

Weighted Tensor Products of Joyal Species, Graphs, and Charades

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Received August 18, 2015, in final form January 14, 2016; Published online January 17, 2016

<http://dx.doi.org/10.3842/SIGMA.2016.005>

Abstract. Motivated by the weighted Hurwitz product on sequences in an algebra, we produce a family of monoidal structures on the category of Joyal species. We suggest a family of tensor products for charades. We begin by seeing weighted derivational algebras and weighted Rota–Baxter algebras as special monoids and special semigroups, respectively, for the same monoidal structure on the category of graphs in a monoidal additive category. Weighted derivations are lifted to the categorical level.

Key words: weighted derivation; Hurwitz series; monoidal category; Joyal species; convolution; Rota–Baxter operator

2010 Mathematics Subject Classification: 18D10; 05A15; 18A32; 18D05; 20H30; 16T30

1 Introduction

The weighted Hurwitz product arises naturally in the study of weighted derivations and weighted Rota–Baxter operators; see [12, 23] and their references. We review these concepts from a categorical perspective in Section 2.

In Section 3 we present a monoidal structure on the category $\text{Gph } \mathcal{V}$ of graphs in a monoidal additive category \mathcal{V} . We show how these derivations and operators on a monoid A in \mathcal{V} can be viewed as monoid and semigroup structures on particular graphs constructed from A .

The main theme of the paper is to lift concepts defined on algebras (or monoids) to concepts defined on monoidal categories¹. The λ -weighted Hurwitz product is traditionally defined on the abelian group $A^{\mathbb{N}}$ of sequences in an algebra A . This lifts in an obvious way, for a set Λ , to a definition of Λ -weighted product on the category $\mathcal{V}^{\mathbb{N}}$ of sequences of objects in a reasonable monoidal \mathcal{V} . Rather than $\mathcal{V}^{\mathbb{N}}$, our interest here is in the category $[\mathfrak{S}, \mathcal{V}]$ of \mathcal{V} -valued Joyal species [13] and a weighted version of the convolution product, called Cauchy product in [2]. This is fully discussed in Sections 4–8 and 10, and motivates a definition of weighted categorical derivation in Section 9. The weighted tensors define an interesting family of tensor products for linear representations of the symmetric groups which, by choosing specific weights, include the Cauchy product and the Heisenberg product [20].

Finally, Sections 11 and 12 suggest and explore a weighted tensor product for species on finite vector spaces (or charades [18, 22]). This generalizes, in particular, the essentially classical tensor product of representations of the general linear groups over a finite field, proved braided in [17].

Further insights into the subject of this paper appear in [11].

¹Some authors call this process “categorification”.

2 Review of weighted derivations and Rota–Baxter operators

The inspiration for the family of tensor products on species came from the λ -weighted product of Hurwitz series as discussed in [12, 23] and their references. They begin by defining a *derivation of weight* λ on an algebra A over a commutative ring k , with given $\lambda \in k$, to be a k -module morphism $d: A \rightarrow A$ satisfying $d(1) = 0$ and

$$d(ab) = d(a)b + ad(b) + \lambda d(a)d(b).$$

They note the generalized Leibnitz rule

$$d^n(ab) = \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k d^{n-j}(a) d^{k+j}(b).$$

However, I prefer to write this in the form

$$d^n(ab) = \sum_{n=r+s+t} \binom{n}{r, s, t} \lambda^t d^{r+t}(a) d^{s+t}(b) \quad (2.1)$$

to emphasise the relationship to the trinomial expansion rule for $(x + y + \lambda xy)^n$. Here

$$\binom{n}{r, s, t} = \frac{n!}{r!s!t!}.$$

The λ -Hurwitz product on $A^{\mathbb{N}}$ can be defined by the clearly related equation

$$(f \cdot^\lambda g)(n) = \sum_{n=r+s+t} \binom{n}{r, s, t} \lambda^t f(r+t)g(s+t). \quad (2.2)$$

Example 2.1. For $\lambda = 0$, $k = \mathbb{R}$ and A the algebra of smooth functions $f: \mathbb{R} \rightarrow \mathbb{R}$ under pointwise addition and multiplication, the differentiation function $d: A \rightarrow A$ is a 0-weighted derivation by the classical Leibnitz rule.

Example 2.2. For λ invertible, $k = \mathbb{R}$ and A the algebra of functions $f: \mathbb{R} \rightarrow \mathbb{R}$ under pointwise addition and multiplication, the function $d: A \rightarrow A$ defined by

$$d(f)(x) = \frac{f(x + \lambda) - f(x)}{\lambda}$$

is a λ -weighted derivation.

Example 2.3. Define $d: A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ by $d(s)(n) = s(n+1) - s(n)$. This d is a 1-weighted derivation when $A^{\mathbb{N}}$ is equipped with the pointwise addition and multiplication.

Example 2.4. Define $d: A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ by $d(f)(n) = f(n+1)$. This d is a λ -weighted derivation when $A^{\mathbb{N}}$ is equipped with the λ -Hurwitz product for any λ . Notice that we have an algebra morphism $d^*: A^{\mathbb{N}} \rightarrow (A^{\mathbb{N}})^{\mathbb{N}}$ defined by $d^*(f)(m)(n) = f(m+n)$. This may motivate the next definition.

Define $d^*: A \rightarrow A^{\mathbb{N}}$ by $d^*(a)(n) = d^n(a)$. We see that the Leibnitz rule (2.1) amounts to

Proposition 2.5. $d^*: A \rightarrow A^{\mathbb{N}}$ is an algebra morphism for all λ -weighted derivations d on A , where $A^{\mathbb{N}}$ has the λ -Hurwitz product.

In fact, $A \mapsto A^{\mathbb{N}}$ is a comonad

$$G = ((-)^{\mathbb{N}}, \varepsilon, \delta) \quad (2.3)$$

on the category Alg_k of k -algebras whose Eilenberg–Moore-coalgebras are k -algebras A equipped with a λ -derivation, so-called λ -derivation algebras; write DA_λ for the category of these. The morphism $d^*: A \rightarrow A^{\mathbb{N}}$ is the coaction of the comonad.

Where there is differentiation, there should also be integration. A *Rota–Baxter operator of weight λ* on a k -algebra A is a k -linear morphism $P: A \rightarrow A$ satisfying

$$P(a)P(b) = P(P(a)b + P(a)b + \lambda ab).$$

The pair (A, P) is called a λ -weighted Rota–Baxter algebra. Write RBA_λ for the category of these.

Example 2.6. For $\lambda = 0$, $k = \mathbb{R}$ and A the algebra of continuous functions $f: \mathbb{R} \rightarrow \mathbb{R}$ under pointwise addition and multiplication, the integration function $P: A \rightarrow A$, defined by $P(f)(x) = \int_0^x f(t)dt$, is a 0-weighted Rota–Baxter operator by the classical integration-by-parts rule.

Example 2.7. For $\lambda = 1$ and any k -algebra A , define $P: A^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$ to take a sequence u in A to its sequence $P(u)$ of partial sums

$$P(u)(n) = \sum_{i=0}^{n-1} u(i).$$

Then P is a 1-weighted Rota–Baxter operator on $A^{\mathbb{N}}$ with pointwise addition and multiplication. See [4, 21]. For d the consecutive difference operator as defined in Example 2.3, notice that $d \circ P = 1_{A^{\mathbb{N}}}$.

Example 2.8. If Q is a 1-weighted Rota–Baxter operator on A then $P(a) = \lambda Q(a)$ defines a λ -weighted Rota–Baxter operator P on A .

A λ -weighted derivation RB-algebra is a k -algebra A equipped with a λ -weighted derivation d and a λ -weighted Rota–Baxter operator P such that $d \circ P = 1_A$. Write DRB_λ for the category of these.

Proposition 2.9 (see [12]). *Let P be a RB-operator of weight λ on A . Then $A^{\mathbb{N}}$ equipped with the λ -Hurwitz product, the derivation d of Example 2.4, and P defined by*

$$P(f)(n) = \begin{cases} P(f(0)) & \text{for } n = 0, \\ f(n-1) & \text{for } n > 0 \end{cases}$$

is a λ -weighted derivation RB-algebra. Moreover, the following square commutes

$$\begin{array}{ccc} A^{\mathbb{N}} & \xrightarrow{P} & A^{\mathbb{N}} \\ \text{ev}_0 \downarrow & & \downarrow \text{ev}_0 \\ A & \xrightarrow{P} & A \end{array}$$

With a little more work following Proposition 2.9, we see that the comonad G (2.3) lifts to RBA_λ . In particular, with V denoting the forgetful functor, we have a comonad \bar{G} and a commutative square

$$\begin{array}{ccc} \text{RBA}_\lambda & \xrightarrow{\bar{G}} & \text{RBA}_\lambda \\ V \downarrow & & \downarrow V \\ \text{Alg}_k & \xrightarrow{G} & \text{Alg}_k \end{array}$$

Write $\text{RBA}_{\lambda-}$ for the category of λ -weighted Rota–Baxter algebras where we do not insist on the algebras having a unit.

Proposition 2.10. *For each $(A, \cdot, P) \in \text{RBA}_{\lambda-}$, there is an associative binary operation $a \diamond b$ defined on A by*

$$a \diamond b = P(a) \cdot b + a \cdot P(b) + \lambda a \cdot b.$$

Moreover, $T(A, \cdot, P) = (A, \diamond, P)$ defines an endofunctor

$$T: \text{RBA}_{\lambda-} \longrightarrow \text{RBA}_{\lambda-},$$

which is (well-)copointed by a natural transformation $\gamma: T \Rightarrow 1_{\text{RBA}_{\lambda-}}$ whose component at (A, \cdot, P) is $P: (A, \diamond, P) \rightarrow (A, \cdot, P)$.

Remark 2.11. The day after my seminar talk of 4 February 2015, on the material of this section and Section 4, Stephen Lack made the following comments:

1. Consider the category $[\Sigma\mathbb{N}, \mathcal{V}]$ whose objects are pairs (M, d) consisting of an object M of a nice monoidal additive category \mathcal{V} and an endomorphism $d: M \rightarrow M$. The forgetful functor

$$U: [\Sigma\mathbb{N}, \mathcal{V}] \longrightarrow \mathcal{V}, \tag{2.4}$$

taking (M, d) to M , has a right adjoint taking M to $(M^{\mathbb{N}}, d)$ where $d(f)(n) = f(n+1)$. There is a monoidal structure on $[\Sigma\mathbb{N}, \mathcal{V}]$ defined by

$$(M, d) \otimes^{\lambda} (N, d) = (M \otimes N, d \otimes 1 + 1 \otimes d + \lambda d \otimes d).$$

The monoids in this monoidal category are precisely λ -derivation algebras. Moreover, U (2.4) and its right adjoint form a monoidal adjunction which therefore defines an adjunction between the categories of monoids. This adjunction generates the comonad G (2.3) on the category $\text{Mon } \mathcal{V}$ of monoids in \mathcal{V} .

2. There is a bialgebra structure on the polynomial algebra $k[x]$ with comultiplication the algebra morphism $\delta: k[x] \rightarrow k[x, y] \cong k[x] \otimes k[x]$ defined by

$$\delta(x) = x + y + \lambda xy.$$

Then the convolution product on the left-hand side of the canonical isomorphism

$$\text{Mod}_k(k[x], A) \cong A^{\mathbb{N}}$$

transports to the λ -Hurwitz product on $A^{\mathbb{N}}$.

3. It feels like there should be a multicategory/promonoidal/substitute structure on $[\Sigma\mathbb{N}, \mathcal{V}]$ for dealing with RB-algebras.

3 Graphs in monoidal additive categories

Let \mathcal{V} be a monoidal additive category. We act as if the monoidal structure were strict.

Let $\text{Gph } \mathcal{V}$ be the category of directed graphs in \mathcal{V} . So an object has the form of a pair of parallel morphisms $s, t: E \rightarrow A$ in \mathcal{V} ; we use s and t for source and target morphisms in all

graphs. A morphism $(f, \phi): (A, E) \rightarrow (B, F)$ in $\text{Gph } \mathcal{V}$ consists of morphisms f and ϕ making the following diagram commute

$$\begin{array}{ccccc} A & \xleftarrow{s} & E & \xrightarrow{t} & A \\ f \downarrow & & \downarrow \phi & & \downarrow f \\ B & \xleftarrow{s} & F & \xrightarrow{t} & B \end{array}$$

Write $\text{ver}: \text{Gph } \mathcal{V} \rightarrow \mathcal{V}$ for the forgetful functor taking (A, E) to A and write $\text{edg}: \text{Gph } \mathcal{V} \rightarrow \mathcal{V}$ for the forgetful functor taking (A, E) to E .

We will use the notation $\langle n \rangle = \{1, 2, \dots, n\}$. For $R \subseteq \langle n \rangle$, write

$$\chi_R: \langle n \rangle \rightarrow \{s, t\}$$

for the characteristic function of R defined by

$$\chi_R(i) = \begin{cases} s & \text{for } i \in R, \\ t & \text{for } i \notin R. \end{cases}$$

Choose an endomorphism $\lambda: I \rightarrow I$ of the tensor unit I in \mathcal{V} . For any $f: A \rightarrow B$ in \mathcal{V} , we define $(\lambda f: A \rightarrow B) = (\lambda \otimes f: I \otimes A \rightarrow I \otimes B)$.

Given a list $(A_1, E_1), \dots, (A_n, E_n)$ of objects of $\text{Gph } \mathcal{V}$, we define an n -fold tensor product

$$\otimes_{1 \leq i \leq n}^{\lambda} (A_i, E_i) = (\otimes_{1 \leq i \leq n} A_i, \otimes_{1 \leq i \leq n} E_i), \quad (3.1)$$

where

$$s = \sum_{\emptyset \neq R \subseteq \langle n \rangle} \lambda^{(\#R-1)} \chi_R(1) \otimes \dots \otimes \chi_R(n) \quad \text{and} \quad t = t \otimes \dots \otimes t.$$

For $n = 2$ this gives a binary tensor product

$$(A, E) \otimes^{\lambda} (B, F) = (A \otimes B, E \otimes F)$$

with

$$s = \lambda s \otimes s + s \otimes t + t \otimes s \quad \text{and} \quad t = t \otimes t.$$

The unit for this tensor is the graph (I, I) with $s = 0: I \rightarrow I$ and $t = 1_I: I \rightarrow I$.

Proposition 3.1. *A monoidal structure on $\text{Gph } \mathcal{V}$ is defined by (3.1) for any given $\lambda \in \mathcal{V}(I, I)$. Both ver and $\text{edg}: \text{Gph } \mathcal{V} \rightarrow \mathcal{V}$ are strict monoidal.*

Proof. Easy calculations of the source morphisms for

$$((A, E) \otimes^{\lambda} (B, F)) \otimes^{\lambda} (C, G) \quad \text{and} \quad (A, E) \otimes^{\lambda} ((B, F) \otimes^{\lambda} (C, G))$$

show they agree with that of the triple tensor product. The target morphisms obviously agree. What this means is that the associativity constraints for \mathcal{V} lift through ver and edg to $\text{Gph } \mathcal{V}$ and are therefore coherent. \blacksquare

Let $[\Sigma\mathbb{N}, \mathcal{V}]$ denote the category whose objects $(A, e: A \rightarrow A)$ consist of an object A of \mathcal{V} equipped with an endomorphism e . Let

$$J: [\Sigma\mathbb{N}, \mathcal{V}] \rightarrow \text{Gph } \mathcal{V} \quad (3.2)$$

be the functor defined by $J(A, e) = (A, A)$ with $s = e$ and $t = 1_A$; and $Jf = (f, f)$. Notice also that a morphism $(f, \phi): (B, F) \rightarrow J(A, e)$ in $\text{Gph } \mathcal{V}$ with codomain in the subcategory amounts to a commutative diagram

$$\begin{array}{ccccc} F & \xrightarrow{t} & B & \xrightarrow{f} & A \\ s \downarrow & & & & \downarrow e \\ B & \xrightarrow{f} & & & A \end{array}$$

where ϕ is forced to be $f \circ t: F \rightarrow A$. Clearly J is fully faithful and the monoidal structure of Proposition 3.1 restricts to a monoidal structure on $[\Sigma\mathbb{N}, \mathcal{V}]$ yielding (3.2) as a strict monoidal functor. Indeed, this is none other than the monoidal structure of Remark 2.11, item 1.

Definition 3.2. A λ -weighted derivational monoid in \mathcal{V} is a monoid (A, d) in $[\Sigma\mathbb{N}, \mathcal{V}]$ equipped with the monoidal structure obtained as the restriction through (3.2) of that of Proposition 3.1 on $\text{Gph } \mathcal{V}$. An object $(A, d) \in [\Sigma\mathbb{N}, \mathcal{V}]$ with an associative binary operation will be called a λ -weighted derivational semigroup.

More explicitly, a λ -weighted derivational monoid is a monoid A in \mathcal{V} equipped with an endomorphism $d: A \rightarrow A$ satisfying the λ -weighted equation

$$d \circ \mu = \mu \circ (\lambda d \otimes d + d \otimes 1 + 1 \otimes d),$$

and the equation $d \circ \eta = 0$ (where $\eta: I \rightarrow A$ is the unit of A).

There is an isomorphism of categories

$$\text{op}: \text{Gph } \mathcal{V} \longrightarrow \text{Gph } \mathcal{V}$$

taking (A, E) to $(A, E)^{\text{op}}$ for which A and E are unchanged but s and t have been interchanged.

Put

$$J^{\text{op}} = ([\Sigma\mathbb{N}, \mathcal{V}] \xrightarrow{J} \text{Gph } \mathcal{V} \xrightarrow{\text{op}} \text{Gph } \mathcal{V}).$$

Like J , this composite J^{op} is fully faithful.

However, the image of J^{op} is *not* closed under the monoidal structure of Proposition 3.1. All we obtain on $[\Sigma\mathbb{N}, \mathcal{V}]$ is a structure of multicategory (sometimes called a ‘‘coloured operad’’). The sets of multimorphisms are defined by

$$P_\lambda((A_1, p_1), \dots, (A_n, p_n); (B, p)) = \text{Gph } \mathcal{V}(\otimes_{1 \leq i \leq n}^{\lambda} J^{\text{op}}(A_i, p_i), J^{\text{op}}(B, p)). \quad (3.3)$$

To be more explicit, for $R \subseteq \langle n \rangle$ and $i \in \langle n \rangle$, put

$$R(i) = \begin{cases} 1_{A_i} & \text{for } i \in R, \\ p_i & \text{for } i \notin R. \end{cases}$$

Then, an element of the set (3.3), a *multimorphism*, is a morphism

$$f: A_1 \otimes \cdots \otimes A_n \longrightarrow B$$

satisfying the equation

$$f \circ (p_1 \otimes \cdots \otimes p_n) = p \circ f \circ \sum_{\emptyset \neq R \subseteq \langle n \rangle} \lambda^{(\#R-1)} R(1) \otimes \cdots \otimes R(n).$$

This is a case of a general process of obtaining a multicategory structure on a category by restriction along a functor into a monoidal category. The notion of monoid makes sense in any multicategory. We have the following special case.

Definition 3.3. A λ -weighted Rota–Baxter monoid in \mathcal{V} is an object (A, p) of $[\Sigma\mathbb{N}, \mathcal{V}]$ (that is, $p: A \rightarrow A$ in \mathcal{V}) equipped with the structure of semigroup on $J^{\text{op}}(A, p)$ in the monoidal category $\text{Gph } \mathcal{V}$ of Proposition 3.1, and a unit $\eta: I \rightarrow A$ for the underlying semigroup A in \mathcal{V} .

This definition should make the calculation of free weighted Rota–Baxter monoids possible; compare [3, 5, 10, 21].

To make Definition 3.3 a little more explicit, as expected, a λ -weighted Rota–Baxter monoid (A, p) is a monoid A in \mathcal{V} equipped with an endomorphism $p: A \rightarrow A$ satisfying

$$\mu \circ (p \otimes p) = p \circ \mu \circ (\lambda 1 \otimes 1 + 1 \otimes p + p \otimes 1).$$

Derivations and Rota–Baxter operators are not the only sources of semigroups and monoids for the monoidal structure of Proposition 3.1. The forgetful functor

$$\text{ve}: \text{Gph } \mathcal{V} \longrightarrow \mathcal{V} \times \mathcal{V},$$

taking the graph (A, E) to the pair (A, E) , is strict monoidal and has a right adjoint R defined by

$$R(X, Y) = (X, X \oplus X \oplus Y)$$

with $s = \text{pr}_1$ (the first projection) and $t = \text{pr}_2$ (the second projection). It follows that R is monoidal and hence takes monoids to monoids.

Example 3.4. Take $\mathcal{V} = \text{Mod}_k$, the category of modules over a commutative ring k . For a graph (A, E) in this \mathcal{V} , we can write $e: a \rightarrow b$ to mean $a, b \in A$, $e \in E$ with $s(e) = a$, $t(e) = b$. For k -algebras A and B , we obtain a monoid $R(A, B)$ in $\text{Gph } \mathcal{V}$: the graph is $\text{pr}_1, \text{pr}_2: A \oplus A \oplus B \rightarrow A$ and the multiplication is defined by:

$$((a_1, a_2, b): a_1 \rightarrow a_2) \cdot ((c_1, c_2, d): c_1 \rightarrow c_2) = (\lambda a_1 c_1 + a_1 c_2 + a_2 c_1, a_2 c_2, bd): a_1 c_1 \rightarrow a_2 c_2.$$

Of course the \mathcal{V} -functor J (3.2) has both adjoints if \mathcal{V} is complete and cocomplete enough. In particular, the right adjoint

$$K: \text{Gph } \mathcal{V} \longrightarrow [\Sigma\mathbb{N}, \mathcal{V}]$$

is defined by taking $K(A, E)$ to be the equalizer of the two morphisms

$$s^{\mathbb{N}}, t^{\text{succ}}: E^{\mathbb{N}} \rightarrow A^{\mathbb{N}}$$

equipped with the endomorphism $e: K(A, E) \rightarrow K(A, E)$ induced by E^{succ} . Here $\text{succ}: \mathbb{N} \rightarrow \mathbb{N}$ is the successor function $n \mapsto n + 1$. Since J (3.2) is strong monoidal for the monoidal structures under discussion, the adjunction $J \dashv K$ is monoidal. So K takes semigroups to semigroups and monoids to monoids.

In particular, if (A, p) is a λ -weighted Rota–Baxter monoid in \mathcal{V} , then K takes the graph (A, A) with $s = 1_A$ and $t = p$ to a λ -weighted derivational semigroup in \mathcal{V} . The underlying object is the limit of the diagram

$$A \xleftarrow{p} A \xleftarrow{p} A \xleftarrow{p} \dots$$

in \mathcal{V} .

Example 3.5. Taking $\mathcal{V} = \text{Vect}_k$ and a λ -weighted Rota–Baxter k -algebra $p: A \rightarrow A$, we have the non-unital λ -weighted derivational k -algebra

$$K(J(A, p)^{\text{op}}) = \{a \in A^{\mathbb{N}} \mid p(a_{n+1}) = a_n\}$$

with $d(a)_n = a_{n+1}$. The multiplication on $K(J(A, p)^{\text{op}})$ is the restriction of the λ -weighted Hurwitz multiplication on $A^{\mathbb{N}}$ arising from the non-unital algebra (A, \diamond) of Proposition 2.10. Moreover, $K(J(A, p)^{\text{op}})$ supports a λ -weighted Rota–Baxter operator p defined by $p(a)_n = p(a_n)$. Notice too that $d \circ p = 1$.

We conclude this section by describing the promonoidal structure in the sense of Day [8] with respect to which the monoidal structure of Proposition 3.1 is convolution.

Let \mathbb{G} denote the category whose only objects are 0 and 1, with the only non-identity morphisms $\sigma, \tau: 1 \rightarrow 0$. Write $I_*\mathbb{G}$ for the free \mathcal{V} -category on \mathbb{G} . Then $\text{Gph } \mathcal{V} = [\mathbb{G}, \mathcal{V}] = [I_*\mathbb{G}, \mathcal{V}]$ where the first set of square brackets means the ordinary functor category while the second means the \mathcal{V} -enriched functor category. The promonoidal structure in question is technically on $I_*\mathbb{G}$ in the \mathcal{V} -enriched sense. However, we can look at it as consisting of an ordinary a functor

$$P: \mathbb{G}^{\text{op}} \times \mathbb{G}^{\text{op}} \longrightarrow \text{Gph } \mathcal{V}$$

and an object $J \in \text{Gph } \mathcal{V}$. Of course J is just the graph $0, 1: I \rightarrow I$ which is the tensor unit. We can regard P as a “cograph of cographs of graphs” (although a cograph looks just like a graph):

$$\begin{array}{ccc}
 I & \xrightarrow{(1,0)} & I \oplus I = 2 \cdot I \\
 \downarrow (1,0) & \searrow (0,1) & \downarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\
 I & & I \\
 \downarrow (1,0) & \searrow (0,1) & \downarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
 2 \cdot I = I \oplus I & \xrightarrow{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}} & ((\lambda, 1, 1, 0), (0, 0, 0, 1)): I \rightarrow 4 \cdot I \\
 & \searrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} &
 \end{array}$$

4 The L -Hurwitz product of species

Let \mathfrak{S} denote the groupoid whose objects are finite sets and whose morphisms are bijective functions. We write $U + V$ for the disjoint union of sets U and V ; this is the binary coproduct as objects of the category Set of sets and all functions. It is not the coproduct in \mathfrak{S} ; yet it does provide the symmetric monoidal structure on \mathfrak{S} of interest here. When we write $X = A + B$ for A and B subsets of a set X , we mean $X = A \cup B$ and $\emptyset = A \cap B$.

We have the particular finite sets $\langle n \rangle = \{1, 2, \dots, n\}$.

Let \mathcal{V} denote a monoidal category with finite coproducts which are preserved by tensoring on either side by an object. The tensor product of $V, W \in \mathcal{V}$ is denoted by $V \otimes W$ and the unit object by I . Justified by coherence theorems (see [16] for example), we write as if the monoidal structure on \mathcal{V} were strictly associative and strictly unital. For any set S , write $S \cdot V$ for the coproduct of S copies of $V \in \mathcal{V}$, when it exists (as it does for S finite).

The *category of \mathcal{V} -valued Joyal species*, after [13, 14], is the functor category $[\mathfrak{S}, \mathcal{V}]$. The objects will simply be called *species* when \mathcal{V} is understood.

Suppose $L: \mathfrak{S} \rightarrow \mathcal{Z}\mathcal{V}$ is a braided strong monoidal functor into the monoidal centre (in the sense of [15]) of \mathcal{V} . We have natural isomorphisms

$$u_{X,V}: LX \otimes V \cong V \otimes LX, \tag{4.1}$$

such that

$$\begin{array}{ccc}
 LX \otimes V \otimes W & \xrightarrow{u_{X,V} \otimes 1_W} & V \otimes LX \otimes W \\
 & \searrow^{u_{X,V} \otimes W} & \swarrow_{1_V \otimes u_{X,W}} \\
 & & V \otimes W \otimes LX
 \end{array}$$

If \mathcal{V} itself is braided (*a fortiori* symmetric), we can take a braided strong monoidal functor $\mathfrak{S} \rightarrow \mathcal{V}$ and compose it with the canonical braided strong monoidal functor $\mathcal{V} \rightarrow \mathcal{Z}\mathcal{V}$ to obtain such an L .

By way of example of an $L: \mathfrak{S} \rightarrow \mathcal{Z}\mathcal{V}$, we could take any finite set Λ and $LX = \Lambda^X \cdot I$ with $L\sigma = \Lambda^{\sigma^{-1}} \cdot I$ for any bijective function σ .

Definition 4.1. The L -Hurwitz product $F \otimes^L G$ of species F and G is defined on objects $X \in \mathfrak{S}$ by

$$(F \otimes^L G)X = \sum_{X=U \cup V} L(U \cap V) \otimes FU \otimes GV. \quad (4.2)$$

The definition of $F \otimes^L G$ on morphisms is clear since any bijective function $\sigma: X \rightarrow Y$ restricts to bijections

$$U \rightarrow \sigma U, \quad V \rightarrow \sigma V, \quad U \cup V \rightarrow \sigma U \cup \sigma V, \quad U \cap V \rightarrow \sigma U \cap \sigma V.$$

Let $J: \mathfrak{S} \rightarrow \mathcal{V}$ be the species whose value at X is the unit I for tensor in \mathcal{V} when X is empty and is initial in \mathcal{V} otherwise. Clearly J is a unit for the L -Hurwitz product in the sense that we have canonical isomorphisms

$$\lambda_G: J \otimes^L G \rightarrow G \quad \text{and} \quad \rho_F: F \rightarrow F \otimes^L J.$$

Associativity isomorphisms

$$\alpha_{F,G,H}: (F \otimes^L G) \otimes^L H \cong F \otimes^L (G \otimes^L H) \quad (4.3)$$

are obtained using the following result easily proved by Venn diagrams.

Lemma 4.2. $(U \cup V) \cap W + U \cap V \cong U \cap (V \cup W) + V \cap W$.

Then, to define (4.3), we use the isomorphisms

$$\begin{aligned}
 & L((U \cup V) \cap W) \otimes L(U \cap V) \otimes FU \otimes GV \otimes HW \\
 & \cong L((U \cup V) \cap W + U \cap V) \otimes FU \otimes GV \otimes HW \\
 & \cong L(U \cap (V \cup W) + V \cap W) \otimes FU \otimes GV \otimes HW \\
 & \cong L(U \cap (V \cup W)) \otimes L(V \cap W) \otimes FU \otimes GV \otimes HW \\
 & \cong L(U \cap (V \cup W)) \otimes FU \otimes L(V \cap W) \otimes GV \otimes HW,
 \end{aligned}$$

the first and third coming from the strong monoidal structure on L , the second from Lemma 4.2, and the fourth from the monoidal centre structure (4.1) on $L(V \cap W)$.

In the case $L = J$, we recover from (4.2) the usual convolution (Cauchy) product of species appearing in [13]. In the case where L is the exponential series $LX = I$ for all X , we recover the Heisenberg product appearing in [1, 20].

For a general L , the term $L(U \cap V)$ can be considered a measure of the failure of U and V to be disjoint.

5 A combinatorial interpretation

We consider the case where $\mathcal{V} = \text{Set}$ so that $[\mathfrak{S}, \text{Set}]$ is the category of species as studied in [13]. Fix any set Λ . Define the species L by

$$LX = \left\{ S = (S_\lambda)_{\lambda \in \Lambda} \mid S_\lambda \subseteq X, \sum_{\lambda \in \Lambda} S_\lambda = X \right\}$$

and $(L\sigma)S = (\sigma S_\lambda)_{\lambda \in \Lambda}$. In other words, a structure of the species L on the set X is a partition of X into a Λ -indexed family of disjoint (possibly empty) subsets.

A structure of the species $F \otimes^L G$ on the set consists of a quintuplet (U, V, S, ϕ, γ) where U, V are subsets of X such that $X = U \cup V$, and S, ϕ, γ are L -, F -, G -structures on $U \cap V, U, V$, respectively.

We write $\#S$ for the cardinality of the set S . We assume Λ is finite and put $\lambda = \#\Lambda$.

The *cardinality sequence* of a species F is the sequence $\#F: \mathbb{N} \rightarrow \mathbb{Z}$ defined by

$$(\#F)(n) = \#F\langle n \rangle.$$

We consider the λ -Hurwitz product (2.2) on $\mathbb{Z}^{\mathbb{N}}$.

Proposition 5.1. $\#(F \otimes^L G) = \#F \cdot^\lambda \#G$.

This result specializes to Theorem 2.7 of [1] when L is the exponential species and $\lambda = 1$.

6 The iterated tensor and coherence

Proposition 6.1. *An alternative definition of $F \otimes^L G$ is*

$$(F \otimes^L G)X = \sum_{X=A+B+C} L(C) \otimes F(A+C) \otimes G(B+C).$$

Proof. Given $X = A + B + C$, put $U = A + C$ and $V = B + C$. Given $X = U \cup V$, put $A = U \setminus V$, $B = V \setminus U$, and $C = U \cap V$. ■

The n -fold version of this tensor product is

$$\begin{aligned} & \otimes_n^L(F_1, \dots, F_n)X \\ &= \sum_{X = \sum_{\emptyset \neq S \subseteq \langle n \rangle} A_S} L\left(\sum_S (\#S - 1) \cdot A_S\right) \otimes F_1\left(\sum_{1 \in S} A_S\right) \otimes \dots \otimes F_n\left(\sum_{n \in S} A_S\right). \end{aligned} \quad (6.1)$$

This yields the formula in Proposition 6.1 for $n = 2$ by taking $A = A_1$, $B = A_2$, $C = A_{\{1,2\}}$. Note that (6.1) is unchanged if we replace $\langle n \rangle$ by any set of cardinality n .

Remark 6.2. As Joachim Kock reminded me, if we replace $\langle n \rangle$ by the ‘ $(n-1)$ -simplex’ $[n-1] = \{0, 1, \dots, n-1\}$, then the non-empty subsets S correspond to the non-degenerate faces of $[n-1]$ and $\#S - 1$ is the dimension of the face.

Let us consider the effect of inserting one pair of parentheses in a multiple tensor (6.1). We look at

$$\otimes_{p+1+r}^L(F_1, \dots, F_p, \otimes_q^L(F_{p+1}, \dots, F_{p+q}), F_{p+q+1}, \dots, F_{p+q+r})X.$$

Using (6.1) twice, once with $n = p + 1 + r$ and once with $n = q$, we obtain the expression

$$\begin{aligned} & L \left(\sum_T (\#T - 1) \cdot B_T \right) \otimes F_1 \left(\sum_{1 \in T} B_T \right) \otimes \cdots \otimes F_p \left(\sum_{p \in T} B_T \right) \\ & \quad \otimes L \left(\sum_R (\#R - 1) \cdot C_R \right) \otimes F_{p+1} \left(\sum_{p+1 \in R} C_R \right) \otimes \cdots \otimes F_{p+q} \left(\sum_{p+q \in R} C_R \right) \\ & \quad \otimes F_{p+q+1} \left(\sum_{p+q+1 \in T} B_T \right) \otimes \cdots \otimes F_{p+q+r} \left(\sum_{p+q+r \in T} B_T \right) \end{aligned}$$

summed over all families

$$B = (B_T \mid \emptyset \neq T \subseteq \{1, \dots, p, \star, p+q+1, \dots, p+q+r\})$$

providing a partition $X = \sum_T B_T$ of X , together with all families

$$C = (C_R \mid \emptyset \neq R \subseteq \{p+1, \dots, p+q\})$$

providing a partition $\sum_{\star \in T} B_T = \sum_R C_R$ of $\sum_{\star \in T}$. Using the fact that L lands in $\mathcal{Z}^{\mathcal{V}}$ and that L is strong monoidal, we obtain the isomorphic expression

$$\begin{aligned} & L \left(\sum_T (\#T - 1) \cdot B_T + \sum_R (\#R - 1) \cdot C_R \right) \otimes F_1 \left(\sum_{1 \in T} B_T \right) \otimes \cdots \otimes F_p \left(\sum_{p \in T} B_T \right) \\ & \quad \otimes F_{p+1} \left(\sum_{p+1 \in R} C_R \right) \otimes \cdots \otimes F_{p+q} \left(\sum_{p+q \in R} C_R \right) \\ & \quad \otimes F_{p+q+1} \left(\sum_{p+q+1 \in T} B_T \right) \otimes \cdots \otimes F_{p+q+r} \left(\sum_{p+q+r \in T} B_T \right) \end{aligned} \tag{6.2}$$

summed over the same families (B, C) . For $\star \in T$, we have $B_T = \sum_R C_R \cap B_T$. On the other hand, $C_R = \sum_{\star \in T} C_R \cap B_T$. Put

$$Q = \{p+1, \dots, p+q\} \quad \text{and} \quad N = \{1, \dots, p\} \cup \{p+q+1, \dots, p+q+r\},$$

and obtain a family

$$A = (A_S \mid \emptyset \neq S \subseteq \langle p+q+r \rangle)$$

partitioning X by defining

$$A_S = \begin{cases} B_S & \text{for } S \cap Q = \emptyset, \\ C_{S \cap Q} \cap B_{(S \cap N) \cup \{\star\}} & \text{for } S \cap Q \neq \emptyset. \end{cases}$$

Then we can recover the B and C families via

$$B_T = \begin{cases} A_T & \text{for } \star \notin T, \\ \sum_R A_{R \cup (T \setminus \star)} & \text{for } \star \in T, \end{cases} \quad \text{and} \quad C_R = \sum_{\star \in T} A_{R \cup (T \setminus \star)}.$$

We have the following equations

$$(i) \quad \sum_S (\#S - 1) \cdot A_S = \sum_T (\#T - 1) \cdot B_T + \sum_R (\#R - 1) \cdot C_R,$$

$$(ii) \quad \sum_{k \in S} A_S = \begin{cases} \sum_{k \in T} B_T & \text{for } 1 \leq k \leq p \text{ or } p+q+1 \leq k \leq p+q+r, \\ \sum_{k \in R} C_R & \text{for } p+1 \leq k \leq p+q. \end{cases}$$

This shows that the sum of the expressions (6.2) over the pairs (B, C) is equal to (6.1) with $n = p + q + r$. Remember however that the tensor product \otimes on \mathfrak{S} is not strict symmetric; the symmetry on \mathfrak{S} provides canonical bijections between the left- and right-hand sides of (i) and (ii). Since L is braided, we have constructed a natural isomorphism

$$a_{p,q,r}: \otimes_n^L (F_1, \dots, F_{p+q+r}) \cong \otimes_{p+1+r}^L (F_1, \dots, F_p, \otimes_q^L (F_{p+1}, \dots, F_{p+q}), F_{p+q+1}, \dots, F_{p+q+r}). \quad (6.3)$$

Now consider the Mac Lane–Stasheff pentagon for 2-fold bracketings of $F_1 \otimes^L F_2 \otimes^L F_3 \otimes^L F_4$ as the vertices. Let $a: H \rightarrow K$ denote one of the edges of the pentagon obtained using the associativity isomorphisms (4.3). There is a composite b of two isomorphisms, each using one instance of an isomorphism (6.3), which goes from $\otimes_4^L (F_1, F_2, F_3, F_4)$ to H , and another one $c: \otimes_4^L (F_1, F_2, F_3, F_4) \rightarrow H$. By coherence of the braided strong monoidal functor L , it follows that $a \circ b = c$. Commutativity of the pentagon is a consequence of commutativity of all these triangular sides of the so-formed pentagonal cone.

7 Promonoidal structures on \mathfrak{S}

For finite sets A, B and X , let $\text{Cov}(A, B; X)$ denote the set of jointly surjective pairs (μ, ν) of injective functions

$$A \xrightarrow{\mu} X \xleftarrow{\nu} B.$$

We write $A \times_X B$ for the pullback of μ and ν .

Define a functor

$$P: \mathfrak{S}^{\text{op}} \times \mathfrak{S}^{\text{op}} \times \mathfrak{S} \longrightarrow \mathcal{V}$$

by

$$P(A, B; X) = \sum_{(\mu, \nu) \in \text{Cov}(A, B; X)} L(A \times_X B).$$

Proposition 7.1. $(F \otimes^L G)X \cong \int^{A, B} P(A, B; X) \otimes FA \otimes GB.$

Proof. A universal dinatural transformation

$$\theta_{A, B}: P(A, B; X) \otimes FA \otimes GB \longrightarrow \sum_{X=U \cup V} L(U \cap V) \otimes FU \otimes GV$$

is defined by taking its composite with the injection at $(\mu, \nu) \in \text{Cov}(A, B; X)$ to be obtained from the $(\mu(A), \nu(B))$ injection and the bijections $A \cong \mu(A)$, $B \cong \nu(B)$, $A \times_X B \cong \mu(A) \cap \nu(B)$, noting $X = \mu(A) \cup \nu(B)$. \blacksquare

By Day’s general theory of promonoidal categories [7, 8], we have

Corollary 7.2. *If moreover \mathcal{V} is (left and right) closed and sufficiently complete then \otimes^L defines a (left and right) closed monoidal structure on $[\mathfrak{S}, \mathcal{V}]$. The monoidal structure coincides with that of Section 4.*

8 The weighted bimonoidale structure on \mathfrak{S}

Lately (for example, in [6]), we have used the term *monoidale* for “pseudomonoid”, also called “monoidal object”, in a monoidal bicategory \mathcal{M} [9]. For example, the monoidales in the cartesian monoidal bicategory Cat are monoidal categories.

When the monoidal bicategory \mathcal{M} is symmetric, the monoidales themselves form a symmetric monoidal bicategory where the morphisms are strong monoidal. With the same tensor product, the opposite bicategory \mathcal{M}^{op} is symmetric monoidal. A *bimonoidale in \mathcal{M}* is a monoidale in \mathcal{M}^{op} . Incidentally, every monoidale in Cat is uniquely a bimonoidale.

Consider the 2-category Cat_+ of (small) categories admitting finite coproducts, and finite-coproduct-preserving functors. This becomes a symmetric closed monoidal bicategory (see [9]) with tensor product $\mathcal{A} \boxtimes \mathcal{B}$ representing functors $H: \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{X}$ for which each $H(A, -)$ and each $H(-, B)$ is finite coproduct preserving. Clearly the monoidal category \mathcal{V} of Section 4 is a monoidale (= pseudomonoid) in Cat_+ .

For any category \mathcal{C} , we write $\text{fam } \mathcal{C}$ for the free finite coproduct completion of \mathcal{C} . That is, fam provides the left biadjoint to the forgetful 2-functor $\text{Cat}_+ \rightarrow \text{Cat}$. Indeed, fam is a strong monoidal pseudofunctor; in particular, there is a canonical equivalence

$$\text{fam } \mathcal{C} \boxtimes \text{fam } \mathcal{D} \simeq \text{fam}(\mathcal{C} \times \mathcal{D}).$$

Every monoidal category \mathcal{C} determines a monoidale $\text{fam } \mathcal{C}$ in Cat_+ .

Explicitly, the objects of $\text{fam } \mathcal{C}$ can be written formally as $\sum_{s \in S} C_s$ where S is a finite set and $C_s \in \mathcal{C}$. Then, if \mathcal{C} is monoidal, the monoidale structure on $\text{fam } \mathcal{C}$ is defined by

$$\sum_{s \in S} C_s \otimes \sum_{t \in T} D_t = \sum_{(s,t) \in S \times T} C_s \otimes D_t.$$

We are interested in $\text{fam } \mathfrak{S}$. By what we have just said, this is a monoidale in Cat_+ :

$$\sum_{s \in S} U_s \otimes \sum_{t \in T} V_t = \sum_{(s,t) \in S \times T} (U_s + V_t).$$

Fix a finite set Λ and define $L: \mathfrak{S} \rightarrow \text{Set}$ by $LX = \Lambda^X$ and $L\sigma = \Lambda^{\sigma^{-1}}$. Define a coproduct-preserving functor

$$\Delta: \text{fam } \mathfrak{S} \longrightarrow \text{fam}(\mathfrak{S} \times \mathfrak{S}) \simeq \text{fam } \mathfrak{S} \boxtimes \text{fam } \mathfrak{S} \tag{8.1}$$

by

$$\Delta(X) = \sum_{X=A+B+C} L(C) \cdot (A + C, B + C)$$

for $X \in \mathfrak{S}$.

Proposition 8.1. *The functor Δ of (8.1) is strong monoidal.*

Proof. In $\Delta(X + Y) = \sum_{X+Y=A+B+C} L(C) \cdot (A + C, B + C)$ we can put

$$\begin{aligned} P &= X \cap A, & Q &= X \cap B, & R &= X \cap C, \\ U &= Y \cap A, & V &= Y \cap B, & W &= Y \cap C \end{aligned}$$

to obtain

$$\begin{aligned} \Delta(X + Y) &= \sum_{X=P+Q+R, Y=U+V+W} L(R+W) \cdot (P+U+R+W, Q+V+R+W) \\ &\cong \sum_{X=P+Q+R} L(R) \cdot (P+R, Q+R) \times \sum_{Y=U+V+W} L(W) \cdot (U+W, V+W) \\ &\cong \Delta X \times \Delta Y, \end{aligned}$$

as required. ■

The relationship between this structure and the promonoidal structure of Section 7 will be examined elsewhere; indeed, see [11].

9 Weighted categorical derivations

Suppose \mathcal{V} is a symmetric monoidal closed category which is complete and cocomplete, and suppose $L: \mathfrak{S} \rightarrow \mathcal{V}$ is a strong monoidal functor.

Harking back to Remark 2.11, we are prompted to consider the 2-category

$$\mathfrak{E} = \text{Hom}(\Sigma\mathfrak{S}, \mathcal{V}\text{-Cat}_{L,+}). \quad (9.1)$$

Here $\Sigma\mathfrak{S}$ denotes the bicategory with one object (denoted \star) whose homcategory is the symmetric groupoid \mathfrak{S} ; composition is provided by the monoidal structure $+$ on \mathfrak{S} . Also $\mathcal{V}\text{-Cat}_{L,+}$ denotes the 2-category of \mathcal{V} -categories admitting finite coproducts and tensoring with the object $L(X)$ of \mathcal{V} ; the morphisms are \mathcal{V} -functors preserving these colimits; the 2-cells are \mathcal{V} -natural transformations. The objects of (9.1) are pseudofunctors $T: \Sigma\mathfrak{S} \rightarrow \mathcal{V}\text{-Cat}_{L,+}$, the morphisms are pseudonatural transformations, and the 2-cells are modifications (in terminology of [19]). Such an object T determines a \mathcal{V} -category $T\star = \mathcal{M} \in \mathcal{V}\text{-Cat}_{L,+}$ and a strong monoidal functor $T_{\star\star}: \mathfrak{S} \rightarrow \mathcal{V}\text{-Cat}_{L,+}(\mathcal{M}, \mathcal{M})$. This $T_{\star\star}$ is determined up to equivalence by an endomorphism $D: \mathcal{M} \rightarrow \mathcal{M}$ in $\mathcal{V}\text{-Cat}_{L,+}$ and an involutive Yang–Baxter² operator $\rho: D \circ D \Rightarrow D \circ D$ on D (for example, see [15] for terminology). Then $T_{\star\star}(n) \cong D^{\circ n}$ and, for the non-identity bijection $\tau: \langle 2 \rangle \rightarrow \langle 2 \rangle$, $T\tau$ transports to ρ . Therefore we shall write the object T of \mathfrak{E} (9.1) as a pair (\mathcal{M}, D^*) where $T\star = \mathcal{M}$ and $T_{\star\star} = D^*$. The morphisms of \mathfrak{E} are then squares

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{K} & \mathcal{N} \\ D^*X \downarrow & \xrightarrow{\kappa_X \cong} & \downarrow E^*X \\ \mathcal{M} & \xrightarrow{K} & \mathcal{N} \end{array}$$

in $\mathcal{V}\text{-Cat}_{L,+}$ which are \mathcal{V} -natural in X and, stacking vertically, respect the tensor in \mathfrak{S} . Generalizing the tensor \boxtimes on Cat_+ as in Section 8, we have a tensor, also denoted by \boxtimes , on $\mathcal{V}\text{-Cat}_{L,+}$, where the tensor product $\mathcal{A} \boxtimes \mathcal{B}$ represents \mathcal{V} -functors $H: \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{X}$ for which each of $H(A, -)$ and $H(-, B)$ preserves finite coproducts and tensoring with each $L(X)$. This makes $\mathcal{V}\text{-Cat}_{L,+}$ into a monoidal bicategory.

This tensor product \boxtimes on $\mathcal{V}\text{-Cat}_{L,+}$ lifts to one, denoted by $\widehat{\boxtimes}$, on \mathfrak{E} (9.1):

$$(\mathcal{M}, D^*) \widehat{\boxtimes} (\mathcal{N}, E^*) = (\mathcal{M} \boxtimes \mathcal{N}, D^* \widehat{\boxtimes} E^*) m,$$

where

$$(D^* \widehat{\boxtimes} E^*)X = \sum_{X=U \cup V} L(U \cap V) \otimes D^*U \boxtimes E^*V.$$

²This is Rodney Baxter http://en.wikipedia.org/wiki/Rodney_Baxter, not the author of [4].

To see that $D^*\widehat{\boxtimes}E^*: \mathfrak{S} \rightarrow \mathcal{V}\text{-Cat}_{L,+}(\mathcal{M} \boxtimes \mathcal{N}, \mathcal{M} \boxtimes \mathcal{N})$ is strong monoidal, we calculate

$$\begin{aligned}
(D^*\widehat{\boxtimes}E^*)(X + Y) &\cong \sum_{X+Y=U \cup V} L(U \cap V) \otimes D^*U \boxtimes E^*V \\
&\cong \sum_{X=U_1 \cup V_1, Y=U_2 \cup V_2} L(U_1 \cap V_1 + U_2 \cap V_2) \otimes (D^*U_1 \circ D^*U_2) \boxtimes (E^*V_1 \circ E^*V_2) \\
&\cong \sum_{X=U_1 \cup V_1, Y=U_2 \cup V_2} L(U_1 \cap V_1) \otimes L(U_2 \cap V_2) \otimes (D^*U_1 \boxtimes E^*V_1) \circ (D^*U_2 \boxtimes E^*V_2) \\
&\cong \sum_{X=U_1 \cup V_1, Y=U_2 \cup V_2} L(U_1 \cap V_1) \otimes (D^*U_1 \boxtimes E^*V_1) \circ L(U_2 \cap V_2) \otimes (D^*U_2 \boxtimes E^*V_2) \\
&\cong (D^*\widehat{\boxtimes}E^*)X \circ (D^*\widehat{\boxtimes}E^*)Y.
\end{aligned}$$

In this way, \mathfrak{E} (9.1) becomes a monoidal bicategory.

Definition 9.1. An L -weighted derivation D^* on a monoidale \mathcal{M} in $\mathcal{V}\text{-Cat}_{L,+}$ is a lifting of the monoidale structure on \mathcal{M} to a monoidale structure on (\mathcal{M}, D^*) in \mathfrak{E} (9.1).

Example 9.2. An L -weighted derivation $D^*: \mathfrak{S} \rightarrow \mathcal{V}\text{-Cat}_{L,+}([\mathfrak{S}, \mathcal{V}], [\mathfrak{S}, \mathcal{V}])$ on the monoidale $([\mathfrak{S}, \mathcal{V}], \otimes^L)$ is defined by $(D^*X)F = F(X + -)$. The main point is the canonical isomorphism below

$$\begin{array}{ccc}
[\mathfrak{S}, \mathcal{V}] \boxtimes [\mathfrak{S}, \mathcal{V}] & \xrightarrow{\otimes^L} & [\mathfrak{S}, \mathcal{V}] \\
(D^*\widehat{\boxtimes}D^*)X \downarrow & \xrightarrow{\cong} & \downarrow D^*X \\
[\mathfrak{S}, \mathcal{V}] \boxtimes [\mathfrak{S}, \mathcal{V}] & \xrightarrow{\otimes^L} & [\mathfrak{S}, \mathcal{V}]
\end{array}$$

Remark 9.3. The first item of Remark 2.11 has a categorical version. The forgetful 2-functor $U: \mathfrak{E} \rightarrow \mathcal{V}\text{-Cat}_{L,+}$ has a right biadjoint JS taking the \mathcal{V} -category \mathcal{A} to the object of \mathfrak{E} determined by the \mathcal{V} -category $\text{JS}\mathcal{A} = [\mathfrak{S}, \mathcal{A}]$ of species in \mathcal{A} , equipped with L -weighted derivation the D^* just as in Example 9.2 with the codomain \mathcal{V} replaced by \mathcal{A} . Since U is strong monoidal, the biadjunction $U \dashv_{\text{bi}} \text{JS}$ is monoidal. Consequently the biadjunction lifts to one between the 2-categories of monoidales in \mathfrak{E} and $\mathcal{V}\text{-Cat}_{L,+}$. Indeed U is pseudocomonadic.

10 The iterated tensor product again

Observe the following simple reindexing of (4.2).

Proposition 10.1. An alternative definition of $F \otimes^L G$ is

$$(F \otimes^L G)X = \sum_{V \subseteq U \subseteq X} L(U \setminus V) \otimes F(U) \otimes G(X \setminus V).$$

This leads us to another formula for the n -fold L -weighted tensor product. Define the *modified n -filtration set*³ on any finite set X by

$$\begin{aligned}
\text{mFil}_n X &= \{(U, V) \mid U = (0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{n-1} \subseteq U_n = X), \\
&\quad V = (V_0, V_1, \dots, V_{n-1}) \text{ with } V_i \subseteq U_i \text{ for } 0 \leq i < n\}.
\end{aligned}$$

³We are “modifying” the filtration U of X by equipping it with the extra subsets V .

Proposition 10.2. *An alternative definition of the n -fold tensor product (6.1) is*

$$\begin{aligned} \otimes_n^L(F_1, \dots, F_n)X &= \sum_{(U,V) \in \text{mFil}_n X} L(U_1 \setminus V_1) \otimes \cdots \otimes L(U_{n-1} \setminus V_{n-1}) \\ &\quad \otimes F_1(U_1 \setminus V_0) \otimes \cdots \otimes F_n(U_n \setminus V_{n-1}). \end{aligned} \quad (10.1)$$

Proof. The formula follows by repeated application of the formula of Proposition 10.1 in evaluating the left bracketing

$$(\cdots (F_1 \otimes^L F_2) \otimes^L \cdots) \otimes^L F_n$$

at X . ■

Let us relate the formulas (6.1) and (10.1) in the case $n = 3$. A modified 3-filtration $(U, V) \in \text{mFil}_3 X$ of X amounts to subsets $U_1 \subseteq U_2 \subseteq X$ and $V_1 \subseteq U_1, V_2 \subseteq U_2$. With this we can define

$$\begin{aligned} A_1 &= V_1 \cap V_2, & A_2 &= V_2 \setminus U_1 \cap V_2, & A_3 &= X \setminus U_2, \\ A_{12} &= U_1 \cap V_2 \setminus A_1, & A_{13} &= V_1 \setminus A_1, & A_{23} &= (U_2 \setminus U_1) \setminus A_2, & A_{123} &= (U_1 \setminus V_1) \setminus A_{12} \end{aligned}$$

and verify that $X = A_1 + A_2 + A_3 + A_{12} + A_{13} + A_{23} + A_{123}$. Conversely, given the partition A of X , we can define

$$U_1 = X \setminus (A_2 + A_3 + A_{23}), \quad U_2 = X \setminus A_3, \quad V_1 = A_1 + A_{13}, \quad V_2 = A_1 + A_2 + A_{12}.$$

11 Tensor products for charades

The term ‘‘charade’’ is intended in the sense of Kapranov [18, Definition 3.2] and is related to Hall algebras (for example, see [22, Section 2]). One has a small abelian (or triangulated) category \mathcal{A} and looks at functors defined on the groupoid \mathcal{A}_g of invertible morphisms in \mathcal{A} . A promonoidal structure is defined on \mathcal{A}_g using the short exact sequences (or triangles) of \mathcal{A} . The functors are tensored using convolution. Here we will only discuss the case where \mathcal{A} is the category of finite vector spaces over a fixed finite field \mathbb{F}_q . We make a conjecture that a family of weighted monoidal structures exists and give some evidence for it.

Motivated by Proposition 10.1, we consider generalizing the tensor product of [17]. Let \mathfrak{G}_q be the groupoid of finite vector spaces over the field \mathbb{F}_q of cardinality q ; the morphisms are linear bijections. We write $V \leq U$ to mean V is an \mathbb{F}_q -linear subspace of U , and we write U/V for the quotient space.

To be specific, take $\mathcal{V} = \text{Vect}_{\mathbb{C}}$ to be the category of complex vector spaces with all linear functions.

Let $L: \mathfrak{G}_q \rightarrow \mathcal{V}$ be a suitable functor: we will consider conditions on it later.

For functors $F, G: \mathfrak{G}_q \rightarrow \mathcal{V}$, define $F \otimes^L G: \mathfrak{G}_q \rightarrow \mathcal{V}$ by

$$(F \otimes^L G)X = \sum_{V \leq U \leq X} L(U/V) \otimes F(U) \otimes G(X/V). \quad (11.1)$$

This leads us to an n -fold tensor product in a manner analogous to (10.1). Define the *modified n -flag set* on any finite \mathbb{F}_q -vector space X by

$$\begin{aligned} \text{mFlg}_n X &= \{(U, V) \mid U = (0 = U_0 \leq U_1 \leq \cdots \leq U_{n-1} \leq U_n = X), \\ &\quad V = (V_0, V_1, \dots, V_{n-1}) \text{ with } V_i \leq U_i \text{ for } 0 \leq i < n\}. \end{aligned}$$

Now we put

$$\begin{aligned} \otimes_n^L(F_1, \dots, F_n)X &= \sum_{(U,V) \in \text{mFlg}_n X} L(U_1 \setminus V_1) \otimes \cdots \otimes L(U_{n-1} \setminus V_{n-1}) \\ &\quad \otimes F_1(U_1 \setminus V_0) \otimes \cdots \otimes F_n(U_n \setminus V_{n-1}). \end{aligned}$$

The formula follows by repeated application of (11.1) in evaluating the left bracketing

$$(\cdots (F_1 \otimes^L F_2) \otimes^L \cdots) \otimes^L F_n$$

at X .

Let us look at the ternary tensor product

$$\otimes_3^L(F, G, H)X = ((F \otimes^L G) \otimes^L H)X$$

It is a direct sum over modified 3-flags (U, V) on X ; that is, subspaces $U_1 \leq U_2 \leq X$, $V_1 \leq U_1$ and $V_2 \leq U_2$. From these we can uniquely define vector spaces A_S for each $\emptyset \neq S \subseteq \langle 3 \rangle$ via the following diagrams of short exact sequences:

$$\begin{array}{ccccc} A_1 & \twoheadrightarrow & U_1 \cap V_2 & \twoheadrightarrow & A_{12} \\ \downarrow & & \downarrow & & \downarrow \\ V_1 & \twoheadrightarrow & U_1 & \twoheadrightarrow & U_1/V_1 \\ \downarrow & & \downarrow & & \downarrow \\ A_{13} & \twoheadrightarrow & U_1/U_1 \cap V_2 & \twoheadrightarrow & A_{123} \\ & & U_1 \cap V_2 & \twoheadrightarrow & V_2 & \twoheadrightarrow & A_2 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & U_1 & \twoheadrightarrow & U_2 & \twoheadrightarrow & U_2/U_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ U_1/U_1 \cap V_2 & \twoheadrightarrow & U_2/V_2 & \twoheadrightarrow & A_{23} \\ & & U_2 & \twoheadrightarrow & X & \twoheadrightarrow & A_3, \end{array}$$

from which we see

$$X \cong A_1 \oplus A_2 \oplus A_3 \oplus A_{12} \oplus A_{23} \oplus A_{13} \oplus A_{123}. \quad (11.2)$$

Note also the isomorphisms

$$\begin{aligned} U_1/V_1 &\cong A_{12} \oplus A_{123}, & U_2/V_2 &\cong A_{13} \oplus A_{23} \oplus A_{123}, \\ U_1 &\cong A_1 \oplus A_{12} \oplus A_{13} \oplus A_{123}, & U_2/V_1 &\cong A_2 \oplus A_{12} \oplus A_{23} \oplus A_{123}, \\ X/V_2 &\cong A_3 \oplus A_{13} \oplus A_{23} \oplus A_{123}. \end{aligned}$$

On the other hand, we can see that the formula for the right bracketing is

$$(F \otimes^L (G \otimes^L H))X = \sum_{M \leq N \leq X, M \leq I \leq J \leq X} L(N/M) \otimes L(J/I) \otimes FN \otimes G(J/M) \otimes H(X/I).$$

We can see that this indexing set also leads to a direct sum decomposition (11.2) from the following diagrams of short exact sequences:

$$\begin{array}{ccccc}
A_1 & \twoheadrightarrow & U_1 \cap V_2 & \twoheadrightarrow & A_{12} \\
\downarrow & & \downarrow & & \downarrow \\
V_1 & \twoheadrightarrow & U_1 & \twoheadrightarrow & U_1/V_1 \\
\downarrow & & \downarrow & & \downarrow \\
A_{13} & \twoheadrightarrow & U_1/U_1 \cap V_2 & \twoheadrightarrow & A_{123} \\
\\
I \cap N & \twoheadrightarrow & I & \twoheadrightarrow & A_2 \\
\downarrow & & \downarrow & & \downarrow \\
J \cap N & \twoheadrightarrow & J & \twoheadrightarrow & J/J \cap N \\
\downarrow & & \downarrow & & \downarrow \\
A_{123} & \hookrightarrow & J/I & \twoheadrightarrow & A_{23} \\
\\
J + N & \twoheadrightarrow & X & \twoheadrightarrow & A_3, \quad J \cap N \twoheadrightarrow N \twoheadrightarrow A_{13}.
\end{array}$$

Note also the isomorphisms

$$\begin{aligned}
N/M &\cong A_{12} \oplus A_{13} \oplus A_{123}, & J/I &\cong A_{23} \oplus A_{123}, & N &\cong A_1 \oplus A_{12} \oplus A_{13} \oplus A_{123}, \\
J/M &\cong A_2 \oplus A_{12} \oplus A_{23} \oplus A_{123}, & X/I &\cong A_3 \oplus A_{13} \oplus A_{23} \oplus A_{123}.
\end{aligned}$$

In order to have an associativity isomorphism we at least need a canonical isomorphism

$$L(A_{12} \oplus A_{123}) \otimes L(A_{13} \oplus A_{23} \oplus A_{123}) \cong L(A_{12} \oplus A_{13} \oplus A_{123}) \otimes L(A_{23} \oplus A_{123}).$$

We do have such an isomorphism if $L: \mathfrak{G}_q \rightarrow \mathcal{V}$ takes direct sums to tensor products; of course, direct sum of \mathbb{F}_q -vector spaces is neither product nor coproduct in \mathfrak{G}_q . This is still merely evidence that the desired associativity isomorphism should exist: it is not a complete definition.

Recall from [17] that \mathfrak{G}_q has a braided promonoidal structure. The convolution structure on $[\mathfrak{G}_q, \mathcal{V}]$ arising from this (as per Day [7, 8]) is precisely the tensor product $F \otimes^J G$ where $JX = I$ for $X = 0$ and $JX = 0$ for $X \neq 0$.

Conjecture 11.1. *If L is braided strong promonoidal then (11.1) defines a monoidal structure \otimes^L on $[\mathfrak{G}_q, \mathcal{V}]$.*

Should this be the case, the tensor \otimes^L on $[\mathfrak{G}_q, \mathcal{V}]$ would be obtained from quite an interesting promonoidal structure on \mathfrak{G}_q . A short sequence

$$A \xrightarrow{f} X \xrightarrow{g} B \tag{11.3}$$

in $\text{Vect}_{\mathbb{F}_q}$ might be called *short pre-exact* when f is a monomorphism, g is an epimorphism and $\ker g \leq \text{im} f$. Write $\text{Spes}(A, B; X)$ for the set of such (f, g) . Put

$$P(A, B; X) = \sum_{(f, g) \in \text{Spes}(A, B; X)} L(\text{im}(g \circ f)).$$

This $P: \mathfrak{G}_q^{\text{op}} \times \mathfrak{G}_q^{\text{op}} \times \mathfrak{G}_q \rightarrow \mathcal{V}$, defined on morphisms in the obvious way, would give the promonoidal structure in question. The term $L(\text{im}(g \circ f))$ measures the failure of the sequence (11.3) to be exact.

12 The dimension sequence

Following on from Section 11, we take $F \in [\mathfrak{G}_q, \text{Vect}_{\mathbb{C}}]$ and define its *dimension sequence* $\dim F \in \mathbb{Z}^{\mathbb{N}}$ by

$$(\dim F)_n = \dim (F(\mathbb{F}_q^n)).$$

This inspires an algebra structure on $A^{\mathbb{N}}$ for any k -algebra A . We assume we have $\lambda \in k$ as before, but also some integer q (not necessarily a prime power). As in [17], we use

$$\phi_n(q) = (q^n - 1)(q^{n-1} - 1) \cdots (q - 1).$$

We define

$$\left[\begin{matrix} n \\ r, s \end{matrix} \right]_q = \frac{\phi_n(q)}{\phi_r(q)\phi_s(q)}, \quad \left[\begin{matrix} n \\ r, s, t \end{matrix} \right]_q = \frac{\phi_n(q)}{\phi_r(q)\phi_s(q)\phi_t(q)}, \quad \cdots$$

For $f, g \in A^{\mathbb{N}}$, put

$$f \cdot_{\lambda, q} g = \sum_{r+s+t=n} \left[\begin{matrix} n \\ r, s, t \end{matrix} \right]_q \lambda^t f(r+t)g(s+t).$$

The calculations of Section 11 show that this is associative at least when $A = \mathbb{Z}$, q is a prime power and $\lambda = \dim L(\mathbb{F})$.

More generally, I claim $A^{\mathbb{N}}$ is an associative k -algebra.

Proposition 12.1. $\dim(F \otimes^L G) = \dim F \cdot_{\lambda, q} \dim G$

Acknowledgements

I am grateful to the referees for their careful work and, in particular, for pointing out the references [1, 3, 20]. The author gratefully acknowledges the support of Australian Research Council Discovery Grant DP130101969.

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