$ of maximal rank if $\alpha_1 - \alpha_1^{-1} \neq 0$. We obtain a unique solution for α_2, δ_2, α_3, δ_3, $\beta_2\gamma_3$, $\gamma_2\beta_3$, and therefore a unique triple (M_1, M_2, M_3) up to conjugation according to the preliminary remark. Note that this triple is not necessarily in SL_2: the compatibility condition corresponds to the Fricke relation.

Now if we begin with the second solution of (2.2a), the new matrix M_1' satisfies $M_1' = PM_1P^{-1}$, where P is the matrix of the transposition. The system (2.2b)–(2.2g) has a unique solution M_i' under the same assumption $\alpha_1 - \alpha^{-1} \neq 0$. Since we know that PM_iP^{-1} is another pre-image for T, we have: $M_i' = PM_iP^{-1}$. Therefore this second solution is conjugated to the first one by P, and we obtain a unique pre-image of a point in F^* in $\chi(S)^*$. For the second part of the statement, if each matrix M_i is a semi-simple one, the trace of M_i characterizes the conjugation class of M_i in SL_2. ■

2.4 The dynamics on $\chi(S)$

We set

$$\chi = \cup_{S\in\mathcal{C}}\chi(S),$$

where S belongs to the space $\mathcal{C}$ of the configurations of 4 distinct points in the plane. This fibration is endowed with a flat connection (the isomonodromic connection) whose local trivialisations are defined by identifying the generators $\gamma_{i,j}(S)$ and $\gamma_{i,j}(S')$, for S' near from S. We want to compute the monodromy of this connection on a fiber $\chi(S)$.

The fundamental group $\pi_1(\mathcal{C}, [S])$ is the pure braid group P_4. It is generated by the 3 elements b_1, b_2, b_3, where b_i is the pure braid between s_i and s_{i+1}, with the relation $b_1b_2b_3 = \mathrm{id}$ (note that the cross ratio induces an isomorphism from $\mathcal{C}$ on $P^1(C)\backslash\{0, 1, \infty\}$).

The generators b_i induce an isomorphism from P_4 to the mapping class group of the disc punctured by 4 holes, with a base point on their boundaries. This interpretation allows us to construct an action from P_4 on the groupoid $\pi_1(X, S)$. We denote by h_1, h_2, h_3 the images of the braids in $\mathrm{Aut}_0(\pi_1(X, S))$.

The automorphisms h_i act on $\mathcal{R}$ by $h_{i*}\colon \rho \mapsto \rho \circ h_i$ and an inner automorphism sends ρ on an equivalent representation. Therefore each $[h_i]$ in $\mathrm{Out}_0^*(\pi_1(X, S))$ acts on $\chi(S)$.

Looking at the picture of the groupoid, we immediately obtain:

Proposition 2.7.

1) $h_1(\gamma_{i,i}) = \gamma_{i,i}$, $i = 1, \ldots, 4$;
2) $h_1(\gamma_{3,2}) = \gamma_{3,2}\gamma_{2,1}\gamma_{1,1}\gamma_{1,2}\gamma_{2,2}$;
3) $h_1(\gamma_{1,2}) = \gamma_{1,2}$, $h_1(\gamma_{3,4}) = \gamma_{3,4}$.

We have similar expressions for h_2 and h_3 by cyclic permutations of the indices.

Now we compute the action h_{i*} on $\chi(S)$ in three steps. Let $[\rho]$ in $\chi(S)$ given by a normalized representation ρ, and therefore by 3 matrices M_i up to a common conjugation. For each generator b_i,

1) we compute $\rho \circ h_i$ on the generating morphisms $\gamma_{i,j}$. The representation $\rho \circ h_i$ is not yet a normalized one;
2) we normalize $\rho \circ h_i$ in a new equivalent representation ρ', by changing the representation of the objects. Let M_i' be the matrices related to ρ';
3) we compute $\mathrm{tr}(M_i'M_j')$ as expressions in the $\mathrm{tr}(M_iM_j)$'s in order to write h_{i*} in the trace coordinates a_i, $x_{1,2}$, $x_{2,3}$, $x_{3,1}$.

For this last step, we will make use of an extended version of the Fricke lemma:

Lemma 2.8 (extended Fricke lemma)**.** *We follow the notations of the Fricke Lemma 2.4. We have*

$$\mathrm{tr}\left(M_1M_2M_1^{-1}M_3\right) = -x_{1,2}x_{1,3} - x_{2,3} + a_1a_4 + a_2a_3,$$
$$\mathrm{tr}\left(M_1M_2M_1M_2^{-1}M_1^{-1}M_3\right) = x_{1,2}^2x_{1,3} + x_{1,2}x_{2,3} - x_{1,3} - x_{1,2}(a_1a_4 + a_2a_3) + (a_1a_3 + a_2a_4).$$

Proof. We only make use of the relation $\mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) = \mathrm{tr}(A)\,\mathrm{tr}(B)$:

$$\begin{aligned}
\mathrm{tr}\left(M_1M_2M_1^{-1}M_3\right) &= \mathrm{tr}\left(M_3M_1M_2M_1^{-1}\right) = a_4a_1 - \mathrm{tr}(M_3M_1M_2M_1)\\
&= a_4a_1 - \left(x_{3,1}x_{1,2} - \mathrm{tr}\left(M_3M_2^{-1}\right)\right) = a_4a_1 - (x_{3,1}x_{1,2} - (a_2a_3 - x_{3,2}))\\
&= -x_{1,2}x_{1,3} - x_{2,3} + a_1a_4 + a_2a_3,\\
\mathrm{tr}\left(M_1M_2M_1M_2^{-1}M_1^{-1}M_3\right) &= \mathrm{tr}\left(M_2^{-1}M_1^{-1}M_3M_1 \cdot M_2M_1\right)\\
&= \mathrm{tr}\left(M_2^{-1}M_1^{-1}M_3 \cdot M_1\right)x_{1,2} - \mathrm{tr}\left(M_2^{-1}M_1^{-1}M_3M_2^{-1}\right)\\
&= \mathrm{tr}(M_2^{-1}M_1^{-1}M_3)a_1x_{1,2} - \mathrm{tr}(M_2^{-1}M_1^{-1} \cdot M_3M_1^{-1})x_{1,2}\\
&\quad - \mathrm{tr}\left(M_2^{-1}M_1^{-1}M_3\right)a_2 + \mathrm{tr}\left(M_2^{-1}M_1^{-1}M_3M_2\right)\\
&= (x_{1,2}a_3 - a_4)a_1x_{1,2} - x_{1,2}^2(a_1a_3 - x_{1,3}) + x_{1,2}x_{2,3} - (x_{1,2}a_3 - a_4)a_2 + (a_1a_3 - x_{1,3})\\
&= x_{1,2}^2x_{1,3} + x_{1,2}x_{2,3} - x_{1,3} - x_{1,2}(a_1a_4 + a_2a_3) + (a_1a_3 + a_2a_4).
\end{aligned}$$

■

Proposition 2.9. *Let $x_{i,j}' = \mathrm{tr}(M_i'M_j')$. In these trace coordinates, h_{1*} is given by*

$$x_{1,2}' = x_{1,2},$$
$$x_{2,3}' = -x_{1,2}x_{1,3} - x_{2,3} + (a_1a_4 + a_2a_3),$$
$$x_{1,3}' = x_{1,2}^2x_{1,3} + x_{1,2}x_{2,3} - x_{1,3} - x_{1,2}(a_1a_4 + a_2a_3) + (a_1a_3 + a_2a_4).$$

We obtain h_{2*} and h_{3*} by a cyclic permutation indices $(+1)$ of the indices 1, 2 and 3.

Proof. We have

$$\rho \circ h_1(\gamma_{1,2}) = \rho(\gamma_{1,2}) = I,$$
$$\rho \circ h_1(\gamma_{3,4}) = \rho(\gamma_{3,4}) = I,$$
$$\rho \circ h_1(\gamma_{3,2}) = \rho(\gamma_{3,2}\gamma_{2,1}\gamma_{1,1}\gamma_{1,2}\gamma_{2,2}) = M_1M_2.$$

Therefore $X_3 = \widetilde{X_2}^{h_1(\gamma_{2,3})}(M_1M_2)^{-1}$, and we normalize $\rho \circ h_1$ by setting: $X_1' = X_1$, $X_2' = X_2$ and $X_3' = X_3 \cdot M_1M_2$ in order to obtain a representation ρ' equivalent to $\rho \circ b_1$ which satisfies $\rho'(\gamma_{2,3}) = I$. This representation ρ' is characterized by the 3 matrices:

$$M_1' = M_1, \qquad M_2' = M_2, \qquad M_3' = (M_1M_2)^{-1}M_3(M_1M_2).$$

The statement of the proposition is obtained from the extended Fricke Lemma 2.8. ■

By this way, we reach the same expressions of the dynamics as S. Cantat and F. Loray in [10]. The study of this dynamics allows them to give a new proof of the irreducibility of the Painlevé VI equation. This is an important motivation to extend the description of this dynamics to the non regular cases.

3 An irregular example (towards P_V)

3.1 Standard facts about irregular singularities

We consider a rank n linear differential system at $z = 0$

$$\Delta\colon\ z^{r+1}\frac{dY}{dz} = A(z)\cdot Y, \qquad A \text{ holomorphic at } 0,$$

where $A(z)$ takes its values in $\mathfrak{gl}(n,\mathbb{C})$.[3] The integer r is positive, and equal to 0 for a Fuchsian system. We suppose here that $r > 0$, and that the eigenvalues of $A_0 = A(0)$ are non vanishing distincts complex numbers. We fix here $\Lambda_0 = \mathrm{diag}(\lambda_i)$ a normal form of A_0 in the Cartan subalgebra $\mathcal{T}_0$ of the diagonal matrices, i.e., we choose an ordering of its eigenvalues. The formal local meromorphic classification is given by

Proposition 3.1.

1. *Up to a local ramified formal meromorphic gauge equivalence, we have*

$$\Delta \sim_0 \frac{dX}{dt} = \left(\frac{dQ}{dt} + \frac{L}{t}\right)\cdot X,$$

where $z = t^\nu$, Q *(the "irregular type")* $= \frac{\Lambda_0}{t^r} + \cdots + \frac{\Lambda_{r-1}}{t}$, *and the matrices* Λ_i *and* L *(the residue matrix) are diagonal matrices. For a fixed* Λ_0, *the pair* (Q, L) *is unique in* $\mathcal{T}_0 \times \mathcal{T}_0/\mathcal{T}_0(\mathbb{Z})$.

2. *Let* F_0 *be a conjugation between* A_0 *and* Λ_0*:* $A_0 = F_0\Lambda_0F_0^{-1}$. *The system* Δ *has a formal fundamental solution*

$$\widehat{X} = \widehat{F}(t)t^L\exp Q \qquad \textit{with} \quad \widehat{F}(0) = F_0.$$

For a fixed Λ_0, there are already two ambiguities in the above writing of $\widehat{X}$:

- the choice of F_0: we may change F_0 with F_0D, and therefore $\widehat{X}$ with $\widehat{X}D$, where D belongs to the centralizer of Λ_0;
- the choice of a branch for the argument, and hence for $\log t$ and t^L.

We suppose now that we are in the unramified case: $\nu = 1$.

Definition 3.2.

1. A *separating ray* is a ray $\arg(z) = \tau$ such that there exists a pair of eigenvalues (λ_j, λ_k) of Λ_0 satisfying: $z^r(\lambda_j - \lambda_k) \in i\mathbb{R}^+$ for $\arg(z) = \tau$.
2. A *singular ray* is a ray $\arg(z) = \sigma$ such that there exists a pair of eigenvalues (λ_j, λ_k) satisfying: $z^r(\lambda_j - \lambda_k) \in \mathbb{R}^-$ for $\arg(z) = \sigma$.[4]

[3] The theory can be extended to any complex reductive Lie algebra, up to some technical complications, see [1].

[4] Since in many references, the definitions of Stokes and anti-Stokes rays are exchanged, we do not use this terminology here.

3. A *regular sector* is an open sector $\mathcal{S}_d$ of angle π/r bisected by d such that d is not a singular ray (or equivalently, such that its edges are not separating rays).

We can remark that:

- If $\arg(z) = \tau$ is a separating (resp. singular) ray then its opposite is also a separating (resp. singular) ray.
- A non singular ray is a ray on which the formal solution admits a unique sum, for the summation theory.[5]
- A separating ray is a ray on which the asymptotic of a general solution (a linear combination of the columns of $\widehat{X}$) changes.
- The knowledge of the μ separating (resp. singular) rays in a regular sector generates the complete knowledge of all the separating (resp. singular) rays, by considering their opposites, and the ramification by z^r.
- The generic case is the situation in which there exists exactly one pair of eigenvalues $(\lambda_j(\nu), \lambda_k(\nu))$ defining each separating ray τ_ν. In this case, we have $\mu = n(n-1)/2$ separating rays in a regular sector, and $m = n(n-1)r = 2r\mu$ separating (resp. singular) rays in S^1.

Theorem 3.3. *On a regular sector S_d containing the μ separating rays $\tau_\nu, \ldots, \tau_{\nu+\mu-1}$, there exists a unique holomorphic fundamental system of solutions X_d admitting the asymptotic expansion $\widehat{X}$. Furthermore X_d can be extended to a solution (with the same asymptotic) on the sector S_ν delimited by the two nearest separating rays $\tau_{\nu-1}$ and $\tau_{\nu+\mu}$ outside S_d.*

There exists two proofs of this fact using either the asymptotic theory (see [2]), or the summation theory (see [26]).

For $\nu = 1$ to m, Theorem 3.3 gives us a unique solution X_ν on the large sector delimited by $\tau_{\nu-1}$ and $\tau_{\nu+\mu}$ admitting $\widehat{X}$ as asymptotic expansion on $\mathcal{S}_\nu$.

Definition 3.4. The m Stokes multipliers U_ν are defined on $\mathcal{S}_\nu \cap \mathcal{S}_{\nu+1}$ by

$$X_\nu = X_{\nu+1} \cdot U_\nu,$$

with a m-periodic indexation.

In the generic case (the support of each singular ray reduces to a unique pair of eigenvalues), the constant matrices U_ν, $\nu = 1, \ldots, m-1$ are transvection matrices: the diagonal entries are equal to 1, and the unique non vanishing coefficient off the diagonal is the coefficient in position $(j(\nu), k(\nu))$ where the separating ray τ_ν (the only ray in $\mathcal{S}_\nu \backslash \mathcal{S}_\nu \cap \mathcal{S}_{\nu+1}$) is defined by the pair (λ_j, λ_k). The diagonal of the matrix U_m is $\exp(2i\pi L)$, where L is the residue coefficient of $A(x)$ after diagonalisation.

3.2 The wild fundamental groupoid for the class of connections $(0, 0, 1)$

We consider a meromorphic $\mathfrak{sl}_2(\mathbb{C})$-connection ∇ on $\mathbb{P}_1(\mathbb{C})$, admitting 2 regular singular points (say 0 and 1) and an irregular one at ∞. This corresponds to an element of the family indexed by $(0, 0, 1)$ in the classification of M. van der Put and K. Saito in [29]. Locally in a coordinate z centered at ∞, ∇ is given by the following system

$$\Delta\colon \quad \frac{dX}{dz} = z^{-2} \sum_{i \geq 0} A_i z^i \cdot X.$$

[5] We only need here k-summation theory, with $k = r$.

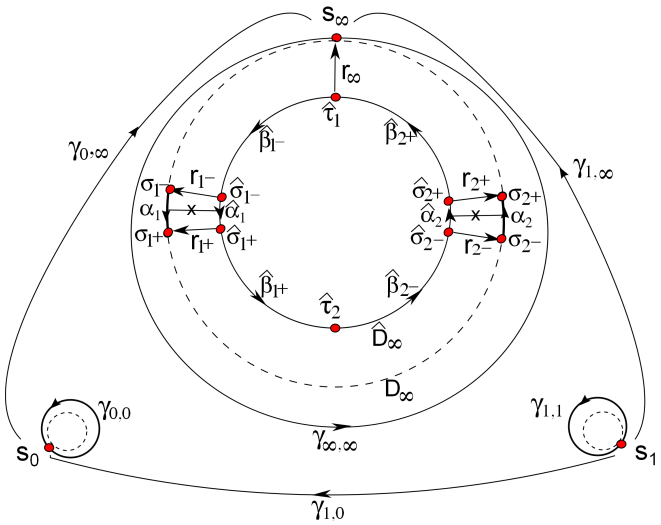

Figure 3. The groupoid $\pi_1(X, S)$ (irregular case).

The initial part $A_0 \neq 0$ is a non trivial semi-simple element of $\mathcal{G} = \mathfrak{sl}_2$. We choose a normal form $\Lambda_0 = \mathrm{diag}(\lambda_0, -\lambda_0)$, $\lambda_0 \neq 0$, in $\mathcal{T}_0$ (the Cartan algebra of diagonal matrices) in the conjucacy class of A_0, i.e., an ordering of its eigenvalues. Its centralizer in $\mathrm{SL}_2(\mathbb{C})$ is

$$C(\Lambda_0) = C(\mathcal{T}_0) = \left\{ D_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha \in \mathbb{C}^* \right\}.$$

We have here two singular rays $\sigma_1(\Lambda_0)$ and $\sigma_2(\Lambda_0) = -\sigma_1(\Lambda_0)$, and two separating rays $\tau_1(\Lambda_0)$ and $\tau_2(\Lambda_0) = -\tau_1(\Lambda_0)$.

In order to construct the wild fundamental groupoid (for a fixed Λ_0), we first use a real blowing up at each singularity in $\mathbb{P}^1$ and we obtain a variety X with 3 exceptional divisors D_0, D_1 and D_∞ (circles in dotted lines in Fig. 3). As in the previous classical case, we choose a base point s_0, s_1 and s_∞ on each of them, and we consider the morphism (path up to homotopy) $\gamma_{i,j}$ joining s_i to s_j. The paths $\gamma_{i,i}$ are homotopic in X to the curves D_i. We choose s_∞ such that it corresponds to a non singular ray.

Since we also have to consider the continuation of $\widehat{X}$ (the formal monodromy along an arc is induced by the substitution $z \mapsto ze^{i\theta}$), we introduce a second copy $\widehat{D}_\infty$ of D_∞ inside the first one, with a base point $\widehat{\tau}_1$ which is the separating ray between σ_2 and σ_1. For each singular direction σ_i (denoted in the picture below by a ray with a cross in the annulus between $\widehat{D}_\infty$ and D_∞), we add two loops delimited by two rays r_i^- and r_i^+ which are non singular and non separating, and two arcs α_i on D_∞ and $\widehat{\alpha}_i$ and $\widehat{D}_\infty$ of opening strictly lower than π, bisected by the singular rays. Let $\widehat{\sigma}_i^-$ and σ_i^- the two points on $\widehat{D}_\infty$ and D_∞ joined by r_i^-, and $\widehat{\sigma}_i^+$ and σ_i^+ joined by r_i^+. Finally we put a ray r_∞ from $\widehat{\tau}_1$ to s_∞.

Remark 3.5. In the picture above, we put arbitrarily the two base points s_∞ on D_∞ and $\widehat{\tau}_1$ on $\widehat{D}_\infty$ in the same direction: it is our initial configuration. Nevertheless, in the dynamical study of the next section, s_∞ will remain fixed, while the separating ray $\widehat{\tau}_1$ (and all the other data related to Λ_0) will move on $\widehat{D}_\infty$.

Definition 3.6. The wild fundamental groupoid $\pi_1(X, S(\Lambda_0))$ is the groupoid defined by

- the objects $S(\Lambda_0)$: the three points s_0, s_1 and s_∞, the points $\widehat{\tau}_i$ (separating rays), $\widehat{\sigma}_{i\pm}$ on $\widehat{D}_\infty$ around the singular rays σ_i (denoted on the figure by a ray with a cross), and the corresponding points $\sigma_{i\pm}$ on D_∞.

- the morphisms: they are generated by the paths $\gamma_{i,j}$ (up to homotopy) between s_0, s_1, and s_∞ in X, the rays r_∞, $r_i^{\pm}$, the arcs α_i on D_∞, and all the arcs on $\widehat{D}_\infty$: $\widehat{\alpha}_i$ from $\widehat{\sigma}_{i^-}$ to $\widehat{\sigma}_{i^+}$, and the connecting arcs $\widehat{\beta}_{i^\pm}$ as indicated on the figure.

The subgroupoid $\pi_1^{\mathrm{loc}}(X, S(\Lambda_0))$ is generated by the morphisms $\gamma_{0,0}$, $\gamma_{1,1}$ and $\widehat{\gamma}_{1,1}$ (the formal loop based in $\widehat{\tau}_1$).

We still have two relations r_{int} and r_{ext} between the generating morphisms:

$$r_{\mathrm{int}}:\ \gamma_{0,0}\gamma_{0,\infty}\gamma_{\infty,\infty}\gamma_{\infty,1}\gamma_{1,1}\gamma_{1,0} = \star_0,$$
$$r_{\mathrm{ext}}:\ \gamma_{0,\infty}\gamma_{\infty,1}\gamma_{1,0} = \star_0$$

and a new one (the wild relation):

$$r_{\mathrm{wild}}:\ \gamma_{\infty,\infty} = (r_\infty)^{-1}\big(\widehat{\beta}_{1^-} r_{1-}\alpha_1 (r_{1^+})^{-1}\widehat{\beta}_{1^+}\widehat{\beta}_{2^-} r_{2^-}\alpha_2 r_{2^+}\widehat{\beta}_{2^+}\big) r_\infty.$$

The *Stokes loops* (based in $\widehat{\sigma}_{i^-}$) are $\mathrm{st}_i := r_{i^-}\alpha_i (r_{i^+})^{-1}\widehat{\alpha}_i^{-1}$, $i = 1, 2$. The *formal loop* (based in $\widehat{\tau}_1$) is $\widehat{\gamma}_{1,1} = \widehat{\beta}_{1^-}\widehat{\alpha}_1\widehat{\beta}_{1^+}\widehat{\beta}_{2^-}\widehat{\alpha}_2\widehat{\beta}_{2^+}$.

A representation ρ of this groupoid induced by the differential system (Δ) is defined in the following way. We first choose a "compatible" representation of the objects:

- We choose analytic fundamental systems $X(s_0)$, $X(s_1)$, at s_0, s_1, as in the regular case (i.e., we choose a logarithmic branch in the corresponding direction);
- We choose a formal fundamental system $\widehat{X}_\infty$ given by Proposition 3.1. For each object $\widehat{\sigma}_{i^\pm}$, $\widehat{\tau}_i$ on $\widehat{D}_\infty$, we choose a formal fundamental system $X(\widehat{\sigma}_i{}^{\pm})$, $X(\widehat{\tau}_i)$ by choosing a determination in the corresponding direction of the formal fundamental solution $\widehat{X}_\infty$.
- For each object $\sigma_{i^\pm}$ on D_∞, we choose an actual sectorial solutions $X(\sigma_{i^\pm})$, given by Theorem 3.3, whose asymptotic expansion is some determinacy of the *same formal fundamental solution* $\widehat{X}_\infty$ (this is the compatibility condition).

Then, we construct the representations of the generating morphisms in the following way:

- We use analytic continuation to represent the morphisms between s_0, s_1, and s_∞, as in the singular regular case (see (2.1)).
- In the same way, we use the analytic continuation of the formal solutions to represent the morphisms $\widehat{\beta}_{i^\pm}$, and $\widehat{\alpha}_i$ on $\widehat{D}_\infty$ between the formal objects: this formal monodromy is defined by the substitution $z \to ze^{i\theta}$ in the formal expressions.
- We also use analytic continuation *preserving the same asymptotic*, to represent the arcs α_i on D_∞. Note that the regular sectors given by Theorem 3.3 centered on r_{i^-} and on r_{i^+} allow us to define this continuation along α_i and α_i^{-1}: indeed the intersection of these two sectors is a sector of opening π delimited by the 2 separating rays, and therefore contains α_i.
- We represent $r_{i^\pm}$ and r_∞ by using Theorem 3.3: starting from the representation $X(\widehat{\sigma}_{i^\pm})$ of the formal objects $\widehat{\sigma}_{i^\pm}$, this theorem gives us an actual solution in this direction, which is denoted by $\widetilde{X(\widehat{\sigma}_{i^\pm})}$. The comparison with the representation $X(\sigma_{i^\pm})$ of the final object $\sigma_{i^\pm}$ defines $\rho(r_{i^\pm})$:

  $$X(\sigma_{i^\pm}) = \widetilde{X(\widehat{\sigma}_{i^\pm})} \cdot \rho(r_{i^\pm}).$$

  The representation of the inverse paths $(r_{i^\pm})^{-1}$ are obtained in the following way: starting from the representation $X(\sigma_{i^\pm})$ of $\sigma_{i^\pm}$ on D_∞ we use its asymptotic expansion $\widehat{X(\sigma_{i^\pm})}$ and compare it with the representation $X(\widehat{\sigma}_{i^\pm})$ of the final object $\widehat{\sigma}_{i^\pm}$ in order to define $\rho((r_{i^\pm})^{-1})$.

Definition 3.7. Two representations of $\pi_1(X, S(\Lambda_0))$ are equivalent if they are obtained by different compatible representations of the objects. The wild character variety $\chi(\Lambda_0)$ is the set of the representations of $\pi_1(X, S(\Lambda_0))$ up to the above equivalence relation. The local wild character variety $\chi^{\text{loc}}(\Lambda_0)$ is the set of the representations of $\pi_1^{\text{loc}}(X, S(\Lambda_0))$ up to the above equivalence relation, and we have a natural fibration

$$\pi\colon\ \chi(\Lambda_0) \longrightarrow \chi^{\text{loc}}(\Lambda_0).$$

3.3 The normalized representations

We first choose a representation $X(\widehat{\tau}_1)$ of $\widehat{\tau}_1$ (our initial object). Now, we can fix the representations of s_0, s_1 and s_∞ in a unique way such that

$$\rho(r_\infty) = \rho(\gamma_{\infty,0}) = \rho(\gamma_{\infty,1}) = I \quad (= \rho(\gamma_{0,1}) \text{ from the exterior relation}).$$

Then we choose $X(\widehat{\tau}_i)$, $X(\widehat{\sigma}_{i\pm})$ and $X(\sigma_{i\pm})$ such that

$$\rho(\widehat{\beta}_{1\pm}) = \rho(r_{i\pm}) = \rho(\widehat{\alpha}_i) = \rho(\widehat{\beta}_{2-}) = I.$$

There remains five matrices

$$M_0 = \rho(\gamma_{0,0}), \qquad M_1 = \rho(\gamma_{1,1}), \qquad \widehat{M} = \rho(\widehat{\beta}_{2+}), \qquad U_1 = \rho(\alpha_1), \qquad U_2 = \rho(\alpha_2).$$

Remark 3.8.

- For such a normalized representation, we also have $U_i = \rho(\text{st}_i)$, U_i is the representation of the Stokes loops. From the above definition of the representation of the paths $r_i^{\pm}$, the matrices U_i are the Stokes multipliers introduced in the previous section. In particular, they are unipotent matrices

$$U_1 = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 0 \\ u_2 & 1 \end{pmatrix}.$$

- We also have $\widehat{M} = \rho(\widehat{\gamma}_{1,1})$. Therefore this matrix is a representation of the formal loop. It is a diagonal matrix

$$\widehat{M} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

- Let $M_\infty := \rho(\gamma_{\infty,\infty})$. From the interior relation r_{int} in the groupoid we have $M_0 M_\infty M_1 = I$ and from the wild relation r_{wild}, $M_\infty = U_1 U_2 \widehat{M}$. Therefore we have

$$M_0 U_1 U_2 \widehat{M} M_1 = I$$

  and ρ is given by a 4-uple of independent matrices $(M_0, U_1, U_2, \widehat{M})$, where the U_i's are upper and lower unipotent matrices and $\widehat{M}$ is a diagonal matrix.

- If we change the choice of the representation of the initial object $X(\widehat{\tau}_1)$ setting $X'(\widehat{\tau}_1) = X(\widehat{\tau}_1) \cdot D_\alpha$, D_α in $C(\mathcal{T}_0)$, the 4-uple $(M_0, U_1, U_2, \widehat{M})$ changes by the common conjugacy with D_α.

Therefore, for a given Λ_0, a representation ρ is characterized by a 4-uple $(M_0, U_1, U_2, \widehat{M})$ up to the conjugation by $C(\mathcal{T}_0)$. According to the previous description, the character variety is

$$\chi(\Lambda_0) = \big\{(M_0, U_1, U_2, \widehat{M})_\sim\big\}.$$

Its dimension is $(3+1+1+1)-1=5$. The character variety of the local datas is

$$\chi^{\mathrm{loc}}(\Lambda_0) = \big\{([M_0]_\sim, [M_1]_\sim, \widehat{M})\big\},$$

where $[M_0]_\sim$ is the conjugation class of M_0, $[M_1]_\sim$ is the (independent) conjugation class of M_1, and $\widehat{M}$ is diagonal. If M_0 and M_1 are semi-simple matrices, it is a 3-dimensional variety, and the fiber of $\chi(\Lambda_0) \to \chi^{\mathrm{loc}}(\Lambda_0)$ is a 2-dimensional variety.

Changing the choice of Λ_0. Let $W := \{\mathrm{id}, w\}$ be the group of permutations of two objects. W is isomorphic to the quotient of the subgroup $\{I, P_w\}$ of SL_2 by $\{\pm I\}$, where

$$P_w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The conjugation by P_w on $\mathfrak{sl}_2$ only depends on the class of P_w in the quotient. Therefore, we denote $c_w(M) = P_w^{-1} M P_w$. Let $w \cdot \Lambda_0 := c_w(\Lambda_0) = -\Lambda_0$.

We consider the fundamental groupoid $\pi_1(X, S(w\cdot\Lambda_0))$ obtained by a new indexation of the singular rays. The objects are

$$S(w\cdot\Lambda_0) = \big\{s_0, s_1, s_\infty, \sigma_{w(i)^\pm}, \widehat{\sigma}_{w(i)^\pm}, \widehat{\tau}_{w(i)}\big\}.$$

The generating morphisms are also re-indexed according to the new indexation of their origin and end-point. We obtain an isomorphism of groupoid Φ_w from $\pi_1(X, S(\Lambda_0))$ to $\pi_1(X, S(w\cdot\Lambda_0))$. Now this new choice of Λ_0 also modify the choice of the initial representation of the object by $X'(\widehat{\tau}_1) = X(\widehat{\tau}_1)\cdot P_w$, since we change F_0 with $F_0 P_w$ (see Proposition 3.1). Therefore the new representation ρ' is obtained from ρ by the conjugation c_w by P_w. Finally we have an isomorphism ψ_w from $\chi(\Lambda_0)$ to $\chi(w\cdot\Lambda_0)$ which send ρ on ρ' defined by the commutative diagram:

$$\begin{array}{ccc} \pi_1(X, S(\Lambda_0) & \xrightarrow{\rho} & \mathrm{SL}_2(\mathbb{C}) \\ \Phi_w \downarrow & & \downarrow c_w \\ \pi_1(X, S(w\cdot\Lambda_0) & \xrightarrow{\rho'} & \mathrm{SL}_2(\mathbb{C}) \end{array}$$

Remark 3.9. Notice that ρ' is characterized by

$$\begin{aligned} \big(M_0', U_1', U_2', \widehat{M}'\big) &= \big(P_w^{-1} M_0 P_w, P_w^{-1} U_{w(1)} P_w, P_w^{-1} U_{w(2)} P_w, P_w^{-1}\widehat{M} P_w\big) \\ &= \big(\big(M_0^{-1}\big)^t, U_2^t, U_1^t, \widehat{M}^{-1}\big), \end{aligned}$$

where M^t denotes the transposed matrix.

3.4 The wild character variety $\chi(\Lambda_0)$ in trace coordinates

Let suppose that, with our choice of Λ_0, U_1 is an upper unipotent matrix, and U_2 a lower one. We have

$$M_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}, \qquad U_1 = \begin{pmatrix} 1 & u_1 \\ 0 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 0 \\ u_2 & 1 \end{pmatrix}, \qquad \widehat{M} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

The action of a diagonal matrix D_α in C is given by

$$D_\alpha M_0 D_\alpha^{-1} = \begin{pmatrix} a_0 & \alpha^2 b_0 \\ \alpha^{-2} c_0 & d_0 \end{pmatrix}, \qquad D_\alpha \widehat{M} D_\alpha^{-1} = \widehat{M},$$

$$D_\alpha U_1 D_\alpha^{-1} = \begin{pmatrix} 1 & \alpha^2 u_1 \\ 0 & 1 \end{pmatrix}, \qquad D_\alpha U_2 D_\alpha^{-1} = \begin{pmatrix} 1 & 0 \\ \alpha^{-2} u_2 & 1 \end{pmatrix}.$$

Lemma 3.10. *Two datas* $(M_0, U_1, U_2, \widehat{M})$ *and* $(M_0', U_1', U_2', \widehat{M'})$ *such that* $u_1u_2 \neq 0 \neq u_1'u_2'$, *are equivalent up to* $C(\mathcal{T}_0)$ *if and only if*

$$\lambda = \lambda', \qquad u_1u_2 = u_1'u_2', \qquad u_1c_0 = u_1'c_0', \qquad u_2b_0 = u_2'b_0', \qquad a_0 = a_0', \qquad d_0 = d_0'.$$

Proof. Clearly, these quantities are invariant. Suppose now that they are equal. We choose α such that

$$\alpha^2 = \frac{u_1'}{u_1} = \frac{u_2}{u_2'} \quad (\neq 0).$$

From the other relations we obtain: $c_0 = \alpha^2 c_0'$, $b_0' = \alpha^2 b_0$ and finally $\widehat{M'} = \widehat{M}$, $U_i' = D_\alpha U_i D_\alpha^{-1}$, and $M_0' = D_\alpha M_0 D_\alpha^{-1}$. ■

We consider the 6 coordinates

$$\lambda \text{ (first eigenvalue of } \widehat{M}), \qquad t_0 = \operatorname{tr}(M_0), \qquad t_1 = \operatorname{tr}(M_1),$$
$$s = \operatorname{tr}(U_1U_2), \qquad x = \operatorname{tr}(M_0U_1U_2), \qquad y = \operatorname{tr}(M_0\widehat{M}).$$

They are invariant by the action of the centralizer $C(\Lambda_0)$ and therefore they induce a map $T\colon \chi(\Lambda_0) \to \mathbb{C}^6$. The Fricke lemma applied on the triple $(M_0, U_1U_2, \widehat{M})$ defines a codimension 1 Fricke variety F given by $t_1^2 - Pt_1 + Q = 0$, with

$$P = t_0\big(\lambda^{-1} - \lambda + \lambda s\big) + sy + \big(\lambda + \lambda^{-1}\big)x,$$
$$Q = t_0^2 + s^2 + \big(\lambda + \lambda^{-1}\big)^2 + x^2 + \big(\lambda^{-1} - \lambda + \lambda s\big)^2 + y^2 + xy\big(\lambda^{-1} - \lambda + \lambda s\big).$$

i.e.,

$$\lambda xys + x^2 + y^2 + \big(1 + \lambda^2\big)s^2 - \big(\lambda - \lambda^{-1}\big)xy - t_1sy - t_1\big(\lambda + \lambda^{-1}\big)x - \big(\lambda t_0t_1 + 2\lambda^2 - 2\big)s$$
$$+ t_0^2 + t_1^2 + t_0t_1\big(\lambda - \lambda^{-1}\big) + 2\lambda^2 - 2\lambda^{-2} = 0.$$

This is a family of cubics parametrized by (t_0, t_1, λ).

Proposition 3.11. *Let* $\chi^*(\Lambda_0) := \{(M_0, U_1, U_2, \widehat{M}),\ u_1u_2 \neq 0,\ \lambda \neq \pm 1\}/C(\mathcal{T}_0)$. *The map* $T\colon \chi(\Lambda_0) \to \mathbb{C}^6$ *defined by the* 6 *coordinates* $(\lambda, t_0, t_1, s, x, y)$ *is an isomorphism between* $\chi^*(\Lambda_0)$ *and the open set* $s \neq 2$, $\lambda \neq \pm 1$ *in the affine variety* F.

Proof. Clearly, from the Fricke lemma, T takes its values in F. In order to check that this map is invertible, we have to solve: $T(M_0, U_1, U_2, \widehat{M}) = (\lambda, t_0, t_1, s, x, y)$. This equation is equivalent to the system

$$a_0 + d_0 = t_0, \tag{3.1a}$$
$$\lambda a_0(1 + u_1u_2) + \lambda u_2b_0 + \lambda^{-1}u_1c_0 + \lambda^{-1}d_0 = t_1, \tag{3.1b}$$
$$2 + u_1u_2 = s, \tag{3.1c}$$
$$t_0 + a_0u_1u_2 + u_2b_0 + u_1c_0 = x, \tag{3.1d}$$
$$\lambda a_0 + \lambda^{-1}d_0 = y. \tag{3.1e}$$

If $\lambda \neq \pm 1$, we obtain from equations (3.1a) and (3.1e) a unique solution for a_0 and d_0

$$a_0 = \frac{y - \lambda^{-1}t_0}{\lambda - \lambda^{-1}}, \qquad d_0 = \frac{\lambda t_0 - y}{\lambda - \lambda^{-1}}.$$

From equations (3.1b) and (3.1d) we obtain a unique solution for u_1c_0 and u_2b_0. Equation (3.1c) gives $u_1u_2 = s - 2$ and we obtain a unique solution for $(\lambda, u_1u_2, u_1c_0, u_2b_0, a_0, d_0)$, which defines a unique solution $(M_0, U, V, \widehat{M})$ up to the action of $C(\mathcal{T}_0)$ according to Lemma 3.10. Note that the solution of the system (3.1) is polynomial in the variables t_0, t_1, s, x, y and rational in λ with poles on $\lambda = \lambda^{-1}$. ■

For $\Lambda_0' = -\Lambda_0$, we have a similar description of $\chi(\Lambda_0')$ as an affine algebraic variety in the coordinates $(\lambda', t_0', t_1', s', x', y')$.

Proposition 3.12. *The change of variables between these two charts describing the character variety χ is given by*

$$\lambda' = \lambda^{-1}, \qquad s' = s,$$
$$x' = x + t_0\frac{\lambda+\lambda^{-1}}{\lambda-\lambda^{-1}}s - \frac{2}{\lambda-\lambda^{-1}}ys + \frac{4}{\lambda-\lambda^{-1}}y - 2t_0\frac{(\lambda+\lambda^{-1})}{\lambda-\lambda^{-1}},$$
$$y' = y,$$

Proof. Since from Remark 3.9 we have

$$(M_0', U_1', U_2', \widehat{M'}) = P_w^{-1}(M_0, U_{w(1)}, U_{w(2)}, \widehat{M})P_w = \left(\left(M_0^{-1}\right)^t, U_2^t, U_1^t, \widehat{M}^{-1}\right),$$

we immediately obtain $\lambda' = \lambda^{-1}$, $s' = s$ and $y' = y$. The only non trivial computation is for $x' = \mathrm{tr}(M_0U_2U_1)$. The Fricke lemma applied on the three matrices U_1, U_2, M_0 gives us

$$x + x' = t_0 s + 2b_0u_2 + 2c_0u_1.$$

From the system (3.1) written in the proof of Proposition 3.11, we have $u_2b_0 + u_1c_0 = x - t_0 - a_0(s-2)$ and $a_0 = \frac{y-\lambda^{-1}t_0}{\lambda-\lambda^{-1}}$. By substitution in the above expression of x', we obtain the result. ∎

We will see in the next section that the natural parameter here is not Λ_0 but the conjugacy class $[A_0]$ of A_0 (see also [7, Remarks (8.5) and (10.6)]). Therefore the character variety $\chi([A_0])$ appears as a gluing of the two affine algebraic charts $\chi(\Lambda_0)$ and $\chi(-\Lambda_0)$ which turns out to be a scheme on our example, from the above proposition. We can conjecture that it is the general case.

3.5 The dynamics on the wild character variety

This part is rather an experimental one. The way used here to encode all the information of irregular representations is not completely well justified. We need other experiments (for example in the P_{II} context) and a very precise presentation of the background to present a definitive version. Nevertheless, we think that a lot of tools are present here to deal with the program presented in the first section.

In the classical previous case, the configuration space $\mathcal{C}$ was the set of the positions of the 4 singularities, identified to $\mathbb{P}_1\backslash\{0,1,\infty\}$ through the cross ratio. In the present context we can't move the three singularities in $\mathbb{P}^1$. We can only move Λ_0 in $\mathcal{T}_0^*$, which is the usual configuration space considered for example in [17]. Ph. Boalch in [5] has introduced a coordinate independent version of this space "through the notion of irregular curve". Its fundamental group is again a pure braid group.

Nevertheless, the most natural configuration space $\mathcal{C}$ is not the set of the Λ_0's but

Definition 3.13. The configuration space $\mathcal{C}$ is the set of the initial terms A_0 up to a conjugation.

Indeed, a gauge equivalence on Δ acts by conjugacy on the initial coefficient A_0. In the same way, Ph. Boalch has introduced a "bare irregular curve" in [7]. We begin with a description of $\mathcal{C}$ in a general framework.

Let G be a reductive algebraic group, with Lie-algebra $\mathcal{G}$. Let

$$\mathcal{D} = \cup\{\mathcal{T},\ \mathcal{T} \text{ Cartan subalgebra of } \mathcal{G}\}.$$

Let $\mathcal{T}_0$ be a fixed Cartan subalgebra (for $\mathfrak{sl}_2$, $\mathcal{D}$ is the set of the *diagonalisable* matrices, and $\mathcal{T}_0$ the Cartan subalgebra of the *diagonal* matrices). Let $\mathcal{T}^{\text{reg}}$ be the subset of $\mathcal{T}$ of matrices with distinct eigenvalues. Recall that all the Cartan subalgebras are conjugated by some element g in G. This element is not unique: if g and g' are two conjugations between $\mathcal{T}_0$ and $\mathcal{T}$, $g'^{-1}g$ keeps (globally) invariant $\mathcal{T}_0$ and therefore belongs to the normalizer $N(T_0)$ of the Cartan torus T_0. We obtain a fibration

$$\mathcal{D} \longrightarrow G/N(T_0),$$

which sends $\mathcal{T}_0$ on the identity element I. Note that the quotient space $G/N(T_0)$ of left classes modulo $N(T_0)$ is not a group.

Lemma 3.14. *If the algebraic group G is connected, and simply connected (which is the case for SL_n), the fundamental group $\pi_1(G/N(T_0), I)$ is the Weyl group $W := N(T_0)/T_0$.*

Proof. From the exact sequence of topological spaces

$$0 \to N(T_0) \to G \to G/N(T_0) \to 0$$

since $\pi_1(G, I) = \pi_0(G) = 0$ we obtain

$$0 \to \pi_1(G/N(T_0), I) \to \pi_0(N(T_0)) \to 0.$$

Now from the exact sequence of groups

$$0 \to T_0 \to N(T_0) \to N(T_0)/T_0 \to 0$$

since $\pi_0(T_0) = 0$ we obtain $\pi_0(N(T_0)) = \pi_0(N(T_0)/T_0) = W$. ■

Remark that the fibration $\mathcal{D} \longrightarrow G/N(T_0)$ has natural local trivialisations: we can lift a path $g(s)$ in $G/N(T_0)$ from A_0 by using the conjugation $g(s)A_0g(s)^{-1}$. The conjugation class $[A_0]$ of A_0 in $\mathcal{D}$ cuts the fiber $\mathcal{T}_0$ in a discrete set which is the orbit of Λ_0 under the action of W by the monodromy of the above fibration. Therefore, an equivalent definition of the configuration space is

Definition 3.15. If the algebraic group G is connected, and simply connected, the configuration space $\mathcal{C}$ is the set of the orbits of the Weyl group W acting on $\mathcal{T}_0^{\text{reg}}$.

For $G = \mathrm{SL}_2$, the action of the generator w (the "Weyl loop") of $\pi_1(G/N(T_0), I) = \{\mathrm{id}, w\}$ can be explicited. Consider the path in SL_2 joining I to P_w by the real rotations

$$w(s) = \begin{pmatrix} \cos s\pi/2 & -\sin s\pi/2 \\ \sin s\pi/2 & \cos s\pi/2 \end{pmatrix}, \qquad s \in [0, 1].$$

It induces a loop w in $G/N(T_0)$ since P belongs to $N(T_0)$. This loop can be lifted in a path δ_w in $\mathcal{D}$ joining Λ_0 until $-\Lambda_0$

$$\delta_w(s) = w(s)\Lambda_0 w(s)^{-1}.$$

This path permutes the eigenvalues of Λ_0 without moving the eigenvalues themselves.

Corollary 3.16. *The whole braid group acts on $\chi(\Lambda_0)$.*

Proof. Clearly the pure braid group $P_n = \pi_1(\mathcal{T}_0^{\mathrm{reg}}, \Lambda_0)$ acts on $\chi(\Lambda_0)$, keeping invariant each element of the orbit $W \cdot \Lambda_0$.

Now let B_n be the whole braid group. From the exact sequence

$$0 \longrightarrow P_n \longrightarrow B_n \xrightarrow{p} W \longrightarrow 0,$$

a braid b sends Λ_0 on $p(b) \cdot \Lambda_0$. Then we can lift the Weyl loop $w = p(b)$ in order to come back to Λ_0. ■

The dynamics of the pure braid group. We come back to the context of $G = \mathrm{SL}(2, \mathbb{C})$. Let b be the (non pure) positive braid which interchanges Λ_0 and $-\Lambda_0$, and b^2 the corresponding pure braid. The braid moves all the objects attached to Λ_0, keeping fixed s_∞. Therefore we have

Proposition 3.17. *The pure braid b^2 induces an automorphism h_{b^2} of $\pi_1(X, S(\Lambda_0))$ which satisfies $h_{b^2}(r_\infty) = (\gamma_{\infty,\infty})^{-1} r_\infty$, and keep invariant all the others generating morphisms.*

Now, since $\rho \circ h_{b^2}(r_\infty) = (U_1 U_2 \widehat{M})^{-1}$, we normalize $\rho \circ h_{b^2}$ in the equivalent representation ρ' by changing the representation of the object s_∞, setting: $X'_\infty = X_\infty \cdot (U_1 U_2 \widehat{M})$. Then, we set $X'_0 = X_0 \cdot (U_1 U_2 \widehat{M})$ and $X'_1 = X_1 \cdot (U_1 U_2 \widehat{M})$ in order to obtain $\rho'(\gamma_{0,\infty}) = \rho'(\gamma_{1,\infty}) = \rho'(\gamma_{0,1}) = I$. The representation ρ' is characterized by the 4-uple of matrices

$$\big(M'_0, U'_1, U'_2, \widehat{M}'\big) = \big(\big(U_1 U_2 \widehat{M}\big)^{-1} M_0 \big(U_1 U_2 \widehat{M}\big), U_1, U_2, \widehat{M}\big).$$

Proposition 3.18. *The automorphism h_{b^2} of $\chi(\Lambda_0)$ fixes λ, t_0 and t_1, and is given in the trace coordinates on the fiber by*

$$\begin{aligned}
s' &= s,\\
x' &= (\lambda^{-1} - \lambda + \lambda s)^2 x + (\lambda^{-1} - \lambda + \lambda s) y - x\\
&\quad - (\lambda^{-1} - \lambda + \lambda s)(\mathrm{st}_1 + \lambda t_0 + \lambda^{-1} t_0) + (\mathrm{st}_0 + \lambda t_1 + \lambda^{-1} t_1),\\
y' &= -(\lambda^{-1} - \lambda + \lambda s) x - y + (\mathrm{st}_1 + \lambda t_0 + \lambda^{-1} t_0).
\end{aligned}$$

Proof. We set $S := U_1 U_2$, $S' := U'_1 U'_2$. Clearly $s' = \mathrm{tr}(S') = \mathrm{tr}(S) = s$. Now we have

$$\begin{aligned}
x' &= \mathrm{tr}(M'_0 S') = \mathrm{tr}\left(\widehat{M}^{-1} S^{-1} M_0 S \widehat{M} S\right) = \mathrm{tr}\left(S \widehat{M} S \widehat{M}^{-1} S^{-1} M_0\right),\\
y' &= \mathrm{tr}\left(M'_0 \widehat{M}'\right) = \mathrm{tr}\left(\widehat{M}^{-1} S^{-1} M_0 S \widehat{M} \widehat{M}\right) = \mathrm{tr}\left(S^{-1} M_0 S \widehat{M}\right).
\end{aligned}$$

We can apply the extended Fricke Lemma 2.8 for $M_1 := S$, $M_2 := \widehat{M}$ and $M_3 := M_0$, i.e., for $a_1 = s$, $a_2 = \lambda + \lambda^{-1}$, $a_3 = t_0$, $a_4 = t_1$, $x_{1,2} = \mathrm{tr}(S\widehat{M}) = \lambda + \lambda^{-1} + \lambda(s-2)$, $x_{1,3} = x$, and $x_{2,3} = y$, which gives the expressions in the statement. ■

The dynamics induced by the whole braid group. The morphism of groupoids $h_b \colon \pi_1(X, S(\Lambda_0)) \to \pi_1(X, S(-\Lambda_0))$ exchanges the opposite objects related to Λ_0, fixing s_∞. Therefore we have

$$h_b(\mathrm{st}_1) = \mathrm{st}_2, \qquad h_b(\mathrm{st}_2) = \mathrm{st}_1, \qquad h_b(r_\infty) = \big(\widehat{\beta}_{1^-} r_{1^-} \alpha_1 \big(r_{1^+}\big)^{-1} \widehat{\beta}_{1^+}\big)^{-1} r_\infty.$$

Since $\rho(h_b)(r_\infty) = U_1^{-1}$ and $\rho(h_b)(\widehat{\beta}_{1^+}) = \widehat{M}$, we normalize $\rho(h_b)$ in ρ' by setting

$$\begin{aligned}
&X'(s_\infty) := X(s_\infty) U_1, \qquad X'(\widehat{\tau}_2) := X(\widehat{\tau}_2) \widehat{M}^{-1},\\
&X'(\widehat{\sigma}_{2^-}) := X(\widehat{\sigma}_{2^-}) \widehat{M}^{-1}, \qquad X'(\widehat{\sigma}_{2^+}) := X(\widehat{\sigma}_{2^+}) \widehat{M}^{-1}.
\end{aligned}$$

The normalized representation ρ' is characterized by the 4-uple of matrices

$$\big(U_1^{-1}M_0U_1, U_2, \widehat{M}U_1\widehat{M}^{-1}, \widehat{M}\big).$$

We can check that the iteration of this action gives us

$$\big(U_2^{-1}U_1^{-1}M_0U_1U_2, \widehat{M}U_1\widehat{M}^{-1}, \widehat{M}U_2\widehat{M}^{-1}, \widehat{M}\big),$$

which is equivalent to

$$\big(\big(U_1U_2\widehat{M}\big)^{-1}M_0\big(U_1U_2\widehat{M}\big), U_1, U_2, \widehat{M}\big)$$

and corresponds to the expression obtained above for h_{b^2}. Now, if we want to obtain the expression of h_b in trace coordinates, we have to compose with the change of charts from $\chi(\Lambda_0)$ to $\chi(-\Lambda_0)$, which interchanges U_1 and U_2 and compose the 4-uple of matrices by P_w (see Remark 3.11). We obtain the following action

$$\big(M_0, U_1, U_2, \widehat{M}\big) \to \big(P_w\big(U_1^{-1}M_0U_1\big)P_w^{-1}, P_w\big(\widehat{M}U_1\widehat{M}^{-1}\big)P_w^{-1}, P_wU_2P_w^{-1}, P_w\widehat{M}P_w^{-1}\big).$$

In the local coordinates on $\chi(\Lambda_0)$

$$\begin{aligned}
&t_0' = t_0, \qquad t_1' = t_1, \qquad \lambda' = \lambda^{-1},\\
&s' = \operatorname{tr}\big(P_w\big(\widehat{M}U_1\widehat{M}^{-1}U_2\big)P_w^{-1}\big) = 2 + \lambda^2(s-2),\\
&x' = \operatorname{tr}\big(P_w\big(U_1^{-1}M_0U_1\widehat{M}U_1\widehat{M}^{-1}U_2\big)P_w^{-1}\big)\\
&\quad = t_0 + \lambda^2a_0(s-2) - \lambda^2(s-2)(u_1c_0) + (s-2)(a_0-d_0) - (s-2)(u_1c_0) + (u_2b_0),\\
&y' = \operatorname{tr}\big(P_w\big(U_1^{-1}M_0U_1\widehat{M}\big)P_w^{-1}\big) = \lambda a_0 - \big(\lambda-\lambda^{-1}\big)(u_1c_0) + d_0\lambda^{-1}.
\end{aligned}$$

By substituting a_0, d_0, u_1c_0 and u_2b_0 by their expressions in λ, t_0, t_1, s, x, y obtained by solving the system (3.1) in the proof of Proposition 3.11, we obtain explicit expressions of the dynamics which are polynomials in the variables t_0, t_1, s, x, y and rational in λ with poles on $\lambda-\lambda^{-1}=0$.

4 Conclusion

In the case of Painlevé VI equation one considers a fibre bundle whose base is a space of configurations and the fibers character varieties. Then isomonodromy can be interpreted as an Ehresmann connection on this bundle and the fundamental group of the base (a pure braid group), induces an algebraic and symplectic dynamics on each fiber. Via the Riemann–Hilbert correspondence this dynamics corresponds to the dynamics of P_{VI}.

Boalch extended partially this picture to the case of deformations of (regular-singular or irregular) G-connections. Introducing a fibre bundle whose base is a (generalized) space of configurations and the fibers (wild) character varieties, he interpreted the isomonodromy as an Ehresmann connection on this bundle. Then the fundamental group of the base (a pure braid group or a braid group) induces a symplectic dynamics on each fiber[6].

In this paper we detailed the P_{V} case, we computed the character varieties[7] and the Boalch dynamics on them. It seems that this dynamics is "too poor" to reflect the actual "wild dynamics" of P_{V} (via the wild Riemann–Hilbert correspondence). The pure braid action corresponds to the branching of the Painlevé functions at the fixed critical points (or holonomy of the Okamoto–Painlevé foliation). But the interpretation of the whole braid group action remains mysterious for us.

[6] One can prove that this dynamics is algebraic in the pure braids case and rational in the braids case as it is proved here for the P_{V} case. We will return to such questions in a future paper.

[7] Using an approach based on Fricke coordinates slightly different of anterior computations by other authors, cf. [29].

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