

Invitation to Random Tensors^{*}

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Received September 21, 2016; Published online September 23, 2016

<http://dx.doi.org/10.3842/SIGMA.2016.094>

Abstract. This article is preface to the SIGMA special issue “Tensor Models, Formalism and Applications”, http://www.emis.de/journals/SIGMA/Tensor_Models.html. The issue is a collection of eight excellent, up to date reviews on random tensor models. The reviews combine pedagogical introductions meant for a general audience with presentations of the most recent developments in the field. This preface aims to give a condensed panoramic overview of random tensors as the natural generalization of random matrices to higher dimensions.

Key words: random tensors

2010 Mathematics Subject Classification: 83C45

Why random tensors?

General relativity, or classical gravity, is a theory of the ambient space-time geometry. The electroweak and strong interactions, that is the other three fundamental forces in nature, are described by perturbatively renormalizable quantum field theory [68, 72, 73, 74, 124, 132, 142, 145] living on this geometric background. The main lesson of general relativity is that the ambient geometry is dynamical, and the main lesson of quantum field theory is that all the dynamical fields must be quantized. Thus, classical gravity predicts its own demise: a more fundamental theory, “quantum gravity”, must come into play at some high energy scale.

While classical general relativity is a field theory, it cannot be quantized the same way the standard model is: general relativity is *not* perturbatively renormalizable [70, 143]. The lack of perturbative renormalizability of general relativity is a clear indicator that “quantum gravity” is quite different from the classical theory of gravity. This is why minimalistic approaches should be taken with a grain of salt: gravity is weakly coupled in the infrared, hence strongly coupled in the ultraviolet. This suggest that the fundamental degrees of freedom of quantum gravity are quite different from the geometric degrees of freedom perceived by low energy observers. It is far more likely that the geometric infrared degrees of freedom are just bound states of the genuine quantum gravity ultraviolet degrees of freedom.

Over the years several candidate quantum gravity theories have been developed, most notably string theory. While much remains to be learned about the elusive fundamental theory of quantum gravity, one thing is certain: whatever this theory may be, it must make sense of an expression like:

$$\sum_{\text{topologies}} \int [\mathcal{D}g][dX] e^{-S_{\text{EH}} - S_{\text{SM}} - S_{\text{other}}}. \quad (1)$$

*This paper is a contribution to the Special Issue on Tensor Models, Formalism and Applications. The full collection is available at http://www.emis.de/journals/SIGMA/Tensor_Models.html

This is how quantum gravity should look like [51]: a path integral over matter fields X and over geometric degrees of freedom (for instance metrics g and topologies) of the exponential of the Einstein–Hilbert action plus the standard model action plus possibly other terms, describing physics yet to be discovered. The sum over topologies in equation (1) can be reduced to a single topology by fine tuning S_{other} , but *a priori* there is no reason to do this: one lesson of quantum field theory is that whatever can happen will happen, and this includes topology change.

As already hinted, the precise meaning of equation (1) is unclear: the sum over topologies and metrics is ill defined and the naive attempts to use this equation as the starting point for a theory of quantum gravity fail. A strategy to make sense of this equation is to replace the sum over (continuum) geometric degrees of freedom with a sum over triangulations:

$$\sum_{\text{topologies}} \int [\mathcal{D}g] \rightarrow \sum_{\text{triangulations}},$$

and to replace the continuum actions by their discretized versions. However, when passing to a discrete setting, several questions arise, most notably:

- what is the weight (probability distribution) one should use to sum over triangulations?
- how does one go back from discrete to continuum geometries?

The answers to these questions are intertwined. In order to go back from discrete to continuum geometries one needs some kind of phase transition. The precise features of this phase transition strongly depend on the probability distribution chosen. In contrast to two or three dimensions, in dimension four information about the smooth structure is lost when going to a discretization. Some of this information could perhaps index the available phases of continuous geometry.

Geometry and topology become increasingly complicated when increasing the topological dimension and it seems reasonable to first address these questions in dimension two, and only afterwards pass to the more realistic dimensions three or four.

In two dimensions the answers to these questions are provided by the theory of random matrices [4, 51, 115, 146, 147]. Random matrices are probability distributions for $N \times N$ random variables \mathbb{M}_{ab} , which are invariant under the conjugation of \mathbb{M} by the unitary group. The moments and partition function of a random matrix model can be evaluated as sums over ribbon Feynman graphs, with weights fixed by the Feynman rules. These ribbon graphs are dual to topological surfaces. As the probability distribution of the surfaces (weights of the graphs) is fixed by the Feynman rules, random matrices yield a *canonical* theory of random two-dimensional topological surfaces and provide an answer to the first question. One still has some freedom in assigning metrics to the triangulated topological surfaces. The simplest choice is to consider the triangulations as equilateral.

As always in quantum field theory, the perturbative Feynman series diverges. However, the situation is much more subtle in matrix models than in usual quantum field theory. A matrix model is endowed with a natural small parameter, $1/N$ (where N is the size of the matrix), which does not exist in usual quantum field theory, and one can reorganize the perturbative expansion of a matrix model as a series in $1/N$. In his seminal work [141] 't Hooft showed that the $1/N$ series is indexed by the genus. This is the fundamental feature of matrix models: the $1/N$ expansion reorganizes the perturbative series into nontrivial but manageable packages of graphs of fixed genus.

At leading order in $1/N$ the planar graphs [36, 48] dominate. While planar graphs can be arbitrarily large (i.e., they can have an arbitrary number of edges), they can be explicitly enumerated [44, 45, 139] and form an exponentially bounded family. This holds order by order in $1/N$: the family of graphs of any fixed genus is exponentially bounded. Tuning the coupling

constants of a matrix model to some critical values, infinite graphs with fixed genus will dominate at any order in $1/N$. At this critical point the model undergoes a phase transition to a continuum theory of random, infinitely refined, surfaces [49, 101]. This answers the second question above. Assuming an equilateral metric assignment, the infinitely refined random geometry emerging at criticality, the Brownian map [109, 110, 111], has average Hausdorff dimension 4 and is widely believed to have average spectral dimension 2.

The behavior of conformal matter coupled to Liouville gravity in two dimensions is very well captured by matrix models [3, 34, 35, 49, 51, 52, 54, 56, 60, 102, 103, 105, 113]. The double scaling limit of matrix models [37, 55, 71] corresponds to a continuum gravity theory with finite renormalized Newton’s constant. Matrix models extend naturally to matrix field theories, which are intimately related to noncommutative quantum field theories [76, 93], in particular to the Grosse Wulkenhaar model, which has been shown asymptotically safe at all orders [53] and solved in the planar sector [77, 78, 79, 80].

Matrix models can be studied using the eigenvalue decomposition or the Schwinger–Dyson equations [8, 57, 58, 59, 81, 114], avoiding the divergent perturbative expansion. However, the link with random surfaces must be revisited in this case. In order to obtain a theory of random surfaces one first performs the perturbative expansion and subsequently reorganizes it in powers of $1/N$. As the perturbative expansion is not summable, it is a non trivial task [100] to establish the connection between random surfaces and matrix models rigorously. Even more tantalizing, the critical point where the continuum limit is reached is at *negative* values of the coupling constants, in a range of parameters where the models are unstable. This is a feature, not an accident: the critical point must correspond to a regime where all the surfaces add up, and not to a point where the sum over surfaces is alternated. The tuning to criticality is meaningful after restricting to a fixed order in $1/N$, but what is the meaning of this tuning to criticality beyond perturbation theory?

Although many non perturbative questions about random matrices are still open, by and large, matrix models are a success story. One lesson can be drawn from their example: in two dimensions, in most cases, *minimal choices suffice*. The relation between random matrices and quantum gravity is already apparent if one studies the simplest matrix model and considers the simplest metric assignment (equilateral) for the triangulations.

The development of matrix models over the past decades is one of the most impressive achievements of modern theoretical and mathematical physics. This success inspired their generalization in the 1990s to random tensor models [1, 75, 136, 137, 138] intended to describe random geometries in higher dimensions. However, for twenty years random tensors essentially failed to match the success of random matrices because, for a long time, a $1/N$ expansion for tensors could not be found.

Random tensors generate Feynman graphs that can be interpreted as topological spaces. However, the spaces generated in this way are quite nontrivial: one obtains not only all the manifolds, but also all the pseudo manifolds of a fixed dimension¹. Several models [6, 7, 33, 116, 117, 118, 121], mainly under the guise of “Group Field Theories”, which are tensor models decorated by extra data, have been proposed in the attempt to tackle this problem. Although different in some important respects, these models did not bring any insight into the problem of the $1/N$ expansion.

Starting in 2009 [82, 83, 85], new results and techniques led to the discovery of the $1/N$ expansion for tensors [31, 84, 88, 94]. These results form the backbone of the modern theory of random tensors. In this modern point of view, random tensors are probability distributions for N^D random variables $\mathbb{T}_{a_1 \dots a_D}$, which are invariant under the independent action of the

¹A further complication arose from the fact that the first models proposed did not generate genuine D -dimensional complexes, but only 2-complexes.

unitary group on each tensor index:

$$\mathbb{T}_{a_1 \dots a_D} \rightarrow \mathbb{T}'_{b_1 \dots b_D} = \sum_{a_1, \dots, a_D} U_{b_1 a_1}^{(1)} \cdots U_{b_D a_D}^{(D)} \mathbb{T}_{a_1 \dots a_D}.$$

The main building blocks of a random tensor model are the invariants one can build out of the tensor \mathbb{T} and its complex conjugate. While for matrices there is essentially one such invariant at any degree, $\text{Tr}[(\mathbf{M}\mathbf{M}^\dagger)^p]$, there are numerous possible choices for tensors. Each of these invariants (generalized traces) can be represented by a bipartite D -regular edge colored graph, and there are as many independent invariants at degree $2p$ as there are non isomorphic such graphs with $2p$ vertices. This is the source of all the richness of random tensors.

The graphs of the new tensor models always encode genuine D -dimensional cellular complexes hence D -dimensional topological spaces. The $1/N$ series [25, 47, 84, 88, 90, 94] is indexed by a positive integer called the *degree*, which plays in higher dimensions the same role the genus played for matrix models. Unlike the genus, however, the degree is *not* a topological invariant, but mixes topological and triangulation dependent information. This is not a drawback, but a quality. Topology is quite complicated in dimensions three and four (or higher): although topologies are not fully classified in arbitrary dimension, the graphs at fixed degree can be [98].

In the large N limit the graphs of degree zero, called *melonic* [29, 91, 97], dominate. The melonic graphs triangulate the D -dimensional sphere in any dimension [84, 88, 94] and are an exponentially bounded family. Like matrix models, tensor models undergo a phase transition to a theory of continuous random D -dimensional topological spaces [29, 31] when tuning to criticality and posses a double scaling limit [32, 46, 98]. In stark contrast to matrix models, for $3 \leq D \leq 5$, in the double scaling regime only an exponentially bounded family of triangulations of the sphere contribute. For $D \geq 6$ (as for $D = 2$) the family selected in the double scaling limit is neither exponentially bounded nor restricted to the spherical topology. The $1/N$ expansion of tensor models exhibits very strong universality properties [87, 89, 91] and has been put on a firm mathematical footing [90] in the sense of constructive field theory [69] by means of the loop vertex expansion [95, 112, 125, 126]. The ensuing theory of random geometries in higher dimensions has been extensively studied [11, 12, 15, 17, 22, 24, 26, 28, 30, 40, 41, 61, 86, 87, 89, 96, 99, 106, 131, 144].

Random tensor models generalize random matrix models and provide a framework for the study of random geometries in any dimension. However, as gravity is quite peculiar in two dimensions (and it is peculiar in a different way in three dimensions, and it is peculiar in yet another different way in four), one should keep an open mind for alternatives.

In their simplest form, tensor models can be seen, like matrix models, as generators of Euclidean dynamical triangulations [2, 50, 92]. There exists a tensor model [29] whose free energy $W_{\text{TM}}(\lambda, N)$ equals an Einstein–Hilbert gravity partition function discretized on an equilateral triangulation with edge length a :

$$\sum_{\text{topologies}} \int [\mathcal{D}g] e^{\frac{1}{16\pi G} \int d^D x \sqrt{g} (R - 2\Lambda)} \rightarrow \sum_{\substack{\text{Triangulations} \\ \text{edge length } a}} e^{-S_{\text{EH}}^{\text{discr.}}(G, \Lambda; a)} = W_{\text{TM}}(\lambda, N),$$

where the coupling constant λ and the parameter N of the tensor model are related to the edge length a , the dimensionfull cosmological constant Λ and Newton's constant G respectively the dimensionless cosmological constant $\tilde{\Lambda}$ and Newton's constant \tilde{G} by the relations:

$$\tilde{G} = \frac{G}{a^{D-2}} = c_1 \frac{1}{\ln N}, \quad \tilde{\Lambda} = \Lambda a^2 = c_2 \frac{G}{a^{D-2}} \ln \left(\frac{1}{\lambda} \right) + c_3 = \tan \alpha \tilde{G} + c_3,$$

with $c_1, c_2, c_3 > 0$ some constants of order 1 (and $c_3 = 0$ for $D = 2$) and $\tan \alpha = c_2 \ln \left(\frac{1}{\lambda} \right)$.

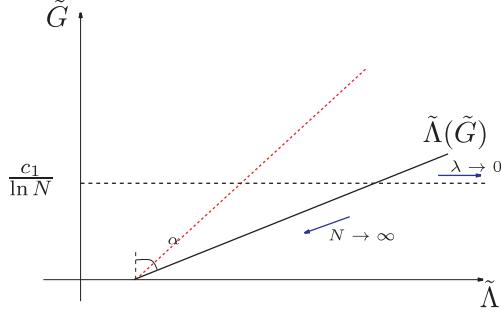


Figure 1. The $(\tilde{G}, \tilde{\Lambda})$ plane.

The $(\tilde{G}, \tilde{\Lambda})$ plane is represented in Fig. 1. The angle α is the *slope* of the (solid) line $\tilde{\Lambda}(\tilde{G}) = \tan \alpha \tilde{G} + c_3$ with respect to the vertical axis. The coupling constant λ controls α , while N controls the height of the dashed line in Fig. 1. The physical constants \tilde{G} and $\tilde{\Lambda}$ are the coordinates of the intersection point of the two lines. For $\lambda \in (0, \infty)$, $\alpha \in (\frac{\pi}{2}, -\frac{\pi}{2})$ and, as $N > 1$, the two parameters λ and N allow one to explore the entire upper half plane of the physical constants.

As long as a is kept fixed, the dimensionfull constants are just a finite rescaling of the dimensionless ones. The correspondence between the various regimes of tensor models and the physical coupling constants can be read out from this figure, observing that $\lambda \rightarrow 0$ at fixed N moves the intersection point to the right ($\alpha \rightarrow \frac{\pi}{2}$) at constant height, while $N \rightarrow \infty$ at fixed λ descends the intersection point along the line $\tilde{\Lambda}(\tilde{G})$ at fixed slope α .

The perturbative expansion. We have a and N fixed, $\lambda \rightarrow 0$. The perturbative expansion is an expansion at *large cosmological constant and fixed Newton's constant*:

$$\tilde{G} \text{ constant (small)}, \quad \alpha \rightarrow \frac{\pi}{2}, \quad \tilde{\Lambda} \rightarrow \infty.$$

This is an expansion around the zero volume state corresponding to $\tilde{\Lambda} = \infty$.

The $1/N$ expansion. We have a and λ fixed, $N \rightarrow \infty$. For $D \geq 3$, the $1/N$ expansion is an expansion at *finite cosmological constant, small Newton's constant* and constant slope:

$$\tilde{G} \rightarrow 0, \quad \alpha \text{ constant}, \quad \tilde{\Lambda} \rightarrow c_3.$$

This is an expansion around geometries of maximal positive curvature at fixed volume. The situation is different in $D = 2$. In that case the $1/N$ expansion is an expansion at *small cosmological constant and small Newton's constant* with constant slope.

The large N limit. We have a , λ fixed, $N = \infty$. In this limit one has a *finite cosmological constant* and *zero Newton's constant*, but this regime is approached along a line with fixed slope:

$$\tilde{G} = 0, \quad \alpha \text{ constant}, \quad \tilde{\Lambda} = c_3.$$

The large N limit projects onto geometries with maximal positive curvature at fixed volume. Among these geometries, one has a second expansion governed by the slope α around the zero volume state. That is, once projected onto geometries with maximal positive curvature, the slope α plays the role of an effective cosmological constant which is large and positive.

The critical regime. We have a fixed, $\lambda \rightarrow \lambda_c$, $N = \infty$. One still has a *finite cosmological constant* and *zero Newton's constant*, but the slope approaches a critical value:

$$\tilde{G} = 0, \quad \alpha \rightarrow \alpha_c, \quad \tilde{\Lambda} = c_3.$$

We represented the line with critical slope α_c in red in Fig. 1. The critical value α_c corresponds to a divergent number of D -simplices hence, as a is kept fixed, these geometries have infinite volume.

The continuum limit. We have $a \rightarrow 0$, $\lambda \rightarrow \lambda_c$, $a^D(\lambda_c - \lambda)^{-1}$ fixed, $N = \infty$. The number of D -simplices diverges but the physical volume is kept fixed by sending a to zero while keeping $a^D(\lambda_c - \lambda)^{-1}$ fixed. The dimensionless and dimensionfull constants have a different behavior:

$$\tilde{G} = 0, \quad \alpha \rightarrow \alpha_c, \quad \tilde{\Lambda} = c_3 \quad | \quad G = 0, \quad \Lambda = \infty.$$

The double scaling regime. We have a fixed, $\lambda \rightarrow \lambda_c$, $N \rightarrow \infty$, $N^{D-2}(\lambda_c - \lambda)$ fixed. This regime corresponds to a well defined trajectory in the plane of physical coupling constants:

$$\tilde{G} \rightarrow 0, \quad \alpha \rightarrow \alpha_c, \quad \tilde{\Lambda} \rightarrow c_3.$$

The continuum double scaling limit. We have $a \rightarrow 0$, $\lambda \rightarrow \lambda_c$, $N \rightarrow \infty$, with $N^{D-2}(\lambda_c - \lambda)$ fixed and $a^D(\lambda_c - \lambda)^{-1}$ fixed. In this regime we have:

$$\tilde{G} \rightarrow 0, \quad \alpha \rightarrow \alpha_c, \quad \tilde{\Lambda} \rightarrow c_3 \quad | \quad G \rightarrow 0, \quad \Lambda \rightarrow \infty.$$

The analytically accessible regimes of this tensor model explore regions with *small positive* (*or zero*) Newton constant and *large (or very large) positive* cosmological constant. Care should be taken when interpreting these values. This theory is supposed to describe the ultraviolet behavior of gravity, therefore the cosmological and Newton's constant we are discussing here are the ultraviolet ones. In order to obtain the infrared coupling constants one needs to follow a renormalization group flow, and both Λ and G will vary substantially as they are dimensionfull in $D \geq 3$.

The geometries of maximal positive curvature at fixed volume are rather simple and fall into the universality class of the continuous random tree (branched polymers). On the one hand this is good news, as one can treat analytically statistical systems in random geometry [22, 24, 27, 28, 30], hence the coupling of gravity with matter fields. On the other, one can only do so much with branched polymers and it is important to find regimes in which the emergent geometries are richer.

The double scaling limit is one attempt to go beyond branched polymers. In this limit larger families of graphs are included and, more importantly, below $D = 6$ it seems likely that the double scaling regime can be followed by a triple scaling regime, revealing an even larger family of graphs. The strategy one employs in this context is to keep the metric interpretation of a graph as dual to an equilateral triangulation, and to extend (in a controlled manner) the family of graphs included in the statistical ensemble.

A second option is to encode metric degrees of freedom in additional data associated to Feynman graphs. This strategy is sometimes called group field theory (GFT) [5, 6, 7, 33, 104, 116, 117, 118, 121] and has proven quite successful. In GFT one considers tensors over some Lie group that are furthermore invariant under the diagonal action of the group. With minimal adaptations the $1/N$ expansion holds for GFTs [84, 88]. Due to the diagonal gauge invariance, the amplitude of a GFT graph is the discretized BF action on the dual triangulation. At least in three dimensions, this is exactly the gravity partition function on the dual triangulation (in higher dimensions the BF action must be supplemented by constraints). The tantalizing fact is that, choosing for instance the group $SU(2)$ in three dimensions, the melonic triangulations (which still dominate) are endowed with a *flat connection*. This, most definitely, is *not* a branched polymer geometry. However, the reader should be aware of the following fact. While a metric

on a (pseudo) manifold is uniquely encoded in the holonomies around all the closed loops in that (pseudo) manifold, when discretizing one loses a *local conformal factor*. While the fact that melonic geometries have a flat connection is very encouraging, before fixing this conformal factor issue one cannot control the emerging random geometry in GFTs. Be that as it may, GFT has successfully been applied to quantum cosmology [62, 63, 64, 65, 66, 67, 119, 120, 122, 123, 140], in what is the most relevant phenomenological application of tensor models to date.

A third option is to extend the framework of tensor models to tensor field theories. As matrix models extend naturally to matrix field theories, tensor models extend naturally to tensor field theories (TFT) [127, 128, 130] by breaking the unitary invariance of the Gaussian part of the measure. The covariance of such models possesses a nontrivial spectrum, which in turn is naturally divided into scales. The integration of the high scales leads to a genuine renormalization group flow [9, 18, 19] and the TFTs explore the *tensor theory space* [129] spanned by all the unitary invariant polynomial interactions. One can combine both tensors with a diagonal gauge invariance and a nontrivial covariance in what could be called tensor group field theory. Such models are also perturbatively renormalizable (which is highly nontrivial in this case, and relies crucially on the ultraviolet dominance of the melonic graphs), and exhibit a very rich phase portrait. While much remains to be done, one could hope that this class of models can fix the local conformal factor problem of the usual GFTs.

The TFTs are examples of *renormalizable, nonlocal* field theories. A regime parallel to the large N limit of tensor models is reached in TFTs via a genuine renormalization group flow [9, 10, 12, 13, 14, 16, 17, 18, 19, 20, 21, 23, 38, 39, 42, 43, 107, 108, 133, 134, 135], and not by sending the parameter N to infinity. The large N behavior of tensor models corresponds to the ultraviolet limit of TFTs. Typically the couplings grow in the infrared which suggests that TFTs develop bound states at low energy. This low energy behavior remains largely to be explored either by analytic [38, 39] or by numeric [21] methods. We emphasize that the natural metric assignment for the triangulations dual to graphs in TFTs is not yet understood. However, the fact that TFTs develop bound states in the infrared is an encouraging sign and establish them as potential ultraviolet completions of gravity.

Random tensors are the straightforward generalization of random matrices in higher dimensions. It should however be stressed that the $D = 2$ case of matrices is very special. Indeed, random tensors behave by and large quite differently from random matrices. This is due to the fact that the melonic family, which dominates in tensor models, is very different from (and in fact much more restricted than) the planar family dominating matrix models.

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