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Null Angular Momentum and Weak KAM Solutions of the Newtonian N-Body Problem

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Abstract. In [*Arch. Ration. Mech. Anal.* **213** (2014), 981–991] it has been proved that in the Newtonian N-body problem, given a minimal central configuration a and an arbitrary configuration x, there exists a completely parabolic orbit starting on x and asymptotic to the homothetic parabolic motion of a, furthermore such an orbit is a free time minimizer of the action functional. In this article we extend this result in abundance of completely parabolic motions by proving that under the same hypothesis it is possible to get that the completely parabolic motion starting at x has zero angular momentum. We achieve this by characterizing the rotation invariant weak KAM solutions as those defining a lamination on the configuration space by free time minimizers with zero angular momentum.

Key words: N-body problem; angular momentum; free time minimizer; Hamilton–Jacobi equation

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1 Introduction

1.1 Preliminaries

Let $E = \mathbb{R}^d$ be the d-dimensional Euclidean space, $d \geq 2$, and consider the N-body problem with Newtonian potential function $U\colon E^N \to]0, +\infty]$,

$$U(x) = \sum_{i<j} \frac{m_i m_j}{r_{ij}},$$

where $x = (r_1, \dots, r_N) \in E^N$ is a configuration of N points having positive masses $m_1, \dots, m_N$ in E, and $r_{ij} = |r_i - r_j|$. Here $|\cdot|$ denotes the Euclidean norm. We will denote by $\|\cdot\|$ to the norm induced by the mass inner product given by

$$x \cdot y = \sum_{i=1}^{N} m_i \langle r_i, s_i \rangle,$$

with $x = (r_1, \dots, r_N)$, $y = (s_1, \dots, s_N) \in E^N$, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. We also introduce the moment of inertia:

$$I(x) = \|x\|^2.$$

In this section we introduce the variational setting of the problem. The Lagrangian function $L\colon E^{2N} \to]0, \infty]$ is given by

$$L(x, v) = \frac{1}{2} I(v) + U(x) = \frac{1}{2} \sum_{i=1}^{N} m_i |v_i|^2 + U(x)$$

and the action of an absolutely continuous curve $\gamma\colon [a,b] \to E^N$ by

$$A_L(\gamma) = \int_a^b L(\gamma(t), \dot\gamma(t))\mathrm{d}t,$$

so that the solutions of the problem are the critical points of the action functional.

For two given configurations $x, y \in E^N$, we will consider minima taken over the set $\mathcal{C}(x,y)$ of absolutely continuous curves binding x and y without any restriction on time,

$$\mathcal{C}(x,y) := \bigcup_{\tau>0} \{\gamma\colon [a,b] \to E^N \text{ absolutely continuous } b-a=\tau,\ \gamma(a)=x,\ \gamma(b)=y\}.$$

The *Mañé critical action potential* $\phi\colon E^n \times E^n \to [0,+\infty)$ is defined as

$$\phi(x,y) := \inf\{A(\gamma)\,|\,\gamma \in \mathcal{C}(x,y)\}.$$

Here the infimum is achieved if and only if $x \neq y$, this is essentially due to the lower semicontinuity of the action. Marchal's theorem asserts that minimizers avoid collisions in the interior of their interval of definition [3, 6, 9].

Let us define the set of configurations without collision

$$\Omega := \big\{x = (r_1, \ldots, r_N) \in E^N\,|\,\text{if } r_i = r_j,\ \text{then } i = j\big\}, \tag{1}$$

and let $M := m_1 + \cdots + m_N$ the total mass of the system.

Given a configuration $x = (r_1, \ldots, r_N) \in E^N$, the center of mass of x is defined by

$$G(x) = \frac{1}{M}\sum_{i=1}^N m_i r_i,$$

it is a standard fact that if $\gamma\colon J \to E^N$ is a minimizer of the action, then $G(\gamma(t))$ has constant velocity.

Definition 1. A *free time minimizer* defined on an interval $J \subset \mathbb{R}$ is an absolutely continuous curve $\gamma\colon J \to E^N$ which satisfies $A(\gamma|_{[a,b]}) = \phi(\gamma(a), \gamma(b))$ for all compact subinterval $[a,b] \subset J$.

An important example is given by A. Da Luz and E. Maderna in [5] where they proved that if a is a *minimal configuration*, i.e., a minimum of the potential restricted to the sphere $I(x) = 1$, then the parabolic homothetic ejection with central configuration a is a free time minimizer. It is not known if there are other central configurations with this property. A. Da Luz and E. Maderna also proved that free time minimizers cannot be defined in the whole line. On the other hand it is also proved that this minimizers are completely parabolic motions.

1.2 Weak KAM solutions

Weak KAM solutions of the Hamilton–Jacobi equation related to the problem are useful tools to study free time minimizers. The Hamiltonian associated to the problem is given by

$$H(x,p) = \tfrac{1}{2}||p||_*^2 - U(x),$$

where $||p||_* = \sum\limits_{i=1}^N m_i^{-1}|p|$.

Definition 2. A *weak KAM solution* of the Hamilton–Jacobi equation

$$\|Du(x)\|_*^2 = 2U(x) \tag{2}$$

is a function $u\colon E^N \to \mathbb{R}$ that satisfies the following conditions:

- u is *dominated*, i.e., $u(y) - u(x) \le \phi(x, y)$ for all $x, y \in E^N$,
- for any $x \in E^N$ there is an absolutely continuous curve $\alpha\colon [0,\infty) \to E^N$ such that $\alpha(0) = x$ and α *calibrates* u, i.e., $u(x) - u(\alpha(t)) = A(\alpha|_{[0,t]})$ for any $t > 0$.

Notice that calibrating curves of weak KAM solutions are indeed free time minimizers. On the other hand, existence of weak KAM solution is proved in [7], where solutions are characterized as fixed points of the so called Lax–Oleinik semigroup. A variety of weak KAM solutions is also obtained by means of Busemann functions used in Riemannian geometry and introduced in weak KAM theory by G. Contreras [4] in the case of regular Hamiltonians; in [11] it is proved the following proposition

Proposition 1. *Let a a minimal central configuration with $\|a\| = 1$, define $U(a) = U_0$ and $c := \left(\frac{9}{2}U_0\right)^{\frac{1}{3}}$, consider the parabolic homothetic ejection with central configuration a given by $\gamma_0(t) = ct^{\frac{2}{3}}a$. Then the Busemann function*

$$u_a(x) = \lim_{t\to+\infty}[\phi(x,\gamma_0(t)) - \phi(0,\gamma_0(t))] \tag{3}$$

is a weak KAM solution of the Hamilton–Jacobi equation (2). Moreover, for any $x \in E^N$ there is a curve $\alpha\colon [0,\infty) \to E^N$ with $\alpha(0) = x$ that calibrates u and

$$\lim_{t\to+\infty} \left\|\alpha(t)t^{-\frac{2}{3}} - cx_0\right\| = 0.$$

A solution defined by identity (3), will be called *Busemann solution*. It is an open problem to determine if there are central configurations, different from minimal configurations, defining Busemann solutions.

On the other hand, due to the symmetries of the potential function, it is interesting to determine if weak KAM solutions are invariant under this symmetries. In the case of translation invariance, E. Maderna proved in [8] that given a weak KAM solution u of (2), then

$$u(r_1, \ldots, r_N) = u(r_1 + r, \ldots, r_N + r)$$

for any configuration $x = (r_1, \ldots, r_N) \in E^N$ and every $r \in E$. The proof is achieved by showing that calibrating curves of weak KAM solutions have constant center of mass.

An important question is to determine if weak KAM solutions are rotation invariant, the main goal of this article is to study this problem. Notice that, there are solutions which are not rotation invariant, Busemann solutions given in (3) for instance. Therefore the problem is to give conditions so that a weak KAM solution is rotation invariant; we achieve this goal by studying the angular momentum for the calibrating curves of rotation invariant solutions and characterizing invariant solutions as those where calibrating curves have zero angular momentum. We obtain rotation invariant solutions by setting

$$\hat{u}_a = \inf_{R\in \mathrm{SO}(d)} u_{Ra}(x),$$

where a is a minimal central configuration and $u_{Ra}(x)$ is the Busemann function associated to Ra.

1.3 Main theorems

We consider the diagonal group action on E^N defined by the special orthogonal group $\mathrm{SO}(d)$, more precisely, the rotation on E^N by an element $\theta \in \mathrm{SO}(d)$ is

$$R_\theta\colon\ E^N \to E^N, \qquad x = (r_1, \ldots, r_N) \mapsto (\theta r_1, \ldots, \theta r_N),$$

where θr_i is the usual group action of $\mathrm{SO}(d)$ on E.

The *Angular momentum* is a first integral closely related to the action of $\mathrm{SO}(d)$ on E^N. If $x=(r_1,\ldots,r_N)\in E^N$ and a vector $v=(v_1,\ldots,v_N)\in E^N$ the angular momentum $C(x,v)$ is defined as

$$C(x,v)=\sum_{j=1}^{N} m_j r_j\wedge v_j.$$

If $d=3$ the $\wedge$ product becomes the usual cross product in E. If $d=2$, by identifying $\mathbb{R}^2$ with $\mathbb{C}$, if $x,v\in\mathbb{C}$ then $r\wedge v=\mathrm{Im}(v\bar{r})$, and $r\wedge v$ is a real number.

Let $u\colon E^N\to\mathbb{R}$ a continuous function, we say that u is *rotation invariant* if for any $x\in E^N$ and any $\theta\in\mathrm{SO}(d)$ we have

$$u(R_\theta(x))=u(x).$$

We have the following characterization of invariant weak KAM solutions of (2) in terms of the angular momentum of their calibrating curves.

Theorem 1. *Let u be a weak KAM solution of the Hamilton–Jacobi equation*

$$\|Du(x)\|_*^2=2U(x).$$

Then u is rotation invariant if and only if all of its calibrating curves have zero angular momentum. That is to say, for any $\theta\in\mathrm{SO}(d)$ and any $x\in E^N$, the identity

$$u(x)=u(R_\theta x)$$

holds if and only if for any $\gamma\colon[0,+\infty[\,\to E^N$ calibrating u, we have

$$C(\gamma(t),\dot\gamma(t))=0.$$

We can give a more general result by considering G a Lie group acting properly on E. Thus we can consider the diagonal action $S\colon G\times E^N\to E^N$ of G on E^N, defined by

$$S_g x=(gr_1,gr_2,\ldots,gr_N),$$

where $g\in G$, $x=(r_1,\ldots,r_N)\in E^N$ and gr is the action of g on r. Let us denote by $\mathfrak{g}$ to the Lie algebra of G. We denote by $[\,,\,]$ to the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Notice that the action S can be lifted to $E^N\times E^N$ by $g(x,v)=(S_gx,T_xS_gv)$ where T_xS_g is the differential of S_g at x. Assume that the Lagrangian is G-invariant, i.e., $g^*L=L$, $g\in G$ and assume also that the action lifts to $E^N\times E^N$ by isometries of the mass inner product.

Under such conditions, the group action defines an *equivariant momentum map*

$$\mu\colon\ E^N\times E^N\to\mathfrak{g}^*,$$

given by

$$[\mu(x,v),\xi]=v\cdot X_\xi(x), \tag{4}$$

where $X_\xi(x)=\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}S_{\exp(t\xi)}x$ is the infinitesimal generator of the one-parameter subgroup action on E^N, associated to $\xi\in\mathfrak{g}$.

A continuous function $u\colon E^N\to\mathbb{R}$ is *G-invariant* if for any $g\in G$ and any $x\in E^N$ we have

$$u(S_gx)=u(x).$$

In a similar way to Theorem 1 we can give a characterization to G-invariant weak KAM solutions in terms of the equivariant momentum map.

Theorem 2. *Let G a connected Lie group acting diagonally on E^N and suppose that the group action satisfies the assumptions above, and let u be a weak KAM solution of the Hamilton–Jacobi equation (2). Then u is G-invariant if and only if for any $\gamma\colon [0,+\infty[\, \to E^N$ that calibrates u, for any $t > 0$, we have*

$$\mu(\gamma(t), \dot\gamma(t)) = 0.$$

2 Rotation invariance

Given $x \in E^N$, consider the orbit of x under $\mathrm{SO}(d)$ given by

$$M_x := \{R_\theta x \mid \theta \in \mathrm{SO}(d)\},$$

let us remind that

$$T_x M_x = \{Ax \mid A \in \mathfrak{so}(d)\}.$$

The key point in the proof of Theorem 1 is the Saari decomposition of the velocities [3, 12]. Define

$$\mathcal{H}_x := \left\{v \in E^N \mid C(x, v) = 0\right\}.$$

Then $T_x M_x \perp \mathcal{H}_x$, with respect to the mass scalar product and

$$E^N = T_x M_x \oplus \mathcal{H}_x.$$

In other words, if $v \in E^N$, then v can be decomposed as

$$v = v_r + v_h,$$

where $v_r \in T_x M_x$, $C(x, v_h) = 0$ and $v_r \cdot v_h = 0$, moreover the components v_r and v_h are uniquely determined by v. For dimensions 2 and 3 this is a direct consequence of the properties of the cross product, for dimensions ≥ 4 it is due to the properties of the "wedge" product.

On the other hand notice that if u is a rotation invariant function and $x \in E^N$, then u is constant on M_x. Thus, if x is a point of differentiability of u, we have that

$$T_x M_x \subset \ker \mathrm{d}_x u.$$

Proof of Theorem 1. Let u be a rotation invariant weak KAM solution and let $x \in E^N$, consider a curve $\gamma\colon [0,+\infty[\, \to E^N$ calibrating u and starting at x. It is known that u is differentiable at $\gamma(t)$ for any $t > 0$, we also know that

$$\mathrm{d}_{\gamma(t)} u(w) = w \cdot \dot\gamma(t), \tag{5}$$

for all $w \in E^N$.

On the other hand, by the previous remark we have that $T_{\gamma(t)} M_{\gamma(t)} \subset \ker \mathrm{d}_{\gamma(t)} u$ and from (5) we get $\dot\gamma(t) \perp T_{\gamma(t)} M_{\gamma(t)}$, therefore $\dot\gamma(t) \in \mathcal{H}_{\gamma(t)}$, then

$$C(\gamma(t), \dot\gamma(t)) = 0$$

for all $t > 0$.

Let us consider now a weak KAM solution u such that all of its calibrating curves have zero angular momentum. Let $x \in E^N$ and let $\theta \in \mathrm{SO}(d)$. We will prove that $u(R_\theta x) = u(x)$.

Clearly if $R_\theta x = x$, the result follows trivially. Suppose that $R_\theta x \neq x$, since u is continuous and the set of collisionless configurations Ω, given in (1), is open, dense and rotation invariant, we can assume $x \in \Omega$.

Since $\mathrm{SO}(d)$ is compact, $\exp\colon \mathfrak{so}(d) \to \mathrm{SO}(d)$ is surjective, thus we can take $\omega \in \mathfrak{so}(d)$ such that $\exp(\omega) = \theta$. Define the curve $\alpha\colon [0,1] \to E^N$ by $\alpha(t) = R_{\exp(t\omega)}(x)$. Let $\varepsilon > 0$ be small enough so that

$$B := \left\{ z \in E^N \mid \langle z, \dot\alpha(0)\rangle = 0,\ \|z - x\| < \varepsilon \right\} \subset \Omega.$$

Notice that the set $W := \{z \in E^N \mid R_\theta z \neq z\}$ is open, therefore we can also choose $\varepsilon > 0$, smaller if necessary, so that $B \subset W$.

We can assume that ε is sufficiently small so that the map

$$B \times [0,1] \to E^N, \qquad (z,t) \mapsto R_{\exp(t\omega)} z \tag{6}$$

is a diffeomorphism onto its image. Indeed, If $\alpha(t) = R_{\exp(t\omega)}(x) \neq x$ for every $t \in (0,1]$, then the curve α is an embedding and, by choosing ε sufficiently small, the map (6) is a diffeomorphism onto its image. Suppose, on the contrary, that $\alpha(\tau) = x$ for some $\tau \in (0,1]$, then α is τ-periodic. Let τ be the minimal period of α, then there exists $s \in (0,\tau)$ such that $\alpha(s) = R_{\exp(s\omega)}(x) = R_{\exp(\omega)}(x)$ and α has no self intersections in the interval $[0,s]$. Replacing ω by $s\omega$, and choosing ε sufficiently small we get as before that (6) is a diffeomorphism onto its image.

Define

$$C := \left\{ R_{\exp(t\omega)}(z) \mid z \in B,\ t \in [0,1] \right\}$$

and denote by C' to the set point in C where u is differentiable. Since u is dominated, u is Lipschitz in C and therefore C' has total measure in C. Notice also that $C \subset \Omega$.

Let $y \in C'$ and let $\gamma_y\colon [0,+\infty[\to E^N$ be a calibrating curve such that $\gamma(0) = y$, then $\mathrm{d}_y u(w) = w \cdot \dot\gamma_y(0)$ for any $w \in E^N$. From the hypothesis $\dot\gamma_y(0) \in \mathcal{H}_q$, thus $\dot\gamma_y(0) \perp T_y M_y$, thus $T_y M_y \subset \ker \mathrm{d}_y u$ for any $y \in C'$.

Let $f\colon B \times [0,1] \to \mathbb{R}$ the Lipschitz continuous function given by

$$f(z,t) = u\big(R_{\exp(t\omega)}(z)\big) - u(z).$$

Since $T_y M_y \subset \ker \mathrm{d}_y u$, we get $\frac{\partial f}{\partial t} = 0$ almost everywhere, by Fubini theorem in $A \times [0,1]$, with A any open subset of B

$$0 = \int_A \int_{[0,1]} \frac{\partial f}{\partial t} \mathrm{d}t\mathrm{d}y = \int_A f(y,1)\mathrm{d}y,$$

therefore $f(y,1) = 0$ for every $y \in B$, in particular we have

$$u(R_\theta(x)) = u(x). \qquad \blacksquare$$

Remark 1. Let us denote by $\mathcal{S}$ to the set of weak KAM solutions of (2) and notice that if $\mathcal{U} \subset \mathcal{S}$ is such that $\inf\limits_{u\in\mathcal{U}} u(x) > -\infty$, then

$$\tilde u(x) = \inf_{u\in\mathcal{U}} \{u(x) \mid u \in \mathcal{U}\}$$

is in $\mathcal{S}$. This is due to the fact that weak KAM solutions are the fixed points of the Lax–Oleinik semigroup [7].

Corollary 1. *Let a be a minimal central configuration with $I(a) = 1$. For any $\theta \in \mathrm{SO}(d)$, let $u_{R_\theta a}$ be the Busemann solution associated to the minimal central configuration $R_\theta a$. Then the function*

$$\hat{u}_a\colon\ E^N \to \mathbb{R}, \qquad \hat{u}_a(x) := \inf_{\theta\in \mathrm{SO}(d)} u_{\theta,a}(x)$$

is a rotation invariant weak KAM solution of the Hamilton–Jacobi equation. Therefore the callibrating curves of $\hat{u}_a$ are free time minimizers having zero angular momentum.

Proof. Let $\mathcal{M}$ be the set of minimal central configurations with moment of inertia one. Let $a \in \mathcal{M}$ and $\theta \in \mathrm{SO}(d)$ let

$$u_{\theta,a}(x) = u_a \circ R_{\theta^{-1}}(x),$$

it is not hard to see that $u_{\theta,a}$ is also a weak KAM solution, furthermore, notice that $u_{R_\theta a} = u_{\theta,a}$, thus

$$\hat{u}_a(x) = \inf_{\theta\in \mathrm{SO}(d)} u_{\theta,a}(x), \tag{7}$$

and from the previous remark the function on the right is a weak KAM solution.

Due to (7), $\hat{u}_a$ is rotation invariant and from Theorem 1, these solutions define laminations by free time minimizer with zero angular momentum. ∎

Given a minimal central configuration a, notice that the rotation invariant weak KAM solution $\hat{u}_a$ given in the previous Corollary in uniquely determined by M_a, the orbit of a under $\mathrm{SO}(d)$, we call this solution *invariant Busemann solution* associated to M_a.

3 G-invariance

Let G be a connected Lie group acting on E^N with the assumptions of Section 1.3, let us notice that in this setting, due to (4) the equivariant momentum map defines a Saari decomposition of the velocity (see [1, 2, 10]), as follows.

For a fixed momentum value, $\mu(x, v) = \mu$, there are orthogonal vectors $v_{\mathcal{H}}$ and $v_{\mathcal{V}}$ such that

$$v = v_{\mathcal{H}} + v_{\mathcal{V}}, \qquad \mu(x, v_{\mathcal{V}}) = \mu \qquad \text{and} \qquad \mu(x, v_{\mathcal{H}}) = 0.$$

Let $x \in E^N$ and let G_x be the orbit of x under the G-action. Consider the subspaces of $T_xE^N = E^N$

$$\mathcal{H}\mathrm{or}_x = \left\{ v \in E^N \mid \mu(x, v) = 0 \right\} \qquad \text{and} \qquad T_xG_x,$$

then these subspaces are orthogonal with respect to the mass inner product and

$$E^N = \mathcal{H}\mathrm{or}_x \oplus T_xG_x.$$

Thus, any $v \in E^N$ can be uniquely decomposed as

$$v = v_{\mathcal{H}} + v_{\mathcal{V}}, \tag{8}$$

where $v_{\mathcal{H}} \in \mathcal{H}\mathrm{or}_x$, $v_{\mathcal{V}} = X_\xi \in T_xG_x$, and $\xi \in \mathfrak{g}$ is the a element such that $\mu(x, v) = \mu(x, X_\xi)$.

Finally notice that if u is a G-invariant function and x a point of differentiability of u, then $T_xG_x \subset \mathrm{d}_x u$.

Proof of Theorem 2. The main difficulty is the surjectivity of the exponential map, nevertheless it can be avoided as follows. Since G is connected, it is well known that for any $g \in G$, there exists $\xi_1, \dots, \xi_n \in \mathfrak{g}$ such that $g = \exp(\xi_1) \cdots \exp(\xi_n)$. Therefore, if we can prove that

$$u\big(S_{\exp(\xi)}x\big) = u(x) \tag{9}$$

for any $x \in E^n$ and any $\xi \in \mathfrak{g}$, we get that $u(S_g x) = u(x)$ for any $x \in E^n$ and any $g \in G$. Given the Saari decomposition of the velocities (8), the proof of (9) follows, as the one of Theorem 1 ■

We can apply Theorem 2 to any connected subgroup $G \subset \mathrm{SO}(d)$ getting that a solution is G-invariant if and only if the corresponding component of the angular momentum of the calibrating curves, in the direction of $\mathfrak{g}^*$ is null at any instant of the motion.

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