

A Family of Finite-Dimensional Representations of Generalized Double Affine Hecke Algebras of Higher Rank

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Abstract. We give explicit constructions of some finite-dimensional representations of generalized double affine Hecke algebras (GDAHA) of higher rank using R -matrices for $U_q(\mathfrak{sl}_N)$. Our construction is motivated by an analogous construction of Silvia Montarani in the rational case. Using the Drinfeld–Kohno theorem for Knizhnik–Zamolodchikov differential equations, we prove that the explicit representations we produce correspond to Montarani’s representations under a monodromy functor introduced by Etingof, Gan, and Oblomkov.

Key words: generalized double affine Hecke algebra; R -matrix; Drinfeld–Kohno theorem

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1 Introduction

Generalized double affine Hecke algebras of higher rank (GDAHA) are a family of algebras that generalize the well-known Cherednik algebras in representation theory and were first introduced by Etingof, Oblomkov, and Rains in [5] (the rank 1 case) and [4] (the rank n case). They have been related to del Pezzo surfaces in algebraic geometry and Calogero–Moser integrable systems. A degenerate version of GDAHA, known as the rational GDAHA, was also introduced in [4]. In [14], Montarani introduces two constructions of finite-dimensional representations of rational GDAHA, one using D -modules and the other using explicit Lie theoretic methods. The latter of these methods, involving an isotypic subspace of a tensor product with a tensor power of the vector representation of \mathfrak{sl}_N , is similar in spirit to the Arakawa–Suzuki functor from the BGG category \mathcal{O} of \mathfrak{sl}_N -modules to the category of finite-dimensional representations of degenerate affine Hecke algebras of type A constructed in [1] and to the construction of representations of affine braid groups given by Orellana and Ram in [15]. There has been much interest in and effort devoted to developing similar techniques for the representation theory of related algebraic objects, for instance by Calaque, Enriquez, and Etingof [2] for degenerate double affine Hecke algebras and by Jordan [9] in the nondegenerate case, among others.

In this paper, we generalize the Montarani’s Lie theoretic construction to the non-degenerate GDAHA using R -matrices for the quantum groups $U_q(\mathfrak{sl}_N)$. Furthermore, we show that the explicit representations we produce are equivalent to the image of Montarani’s representations under a monodromy functor constructed in [4] which introduces an action of a nondegenerate GDAHA on a finite-dimensional representation of a rational GDAHA. In the same sense that Jordan’s work [9] is a generalization of the work of Calaque, Enriquez, and Etingof [2] to the nondegenerate case, this paper generalizes Montarani’s construction to the nondegenerate case.

The structure of this paper is as follows. In Section 2 we fix the notation we use for quantum groups and review their basic properties and also introduce the definition of GDAHA. Following

that, our explicit construction of representations is given in Section 3, along with its proof. The calculation that relates this representation with the monodromy representation given in [4] is done in Section 4.

2 Background

2.1 Generalized double affine Hecke algebras of higher rank (GDAHA)

We recall the definition of GDAHA in [4]. Let D be a star-shaped finite graph that is not finite Dynkin with m legs and leg lengths d_1, \dots, d_m (number of vertices on each leg, including the center).

Definition 2.1 (GDAHA). $H_n(D)$, the generalized double affine Hecke algebra of rank n associated with graph D , is the associative algebra generated over $\mathbb{C}[u_{1,1}^{\pm 1}, \dots, u_{1,d_1}^{\pm 1}, u_{2,1}^{\pm 1}, \dots, u_{m,d_m}^{\pm 1}, t^{\pm 1}]$ by invertible generators $U_1, \dots, U_m, T_1, \dots, T_{n-1}$ and the following relations:

1. $U_1 U_2 \cdots U_m T_1 T_2 \cdots T_{n-2} T_{n-1} T_{n-1} T_{n-2} \cdots T_2 T_1 = 1$;
2. $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for $1 \leq i < n-1$;
3. $[T_i, T_j] = 0$ for $|i-j| > 1$;
4. $[U_i, T_j] = 0$ for $1 < i \leq n-1, 1 < j \leq m$;
5. $[U_i, T_1 U_i T_1] = 0$ for $1 \leq i \leq m$;
6. $[U_i, T_1^{-1} U_j T_1] = 0$ for $1 \leq i < j \leq m$;
7. $\prod_{j=1}^{d_k} (U_k - u_{k,j}) = 0$ for $1 \leq k \leq m$;
8. $T_i - T_i^{-1} = t - t^{-1}$ for $1 \leq i \leq n-1$.

The rank- n GDAHA $H_{n,m}$ can be seen as a quotient of the group algebra of the fundamental group of the configuration space of n unordered points on the m -punctured sphere, where the quotient is by the eigenvalue relations (7) and (8). From this perspective, U_i is represented by a path in the configuration space in which one of the points loops around a missing point α_i , and T_i is represented by a typical braid group generator exchanging the positions of two points in the configuration. Note that the generators T_i satisfy the relations of the finite-type Hecke algebra of type A_{n-1} .

2.2 The quantum group $U_q(\mathfrak{sl}_N)$

Due to the large volume of literature and varying conventions regarding quantum groups, it is necessary to fix the notations that we work with. We use conventions compatible with, for example, [6, 8, 12]. Throughout this paper, it is assumed that q is a nonzero complex number that is not a root of unity.

For every $n \in \mathbb{Z}$, let $[n]_q$ denote the associated symmetrized q -number

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

In particular, we have

$$[n]_q = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}$$

for $n > 0$ and $[n]_q = -[-n]_q$ for all n . Define q -binomial coefficients by

$$\begin{bmatrix} a \\ n \end{bmatrix}_q := \frac{[a]_q [a-1]_q \cdots [a-n+1]_q}{[1]_q [2]_q \cdots [n]_q}$$

for all $a \in \mathbb{Z}$, $n \in \mathbb{N}$, so in particular $\begin{bmatrix} a \\ 0 \end{bmatrix}_q = 1$ by the usual convention on empty products.

Define q -factorials $[a]_q!$ for $a \in \mathbb{N}$ by

$$[a]_q! = [1]_q [2]_q \cdots [a]_q.$$

Note that $[0]_q! = 1$.

Definition 2.2 (quantum group $U_q(\mathfrak{sl}_N)$). Let $A = (a_{ij})$ be the Cartan matrix for the Lie algebra \mathfrak{sl}_N . The quantum group $U_q(\mathfrak{sl}_N)$ is the associative \mathbb{C} -algebra with generators E_i, F_i, K_i, K_i^{-1} ($1 \leq i < N$) and relations:

$$\begin{aligned} [K_i, K_j] &= 0, \\ K_i K_i^{-1} &= 1 = K_i^{-1} K_i, \\ K_i E_j &= q^{a_{ij}} E_j K_i, \\ K_i F_j &= q^{-a_{ij}} F_j K_i, \\ [E_i, F_j] &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_q E_i^{1-a_{ij}-r} E_j E_i^r &= 0, \quad \text{and} \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_q F_i^{1-a_{ij}-r} F_j F_i^r &= 0 \quad \text{if } i \neq j, \end{aligned}$$

where δ_{ij} is the Kronecker delta and $[a, b] = ab - ba$.

We will use the Hopf algebra structure on $U_q(\mathfrak{sl}_N)$ specified in the following standard proposition:

Proposition 2.3 (Hopf algebra structure on $U_q(\mathfrak{sl}_N)$). *Denote $U = U_q(\mathfrak{sl}_N)$. The assignments below extend to unique algebra homomorphisms $(\Delta: U \rightarrow U \otimes U, \varepsilon: U \rightarrow \mathbb{C}, \eta: \mathbb{C} \rightarrow U, S: U \rightarrow U^{\text{op}})$ that give U a Hopf algebra structure:*

$$\begin{aligned} \Delta(K_i) &= K_i \otimes K_i, & \Delta(E_i) &= E_i \otimes K_i + 1 \otimes E_i, & \Delta(F_i) &= F_i \otimes 1 + K_i^{-1} \otimes F_i, \\ \varepsilon(E_i) &= \varepsilon(F_i) = 0, & \varepsilon(K_i) &= 1, & S(K_i) &= K_i^{-1}, & S(E_i) &= -E_i K_i^{-1}, \\ S(F_i) &= -K_i F_i. \end{aligned}$$

2.3 R -matrices

In this section we will fix our conventions for R -matrices for $U_q(\mathfrak{sl}_N)$. All statements below are well-known and their proofs can be found in [12] or [10].

Definition 2.4. For a given pair (V, V') of finite-dimensional $U_q(\mathfrak{sl}_N)$ -modules, define the linear map $f: V \otimes V' \rightarrow V \otimes V'$ such that $f(v \otimes w) = q^{(\lambda, \mu)}(v \otimes w)$ if v and w have weights λ and μ , respectively, where (\cdot, \cdot) is the standard pairing on the weight lattice.

Proposition 2.5. *There exists an element*

$$\tilde{\mathfrak{R}} \in 1 + U_q(\mathfrak{sl}_n)_{>0} \hat{\otimes} U_q(\mathfrak{sl}_n)_{<0}$$

in an appropriate completion $U_q(\mathfrak{sl}_N) \hat{\otimes} U_q(\mathfrak{sl}_N)$ such that the operators

$$\mathfrak{R} := f \circ \tilde{\mathfrak{R}}, \quad R := P \circ \mathfrak{R} = P \circ f \circ \tilde{\mathfrak{R}}$$

on tensor products of finite-dimensional $U_q(\mathfrak{sl}_N)$ -modules satisfy, for any finite-dimensional modules V_1, V_2, V_3 :

- $R: V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$ is an isomorphism of $U_q(\mathfrak{sl}_n)$ -modules,
- $\mathfrak{R}_{12,3} = \mathfrak{R}_{13}\mathfrak{R}_{23}$,
- $\mathfrak{R}_{1,23} = \mathfrak{R}_{13}\mathfrak{R}_{12}$,

where P denotes the operator exchanging tensor factors and the subscripts indicate the tensor factor positions on which \mathfrak{R} acts.

For an explicit description of the element $\tilde{\mathfrak{R}}$, see [12, Section 8.3.3].

The operator \mathfrak{R} satisfies the quantum Yang–Baxter equation:

Proposition 2.6 (QYBE). $\mathfrak{R}_{12}\mathfrak{R}_{13}\mathfrak{R}_{23} = \mathfrak{R}_{23}\mathfrak{R}_{13}\mathfrak{R}_{12}$.

Corollary 2.7. *For $1 \leq i \neq j < n$, let $R_i = P_{i,i+1} \circ \mathfrak{R}_{i,i+1}$, then we have $R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1}$ and $R_i R_j = R_j R_i$ for $|i - j| > 1$, and in particular the R_i operators yield a representation of the braid group B_n on $V^{\otimes n}$ for any finite-dimensional $U_q(\mathfrak{sl}_N)$ -representation V , and this representation is functorial in V .*

Proposition 2.8. *Let \mathbb{C}^N be the N -dimensional vector representation of $U_q(\mathfrak{sl}_N)$. Then R_i ($1 \leq i < n$) act on the space $(\mathbb{C}^N)^{\otimes n}$. The operators $q^{1/N} R_i$ on $(\mathbb{C}^N)^{\otimes n}$ act with eigenvalues $q, -q^{-1}$ and in particular define a representation of the Hecke algebra of type A_{n-1} with parameter q .*

Proof. This is Proposition 23 in Section 8.4.3 of [12]. ■

2.4 Ribbon category structure on $U_q(\mathfrak{sl}_N)\text{-mod}_{\text{f.d.}}$

Recall that a type-1 representation of $U_q(\mathfrak{sl}_N)$ is a representation V such that V has a weight decomposition $V = \bigoplus_{\lambda} V_{\lambda}$ where the direct sum is over the weight lattice for \mathfrak{sl}_N and V_{λ} is the subspace

$$V_{\lambda} := \{v \in V : K_{\mu}(v) = q^{(\lambda, \mu)} v \text{ for all roots } \mu\}.$$

Let $U_q(\mathfrak{sl}_N)\text{-mod}_{\text{f.d.}}$ denote the category of finite-dimensional type-1 representations of $U_q(\mathfrak{sl}_N)$. We will only consider representations in this category.

The operator R , along with the Hopf algebra structure on $U_q(\mathfrak{sl}_N)$, gives the category $U_q(\mathfrak{sl}_N)\text{-mod}_{\text{f.d.}}$ of finite-dimensional type-1 representations of $U_q(\mathfrak{sl}_N)$ the structure of a braided rigid tensor category. In fact, $U_q(\mathfrak{sl}_N)\text{-mod}_{\text{f.d.}}$ has even richer structure – it is a *ribbon category*. In particular, there exists an automorphism θ of the identity functor on $U_q(\mathfrak{sl}_N)\text{-mod}_{\text{f.d.}}$ which is given by multiplication by $q^{(\lambda, \lambda + 2\rho)}$ (where, as usual, ρ is the half-sum of the positive roots) on any irreducible representation with highest weight λ , and θ satisfies the compatibility

$$\theta_{V(\lambda) \otimes V(\mu)} = R^2 \circ (\theta_{V(\lambda)} \otimes \theta_{V(\mu)}), \tag{2.1}$$

where $V(\lambda)$ and $V(\mu)$ are the irreducible highest weight representations with highest weights λ and μ , respectively, and R is the R -matrix introduced above. An h -adic version of these statements can be found in Proposition 21 in Section 8.4.3 of [12], and it is routine to translate the result into the setting of numeric q .

3 Representations of GDAHA via R -matrices

3.1 The construction

Again, we assume q is a nonzero complex number which is not a root of unity, and all representations of $U_q(\mathfrak{sl}_N)$ we consider will be type-1 and finite-dimensional.

Let $n \geq 1$ be an integer, let V_1, \dots, V_m be irreducible finite-dimensional highest weight modules for $U_q(\mathfrak{sl}_N)$, let $V_{m+1} = V_{m+2} = \dots = V_{m+n} = \mathbb{C}^N$ be copies of its vector representation, and let

$$V = V_1 \otimes \dots \otimes V_m \otimes V_{m+1} \otimes \dots \otimes V_{m+n}.$$

Let E be the 0-isotypic component of V . For $1 \leq i < m+n$ let R_i denote $\text{id}^{\otimes i-1} \otimes R \otimes \text{id}^{\otimes m+n-i-1}$.

Theorem 3.1 (main theorem). *Let $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ be any complex numbers and let $c = (n + \sum_k \lambda_k)/N$. Then for $t = q$, some specific values of u_{kj} , and some graph D , the following formulas for $1 \leq i < n$ and $1 \leq k \leq m$ define a representation of the GDAHA of rank n attached to D on E :*

- $T_i = q^{1/N} R_{m+i}$, and
- $U_k = q^{2[(N-c)/m + \lambda_k/N]} R_m R_{m-1} \dots R_{k+1} R_k^2 R_{k+1}^{-1} \dots R_{m-1}^{-1} R_m^{-1}$.

Note that the T_i are endomorphisms of E because V_{m+1}, \dots, V_{m+n} are all copies of \mathbb{C}^N and R_{m+i} acts as a $U_q(\mathfrak{sl}_N)$ -module isomorphism. The U_k act on E for similar reasons. The dependence of the graph D and the parameters u_{kj} on the representations V_1, \dots, V_m and the scalars $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ is given later in equation (3.1).

3.2 Validity of the defining relations

We need to verify that the relations (1)–(8) defining GDAHA hold for the operators defined in the theorem above. The relations (2), (3) and (8) among T_i are the defining relations of the type A_{n-1} Hecke algebra and hold by Proposition 2.8. Relation (4) holds trivially because the operators act on different tensor factors. Relations (5) and (6) follow from routine calculations using that the R -matrices R_i satisfy the braid relations, and so we need only check relations (1) and (7). We begin with relation (7), which restricts the eigenvalues of the operators U_i .

3.2.1 Eigenvalues of U_k

Recall from Section 2 the functorial operator θ defined on any $U_q(\mathfrak{sl}_N)$ -module and acting on any copy of the highest weight module $V(\lambda)$ by $q^{(\lambda, \lambda + 2\rho)}$. From the compatibility between the R matrix and θ given in (2.1) that R^2 acts as a scalar on each highest weight submodule of the tensor product of two highest weight modules. More precisely,

Lemma 3.2. *Let $V(\mu)$ and $V(\mu')$ be highest weight representations of U and let $V(\lambda) \subset V(\mu) \otimes V(\mu')$ be a copy of the simple highest weight module with highest weight λ . Then R^2 acts on $V(\lambda)$ by the scalar $q^{-(\mu, \mu + 2\rho) - (\mu', \mu' + 2\rho) + (\lambda, \lambda + 2\rho)}$.*

Proof. Immediate from $\theta_{V(\mu) \otimes V(\mu')} = R^2(\theta_{V(\mu)} \otimes \theta_{V(\mu')})$. ■

For $1 \leq k \leq m$, let $\tilde{U}_k := R_m \dots R_{k+1} R_k^2 R_{k+1}^{-1} \dots R_m^{-1}$. Relation (7) requires that the eigenvalues of U_k counted with multiplicity are among $u_{k,1}, \dots, u_{k,d_k}$, so it is equivalent to show that \tilde{U}_k has all its eigenvalues counted with multiplicity appearing in

$$q^{-2[(N-c)/m + \lambda_k/N]} u_{k,1}, \quad \dots, \quad q^{-2[(N-c)/m + \lambda_k/N]} u_{k,d_k}.$$

As \tilde{U}_k is conjugate to R_k^2 via $R_m \cdots R_{k+1}$, we need only compute the eigenvalues of R_k^2 acting on $V_k \otimes \mathbb{C}^N$.

Suppose V_k is of highest weight μ_k . As \mathbb{C}^N is of highest weight $\epsilon_1 := (1, 0, \dots, 0)$, it follows from Lemma 3.2 that the eigenvalues of R_k^2 on $V_k \otimes \mathbb{C}^N$ are $q^{-(\mu_k, \mu_k + 2\rho) - (\epsilon_1, \epsilon_1 + 2\rho) + (\eta, \eta + 2\rho)}$, where η ranges over the highest weights of the irreducible constituents appearing in the direct sum decomposition of $V_k \otimes \mathbb{C}^N$. Let d_k be the number of non-isomorphic irreducibles appearing in $V_k \otimes \mathbb{C}^N$, and suppose their distinct highest weights are $\eta_{k,1}, \dots, \eta_{k,d_k}$. Then, setting

$$u_{k,j} = q^{2[(N-c)/m + \lambda_k/N] - (\mu_k, \mu_k + 2\rho) + (\eta_{k,j}, \eta_{k,j} + 2\rho) - N + 1/N} \quad (3.1)$$

provides the choice of d_1, \dots, d_m and u_{kj} so that relation (7) holds (note that $(\epsilon_1, \epsilon_1 + 2\rho) = N - 1/N$). In particular, the graph D is one with m legs and leg lengths d_1, \dots, d_m .

3.2.2 The first relation

If we expand relation (1) of the GDAHA, i.e., $U_1 \cdots U_m T_1 \cdots T_{n-1} T_{n-1} \cdots T_1 = 1$, in terms of the operators given in Theorem 3.1, we see that it simplifies to the following equality:

$$q^{2N-2/N} R_m R_{m-1} \cdots R_1 (R_1 R_2 \cdots R_m R_{m+1} \cdots R_{m+n-1}) R_{m+n-1} R_{m+n-2} \cdots R_{m+1} = \text{id}_E.$$

Now, suppose we have the following identity:

$$R_1 R_2 \cdots R_{m+n-1} R_{m+n-1} R_{m+n-2} \cdots R_2 R_1 = q^{-2N+2/N} \text{id}_{E'},$$

where E' is the 0-isotypic subspace of $V' := V_{m+1} \otimes V_1 \otimes \cdots \otimes V_m \otimes V_{m+2} \otimes \cdots \otimes V_{m+n}$, then one can write

$$R_1 R_2 \cdots R_{m+n-1} = q^{-2N+2/N} R_1^{-1} R_2^{-1} \cdots R_{m+n-1}^{-1}$$

as maps from the 0-isotypic component of $V_1 \otimes \cdots \otimes V_m \otimes V_{m+2} \otimes \cdots \otimes V_{m+n} \otimes V_{m+1}$ to the 0-isotypic component E' of V' . Substituting this into the parenthesized part in the expression above, we see it immediately proves relation (1). Thus we have reduced relation (1) to the following lemma:

Lemma 3.3. *The operator $R_1 R_2 \cdots R_{m+n-1} R_{m+n-1} R_{m+n-2} \cdots R_2 R_1: V' \rightarrow V'$ acts on the zero isotypic component of V' by $q^{-2N+2/N}$.*

Proof. It follows from properties (2) and (3) of R -matrices in Proposition 2.5 that we have

$$R_1 \cdots R_{m+n-1} = R_{12 \dots (m+n-1), m+n}$$

and

$$R_{m+n-1} \cdots R_1 = R_{1,23 \dots (m+n)},$$

where the notations $R_{12 \dots (m+n-1), m+n}$ and $R_1 = R_{1,23 \dots (m+n)}$ are as in Proposition 2.5, i.e., these represent the R -matrices with flip associated to the bracketings $(\bullet^{m+n-1})\bullet$ and $\bullet(\bullet^{m+n-1})$ respectively. Let

$$V_1 \otimes \cdots \otimes V_m \otimes V_{m+2} \otimes \cdots \otimes V_{m+n} = \bigoplus_i W_i$$

be a decomposition of $V_1 \otimes \cdots \otimes V_m \otimes V_{m+2} \otimes \cdots \otimes V_{m+n}$ into irreducibles and similarly let

$$\mathbb{C}^N \otimes W_i = \bigoplus_j Z_{ij}$$

be a decomposition of $\mathbb{C}^N \otimes W_i$ into irreducibles. If W_i is of highest weight μ_i and if Z_{ij} is of highest weight ν_{ij} , by Lemma 3.2 and the first sentence of this proof, $R_1 \cdots R_{m+n-1} R_{m+n-1} \cdots R_1$ acts on Z_{ij} by the scalar $q^{-(\epsilon_1, \epsilon_1 + 2\rho) - (\mu_i, \mu_i + 2\rho) + (\nu_{ij}, \nu_{ij} + 2\rho)}$. However, we are only concerned with those Z_{ij} with highest weight 0. By Pieri's rule, $V(0)$ can appear as a constituent of $\mathbb{C}^N \otimes V(\mu_i)$ only if μ_i is the weight $(1, \dots, 1, 0)$, i.e., when $V(\mu_i)$ is labeled by the Young diagram associated to the partition 1^{N-1} [7]. In this case the power of q we just computed becomes $q^{-2N+2/N}$, as needed. The lemma, and hence Theorem 3.1, follow. \blacksquare

4 Equivalence with the monodromy representation

4.1 The monodromy functor

In [4], the authors introduced a certain connection of Knizhnik–Zamolodchikov type whose monodromy defines a functor from the category of finite-dimensional representations of a rational GDAHA (rGDAHA) B_n to the category of finite-dimensional representations of a corresponding non-degenerate GDAHA H_n attached to the same diagram D with m legs of lengths d_1, \dots, d_m . Recall that the rGDAHA B_n is the algebra generated over $\mathbb{C}[\gamma_{1,1}, \dots, \gamma_{m,d_m}, \nu]$ by elements $Y_{i,k}$ ($1 \leq i \leq n, 1 \leq k \leq m$) and the symmetric group S_n , such that the following hold for every $i, j, h \in [1, n]$ with $i \neq j$ and every $k, l \in [1, m]$:

1. $s_{ij} Y_{i,k} = Y_{j,k} s_{ij}$;
2. $s_{ij} Y_{h,k} = Y_{h,k} s_{ij}$ if $h \neq i, j$;
3. $\prod_{j=1}^{d_k} (Y_{i,k} - \gamma_{k,j}) = 0$;
4. $\sum_{j=1}^m Y_{i,j} = \nu \sum_{j \neq i} s_{ij}$;
5. $[Y_{i,k}, Y_{j,k}] = \nu(Y_{i,k} - Y_{j,k}) s_{ij}$;
6. $[Y_{i,k}, Y_{j,l}] = 0$ if $k \neq l$.

Let us recall the monodromy functor from [4]. Fix a finite-dimensional B_n module M . For convenience, let $\alpha_1, \dots, \alpha_m \in \mathbb{C}$ be distinct points defined by $\alpha_i = -m - 1 + i$ and choose the basepoint $\mathbf{z}_0 = (1, \dots, n)$ in the ordered configuration space $\text{Conf}_n(\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_m\})$ of n -points in the m -punctured plane. In [4] the Knizhnik–Zamolodchikov-style connection

$$\nabla_{\text{EGO}} := d - \sum_{i=1}^n \left(\sum_{k=1}^m \frac{Y_{i,k}}{z_i - \alpha_k} - \sum_{j \neq i} \frac{\nu s_{ij}}{z_i - z_j} \right) dz_i \quad (4.1)$$

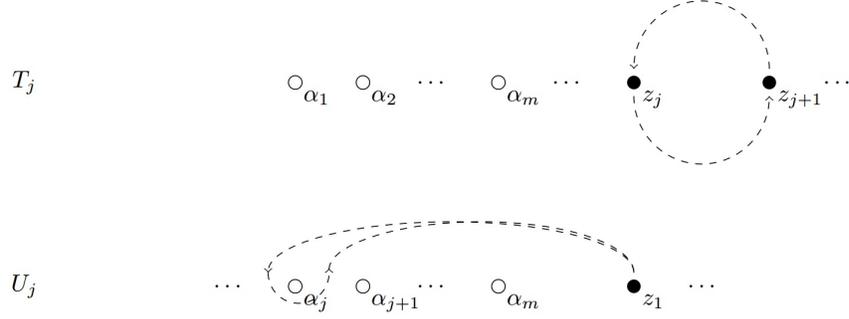
is introduced on the trivial vector bundle E_M with fiber M over $\text{Conf}_n(\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_m\})$. It follows readily from the defining relations of B_n and a calculation that the connection ∇ is flat and has trivial residue at ∞ . This connection is visibly S_n -equivariant and so descends to a connection on the unordered configuration space $\text{UConf}_n(\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_m\})$. The residue condition implies that the monodromy of this connection at \mathbf{z}_0 defines a representation

$$\rho_M: \pi_1(\text{UConf}_n(\mathbb{CP}^1 \setminus \{\alpha_1, \dots, \alpha_m\}), \mathbf{z}_0) \rightarrow \text{Aut}(M).$$

The algebra H_n may be interpreted as the quotient of the group algebra

$$\mathbb{C}\pi_1(\text{UConf}_n(\mathbb{CP}^1 \setminus \{\alpha_1, \dots, \alpha_m\}), \mathbf{z}_0)$$

by the eigenvalue relations (7) and (8) of Definition 2.1, where U_i and T_i are the generators represented by the following loops:



In [4, Section 4.2] it is shown that the monodromy operators $\rho_M(T_i)$ and $\rho_M(U_k)$ satisfy the eigenvalue conditions (7) and (8) for $t = e^{-\pi i \nu}$, $u_{k,j} = e^{2\pi i \gamma_{k,j}}$ and therefore define a finite-dimensional representation of H_n with these parameter values. This construction is clearly functorial in M and defines a functor

$$F: B_n\text{-mod}_{\text{f.d.}} \rightarrow H_n\text{-mod}_{\text{f.d.}},$$

where as before $\text{-mod}_{\text{f.d.}}$ denotes the category of finite-dimensional representations. Note that this functor is the identity at the level of vector spaces.

4.2 Montarani's rGDAHA representation

We now recall the construction of a family of finite-dimensional representations of the rGDAHA B_n given in [14, Section 5]. Let V_1, \dots, V_m be irreducible finite-dimensional representations of \mathfrak{gl}_N , and let $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ be the scalars by which the identity matrix $I \in \mathfrak{gl}_N$ acts, respectively. Let χ be a character of \mathfrak{gl}_N , and denote the χ -isotypic subspace of $V_1 \otimes \dots \otimes V_m \otimes (\mathbb{C}^N)^{\otimes n}$ by

$$E_{n,\chi} := \{v \in V_1 \otimes \dots \otimes V_m \otimes (\mathbb{C}^N)^{\otimes n} \mid xv = \chi'(x)v \ \forall x \in \mathfrak{gl}_N\}.$$

Let $c \in \mathbb{C}$ be the scalar such that $\chi = c \text{Tr}_{\mathfrak{gl}_N}$. Note that if $E_{n,\chi} \neq 0$ then we have the relation $n + \sum_{j=1}^m \lambda_j = cN$ defining c in Theorem 3.1.

Let $\Omega^{\mathfrak{gl}_N} \in \mathfrak{gl}_N \otimes \mathfrak{gl}_N$ be the Casimir tensor of \mathfrak{gl}_N , and let $\Omega_{ij}^{\mathfrak{gl}_N}$ represent the tensor that acts as $\Omega^{\mathfrak{gl}_N}$ on the i th and j th tensor factors and as identity on other factors.

Theorem 4.1 ([14, Theorem 5.1]). *For any choice of $\nu \in \mathbb{C}$, the assignments $s_{ij} = \Omega_{m+i, m+j}^{\mathfrak{gl}_N}$, and $Y_{i,k} = -\nu(\Omega_{k, m+i}^{\mathfrak{gl}_N} + \frac{N-c}{m})$ define a representation of the rGDAHA B_n on $E_{n,\chi'}$ for an appropriate parameter value γ .*

4.3 Equivalence of the representations

We may apply the monodromy functor of [4] to the representation of the rGDAHA B_n in the previous theorem to produce a representation $F(E_{n,\chi})$ of a corresponding GDAHA H_n . On the other hand, given the representations V_1, \dots, V_{m+n} of \mathfrak{gl}_N , we can extract the constants $\lambda_1, \dots, \lambda_n$ as above. When $\nu \notin \mathbb{Q}$, the associated parameter $t = e^{-\pi i \nu}$ is a nonzero complex number which is not a root of unity. Let $q = t$. If V_i^q denotes the irreducible representation of $U_q(\mathfrak{sl}_N)$ with highest weight corresponding to the highest weight of V_i , then from the V_i^q and the λ_i we may produce, using Theorem 3.1, a finite-dimensional representation E of an associated non-degenerate GDAHA H'_n .

Theorem 4.2. *For $\nu \notin \mathbb{Q}$, the parameters of the GDAHAs H_n and H'_n agree, and the representations $F(E_{n,\chi})$ and E are equivalent.*

Proof. Let us first check that the eigenvalue parameters agree. The parameters t for H_n and H'_n agree and equal $e^{-\pi i \nu}$ by definition. Let μ_k denote the highest weight of V_k , let d_k be the number of non-isomorphic irreducible subrepresentations in $V_k \otimes \mathbb{C}^N = \bigoplus_j W_j$, and let their highest weights be $\eta_{k,1}, \dots, \eta_{k,d_k}$. Then, by [14, Lemma 5.2], the graph D attached to the rGAHA B_n in Theorem 4.1 has leg lengths d_1, \dots, d_m and parameter values $\gamma_{k,j} = -\nu(w_j + (N-c)/m)$ where w_j is the eigenvalue of $\Omega^{\mathfrak{gl}_N}$ on W_j . Therefore, with $q = e^{-i\pi\nu}$, the u_{kj} parameter for the algebra H_n is given by

$$u_{kj} := e^{2\pi i \gamma_{k,j}} = q^{2(N-c)/m + (\eta_{k,j}, \eta_{k,j} + 2\rho) - (\mu_k, \mu_k + 2\rho) - (N-1/N) + 2\lambda_j/N},$$

where the term $2\lambda_j/N$ comes from the discrepancy $\Omega^{\mathfrak{gl}_N} = \Omega^{\mathfrak{sl}_N} + \frac{1}{N}I \otimes I$ between the Casimir tensors for \mathfrak{gl}_N and \mathfrak{sl}_N . This agrees with the parameter values for H'_n obtained in Section 3.2.1.

The strategy for proving the equivalence statement in Theorem 4.2 is to relate the connection ∇_{EGO} to the classical KZ connection and to use the Drinfeld–Kohno theorem to relate the monodromy of the latter connection to R -matrices for $U_q(\mathfrak{sl}_N)$. Let V_1, \dots, V_{m+n} be finite-dimensional irreducible \mathfrak{gl}_N -representations as in Montarani’s construction, with V_{m+i} a copy of the vector representation \mathbb{C}^N for $1 \leq i \leq n$. We have the associated KZ-connection

$$\nabla_{\text{KZ}} := d + \nu \sum_{1 \leq j \leq m+n, j \neq i} \frac{\Omega_{ij}^{\mathfrak{sl}_N}}{z_i - z_j} dz_i$$

on the trivial vector bundle $E_{Y_{m+n}}$ with fiber $E_{n,\chi}$ over the unordered configuration space

$$Y_{m+n} := \{(z_1, \dots, z_{m+n}) \in \mathbb{C}^{m+n} : z_i \neq z_j \text{ for all } i \neq j\}$$

of $m+n$ points in \mathbb{C} . Here $\Omega_{ij}^{\mathfrak{sl}_N}$ denotes the Casimir tensor for \mathfrak{sl}_N acting on the i^{th} and j^{th} tensor factors and we view the \mathfrak{gl}_N representations as \mathfrak{sl}_N representations. It is well-known, and easy to check, that this connection is flat. Observe that ∇_{KZ} is S_n -equivariant, where S_n acts on Y_{m+n} by permuting the last n coordinates and on the fiber by permuting the last n tensor factors.

Let $Y := \text{Conf}_n(\mathbb{C} \setminus \{\alpha_1, \dots, \alpha_m\})$ be the space on which ∇_{EGO} is defined. There is a natural S_n -equivariant map

$$r: Y \rightarrow Y_{m+n}$$

given by $(z_1, \dots, z_n) \mapsto (\alpha_1, \dots, \alpha_m, z_1, \dots, z_n)$. Pulling back the connection ∇_{KZ} along r we obtain the S_n -equivariant flat connection

$$r^* \nabla_{\text{KZ}} = d + \nu \sum_{i=1}^n \left(\sum_{k=1}^m \frac{\Omega_{k,m+i}^{\mathfrak{sl}_N}}{z_i - \alpha_k} + \sum_{1 \leq j \neq i \leq n} \frac{\Omega_{m+i,m+j}^{\mathfrak{sl}_N}}{z_i - z_j} \right) dz_i$$

on the trivial vector bundle E_Y with fiber $E_{n,\chi}$ over Y . On the other hand, inserting the operators $Y_{i,k}$ and $s_{i,j}$ on $E_{n,\chi}$ defined in Theorem 4.1 into the connection ∇_{EGO} defined in equation (4.1), we obtain

$$\nabla_{\text{EGO}} = d + \nu \sum_{i=1}^n \left(\sum_{k=1}^m \frac{\Omega_{k,m+i}^{\mathfrak{gl}_N} + \frac{N-c}{m}}{z_i - \alpha_k} + \sum_{1 \leq j \neq i \leq n} \frac{\Omega_{m+i,m+j}^{\mathfrak{gl}_N}}{z_i - z_j} \right) dz_i.$$

Both connections above are flat S_n -equivariant connections on the trivial vector bundle E_Y over Y . To relate their monodromy, first recall that $\Omega^{\mathfrak{sl}_N}$ and $\Omega^{\mathfrak{gl}_N}$ are related by the equation

$\Omega^{\mathfrak{gl}_N} = \Omega^{\mathfrak{sl}_N} + \frac{1}{N}I \otimes I$ where $I \in \mathfrak{gl}_N$ is the identity matrix. As I acts on V_i by λ_i for $1 \leq i \leq m$ and as 1 for $m+1 \leq i \leq m+n$, we have

$$\nabla_{\text{EGO}} = r^* \nabla_{\text{KZ}} + \nu \sum_{i=1}^n \left(\sum_{k=1}^m \frac{\lambda_k/N + (N-c)/m}{z_i - \alpha_k} + \sum_{1 \leq j \neq i \leq n} \frac{1/N}{z_i - z_j} \right) dz_i. \quad (4.2)$$

Note that the connection

$$\nabla_{\text{diff}} := d + \nu \sum_{i=1}^n \left(\sum_{k=1}^m \frac{\lambda_k/N + (N-c)/m}{z_i - \alpha_k} + \sum_{1 \leq j \neq i \leq n} \frac{1/N}{z_i - z_j} \right) dz_i \quad (4.3)$$

on E_Y is itself flat and scalar-valued, so it follows from equation (4.2) that the parallel transport operator associated to a path γ in Y for the connection $(E_Y, \nabla_{\text{EGO}})$ is obtained by multiplying the parallel transport operator associated to γ for the connection $(E_Y, r^* \nabla_{\text{KZ}})$ by the scalar-valued parallel transport operator associated to γ for the connection $(E_Y, \nabla_{\text{diff}})$. By inspection of the residues in ∇_{diff} in equation (4.3), it follows that for the loops U_k and T_i about \mathbf{z}_0 in Y/S_n , the monodromies $\mu_{\nabla_{\text{EGO}}}$ and $\mu_{\nabla_{r^* \text{KZ}}}$ for the connections ∇_{EGO} and $r^* \nabla_{\text{KZ}}$ are related by

$$\mu_{\nabla_{\text{EGO}}}(U_k) = q^{2(\lambda_k/m + (N-c)/m)} \mu_{\nabla_{r^* \text{KZ}}}(U_k) \quad (4.4)$$

and

$$\mu_{\nabla_{\text{EGO}}}(T_i) = q^{1/N} \mu_{\nabla_{r^* \text{KZ}}}(T_i), \quad (4.5)$$

where as before q is defined by $q = e^{-\pi i \nu}$.

All that remains is to relate the monodromy operators $\mu_{\text{KZ}}(U_k)$ and $\mu_{\text{KZ}}(T_i)$ to the R -matrix expressions appearing in Theorem 3.1 using the Drinfeld–Kohno theorem. The original formulation of the Drinfeld–Kohno theorem as stated in [3] was for the \hbar -adic quantum group $U_{\hbar}(\mathfrak{g})$, but here we need a version of this theorem for $U_q(\mathfrak{sl}_N)$ for q a nonzero complex number which is not a root of unity (Drinfeld’s work in [3] was a generalization of previous results obtained by Kohno [13] for \mathfrak{sl}_N). Such a result was obtained in [11] and an exposition can be found, for example, in [6, Theorem 8.6.4]. Similarly to [6, Corollary 8.6.5], this theorem immediately implies in particular that for $\nu \notin \mathbb{Q}$, the monodromy representation $\mu_{\nabla_{\text{KZ}}}$ of $\pi_1(Y_{m+n}/S_n)$ on $E_{n,\chi}$ at \mathbf{z}_0 is equivalent to the representation given by R -matrix expressions in which the class of a loop γ at \mathbf{z}_0 acts by the product of R -matrices $R_{i_1} \cdots R_{i_l}$ whenever $[\pi_* \gamma]$ factors as $\sigma_{i_1} \cdots \sigma_{i_l}$ in the braid group $B_{n+m} := \pi_1(Y_{m+n}/S_{m+n})$. Here σ_i is the standard i^{th} generator of B_i (counterclockwise half-loop around the hyperplane $z_i = z_{i+1}$), R_i is the R -matrix for $U_q(\mathfrak{sl}_N)$ with $q = e^{-\pi i \nu}$, and $\pi: Y_{m+n}/S_n \rightarrow Y_{m+n}/S_{m+n}$ is the natural projection. As $\mu_{\nabla_{r^* \text{KZ}}}(\gamma) = \mu_{\nabla_{\text{KZ}}}(r_* \gamma)$ for any loop γ , Theorem 4.2 now follows from equations (4.4) and (4.5) and the observation that $[\pi_* r_* T_i] = \sigma_{m+i}$ and $[\pi_* r_* U_k] = \sigma_m \cdots \sigma_k \sigma_k \sigma_{k+1}^{-1} \cdots \sigma_m^{-1}$. ■

Remark 4.3. A similar approach may be used for some $\nu \in \mathbb{Q}$ using Part IV of [11], but the analogous statements more complicated due to the failure of $U_q(\mathfrak{sl}_N)\text{-mod}_{f,d}$ to be semisimple in this case. If the representations V_1, \dots, V_m are fixed, then there is an analogous construction via R -matrices for $\nu \in \mathbb{Q}$ as long as the denominator of ν is sufficiently large.

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