Symmetry, Integrability and Geometry: Methods and Applications SIGMA **14** (2018), 012, 33 pages

# *k*-Dirac Complexes

*Tomáš SALAČ*

*Mathematical Institute, Charles University, Sokolovská 49/83, Prague, Czech Republic*
E-mail: *salac@karlin.mff.cuni.cz*
URL: http://www.karlin.mff.cuni.cz/~salac/

Received June 01, 2017, in final form February 06, 2018; Published online February 16, 2018
https://doi.org/10.3842/SIGMA.2018.012

**Abstract.** This is the first paper in a series of two papers. In this paper we construct complexes of invariant differential operators which live on homogeneous spaces of $|2|$-graded parabolic geometries of some particular type. We call them $k$-Dirac complexes. More explicitly, we will show that each $k$-Dirac complex arises as the direct image of a relative BGG sequence and so this fits into the scheme of the Penrose transform. We will also prove that each $k$-Dirac complex is formally exact, i.e., it induces a long exact sequence of infinite (weighted) jets at any fixed point. In the second part of the series we use this information to show that each $k$-Dirac complex is exact at the level of formal power series at any point and that it descends to a resolution of the $k$-Dirac operator studied in Clifford analysis.

*Key words:* Penrose transform; complexes of invariant differential operators; relative BGG complexes; formal exactness; weighted jets

*2010 Mathematics Subject Classification:* 58J10; 32N05; 32L25; 35A22; 53C28; 58A20

## 1 Introduction

Let $h$ be a non-degenerate, symmetric and $\mathbb{C}$-bilinear form on $\mathbb{C}^{2m}$. The Grassmannian variety $M$ of totally isotropic $k$-dimensional subspaces in $\mathbb{C}^{2m}$ is a homogeneous space of a $|2|$-graded parabolic geometry. We assume throughout the paper that $n := m - k \geq k \geq 2$. We will show that on $M$ there is a complex of invariant differential operators which we call the *$k$-Dirac complex*. The main result of this article is (see Theorem 7.14) that the complex is formally exact (as explained above in the abstract) in the sense of [25].

This result is crucial for an application in [24] where it is shown that the complex is exact with formal power series at any fixed point and that it descends (as outlined in the recent series [5, 6, 7]) to a resolution of the $k$-Dirac operator studied in Clifford analysis (see [12, 20]). As a potential application of the resolution, there is an open problem of characterizing the domains of monogenicity, i.e., an open set $\mathcal{U}$ is a domain of monogenicity if there is no open set $\mathcal{U}'$ with $\mathcal{U} \subsetneq \mathcal{U}'$ such that each monogenic function[1] in $\mathcal{U}$ extends to a monogenic function in $\mathcal{U}'$. Recall from [16, Section 4] that the Dolbeault resolution together with some $L^2$ estimates are crucial in a proof of the statement that any pseudoconvex domain is a domain of holomorphy.

The $k$-Dirac complexes belong to the singular infinitesimal character and so the BGG machinery introduced in [9] is not available. However, we will show that each $k$-Dirac complex arises as the direct image of a relative BGG sequence (see [10, 11] for a recent publication on this topic) and so, this paper fits into the scheme of the Penrose transform (see [2, 26]). In particular, we will work here only in the setting of complex parabolic geometries.

The machinery of the Penrose transform is a main tool used in [1]. The main result of that article is a construction of families of locally exact complexes of invariant differential operators on quaternionic manifolds. One of these quaternionic complexes can (see [3, 4, 13]) be identified

[1] A monogenic function is a null solution of the $k$-Dirac operator.

with a resolution of the $k$-*Cauchy–Fueter operator* which has been intensively studied in Clifford analysis (see again [12, 20]). In order to prove the local exactness of this quaternionic complex, one uses that an almost quaternionic structure is a $|1|$-graded parabolic geometry and the theory of constant coefficient operators from [19].

Unfortunately, the parabolic geometry on $M$ is $|2|$-graded and so there is canonical a 2-step filtration of the tangent bundle of $M$ given by a bracket generating distribution. With such a structure, it is more natural to work with weighted jets (see [18]) rather than usual jets and we use this concept also here, i.e., we prove the formal exactness of the $k$-Dirac complexes with respect to the weighted jets. Nevertheless, we will prove in [24] that the formal exactness of the $k$-Dirac complex (or more precisely the exactness of (7.16) for each $\ell + j \geq 0$) is enough to conclude that it descends to a resolution of the $k$-Dirac operator.

We consider here only the even case $\mathbb{C}^{2m}$ as, due to the representation theory, the Penrose transform does not work in odd dimension $\mathbb{C}^{2m+1}$ and it seems that this case has to be treated by completely different methods. The assumption $n \geq k$ is (see [12]) called the stable range. This assumption is needed only in Proposition 5.7 where we compute direct images of sheaves that appear in the relative BGG sequences. Hence, it is reasonable to expect that (see also [17]) the machinery of the Penrose transform provides formally exact complexes also in the unstable range $n < k$.

For the application in [24], we need to show that the $k$-Dirac complexes constructed in this paper give rise to complexes from [22] which live on the corresponding real parabolic geometries. This turns out to be rather easy since any linear G-invariant operator is determined by a certain P-equivariant homomorphism. As this correspondence works also in the smooth setting, passing from the holomorphic setting to the smooth setting is rather elementary.

The abstract approach of the Penrose transform is not very helpful when one is interested in local formulae of differential operators. Local formulae of the operators in the $k$-Dirac complexes can be found in [22]. Notice that in this article we construct only one half of each complex from [22]. This is due to the fact that the complex space of spinors decomposes into two irreducible $\mathfrak{so}(2n, \mathbb{C})$-sub-representations. The other half of each $k$-Dirac complex can (see Remark 5.8) be easily obtained by adapting results from this paper.

Finally, let us mention few more articles which deal with the $k$-Dirac complexes. The null solutions of the first operator in the $k$-Dirac complex were studied in [21, 23]. The singular Hasse graphs and the corresponding homomorphisms of generalized Verma modules were computed in [14].

### Notation

- $M(n, k, \mathbb{C})$ matrices of size $n \times k$ with complex coefficients;
- $M(n, \mathbb{C}) := M(n, n, \mathbb{C})$;
- $A(n, \mathbb{C}) := \{A \in M(n, \mathbb{C}) \mid A^T = -A\}$;
- $1_n$ identity $n \times n$-matrix;
- $[v_1, \dots, v_\ell]$ linear span of vectors $v_1, \dots, v_\ell$.

## 2 Preliminaries

In Section 2 we will review some well known material. Namely, in Section 2.1 we will summarize some theory of complex parabolic geometries. We will recall in Section 2.2 the concept of weighted jets on filtered manifolds and in Section 2.3 the definition of the normal bundle associated to analytic subvariety and the formal neighborhood. In Section 2.4 we will give a short summary of the Penrose transform.

See [8] for a thorough introduction into the theory of parabolic geometries. The concept of weighted jets was originally introduced in the smooth setting by Tohru Morimoto, see for example [18]. Sections 2.3 and 2.4 were taken mostly from [2, 26].

### 2.1 Review of parabolic geometries

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra, $\triangle$ be the associated set of roots, $\triangle^+$ be a set of positive roots and $\triangle^0 = \{\alpha_1, \dots, \alpha_m\}$ be the associated set of (pairwise distinct) simple roots. We will denote by $\mathfrak{g}_\alpha$ the root space associated to $\alpha \in \triangle$ and we will write $\alpha > 0$ if $\alpha \in \triangle^+$ and $\alpha < 0$ if $-\alpha \in \triangle^+$. Given $\alpha \in \triangle$, there are unique integers $\lambda_1, \dots, \lambda_m$ such that $\alpha = \lambda_1\alpha_1 + \dots + \lambda_m\alpha_m$. If $\Sigma \subset \triangle^0$, then the integer $ht_\Sigma(\alpha) := \sum_{i\colon \alpha_i \in \Sigma} \lambda_i$ is called the $\Sigma$-*height* of $\alpha$. We put $\mathfrak{g}_i := \bigoplus_{\alpha\colon ht_\Sigma(\alpha)=i} \mathfrak{g}_\alpha$. Then there is an integer $k \geq 0$ such that $\mathfrak{g}_k \neq \{0\}$, $\mathfrak{g}_\ell = \{0\}$ whenever $|\ell| > k$ and

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \mathfrak{g}_{-k+1} \oplus \dots \oplus \mathfrak{g}_{k-1} \oplus \mathfrak{g}_k. \tag{2.1}$$

The direct sum decomposition (2.1) is the $|k|$-*grading* on $\mathfrak{g}$ associated to $\Sigma$.

Since $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ for every $\alpha, \beta \in \triangle$, it follows that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ for each $i, j \in \mathbb{Z}$. In particular, it follows that $\mathfrak{g}_0$ is a subalgebra and it can be shown that it is always reductive, i.e., $\mathfrak{g}_0 = \mathfrak{g}_0^{ss} \oplus \mathfrak{z}(\mathfrak{g}_0)$ where $\mathfrak{g}_0^{ss} := [\mathfrak{g}_0, \mathfrak{g}_0]$ is semi-simple and $\mathfrak{z}(\mathfrak{g}_0)$ is the center (see [8, Corollary 2.1.6]). Moreover, each subspace $\mathfrak{g}_i$ is $\mathfrak{g}_0$-invariant and the Killing form of $\mathfrak{g}$ induces an isomorphism $\mathfrak{g}_{-i} \cong \mathfrak{g}_i^*$ of $\mathfrak{g}_0$-modules where $^*$ denotes the dual module. We put

$$\mathfrak{g}^i := \bigoplus_{j \geq i} \mathfrak{g}_j, \qquad \mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}, \qquad \mathfrak{p}_\Sigma := \mathfrak{g}^0 \qquad \text{and} \qquad \mathfrak{p}_+ := \mathfrak{g}^1.$$

Then $\mathfrak{p}_\Sigma$ is the *parabolic subalgebra* associated to the $|k|$-grading and $\mathfrak{p}_\Sigma = \mathfrak{g}_0 \oplus \mathfrak{p}_+$ is known as the Levi decomposition (see [2, Section 2.2]). This means that $\mathfrak{p}_+$ is the nilradical[2] and that $\mathfrak{g}_0$ is a maximal reductive subalgebra called the *Levi factor*. It is clear that each subspace $\mathfrak{g}^i$ is $\mathfrak{p}_\Sigma$-invariant and that $\mathfrak{g}_-$ is a nilpotent subalgebra. Moreover, it can be shown that, as a Lie algebra, $\mathfrak{g}_-$ is generated by $\mathfrak{g}_{-1}$.

The algebra $\mathfrak{b} := \mathfrak{p}_{\triangle^0}$ is called the *standard Borel subalgebra*. A subalgebra of $\mathfrak{g}$ is called *standard parabolic* if it contains $\mathfrak{b}$ and in particular, $\mathfrak{p}_\Sigma$ is such an algebra. More generally, a subalgebra of $\mathfrak{g}$ is called a *Borel subalgebra* and a *parabolic subalgebra* if it is conjugated to the standard Borel subalgebra and to a standard parabolic subalgebra, respectively. We will for brevity sometimes write $\mathfrak{p}$ instead of $\mathfrak{p}_\Sigma$.

Let $s_i$ be the simple reflection associated to $\alpha_i$, $i = 1, 2, \dots, m$, $W_\mathfrak{g}$ be the Weyl group of $\mathfrak{g}$ and $W_\mathfrak{p}$ be the subgroup of $W_\mathfrak{g}$ generated by $\{s_i\colon \alpha_i \notin \Sigma\}$. Then $W_\mathfrak{p}$ is isomorphic to the Weyl group of $\mathfrak{g}_0^{ss}$ and the directed graph that encodes the Bruhat order on $W_\mathfrak{g}$ contains a subgraph called the *Hasse diagram* $W^\mathfrak{p}$ *attached to* $\mathfrak{p}$ (see [2, Section 4.3]). The vertices of $W^\mathfrak{p}$ consist of those $w \in W_\mathfrak{g}$ such that $w.\lambda$ is $\mathfrak{p}$-dominant for any $\mathfrak{g}$-dominant weight $\lambda$ where the dot stands for the affine action, namely, $w.\lambda = w(\lambda + \rho) - \rho$ where $\rho$ is the lowest form. It turns out that each right coset of $W_\mathfrak{p}$ in $W_\mathfrak{g}$ contains a unique element from $W^\mathfrak{p}$ and it can be shown that this is the element of minimal length (see [2, Lemma 4.3.3]). This identifies $W^\mathfrak{p}$ with $W_\mathfrak{p}\backslash W_\mathfrak{g}$.

We will need also the relative case. Assume that $\Sigma' \subset \triangle^0$ and put $\mathfrak{r} := \mathfrak{p}_{\Sigma'}$. Then $\mathfrak{q} := \mathfrak{r} \cap \mathfrak{p} = \mathfrak{p}_{\Sigma \cup \Sigma'}$ is a standard parabolic subalgebra and $\mathfrak{l} := \mathfrak{g}_0^{ss} \cap \mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}_0^{ss}$ (see [2, Section 2.4]). The definition of the Hasse diagram attached to $\mathfrak{p}$ applies also to the pair $(\mathfrak{g}_0^{ss}, \mathfrak{l})$, namely an element $w \in W_\mathfrak{p}$ (as [2, Section 4.4]) belongs to the *relative Hasse diagram* $W_\mathfrak{p}^\mathfrak{q}$ if it

[2]Recall that the nilradical is a maximal nilpotent ideal and that it is unique.

is the element of minimal length in its right coset of $W_{\mathfrak{q}}$ in $W_{\mathfrak{p}}$. Hence, $W_{\mathfrak{p}}^{\mathfrak{q}}$ is a subset of $W_{\mathfrak{g}}$ which can be naturally identified with $W_{\mathfrak{q}}\backslash W_{\mathfrak{p}}$.

There is (up to isomorphism) a unique connected and simply connected complex Lie group G with Lie algebra $\mathfrak{g}$. Assume that $\Sigma = \{\alpha_{i_1}, \dots, \alpha_{i_j}\}$. Let $\omega_1, \dots, \omega_m$ be the fundamental weights associated to the simple roots and $\mathbb{V}$ be an irreducible $\mathfrak{g}$-module with highest weight $\lambda := \omega_{i_1} + \dots + \omega_{i_j}$. Since any $\mathfrak{g}$-representation integrates to G, $\mathbb{V}$ is also a G-module. The action descends to the projective space $\mathbb{P}(\mathbb{V})$ and the stabilizer of the line spanned by a non-zero highest weight vector $v$ is the associated *parabolic subgroup* P. This is by definition a closed subgroup of G and its Lie algebra is $\mathfrak{p}$. The homogeneous space G/P is biholomorphic to the G-orbit of $[v] \in \mathbb{P}(\mathbb{V})$ and since it is completely determined by $\Sigma$, we denote it by crossing in the Dynkin diagram of $\mathfrak{g}$ the simple roots from $\Sigma$. We will for brevity put $M := \mathrm{G/P}$ and denote by $\mathbf{p}\colon \mathrm{G} \to M$ the canonical projection.

On G lives a tautological $\mathfrak{g}$-valued 1-form $\omega$ which is known as the *Maurer-Cartan form*. This form is P-equivariant in the sense that for each $p \in \mathrm{P}\colon (r^p)^*\omega = \mathrm{Ad}(p^{-1}) \circ \omega$ where Ad is the adjoint representation and $r^p$ is the principal action of $p$. If $\mathbb{V}$ is a subspace of $\mathfrak{g}$ and $g \in \mathrm{G}$, then $\omega_g^{-1}(\mathbb{V})$ is a subspace of $T_g\mathrm{G}$ and the disjoint union $\bigcup_{g\in \mathrm{G}} \omega_g^{-1}(\mathbb{V})$ determines a distribution on G which we for brevity denote by $\omega^{-1}(\mathbb{V})$. Since $T\mathbf{p} = T\mathbf{p} \circ Tr^p$, it follows that $T\mathbf{p}(\omega^{-1}(\mathbb{V}))$ is a well-defined distribution on $M$ provided that $\mathbb{V}$ is P-invariant. In particular, this applies to $\mathfrak{g}^i$ and we put $F_i := T\mathbf{p}(\omega^{-1}(\mathfrak{g}^i))$. Since $\ker(T\mathbf{p}) = \omega^{-1}(\mathfrak{p})$, it follows that the filtration $\mathfrak{g}^{-k}/\mathfrak{p} \supset \dots \supset \mathfrak{g}^{-1}/\mathfrak{p} \supset \mathfrak{g}^0/\mathfrak{p}$ gives a filtration $TM = F_{-k} \supset F_{-k+1} \supset \dots \supset F_{-1} \supset F_0 = \{0\}$ of the tangent bundle $TM$ where $\{0\}$ is the zero section. The *graded tangent bundle* associated to the filtration $\{F_{-i}\colon i = k, \dots, 0\}$ is $gr(TM) := \bigoplus_{i=-k}^{-1} gr_i(TM)$ where $gr_i(TM) := F_i/F_{i+1}$. Since $M$ is the homogeneous model, we have the following:

- the filtration is compatible[3] with the Lie bracket of vector fields in the sense that the commutator of a section of $F_i$ and a section of $F_j$ is a section of $F_{i+j}$,
- the Lie bracket descends to a vector bundle map $\mathcal{L}\colon \Lambda^2 gr(TM) \to gr(TM)$, called the *Levi form*, which is homogeneous of degree zero and
- $(gr(T_xM), \mathcal{L}_x)$, $x \in M$ is a nilpotent Lie algebra isomorphic to $\mathfrak{g}_-$.

Hence, $(gr(TM), \mathcal{L})$ is a locally trivial bundle of nilpotent Lie algebras with typical fiber $\mathfrak{g}_-$ and it follows that $F_{-1}$ is a bracket generating distribution.

We denote by $T^*M = \Lambda^{1,0}T^*M$ the vector bundle dual to $TM$, i.e., the fiber over $x \in M$ is the space of $\mathbb{C}$-linear maps $T_xM \to \mathbb{C}$. The filtration of $TM$ induces a filtration $T^*M = F_1 \supset F_2 \supset \dots \supset F_{k+1} = \{0\}$ where $F_i := F_{-i+1}^{\perp}$ is the annihilator of $F_{-i+1}$. We put $gr_i(T^*M) := F_i/F_{i+1}$, $i = 1, 2, \dots, k$ so that $gr(T^*M) = \bigoplus_{i=1}^{k} gr_i(T^*M)$ is the associated graded vector bundle and $gr_i(T^*M)$ is isomorphic to the dual of $gr_i(TM)$.

## 2.2 Weighted differential operators

Let $M$ be the homogeneous space with the regular filtration $\{F_{-j}\colon j = 0, \dots, k\}$ as in Section 2.1. As $M$ is a complex manifold, $TM_{\mathbb{C}} := TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M$ where the first and the second summand is the holomorphic and the anti-holomorphic part[4], respectively. As each vector bundle $F_{-j}$ is a holomorhic sub-bundle of $TM$, we have $F_{-j} \otimes \mathbb{C} = F_{-j}^{1,0} \oplus F_{-j}^{0,1}$ as above.

Let $\mathcal{U}$ be an open subset of $M$ and $X$ be a holomorphic vector field on $\mathcal{U}$. The *weighted order* $\mathfrak{wo}(X)$ of $X$ is the smallest integer $j$ such that $X \in \Gamma(F_{-j}^{1,0}|_{\mathcal{U}})$. A differential operator $D$ acting on the space $\mathcal{O}(\mathcal{U})$ of holomorphic functions on $\mathcal{U}$ is called a *differential operator of weighted*

[3] Filtrations which satisfy this property are called *regular*.

[4] The holomorphic and anti-holomorphic part is the $-i$ and the $+i$-eigenspace, respectively, for the canonical almost complex structure on $TM_{\mathbb{C}}$.

*order at most* $r$ if for each $x \in \mathcal{U}$ there is an open neighborhood $\mathcal{U}_x$ of $x$ with a local framing[5] $\{X_1, \dots, X_p\}$ by holomorphic vector fields such that

$$D|_{\mathcal{U}_x} = \sum_{a \in \mathbb{N}_0^p} f_a X_1^{a_1} \cdots X_p^{a_p},$$

where $\mathbb{N}_0^p := \{a = (a_1, \dots, a_p) \colon a_i \in \mathbb{Z},\ a_i \geq 0,\ i = 1, \dots, p\}$, $f_a \in \mathcal{O}(\mathcal{U}_x)$ and for all $a$ in the sum with $f_a$ non-zero: $\sum_{i=1}^{p} a_i.\mathfrak{wo}(X_i) \leq r$. We write $\mathfrak{wo}(D) \leq r$.

Let $\mathcal{O}_x$ be the space of germs of holomorphic functions at $x$. We denote by $\mathfrak{F}^i_x$ the space of those germs $f \in \mathcal{O}_x$ such that $Df(x) = 0$ for every differential operator $D$ which is defined on an open neighborhood of $x$ and $\mathfrak{wo}(D) \leq i$. We put $\mathfrak{J}^i_x := \mathcal{O}_x/\mathfrak{F}^{i+1}_x$, denote by $\mathfrak{j}^i_x f \in \mathfrak{J}^i_x$ the class of $f$ and call it the *i-th weighted jet of f*. Then the disjoint union $\mathfrak{J}^i := \cup_{x \in M} \mathfrak{J}^i_x$ is naturally a holomorphic vector bundle over $M$, the canonical vector bundle map $\mathfrak{J}^i \xrightarrow{\pi_i} \mathfrak{J}^{i-1}$ has constant rank and thus, its kernel $\mathfrak{gr}^i$ is again a holomorphic vector bundle with fiber $\mathfrak{gr}^i_x$ over $x$. Notice that for each integer $i \geq 0$ there is a short exact sequence $0 \to \mathfrak{F}^{i+1}_x \to \mathfrak{F}^i_x \to \mathfrak{gr}^{i+1}_x \to 0$ of vector spaces.

Assume that $V$ is a holomorphic vector bundle over $M$. We denote by $V^*$ the dual bundle, by $\langle -, - \rangle$ the canonical pairing between $V$ and $V^*$ and finally, by $\mathcal{O}(V)_x$ the space of germs of holomorphic sections of $V$ at $x$. We define $\mathfrak{F}^i_x V$ as the space of germs $s \in \mathcal{O}(V)_x$ such that $\langle \lambda, s \rangle \in \mathfrak{F}^i_x$ for each $\lambda \in \mathcal{O}(V^*)_x$. We put $\mathfrak{J}^i_x := \mathcal{O}(V)_x/\mathfrak{F}^{i+1}_x V$, denote by $\mathfrak{j}^i_x s \in \mathfrak{J}^i_x V$ the equivalence class of $s$ and call it the *i-th weighted jet of s*. Then the disjoint union $\mathfrak{J}^i V := \bigcup_{x \in M} \mathfrak{J}^i_x M$ is naturally a holomorphic vector bundle over $M$, the canonical bundle map $\mathfrak{J}^i V \xrightarrow{\pi_i} \mathfrak{J}^{i-1} V$ has constant rank and thus, its kernel $\mathfrak{gr}^i V$ is again a holomorphic vector bundle and we denote by $\mathfrak{gr}^i_x V$ its fiber over $x$. As above, there is for each integer $i \geq 0$ a short exact sequence $0 \to \mathfrak{F}^{i+1}_x V \to \mathfrak{F}^i_x V \to \mathfrak{gr}^{i+1}_x V \to 0$ and just as in the smooth case, there is a canonical linear isomorphism $\mathfrak{gr}^i_x \otimes V_x \to \mathfrak{gr}^i_x V$.

**Remark 2.1.** If the filtration is trivial, i.e., $F_{-1} = TM$, then the concept of weighted jets agrees with that of usual jets. In this case we will use calligraphic letters instead of Gothic letters, i.e., we write $\mathcal{F}^i$ and $\mathcal{J}^i$ and $gr^i$ and $j^i_x f$ instead of $\mathfrak{F}^i$ and $\mathfrak{J}^i$ and $\mathfrak{gr}^i$ and $\mathfrak{j}^i_x f$, respectively. The vector bundle $gr^i$ is canonically isomorphic to the $i$-th symmetric power $S^i T^* M$.

Assume that there is a P-module $\mathbb{V}$ such that $V$ is isomorphic to the G-homogeneous vector bundle $\mathrm{G} \times_{\mathrm{P}} \mathbb{V}$. Let $e$ be the identity element of G. Then we call the point $x_0 := e\mathrm{P}$ the origin of $M$ and we put

$$\mathfrak{J}^i \mathbb{V} := \mathfrak{J}^i_{x_0} V \qquad \text{and} \qquad \mathfrak{gr}^i \mathbb{V} := \mathfrak{gr}^i_{x_0} V. \tag{2.2}$$

There are linear isomorphisms

$$\mathfrak{gr}^r \mathbb{V} \cong \mathfrak{gr}^r_{x_0} \otimes \mathbb{V} \cong \bigoplus_{i_1 + 2i_2 + \cdots + ki_k = r} S^{i_1}\mathfrak{g}_1 \otimes S^{i_2}\mathfrak{g}_2 \otimes \cdots \otimes S^{i_k}\mathfrak{g}_k \otimes \mathbb{V}. \tag{2.3}$$

We will be interested in the sub-bundle $S^i gr_1(T^*M) \otimes V$ of $\mathfrak{gr}^i V$. Notice that the fiber of this sub-bundle over $x \in M$ is $\{\mathfrak{j}^i_x f \colon f \in \mathcal{O}(V)_x,\ j^{i-1}_x f = 0\}$, i.e., the vector space of all weighted $i$-th jets of germs of holomorphic sections at $x$ whose usual $(i-1)$-th jet vanishes. The fiber of this bundle over $x_0$ is isomorphic to $S^i \mathfrak{g}_1 \otimes \mathbb{V}$ and we denote it for brevity by $gr^i \mathbb{V}$.

Suppose that $\mathbb{W}$ is another P-module and $W := \mathrm{G} \times_{\mathrm{P}} \mathbb{W}$ be the associated homogeneous vector bundle. We say that the *weighted order* of a linear differential operator $D \colon \Gamma(V) \to \Gamma(W)$ is at most $r$ if for each $x \in M$, $s \in \mathcal{O}(V)_x \colon \mathfrak{j}^r_x s = 0 \Rightarrow Ds(x) = 0$. It is well known (see [18]) that $D$

[5] This means that the holomorphic vector fields $X_1, \dots, X_p$ trivialize $T^{1,0}M$ over $\mathcal{U}_x$.

induces for each $i \geq 0$ a vector bundle map $\mathfrak{gr}^i V \to \mathfrak{gr}^{i-r} W$ where we agree that $\mathfrak{gr}^\ell W = 0$ if $\ell < 0$. The restriction of this map to the fibers over the origin is a linear map

$$\mathfrak{gr} D\colon \ \mathfrak{gr}^i \mathbb{V} \to \mathfrak{gr}^{i-r} \mathbb{W}.$$

### 2.3 Ideal sheaf of an analytic subvariety

Let us first recall some basics from the theory of sheaves (see for example [26]). Suppose that $\mathcal{F}$ and $\mathcal{G}$ are sheaves on topological spaces $X$ and $Y$, respectively, and that $\iota\colon X \to Y$ is a continuous map. We denote by $\mathcal{F}_x$ the stalk of $\mathcal{F}$ at $x \in X$ and by $\mathcal{F}(\mathcal{U})$ or by $\Gamma(\mathcal{U}, \mathcal{F})$ the space of sections of $\mathcal{F}$ over an open set $\mathcal{U}$. Then the pullback sheaf $\iota^{-1}\mathcal{G}$ is a sheaf on $X$ and the direct image $\iota_* \mathcal{F}$ is a sheaf on $Y$. The $q$-th direct image $\iota^q_* \mathcal{F}$ is a sheaf on $Y$, it is defined as the sheafification of the pre-sheaf $\mathcal{V} \mapsto H^q(\iota^{-1}(\mathcal{V}), \mathcal{F})$ where $\mathcal{V}$ is open in $Y$.

Suppose now that $X$ and $Y$ are complex manifolds with structure sheaves of holomorphic functions $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively, that $\iota$ is holomorphic and that $\mathcal{G}$ is a sheaf of $\mathcal{O}_Y$-modules. Then $\iota^{-1}\mathcal{G}$ is in general not a sheaf of $\mathcal{O}_X$-modules. To fix this problem, we use that $\iota^{-1}\mathcal{O}_Y$ is naturally a sub-sheaf of $\mathcal{O}_X$ and define a new sheaf $\iota^*\mathcal{G} := \mathcal{O}_X \otimes_{\iota^{-1}\mathcal{O}_Y} \iota^{-1}\mathcal{G}$. Then $\iota^*\mathcal{G}$ is by construction a sheaf of $\mathcal{O}_X$-modules.

Now we can continue with the definition of the ideal sheaf. Suppose that the holomorphic map $\iota$ is an embedding. The restriction $TY|_X$ contains the tangent bundle $TX$ of $X$. The normal bundle $N_X$ of $X$ in $Y$ is simply the quotient bundle, i.e., it fits into the short exact sequence $0 \to TX \to TY|_X \to NX \to 0$ of holomorhic vector bundles. Dually, the co-normal bundle $N^*$ fits into the short exact sequence $0 \to N^* \to T^*Y|_X \to T^*X \to 0$.

The structure sheaf $\mathcal{O}_Y$ contains a sub-sheaf called the *ideal sheaf* $\mathcal{I}_X$. If $\mathcal{V}$ is an open subset of $Y$, then $\mathcal{I}_X(\mathcal{V}) = \{f \in \mathcal{O}_Y(\mathcal{V})\colon f = 0 \text{ on } \mathcal{V} \cap X\}$. Notice that $\mathcal{I}_X(\mathcal{V})$ is an ideal in the ring $\mathcal{O}_Y(\mathcal{V})$ and hence, for each positive integer $i$ there is the sheaf $\mathcal{I}^i_X$ whose space of sections over $\mathcal{V}$ is $(\mathcal{I}_X(\mathcal{V}))^i$. Then there are short exact sequences of sheaves

$$0 \to \mathcal{I}_X \to \mathcal{O}_Y \to \iota_*\mathcal{O}_X \to 0$$

and

$$0 \to \mathcal{I}^{i+1}_X \to \mathcal{I}^i_X \to \iota_*\mathcal{O}(S^i N^*) \to 0,$$

where $S^i N^*$ is the $i$-th symmetric power of $N^*$ and we agree that $\mathcal{I}^0_X = \mathcal{O}_Y$. We put $\mathcal{F}^i_X := \iota^{-1}\mathcal{I}^i_X$. As $\iota^{-1}$ is an exact functor, we get short exact sequences

$$0 \to \mathcal{F}_X \to \iota^{-1}\mathcal{O}_Y \to \mathcal{O}_X \to 0$$

and

$$0 \to \mathcal{F}^{i+1}_X \to \mathcal{F}^i_X \to \mathcal{O}(S^i N^*) \to 0$$

of sheaves on $X$. Here we use that the adjunction morphism $\iota^{-1}\iota_*\mathcal{F} \to \mathcal{F}$ is an isomorphism when $\mathcal{F} = \mathcal{O}_X$ or $\mathcal{O}(S^i N^*)$.

Put $\mathcal{O}^{(i)}_X := \mathcal{O}_Y/\mathcal{I}^{i+1}_X$. The pair $(X, \mathcal{O}^{(i)}_X)$ is called the *$i$-th formal neighborhood of $X$ in $Y$*. Then $\iota^{-1}\mathcal{O}^{(i)}_X \cong \iota^{-1}\mathcal{O}_Y/\mathcal{F}^{(i+1)}$ and since the support of $\mathcal{O}^{(i)}_X$ is contained in $X$, the sheaf $\iota^{-1}\mathcal{O}^{(i)}_X$ contains basically the same information as the sheaf $\mathcal{O}^{(i)}_X$. These sheaves will be crucial in this article.

**Remark 2.2.** The stalk of $\mathcal{F}^i_X$ at $x \in X$ is equal to $\{f \in \mathcal{F}_x\colon j^i_x f = 0\}$. Hence, if $X = x$ is a point, the stalk of $\mathcal{F}^i_X$ at $x$ is $\{f \in (\mathcal{O}_Y)_x\colon j^i_x f = 0\}$. Since any sheaf over a point is completely determined by its stalk, there is no risk of confusion with the notation set in Remark 2.1.

### 2.4 The Penrose transform

Let us first set notation. Suppose that $\lambda \in \mathfrak{h}^*$ is a $\mathfrak{g}$-integral and $\mathfrak{p}$-dominant weight. Then there is (see [2, Remark 3.1.6]) an irreducible P-module $\mathbb{V}_\lambda$ with lowest weight $-\lambda$. We denote by $V_\lambda := \mathrm{G} \times_{\mathrm{P}} \mathbb{V}_\lambda$ the induced vector bundle and by $\mathcal{O}_{\mathfrak{p}}(\lambda)$ the associated sheaf of holomorphic sections.

Suppose that $\mathfrak{p}$, $\mathfrak{r}$ are standard parabolic subalgebras. Then $\mathfrak{q} := \mathfrak{r} \cap \mathfrak{p}$ is also a standard parabolic subalgebra and we denote by P and R and Q the associated parabolic subgroups with Lie algebras $\mathfrak{p}$ and $\mathfrak{r}$ and $\mathfrak{q}$, respectively, as explained in Section 2.1. Then $\mathrm{Q} = \mathrm{R} \cap \mathrm{P}$ and there is a double fibration diagram

$$\begin{tikzcd} & \mathrm{G/Q} \arrow[dl, "\eta"'] \arrow[dr, "\tau"] & \\ \mathrm{G/R} & & \mathrm{G/P}, \end{tikzcd}$$

where $\eta$ and $\tau$ are the canonical projections. The space G/R is called the *twistor space* $TS$ and G/Q the *correspondence space* $CS$. Such a diagram is a starting point for the Penrose transform.

Next we need to fix an $\mathfrak{r}$-dominant and integral weight $\lambda \in \mathfrak{h}^*$. Then there is a relative BGG sequence $\blacktriangle^*(\lambda)$ which is an exact sequence of holomorphic sections of associated vector bundles over $CS$ and linear G-invariant differential operators such that $\eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda)$ is the kernel sheaf of the first operator in the sequence. In other words, there is a long exact sequence of sheaves

$$0 \to \eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda) \to \blacktriangle^*(\lambda).$$

The upshot of this is that although the pullback sheaf $\eta^{-1}\mathcal{O}_{\mathfrak{r}}(\lambda)$ is not a sheaf of holomorphic sections of an associated vector bundle over $CS$, it is naturally a sub-sheaf of $\mathcal{O}_{\mathfrak{q}}(\lambda)$ which is cut out by an invariant differential equation. Moreover, the graph of the relative BGG sequence is [2, Section 8.7] completely determined by the $W^{\mathfrak{q}}_{\mathfrak{r}}$-orbit of $\lambda$.

Then we push down the relative BGG sequence by the direct image functor $\tau_*$. Computing higher direct images of sheaves in the relative BGG sequence is completely algorithmic and algebraic (see [2, Section 5.3]). On the other hand, there is no general algorithm which computes direct images of differential operators and it seems that this has to be treated in each case separately. Nevertheless, in this way one obtains a complex of operators on G/P.

## 3 Lie theory

In Section 3 we will provide an algebraic background which is needed in the construction of the $k$-Dirac complexes via the Penrose transform. We will work with complex parabolic geometries which are associated to gradings on the simple Lie algebra $\mathfrak{g} = \mathfrak{so}(2m, \mathbb{C})$. Section 3 is organized as follows: in Section 3.1 we will set notation and study the gradings on $\mathfrak{g}$. In Section 3.2 we will compute the relative Hasse diagram $W^{\mathfrak{q}}_{\mathfrak{r}}$.

### 3.1 Lie algebra $\mathfrak{g}$ and parabolic subalgebras

Let $\{e_1, \dots, e_m, e^*_{k+1}, \dots, e^*_m, e^*_1, \dots, e^*_k\}$ be the standard basis of $\mathbb{C}^{2m}$, $\delta$ be the Kronecker delta and $h$ be the complex bilinear form that satisfies $h(e_i, e^*_j) = \delta_{ij}$, $h(e_i, e_j) = h(e^*_i, e^*_j) = 0$ for all $i, j = 1, \dots, m$. A matrix belongs to the associated Lie algebra $\mathfrak{g} := \mathfrak{so}(h) \cong \mathfrak{so}(2m, \mathbb{C})$ if and only if it is of the form

$$\begin{pmatrix} A & Z_1 & Z_2 & W \\ X_1 & B & D & -Z_2^T \\ X_2 & C & -B^T & -Z_1^T \\ Y & -X_2^T & -X_1^T & -A^T \end{pmatrix}, \tag{3.1}$$

where $A \in M(k,\mathbb{C})$, $B \in M(n,\mathbb{C})$, $C, D \in A(n,\mathbb{C})$, $X_1, X_2, Z_1^T, Z_2^T \in M(n,k,\mathbb{C})$, $W, Y \in A(k,\mathbb{C})$.

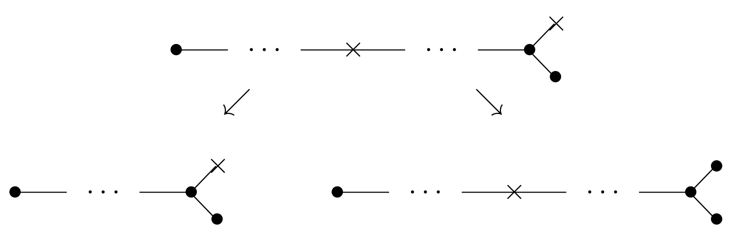
The subspace of diagonal matrices $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. We denote by $\epsilon_i$ the linear form on $\mathfrak{h}$ defined by $\varepsilon_i \colon H = (h_{kl}) \mapsto h_{ii}$. Then $\{\epsilon_1, \dots, \epsilon_m\}$ is a basis of $\mathfrak{h}^*$ and $\triangle = \{\pm\epsilon_i \pm \epsilon_j \colon i, j = 1, \dots, m\}$. If we choose $\triangle^+ = \{\epsilon_i \pm \epsilon_j \colon 1 \le i < j \le m\}$ as positive roots, then the simple roots are $\alpha_i := \epsilon_i - \epsilon_{i+1}$, $i = 1, \dots, m-1$ and $\alpha_m := \epsilon_{m-1} + \epsilon_m$. The associated fundamental weights are $\omega_i = \epsilon_1 + \dots + \epsilon_i$, $i = 1, \dots, m-2$, $\omega_{m-1} = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_{m-1} - \epsilon_m)$ and $\omega_m = \frac{1}{2}(\epsilon_1 + \dots + \epsilon_m)$. The *lowest form* $\rho$ is equal to $\frac{1}{2} \sum_{\alpha \in \triangle^+} \alpha = \omega_1 + \dots + \omega_m = (m-1, \dots, 1, 0)$. If $\lambda = \sum_{i=1}^{m} \lambda_i \epsilon_i$ where $\lambda_i \in \mathbb{C}$, then we will also write $\lambda = (\lambda_1, \dots, \lambda_m)$. The simple reflection $s_i \in W_{\mathfrak{g}}$ associated to $\alpha_i$ acts on $\mathfrak{h}^*$ by

$$s_i(\lambda) = (\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \lambda_i, \lambda_{i+2}, \dots, \lambda_m), \qquad i = 1, \dots, m-1 \tag{3.2}$$

and

$$s_m(\lambda) = (\lambda_1, \dots, \lambda_{m-2}, -\lambda_m, -\lambda_{m-1}).$$

We will be interested in the double fibration diagram



(3.3)

where, going from left to right, the sets of simple roots are $\{\alpha_m\}$, $\{\alpha_k, \alpha_m\}$ and $\{\alpha_k\}$ and the associated gradings are

$$\mathfrak{r}_{-1} \oplus \mathfrak{r}_0 \oplus \mathfrak{r}_1, \qquad \mathfrak{q}_{-3} \oplus \mathfrak{q}_{-2} \oplus \dots \oplus \mathfrak{q}_3 \qquad \text{and} \qquad \mathfrak{g}_{-2} \oplus \dots \oplus \mathfrak{g}_2,$$

respectively. With respect to the block decomposition from (3.1), we have[6]

$$\begin{array}{ccc} & \begin{pmatrix} \mathfrak{q}_0 & \mathfrak{q}_1 & \mathfrak{q}_2 & \mathfrak{q}_3 \\ \mathfrak{q}_{-1} & \mathfrak{q}_0 & \mathfrak{q}_1 & \mathfrak{q}_2 \\ \mathfrak{q}_{-2} & \mathfrak{q}_{-1} & \mathfrak{q}_0 & \mathfrak{q}_1 \\ \mathfrak{q}_{-3} & \mathfrak{q}_{-2} & \mathfrak{q}_{-1} & \mathfrak{q}_0 \end{pmatrix} & \\ \swarrow & & \searrow \\ \begin{pmatrix} \mathfrak{r}_0 & \mathfrak{r}_0 & \mathfrak{r}_1 & \mathfrak{r}_1 \\ \mathfrak{r}_0 & \mathfrak{r}_0 & \mathfrak{r}_1 & \mathfrak{r}_1 \\ \mathfrak{r}_{-1} & \mathfrak{r}_{-1} & \mathfrak{r}_0 & \mathfrak{r}_0 \\ \mathfrak{r}_{-1} & \mathfrak{r}_{-1} & \mathfrak{r}_0 & \mathfrak{r}_0 \end{pmatrix} & & \begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & \mathfrak{g}_1 & \mathfrak{g}_2 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 & \mathfrak{g}_1 \\ \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix} \end{array}.$$

[6]Here we mean $\mathfrak{q}_0$ is the subspace of block diagonal matrices, $\mathfrak{q}_1$ is the subspace of those block matrices where only the matrices $Z_1$, $D$ are non-zero, etc.

The associated standard parabolic subalgebras are

$$\mathfrak{r} = \mathfrak{r}_0 \oplus \mathfrak{r}_1, \qquad \mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1 \oplus \mathfrak{q}_2 \oplus \mathfrak{q}_3 \qquad \text{and} \qquad \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2,$$

respectively, and we have the following isomorphisms

$$\mathfrak{r}_0 \cong M(m, \mathbb{C}), \qquad \mathfrak{q}_0 \cong M(k, \mathbb{C}) \oplus M(n, \mathbb{C}) \qquad \text{and} \qquad \mathfrak{g}_0 \cong M(k, \mathbb{C}) \oplus \mathfrak{so}(2n, \mathbb{C}).$$

We for brevity put

$$\begin{aligned} &\mathbb{C}^k := [e_1, \dots, e_k], \qquad \mathbb{C}^{k*} := [e_1^*, \dots, e_k^*], \\ &\mathbb{C}^n := [e_{k+1}, \dots, e_m], \qquad \mathbb{C}^{n*} := [e_{k+1}^*, \dots, e_m^*], \\ &\mathbb{C}^{2n} := \mathbb{C}^n \oplus \mathbb{C}^{n*} \qquad \text{and} \qquad \mathbb{C}^m := \mathbb{C}^k \oplus \mathbb{C}^n. \end{aligned} \tag{3.4}$$

Notice that the bilinear form $h$ induces dualities between $\mathbb{C}^k$ and $\mathbb{C}^{k*}$ and between $\mathbb{C}^n$ and $\mathbb{C}^{n*}$ which justifies the notation, that $\mathbb{C}^m$ is a maximal, totally isotropic and $\mathfrak{r}_0$-invariant subspace, that $\mathbb{C}^k$, $\mathbb{C}^{k*}$, $\mathbb{C}^n$ and $\mathbb{C}^{n*}$ are $\mathfrak{q}_0$-invariant, that $\mathbb{C}^k$, $\mathbb{C}^{2n}$ and $\mathbb{C}^{k*}$ are $\mathfrak{g}_0$-invariant and finally, that $h|_{\mathbb{C}^{2n}}$ is a non-degenerate, symmetric and $\mathfrak{g}_0$-invariant bilinear form. We will for brevity write only $h$ instead of $h|_{\mathbb{C}^{2n}}$ as it will be always clear from the context what is meant.

Let us now consider the associated nilpotent subalgebras

$$\mathfrak{r}_- = \mathfrak{r}_{-1}, \qquad \mathfrak{q}_- = \mathfrak{q}_{-3} \oplus \mathfrak{q}_{-2} \oplus \mathfrak{q}_{-1} \qquad \text{and} \qquad \mathfrak{g}_- = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}.$$

By the Jacobi identity, the Lie bracket is equivariant with respect to the adjoint action of the corresponding Levi factor and by the grading property following equation (2.1), it is homogeneous of degree zero. Hence, we can consider the Lie bracket in each homogeneity separately.

The first algebra $\mathfrak{r}_-$ is abelian and so there is nothing to add.

On the other hand, $\mathfrak{q}_-$ is 3-graded and, as $\mathfrak{q}_0$-modules, we have $\mathfrak{q}_{-1} \cong \mathbb{E} \oplus \mathbb{F}$, $\mathfrak{q}_{-2} \cong \mathbb{C}^{k*} \otimes \mathbb{C}^{n*}$, $\mathfrak{q}_{-3} \cong \Lambda^2 \mathbb{C}^{k*}$ where we put $\mathbb{E} := \mathbb{C}^{k*} \otimes \mathbb{C}^n$ and $\mathbb{F} := \Lambda^2 \mathbb{C}^{n*}$. Using these isomorphisms, the Lie brackets in homogeneity $-2$ and $-3$ are the compositions of the canonical projections

$$\Lambda^2 \mathfrak{q}_{-1} \to \mathbb{E} \otimes \mathbb{F} = \big(\mathbb{C}^{k*} \otimes \mathbb{C}^n\big) \otimes \Lambda^2 \mathbb{C}^{n*} \to \mathbb{C}^{k*} \otimes \mathbb{C}^{n*} = \mathfrak{q}_{-2} \tag{3.5}$$

and

$$\mathfrak{q}_{-1} \otimes \mathfrak{q}_{-2} \to \mathbb{E} \otimes \mathfrak{q}_{-2} = \big(\mathbb{C}^{k*} \otimes \mathbb{C}^n\big) \otimes \big(\mathbb{C}^{k*} \otimes \mathbb{C}^{n*}\big) \to \Lambda^2 \mathbb{C}^{k*} = \mathfrak{q}_{-3},$$

respectively. Here we use the canonical pairing $\mathbb{C}^n \otimes \mathbb{C}^{n*} \to \mathbb{C}$. Notice that $\Lambda^2 \mathbb{E} \oplus \Lambda^2 \mathbb{F}$ is contained in the kernel of (3.5).

In order to understand the Lie bracket on $\mathfrak{g}_-$, first notice that there are isomorphisms $\mathfrak{g}_{-1} \cong \mathbb{C}^{k*} \otimes \mathbb{C}^{2n}$ and $\mathfrak{g}_{-2} \cong \Lambda^2 \mathbb{C}^{k*} \otimes \mathbb{C}$ of irreducible $\mathfrak{g}_0$-modules where $\mathbb{C}$ is the trivial representation of $\mathfrak{so}(2n, \mathbb{C})$. As $\mathfrak{g}_-$ is 2-graded, the Lie bracket is non-zero only in homogeneity $-2$. It is given by

$$\Lambda^2 \mathfrak{g}_{-1} = \Lambda^2 \big(\mathbb{C}^{k*} \otimes \mathbb{C}^{2n}\big) \to \Lambda^2 \mathbb{C}^{k*} \otimes S^2 \mathbb{C}^{2n} \to \Lambda^2 \mathbb{C}^{k*} \otimes \mathbb{C} = \mathfrak{g}_{-2},$$

where in the last map we take the trace with respect to $h$.

In the table below we specify when $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathfrak{h}^*$ is dominant for each parabolic subalgebra $\mathfrak{p}, \mathfrak{q}$ and $\mathfrak{r}$. We put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

**Table 1.** Dominant weights.

| algebra | dominant and integral weights |
|---|---|
| $\mathfrak{p}$ | $\lambda_i - \lambda_{i+1} \in \mathbb{N}_0,\ i \neq k,\ 2\lambda_m \in \mathbb{Z},\ \lambda_{m-1} \geq \lvert\lambda_m\rvert$ |
| $\mathfrak{r}$ | $\lambda_i - \lambda_{i+1} \in \mathbb{N}_0$ |
| $\mathfrak{q}$ | $\lambda_i - \lambda_{i+1} \in \mathbb{N}_0, i \neq k$ |

### 3.2 Relative Hasse diagram $W_{\mathfrak{r}}^{\mathfrak{q}}$

Let us first set notation. By a *partition* we will mean an element of $\mathbb{N}_{++}^{k,n} := \{(a_1, \dots, a_k) \colon a_i \in \mathbb{Z},\ n \geq a_1 \geq a_2 \geq \dots \geq a_k \geq 0\}$. For two partitions $a = (a_1, \dots, a_k)$ and $a' = (a'_1, \dots, a'_k)$ we write $a \leq a'$ if $a_i \leq a'_i$ for all $i = 1, \dots, k$ and $a < a'$ if $a \leq a'$ and $a \neq a'$. If $a < a'$ does not hold, then we write $a \not< a'$. We put

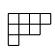
$$\begin{aligned} &|a| = a_1 + \dots + a_k, \qquad d(a) := \max\{i \colon a_i \geq i\}, \\ &q(a) = \sum_{i=1}^{k} \max\{a_i - i, 0\} \qquad \text{and} \qquad r(a) := d(a) + q(a). \end{aligned} \tag{3.6}$$

To the partition $a$ we associate the *Young diagram* (or the *Ferrers diagram*) Y consisting of $k$ left-justified rows with $a_i$-boxes in the $i$-th row. Let $b_i$ be the number of boxes in the $i$-th column of Y. Then we call $b = (b_1, \dots, b_n) \in \mathbb{N}_{++}^{n,k}$ the *partition conjugated to* $a$ and we say that $a$ is *symmetric* if $a_i = b_i$, $i = 1, \dots, k$ and $b_{k+1} = \dots = b_n = 0$. As we assume $n \geq k$, the set of symmetric partitions in $\mathbb{N}_{++}^{k,n}$ depends only on $k$, and thus, we denote it for simplicity by $S^k$ and put $S_j^k := \{a \in S^k \colon r(a) = j\}$.

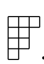
**Example 3.1.**

(1) The empty partition is by definition always symmetric.

(2) The Young diagram of $a = (4, 3, 1, 0, 0) \in \mathbb{N}_{++}^{5,6}$ is

 (3.7)

and we find that $d(a) = 2$, $q(a) = 4$ and $r(a) = 6$. The conjugated partition is $b = (3, 2, 2, 1, 0, 0)$ with $d(b) = 2$, $q(b) = 2$ and $r(b) = 4$. The Young diagram of $b$ is

.

We see that the partition is not symmetric.

Notice that $d(a)$ and $q(a)$ are equal to the number of boxes in the associated Young diagram that are on and above the main diagonal, respectively and that a partition is symmetric if and only if its Young diagram is symmetric with respect to the reflection along the main diagonal.

We can now continue by investigating the relative Hasse graph $W_{\mathfrak{r}}^{\mathfrak{q}}$. The group $W_{\mathfrak{r}}$ is generated by $s_1, \dots, s_{m-1}$ while $W_{\mathfrak{q}}$ is generated by elements $s_1, \dots, s_{k-1}, s_{k+1}, \dots, s_{m-1}$. By (3.2), it follows that $W_{\mathfrak{r}}$ is the permutation group $S_m$ on $\{1, \dots, m\}$ and that $W_{\mathfrak{q}} \cong S_k \times S_n$ is the stabilizer of $\{1, \dots, k\}$. Recall from Section 2.1 that in each left coset of $W_{\mathfrak{q}}$ in $W_{\mathfrak{r}}$ there is a unique element of minimal length and that we denote the set of all such distinguished representatives by $W_{\mathfrak{r}}^{\mathfrak{q}}$. Moreover, the Bruhat order on $W_{\mathfrak{g}}$ descends to a partial order on $W_{\mathfrak{r}}$ and on $W_{\mathfrak{q}}^{\mathfrak{r}}$. We will now show that there is an isomorphism $\mathbb{N}_{++}^{k,n} \to W_{\mathfrak{r}}^{\mathfrak{q}}$ of partially ordered sets.

Let $a = (a_1, \dots, a_k) \in \mathbb{N}_{++}^{k,n}$ and Y be the associated Young diagram. We will call the box in the $i$-th row and the $j$-th column of $Y$ an $(i, j)$-*box* and we write into this box the number $\sharp(i,j) := k - i + j$. Notice that $1 \le \sharp(i,j) \le m$. Then the set of boxes in $Y$ is indexed by $\Xi_a := \{(i,j)\colon i = 1,\dots,k,\ j = 1,\dots,a_i\}$ and we order this set lexicographically, i.e., $(i,j) < (i',j')$ if $i < i'$ or $i = i'$ and $j < j'$. Then

$$w_a := s_{\sharp(\Psi(1))} s_{\sharp(\Psi(2))} \dots s_{\sharp(\Psi(|a|))} \in S_m, \tag{3.8}$$

where $\Psi\colon \{1,2,\dots,|a|\} \to \Xi_a$ is the unique isomorphism of ordered sets. Let us now look at an example.

**Example 3.2.** The Young diagram from (3.7) is filled as

$$\begin{matrix} k & k+1 & k+2 & k+3 \\ k-1 & k & k+1 & \\ k-2 & & & \end{matrix}$$

and so $w_a := s_k s_{k+1} s_{k+2} s_{k+3} s_{k-1} s_k s_{k+1} s_{k-2}$.

We have the following preliminary observation.

**Lemma 3.3.** *Let $a = (a_1,\dots,a_k)$ be as above and $b = (b_1,\dots,b_n)$ be the conjugated partition. Then the permutation $w_a \in S_m$ from (3.8) satisfies*

$$w_a(k - i + 1 + a_i) = k - i + 1 \qquad \text{and} \qquad w_a(k + j - b_j) = k + j \tag{3.9}$$

*for each $i = 1,\dots,k$ and $j = 1,\dots,n$.*

**Proof.** Fix $i = 1,\dots,k$. If $a_i > 0$, there is $r_i := k - i + 1$ written in the $(i,1)$-box and $r^i := k - i + a_i = r_i + a_i - 1$ in the $(i,a_i)$-box. We put $r^i - 1 = r_i := k - i + 1$ if $a_i = 0$. Similarly, if $j = 1,\dots,n$ and $b_j > 0$, then there is $c_j := k + j - 1$ in the $(1,j)$-box and $c^j := k + j - b_j = c_j - b_j + 1$ in the $(b_j,j)$-box. We put $c^j - 1 = c_j := k + j - 1$ if $b_j = 0$. Then it is easy to check that $w_a(r^i + 1) = r_i$ and $w_a(c^j) = c_j + 1$ which completes the proof. ∎

Notice that the sets $\{k - i + 1 + a_i\colon i = 1,\dots,k\}$ and $\{k + j - b_j\colon j = 1,\dots,n\}$ are disjoint and that their union is $\{1,2,\dots,m\}$. By (3.9), it follows that

$$w_a\rho = \rho + (-a_k,\dots,-a_1 \,|\, b_1,\dots,b_n), \tag{3.10}$$

where $\rho = (m-1,\dots,1,0)$ is the lowest form of $\mathfrak{g}$ and for clarity, we separate the first $k$ and last $n$ coefficients by $|$. Comparing this with Table 1, we see that $w_a\rho$ is $\mathfrak{q}$-dominant. As the same holds for any $\mathfrak{r}$-dominant weight, it follows that $w_a \in W^{\mathfrak{q}}_{\mathfrak{r}}$.

**Lemma 3.4.** *The map $\mathbb{N}^{k,n}_{++} \to W^{\mathfrak{q}}_{\mathfrak{r}}$, $a \mapsto w_a$ is an isomorphism of partially ordered sets.*

**Proof.** The map $a \mapsto w_a$ is by (3.10) clearly injective. To show surjectivity, fix $w \in W^{\mathfrak{q}}_{\mathfrak{r}}$. Then the sequence $c_1,\dots,c_k$ where $c_i = w^{-1}(i)$, $i = 1,\dots,k$ is increasing. By [8, Proposition 3.2.16], the map $w \in W^{\mathfrak{q}}_{\mathfrak{r}} \mapsto w^{-1}\omega_k$ is injective. It follows that $w^{-1}\omega_k$ is uniquely determined by the sequence $c_1,\dots,c_k$. Then $a := (a_1,\dots,a_k) \in \mathbb{N}^{k,n}_{++}$ where $a_i := c_{k-i+1} + i - k - 1$, $i = 1,\dots,k$ and from (3.9), it follows that $w_a^{-1}(k - i + 1) = k - i + 1 + a_i = c_{k-i+1}$, $i = 1,\dots,k$. This shows that $w_a^{-1}\omega_k = w^{-1}\omega_k$ and thus, $w = w_a$. Now it remains to show that the map is compatible with the orders.

Assume that $a = (a_1,\dots,a_k)$, $a' = (a'_1,\dots,a'_k) \in \mathbb{N}^{k,n}_{++}$ satisfy $|a'| = |a| + 1$ and $a < a'$. Then there is a unique integer $i \le k$ such that $a'_i = a_i + 1$ and so $w_{a'} = w_a s_{k-i+a'_i}$. By (3.9), we have that $w_a\alpha_{k-i+a'_i} > 0$ and thus by [8, Proposition 3.2.16], there is an arrow $w_a \to w_{a'}$ in $W^{\mathfrak{q}}_{\mathfrak{r}}$.

On the other hand, suppose that $a'' = (a''_1,\dots,a''_k) \in \mathbb{N}^{k,n}_{++}$ satisfies $a' \not< a''$. In order to complete the proof, it is enough to show that there is no arrow $w_{a'} \to w_{a''}$. By assumptions, there is $j$ such that $a'_1 \leq a''_1, \ldots, a'_{j-1} \leq a''_{j-1}$ and $a'_j > a''_j$. Without loss of generality we may assume that $i = j$. Then $w_{a'}^{-1}(w_a \alpha_{k+a_i-i}) = s_{k+a_i-i}\alpha_{k+a_i-i} < 0$. On the other hand by (3.9), it follows that $w_{a''}^{-1}(w_a \alpha_{k+a_i-i}) > 0$. We proved that $\Phi_{w'_a} \not\subset \Phi_{w_{a''}}$ and thus by [8, Proposition 3.2.17], there cannot be any arrow $w_{a'} \to w_{a''}$. ■

We will later need the following two observations. A permutation $w \in S_m$ is *$k$-balanced*, if the following is true: if $w(k-i) > k$ for some $i = 0, \ldots, k-1$, then $w(k+i+1) \leq k$.

**Lemma 3.5.** *The permutation $w_a$ associated to $a \in \mathbb{N}^{k,n}_{++}$ is $k$-balanced if and only if $a \in S^k$.*

**Proof.** Let $b = (b_1, \ldots, b_n)$ be the partition conjugated to $a = (a_1, \ldots, a_k)$. First notice that if $w_a(k-i) = k+j > k$, then by (3.9) we have $i = b_j - j$.

If $a \in S^k$, then $w_a(k+i+1) = w_a(k-j+b_j+1) = w_a(k-j+a_j+1) = k-j+1 \leq k$ and so $a$ is $k$-balanced.

If $a \notin S^k$, then there is $j$ such that $a_1 = b_1, \ldots, a_{j-1} = b_{j-1}$ and $a_j \neq b_j$. It follows that $b_j \geq j$ and so $i := b_j - j \geq 0$. Then $w_a(k-i) = w_a(k-b_j+j) = k+j > k$. If $a_j > b_j$, then $w_a(k+i+1) = k+b_j+1 > k$. If $a_j < b_j$, then $w_a(k+i+1) = k+b_j > k$. ■

Recall from [2] that given $w \in S_m$, there exists a minimal integer $\ell(w)$, called the *length* of $w$, such that $w$ can be expressed as a product of $\ell(w)$ simple reflections $s_1, \ldots, s_m$. It is well known that $\ell(w)$ is equal to the number of pairs $1 \leq i < j \leq m$ such that $w(i) > w(j)$.

**Lemma 3.6.** *Let $w_a \in S_m$. Then $\ell(w_a) = |a|$.*

**Proof.** By the definition of $w_a$, it follows that $\ell(w_a) \leq |a|$. On the other hand, if $a < a'$, then $w_a \to w_{a'}$ and thus also $\ell(w_a) < \ell(w_{a'})$. By induction on $|a|$, we have that $\ell(w_a) \geq |a|$. ■

# 4 Geometric structures attached to (3.3)

In Section 4 we will consider different geometric structures associated to (3.3). Namely, we will consider in Section 4.1 the associated homogeneous spaces, in Section 4.2 the filtrations of tangent bundles of these parabolic geometries and in Section 4.3 the projections $\eta$ and $\tau$.

## 4.1 Homogeneous spaces

A connected and simply connected Lie group G with Lie algebra $\mathfrak{g}$ is isomorphic to $\mathrm{Spin}(2m, \mathbb{C})$. Let R, Q and P be the parabolic subgroups of G with Lie algebras $\mathfrak{r}$, $\mathfrak{q}$ and $\mathfrak{p}$ that are associated to $\{\alpha_m\}$, $\{\alpha_k, \alpha_m\}$ and $\{\alpha_k\}$, respectively, as explained in Section 2.1. We for brevity put $TS := \mathrm{G}/\mathrm{R}$, $CS := \mathrm{G}/\mathrm{Q}$ and $M := \mathrm{G}/\mathrm{P}$. Recall from Section 2.4 that we call $TS$ the twistor space and $CS$ the correspondence space.

*The twistor space $TS$.* Let us first recall (see [15, Section 6]) some well known facts about spinors. Recall from (3.4) that $\mathbb{W} := \mathbb{C}^m$ is a maximal totally isotropic subspace of $\mathbb{C}^{2m}$. We can (via $h$) identify the dual space $\mathbb{W}^*$ with the subspace $[e^*_1, \ldots, e^*_m]$. Put $\mathbb{S} := \bigoplus_{i=0}^m \Lambda^i \mathbb{W}^*$. There is a canonical linear map $\mathbb{C}^{2m} \to \mathrm{End}(\mathbb{S})$ which is determined by $w \cdot \psi = i_w \psi$ and $w^* \cdot \psi = w^* \wedge \psi$ where $w \in \mathbb{W}$, $w^* \in \mathbb{W}^*$, $\psi \in \mathbb{S}$ and $i_w$ stands for the contraction by $w$. If $\psi \in \mathbb{S}$, then we put $T_\psi := \{v \in \mathbb{C}^{2m} \colon v \cdot \psi = 0\}$. If $\psi \neq 0$, then $T_\psi$ is a totally isotropic subspace and we call $\psi$ a *pure spinor* if $\dim T_\psi = m$ (which is equivalent to saying that $T_\psi$ is a maximal totally isotropic subspace).

The standard linear isomorphism $\Lambda^2 \mathbb{C}^{2m} \cong \mathfrak{g}$ gives an injective linear map $\mathfrak{g} \to \mathrm{End}(\mathbb{S})$. It is straightforward to verify that the map is a homomorphism of Lie algebras where the commutator in the associative algebra $\mathrm{End}(\mathbb{S})$ is the standard one. Hence, $\mathfrak{g}$ is a Lie subalgebra of $\mathrm{End}(\mathbb{S})$ and it turns out that $\mathbb{S}$ is no longer irreducible under $\mathfrak{g}$ but it decomposes as $\mathbb{S}_+ \oplus \mathbb{S}_-$ where

$\mathbb{S}_+ := \bigoplus_{i=0}^m \Lambda^{2i}\mathbb{W}^*$ and $\mathbb{S}_- := \bigoplus_{i=0}^m \Lambda^{2i+1}\mathbb{W}^*$. Then $\mathbb{S}_+$ and $\mathbb{S}_-$ are irreducible non-isomorphic complex spinor representations of $\mathfrak{g}$ with highest weights $\omega_m$ and $\omega_{m-1}$, respectively. It is well known that any pure spinor belongs to $\mathbb{S}_+$ or to $\mathbb{S}_-$ (which explains why the Grassmannian of maximal totally isotropic subspaces in $\mathbb{C}^{2m}$ has two connected components).

Now we can easily describe the twistor space. The spinor $1 \in \mathbb{S}_+$ is annihilated by all positive roots in $\mathfrak{g}$ and hence, it is a highest weight vector. Recall from Section 2.1 that the line spanned by 1 is invariant under R and since $T_1 = \mathbb{W}$, we find that R is the stabilizer of $\mathbb{W}$ inside G. As G is connected, we conclude that $TS$ is the connected component of $\mathbb{W}$ in the Grassmannian of maximal totally isotropic subspaces in $\mathbb{C}^{2m}$.

*The isotropic Grassmannian $M$.* An irreducible $\mathfrak{g}$-module with highest weight $\omega_k$ is isomorphic to $\Lambda^k\mathbb{C}^{2m}$. Then $e_1 \wedge e_2 \wedge \cdots \wedge e_k$ is clearly a highest weight vector and the corresponding point in $\mathbb{P}(\Lambda^k\mathbb{C}^{2m})$ can be viewed as the totally isotropic subspace $x_0 := \mathbb{C}^k$. We see that $M$ is the Grassmannian of totally isotropic $k$-dimensional subspaces in $\mathbb{C}^{2m}$. We denote by $\mathbf{p}\colon \mathrm{G} \to M$ the canonical projection.

*The correspondence space $CS$.* The correspondence space $CS$ is the generalized flag manifold of nested subspaces $\{(z,x)\colon z \in TS,\ x \in M,\ x \subset z\}$ and Q is the stabilizer of $(\mathbb{W}, x_0)$. Let $\mathbf{q}\colon \mathrm{G} \to CS$ be the canonical projection.

### 4.2 Filtrations of the tangent bundles of $M$ and $CS$

Recall from Section 2.1 that the $|2|$-grading $\mathfrak{g} = \mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\cdots\oplus\mathfrak{g}_2$ associated to $\{\alpha_k\}$ determines a 2-step filtration $\{0\} = F^M_0 \subset F^M_{-1} \subset F^M_{-2} = TM$ of the tangent bundle of $M$ where $\{0\}$ is the zero section. We put $G^M_i := F^M_i/F^M_{i+1}$, $i = -2,-1$ so that the associated graded bundle $gr(TM) = G^M_{-2} \oplus G^M_{-1}$ is a locally trivial bundle of graded nilpotent Lie algebras with typical fiber $\mathfrak{g}_-$. Dually, there is a filtration $T^*M = F^M_1 \supset F^M_2 \supset F^M_3 = \{0\}$ where $F^M_i \cong (F^M_{-i+1})^\perp$. We put $G^M_i := F^M_i/F^M_{i+1}$ so that $G^M_i \cong (G^M_{-i})^*$. There are linear isomorphisms

$$\mathfrak{g}_i \cong \left(G^M_i\right)_{x_0}, \qquad i = -2,-1,1,2. \tag{4.1}$$

Recall from Section 2.2 that $\mathfrak{gr}^r_{x_0}$ denotes the vector space of weighted $r$-jets of germs of holomorhic functions at $x_0$ whose weighted $(r-1)$-jet vanishes. Then the isomorphisms from (2.3) are

$$\mathfrak{gr}^1_{x_0} \cong \mathfrak{g}_1, \qquad \mathfrak{gr}^2_{x_0} \cong S^2\mathfrak{g}_1 \oplus \mathfrak{g}_2, \qquad \mathfrak{gr}^3_{x_0} \cong S^3\mathfrak{g}_1 \oplus \mathfrak{g}_1 \otimes \mathfrak{g}_2, \qquad \ldots$$

for small $r$ and in general

$$\mathfrak{gr}^r_{x_0} \cong \bigoplus_{i+2j=r} S^i\mathfrak{g}_1 \otimes S^j\mathfrak{g}_2. \tag{4.2}$$

The $|3|$-grading $\mathfrak{g} = \bigoplus_{i=-3}^3 \mathfrak{q}_i$ determined by $\{\alpha_k, \alpha_m\}$ induces a 3-step filtration $TCS = F^{CS}_{-3} \supset F^{CS}_{-2} \supset F^{CS}_{-1} \supset F^{CS}_0 = \{0\}$. We put $G^{CS}_i := F^{CS}_i/F^{CS}_{i+1}$ so that $gr(TCS) = \bigoplus_{i=-3}^{-1} G^{CS}_i$ is a locally trivial vector bundle of graded nilpotent Lie algebras with typical fiber $\mathfrak{q}_-$. Dually, we get a filtration $T^*CS = F^{CS}_1 \supset F^{CS}_2 \supset F^{CS}_3 \supset F^{CS}_4 = \{0\}$ where $F^{CS}_i := (F^{CS}_{-i+1})^\perp$. The associated graded vector bundle is $gr(T^*CS) = \bigoplus_{i=1}^3 G^{CS}_i$ where we put $G^{CS}_i := F^{CS}_i/F^{CS}_{i+1}$. Then as above, $G^{CS}_i \cong (G^{CS}_{-i})^*$.

The Q-invariant subspaces $\mathbb{E} \oplus \mathfrak{q}$ and $\mathbb{F} \oplus \mathfrak{q}$ give a finer filtration of the tangent bundle, namely $F^{CS}_{-1} = E^{CS} \oplus F^{CS}$. Since the Lie bracket $\Lambda^2\mathfrak{q}_{-1} \to \mathfrak{q}_{-2}$ vanishes on $\Lambda^2\mathbb{E} \oplus \Lambda^2\mathbb{F}$, it follows that $E^{CS}$ and $F^{CS}$ are integrable distributions. This can be deduced also from the short exact sequences

$$0 \to E^{CS} \to TCS \xrightarrow{T\eta} T(TS) \to 0 \qquad \text{and} \qquad 0 \to F^{CS} \to TCS \xrightarrow{T\tau} TM \to 0, \tag{4.3}$$

i.e., $E^{CS} = \ker(T\eta)$ and $F^{CS} = \ker(T\tau)$. Notice that $(T\tau)^{-1}(F^M_{-1}) = F^{CS}_{-2}$.

### 4.3 Projections $\tau$ and $\eta$

Recall from (3.4) that $\mathbb{C}^{2n} := [e_{k+1},\dots,e_m,e^*_{k+1},\dots,e^*_m]$ and $\mathbb{C}^n := [e_{k+1},\dots,e_m]$, i.e., we view $\mathbb{C}^{2n}$ and $\mathbb{C}^n$ as subspaces of $\mathbb{C}^{2m}$. On $\mathbb{C}^{2n}$ we consider the non-degenerate bilinear form $h|_{\mathbb{C}^{2n}}$ which we for brevity denote by $h$. Then $\mathbb{C}^n$ is a maximal totally isotropic subspace of $\mathbb{C}^{2n}$.

The fibers of $\tau$ and $\eta$ are homogeneous spaces of parabolic geometries which (see [2]) can be recovered from the Dynkin diagrams given in (3.3).

**Lemma 4.1.**

(a) *The fibers of $\tau$ are biholomorphic to the Grassmannian of $k$-dimensional subspaces in $\mathbb{C}^{n+k}$.*

(b) *The fibers of $\eta$ are biholomorphic to the connected component $\mathrm{Gr}^+_h(n,n)$ of $\mathbb{C}^n$ in the Grassmannian of maximal totally isotropic subspaces in $\mathbb{C}^{2n}$.*

**Proof.** As the fibers over distinct points are biholomorphic, it suffices to look at the fibers of $\eta$ and $\tau$ over $\mathbb{W}$ and $x_0$, respectively.

(a) By definition, $\eta^{-1}(\mathbb{W})$ is the set of $k$-dimensional totally isotropic subspaces in $\mathbb{W}$. As $\mathbb{W}$ is already totally isotropic, the first claim follows.

(b) Notice that $x_0^\perp = x_0 \oplus \mathbb{C}^{2n}$. Then it is easy to see that $y \in \mathrm{Gr}^+_h(n,n) \mapsto (x_0 \oplus y, x_0) \in \tau^{-1}(x_0)$ is a biholomorphism. ∎

We will use the following notation. Assume that $X \in M(2m,k,\mathbb{C})$ and $Y \in M(2m,n,\mathbb{C})$ have maximal rank. Then we denote by $[X]$ the $k$-dimensional subspace of $\mathbb{C}^{2m}$ that is spanned by the columns of the matrix and by $[X|Y]$ the flag of nested subspaces $[X] \subset [X] \oplus [Y]$.

It is straightforward to verify that

$$\begin{aligned}
&(\mathbf{p}\circ\exp)\colon\ \mathfrak{g}_- \to M,\\
&(\mathbf{p}\circ\exp)\begin{pmatrix} 0 & 0 & 0 & 0\\ X_1 & 0 & 0 & 0\\ X_2 & 0 & 0 & 0\\ Y & -X_2^T & -X_1^T & 0\end{pmatrix} = \begin{bmatrix} 1_k\\ X_1\\ X_2\\ Y-\frac{1}{2}(X_1^TX_2+X_2^TX_1)\end{bmatrix}.
\end{aligned}\tag{4.4}$$

We see that $\mathcal{X} := \mathbf{p}\circ\exp(\mathfrak{g}_-)$ is an open, dense and affine subset of $M$ and that any $(z,x) \in \tau^{-1}(\mathcal{X})$ can be represented by

$$\left[\begin{array}{c|c} 1_k & 0\\ X_1 & A\\ X_2 & B\\ Y-\frac{1}{2}(X_1^TX_2+X_2^TX_1) & C\end{array}\right],\tag{4.5}$$

where $A,B \in M(n,\mathbb{C})$, $C \in M(k,n,\mathbb{C})$ are such that $\begin{bmatrix} A\\ B\end{bmatrix} \in \mathrm{Gr}^+_h(n,n)$ and $C = -(X_1^TB+X_2^TA)$. We immediately get the following observation.

**Lemma 4.2.** *The set $\tau^{-1}(\mathcal{X})$ is biholomorphic to $\mathcal{X}\times\tau^{-1}(x_0)$. The restriction of $\tau$ to this set is then the projection onto the first factor.*

The set $\tau^{-1}(\mathcal{X})$ is not affine as different choices of $A$ and $B$ might lead to the same element in $\tau^{-1}(\mathcal{X})$. Let $\mathcal{Y}$ be the subset of $\tau^{-1}(\mathcal{X})$ of those nested flags $x \subset z$ which can be represented by a matrix as above with $A$ regular. In that case we may assume $A = 1_n$ which uniquely pins down $B$. It is straightforward to find that $B = -B^T$ and conversely, any skew-symmetric $n\times n$ matrix determines a totally isotropic $n$-dimensional subspace in $\mathbb{C}^{2n}$. We see that $\mathcal{Y}$ is an open and affine set which is biholomorphic to $\mathfrak{g}_- \times A(n,\mathbb{C})$. In order to write down also $\eta$ as a canonical projection $\mathbb{C}^{\binom{m}{2}+nk} \to \mathbb{C}^{\binom{m}{2}}$, it will be convenient to choose a different coordinate system on $\mathcal{Y}$.

**Lemma 4.3.** *Let $\mathcal{Y}$ be as above and put $\mathcal{Z} := \eta(\mathcal{Y})$. Then $\mathcal{Y}$ and $\mathcal{Z}$ are open affine sets and there is a commutative diagram of holomorphic maps*

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d, "\eta|_{\mathcal{Y}}"] & A(m,\mathbb{C}) \times M(n,k,\mathbb{C}) \arrow[d, "pr_1"] \\ \mathcal{Z} \arrow[r] & A(m,\mathbb{C}), \end{tikzcd} \tag{4.6}$$

*where $pr_1$ is the canonical projection and the horizontal arrows are biholomorphisms.*

**Proof.** Let $(z, x)$ be the nested flag corresponding to (4.5) where $A = 1_n$ so that $B = -B^T$. Put for brevity $Y' := Y - \frac{1}{2}(X_1^T X_2 + X_2^T X_1)$ and $C := -X_2^T - X_1^T B$. The map in the first row in (4.6) is $(z, x) \mapsto (W, Z)$ where

$$W = \begin{pmatrix} W_1 & W_2 \\ W_0 & -W_1^T \end{pmatrix}$$

and $Z := X_1$, $W_1 := X_2 - BX_1$, $W_0 := Y' - CX_1$, $W_2 := B$. Then $W = -W^T$ and the map $\mathcal{Y} \to A(m,\mathbb{C}) \times M(n,k,\mathbb{C})$ is clearly a biholomorphism. In order to have a geometric interpretation of the map, consider the following. Using Gaussian elimination on the columns of the matrix (4.5), we can eliminate the $X_1$-block and get a new matrix

$$\begin{pmatrix} 1_k & 0 \\ 0 & 1_n \\ X_2 - BX_1 & B \\ Y' - CX_1 & C \end{pmatrix} = \begin{pmatrix} 1_m \\ W \end{pmatrix}.$$

The columns of the matrix span the same totally isotropic subspace $z$ as the original matrix. Moreover, it is clear that $z$ admits a unique basis of this form. From this we easily see that $\mathcal{Z}$ is indeed an open affine subset of $TS$ which is biholomorphic to $A(m,\mathbb{C})$. In these coordinate systems, the restriction of $\eta$ is the projection onto the first factor. ■

## 5 The Penrose transform for the $k$-Dirac complexes

In Section 5 we will consider the relative BGG sequence associated to a particular $\mathfrak{r}$-dominant and integral weight as explained in Section 2.4. More explicitly, we will define in Section 5.1 for each $p, q \geq 0$ a sheaf of relative $(p,q)$-forms and we get a Dolbeault-like double complex. Then we will show (see Section 5.2) that this double complex contains a relative holomorphic de Rham complex. Then in Section 5.3 we will twist each sheaf of relative $(p,q)$-forms as well as the holomorhic de Rham complex by a certain pullback sheaf. Using some elementary representation theory, we will turn (see Section 5.4) the twisted relative de Rham complex into the relative BGG sequence. In Section 5.5 we will compute direct images of sheaves in the relative BGG sequence.

We will use the following notation. We denote by $\mathcal{O}_\mathfrak{q}$ and $\mathcal{E}^{p,q}_\mathfrak{q}$ the structure sheaf and the sheaf of smooth $(p,q)$-forms, respectively, over $CS$. We denote the corresponding sheaves over $TS$ by the subscript $\mathfrak{r}$. If $W$ is a holomorphic vector bundle over $CS$, then we denote by $\mathcal{O}_\mathfrak{q}(W)$ the sheaf of holomorphic sections of $W$ and by $\mathcal{E}^{p,q}_\mathfrak{q}(W)$ the sheaf of smooth $(p,q)$-forms with values in $W$. We for brevity put $\mathcal{E}_* := \mathcal{E}^{0,0}_*$ and $\mathcal{E}^{p,q}_*(\mathcal{U}) := \Gamma(\mathcal{U}, \mathcal{E}^{p,q}_*)$ where $\mathcal{U}$ is an open set and $* = \mathfrak{q}$ or $\mathfrak{r}$. Moreover, we put $\eta^*\mathcal{E}^{p,q}_\mathfrak{r} := \mathcal{E}_\mathfrak{q} \otimes_{\eta^{-1}\mathcal{E}_\mathfrak{r}} \mathcal{E}^{p,q}_\mathfrak{r}$ where we use that $\eta^{-1}\mathcal{E}_\mathfrak{r}$ is naturally a sub-sheaf of $\mathcal{E}_\mathfrak{q}$.

### 5.1 Double complex of relative forms

Recall from Lemma 4.3 that $\mathcal{Y}$ is biholomorphic to $A(m,\mathbb{C})\times M(n,k,\mathbb{C})$, that $\eta(\mathcal{Y})=\mathcal{Z}$ is biholomorphic to $A(m,\mathbb{C})$ and that the canonical map $\mathcal{Y}\to\mathcal{Z}$ is the projection onto the first factor. In this way we can use matrix coefficients on $M(n,k,\mathbb{C})$ as coordinates on the fibers of $\eta$. We will write $Z=(z_{\alpha i})\in M(n,k,\mathbb{C})$ and if $I=((\alpha_1,i_1),\dots,(\alpha_p,i_p))$ is a multi-index where $(\alpha_j,i_j)\in I(n,k):=\{(\alpha,i)\colon \alpha=1,\dots,n,\ i=1,\dots,k\}$, $j=1,\dots,p$, then we put $dz_I:=dz_{(\alpha_1,i_1)}\wedge\dots\wedge dz_{(\alpha_p,i_p)}$ and $|I|=p$.

We call

$$\mathcal{E}^{p+1,q}_\eta:=\mathcal{E}^{p+1,q}_{\mathfrak{q}}/(\eta^*\mathcal{E}^{1,0}_{\mathfrak{r}}\wedge\mathcal{E}^{p,q}_{\mathfrak{q}}),\qquad p,q\geq 0$$

the *sheaf of relative* $(p+1,q)$*-forms*. By the G-action, it is clearly enough to understand the space of sections of this sheaf over the open set $\mathcal{Y}$ from Section 4.3. Given $\omega\in\mathcal{E}^{p,q}_\eta(\mathcal{Y})$, it is easy to see that there is a unique $(p,q)$-form cohomologous to $\omega$ which can be written in the form

$$\sum_{|I|=p}{}' dz_I\wedge\omega_I, \tag{5.1}$$

where $\Sigma'$ denote that the summation is performed only over strictly increasing multi-indeces[7] and each $\omega_I\in\mathcal{E}^{0,q}_{\mathfrak{q}}(\mathcal{Y})$.

As $\eta$ is holomorphic, $\partial$ and $\bar\partial$ commute with the pullback map $\eta^*$. We see that $\partial(\eta^*\mathcal{E}^{1,0}_{\mathfrak{r}}\wedge\mathcal{E}^{p,q}_{\mathfrak{q}})\subset\eta^*\mathcal{E}^{1,0}_{\mathfrak{r}}\wedge\mathcal{E}^{p+1,q}_{\mathfrak{q}}$ and $\bar\partial(\eta^*\mathcal{E}^{1,0}_{\mathfrak{r}}\wedge\mathcal{E}^{p,q}_{\mathfrak{q}})\subset\eta^*\mathcal{E}^{1,0}_{\mathfrak{r}}\wedge\mathcal{E}^{p,q+1}_{\mathfrak{q}}$ and thus, $\partial$ and $\bar\partial$ descend to differential operators

$$\partial_\eta\colon\ \mathcal{E}^{p,q}_\eta\to\mathcal{E}^{p+1,q}_\eta\qquad\text{and}\qquad\bar\partial\colon\ \mathcal{E}^{p,q}_\eta\to\mathcal{E}^{p,q+1}_\eta,$$

respectively. From the definitions it easily follows that:

$$\begin{aligned}&\partial_\eta(\omega\wedge\omega')=(\partial_\eta\omega)\wedge\omega'+(-1)^{p+q}\omega\wedge\partial_\eta\omega',\\ &\partial_\eta f=\sum_{(\alpha,i)\in I(n,k)}\frac{\partial f}{\partial z_{\alpha i}}dz_{\alpha i},\end{aligned} \tag{5.2}$$

where $\omega\in\mathcal{E}^{p,q}_\eta(\mathcal{Y})$, $\omega'\in\mathcal{E}^{p',q'}_\eta(\mathcal{Y})$ and $f\in\mathcal{E}(\mathcal{Y})$. Recall from Section 4.2 that $\partial_{z_{\alpha i}}\in\Gamma(E^{1,0}|_{\mathcal{Y}})$ and thus, $\partial_\eta f(x)$, $x\in\mathcal{Y}$ depends only on the first weighted jet $\mathrm{j}^1_x f$ of $f$ at $x$ (see Section 2.2).

Recall from (4.3) that the distribution $E^{CS}$ is equal to $\ker(T\eta)$.

**Proposition 5.1.**

(*i*) *The sheaf* $\mathcal{E}^{p,q}_\eta$ *is naturally isomorphic to the sheaf* $\mathcal{E}^{0,q}_{\mathfrak{q}}(\Lambda^pE^{CS*})$ *of smooth* $(0,q)$*-forms with values in the vector bundle* $\Lambda^pE^{CS*}$ *and* $(\mathcal{E}^{p,*}_\eta,\bar\partial)$ *is a resolution of* $\mathcal{O}_{\mathfrak{q}}(\Lambda^pE^{CS*})$ *by fine sheaves.*

(*ii*) $\partial_\eta$ *is a linear* G*-invariant differential operator of weighted order one and the sequence of sheaves* $(\mathcal{E}^{*,q}_\eta,\partial_\eta)$, $q\geq 0$ *is exact.*

(*iii*) *The data* $(\mathcal{E}^{p,q}_\eta,(-1)^p\bar\partial,\partial_\eta)$ *define a double complex of fine sheaves with exact rows and columns.*

**Proof.** (*i*) By definition, the sequence of vector bundles $0\to E^{CS\perp}\to T^*CS\to E^{CS*}\to 0$ is short exact. Hence, also the sequence $0\to E^{CS\perp}\wedge\Lambda^pT^*CS\to\Lambda^{p+1}T^*CS\to\Lambda^{p+1}E^{CS*}\to 0$, $p\geq 1$ is short exact. In view of the isomorphism $\mathcal{E}^{p+1,q}_{\mathfrak{q}}\cong\mathcal{E}^{0,q}_{\mathfrak{q}}(\Lambda^{p+1}T^*CS)$, it is enough to show

[7] We order the set $I(n,k)$ lexicographically, i.e., $(\alpha,i)<(\alpha',i')$ if $\alpha<\alpha'$ or $\alpha=\alpha'$ and $i<i'$.

that $\eta^*\mathcal{E}^{1,0}_{\mathfrak{r}} \wedge \mathcal{E}^{p,q}_{\mathfrak{q}}$ is isomorphic to $\mathcal{E}^{0,q}_{\mathfrak{q}}(E^{CS\perp} \wedge \Lambda^p T^*CS) \cong \mathcal{E}_{\mathfrak{q}}(E^{CS\perp}) \wedge \mathcal{E}^{0,q}_{\mathfrak{q}}(\Lambda^p T^*CS)$. Now $\eta^*\mathcal{E}^{1,0}_{\mathfrak{r}}$ is a sub-sheaf of $\mathcal{E}^{1,0}_{\mathfrak{q}} = \mathcal{E}_{\mathfrak{q}}(T^*CS)$ and since $\ker(T\eta) = E^{CS}$, it is contained in $\mathcal{E}_{\mathfrak{q}}(E^{CS\perp})$. Using that $\ker(T\eta) = E^{CS}$ again, it is easy to see that the map $\eta^*\mathcal{E}^{1,0}_{\mathfrak{r}} \to \mathcal{E}_{\mathfrak{q}}(E^{CS\perp})$ induces an isomorphism of stalks at any point. Hence, $\eta^*\mathcal{E}^{1,0}_{\mathfrak{r}} \cong \mathcal{E}_{\mathfrak{q}}(E^{CS\perp})$ and the proof of the first claim is complete. The second claim is clear.

(ii) It is clear that $\partial_\eta$ is $\mathbb{C}$-linear. It is G-invariant as $\partial$ commutes with the pullback of any holomorphic map and since $\eta$ is G-equivariant. As we already observed above that $\partial_\eta f(x)$ depends only on $\mathrm{j}^1_x f$ when $x \in \mathcal{Y}$, the G-invariance of $\partial_\eta$ shows that the same holds on $CS$ and thus, $\partial_\eta$ is a differential operator of weighted order one. It remains to check the exactness of the complex and using the G-invariance, it is enough to do this at $x \in \mathcal{Y}$. By Lemma 4.3, $\mathcal{Y}$ is biholomorphic to $\mathbb{C}^\ell$ where $\ell = \binom{m}{2} + nk$. Hence, we can view the standard coordinates $w_1, \dots, w_\ell$ on $\mathbb{C}^\ell$ as coordinates on $\mathcal{Y}$. If $J = (j_1, \dots, j_q)$ where $j_1, \dots, j_q \in \{1, \dots, \ell\}$, then we put $d\bar{w}_J = d\bar{w}_{j_1} \wedge \dots \wedge d\bar{w}_{j_q}$ and $|J| = q$. Let $\omega = \sum'_{|I|=p} dz_I \wedge \omega_I \in \mathcal{E}^{p,q}_\eta(\mathcal{Y})$ be the relative form as in (5.1). Then there are unique functions $f_{I,J} \in \mathcal{E}_{\mathfrak{q}}(\mathcal{Y})$ so that $\omega_I = \sum'_{|J|=q} f_{I,J} d\bar{w}_J$. Assume that $\partial_\eta\omega = 0$ on some open neighborhood $\mathcal{U}_x$ of $x$. This is equivalent to saying that for each increasing multi-index $J$: $\partial_\eta \sigma_J = 0$ on $\mathcal{U}_x$ where $\sigma_J := \sum'_I f_{I,J} dz_I$. Now using the same arguments as in the proof of the Dolbeault lemma, see [16, Theorem 2.3.3], we can for each $J$ find a $(p-1,0)$-form $\phi_J$ such that $\partial_\eta \phi_J = \sigma_J$ on some open neighborhood of $x$. Then $\partial_\eta\big(\sum'_J \phi_J \wedge d\bar{w}_J\big) = \sum'_J \sigma_J \wedge d\bar{w}_J = \omega$ on some open neighborhood of $x$ and the proof is complete.

(iii) This follows from $[\bar\partial, \partial_\eta] = 0$ and the observations made above. ■

## 5.2 Relative de Rham complex

By definition, $\Omega^*_\eta := \mathcal{E}^{*,0}_\eta \cap \ker\bar\partial$ is a sheaf of holomorphic sections. Since $[\bar\partial, \partial_\eta] = 0$, there is a complex of sheaves $(\Omega^*_\eta, \partial_\eta)$ and we call it the *relative de Rham complex.*

**Proposition 5.2.**

*(i) The relative de Rham complex is an exact sequence of sheaves which resolves the sheaf $\eta^{-1}\mathcal{O}_{\mathfrak{r}}$.*

*(ii) The relative de Rham complex induces for each $r := \ell + j \geq 0$ a long exact sequence of vector bundles*

$$\begin{aligned}&\mathfrak{gr}^{\ell+j} \xrightarrow{\mathfrak{gr}\partial_\eta} E^{CS*} \otimes \mathfrak{gr}^{\ell+j-1} \to \cdots \\ &\qquad \to \Lambda^j E^{CS*} \otimes \mathfrak{gr}^{\ell} \xrightarrow{\mathfrak{gr}\partial_\eta} \Lambda^{j+1} E^{CS*} \otimes \mathfrak{gr}^{\ell+j-1} \to \cdots . \end{aligned} \tag{5.3}$$

*Let $s_0 > 0$, $s_1, s_2, s_3 \geq 0$ be integers such that $s_0 + s_1 + 2s_2 + 3s_3 = r$. Then the sequence (5.3) contains a long exact subsequence*

$$\begin{aligned}&0 \to S^{s_0}E^{CS*} \otimes S^{s_1,s_2,s_3} \to E^{CS*} \otimes S^{s_0-1}E^{CS*} \otimes S^{s_1,s_2,s_3} \to \cdots \\ &\to \Lambda^j E^{CS*} \otimes S^{s_0-j}E^{CS*} \otimes S^{s_1,s_2,s_3} \to \Lambda^{j+1}E^{CS*} \otimes S^{s_0-j-1}E^{CS*} \otimes S^{s_1,s_2,s_3} \to \cdots ,\end{aligned} \tag{5.4}$$

*where $S^{s_1,s_2,s_3} := S^{s_1}F^{CS*} \otimes S^{s_2}G^{CS}_2 \otimes S^{s_3}G^{CS}_3$.*

*(iii) The kernel of the first map in (5.3) is $\bigoplus_{s_1+2s_2+3s_3=r} S^{s_1,s_2,s_3}$.*

**Proof.** (i) Since $[\partial_\eta, \bar\partial] = 0$, the relative de Rham complex is a sub-complex of the zero-th row $(\mathcal{E}^{*,0}_{\mathfrak{q}}, \partial_\eta)$ of the double complex from Proposition 5.3. By diagram chasing and using the exactness of columns and rows in the double complex, one easily proves the exactness of the relative de Rham complex. By (5.2), it easily follows that $\eta^{-1}\mathcal{O}_{\mathfrak{r}} = \mathcal{E}^{0,0}_\eta \cap \ker(\partial_\eta) \cap \ker(\bar\partial)$.

(ii) The standard de Rham complex induces the Spencer complex (see [25]) which is known to be exact. As the complex $(\Omega^*_\eta, \partial_\eta)$ is just a relative version of the (holomorphic) de Rham complex and $\partial_\eta$ satisfies the usual properties of $\partial$, it is clear that the relative de Rham complex induces for each $s_0 > 0$, $s_1$, $s_2$ and $s_3$ the long exact sequence (5.4). The sequence (5.3) is the direct sum of all such sequences as $s_0$, $s_1$, $s_2$, and $s_3$ ranges over all quadruples of non-negative integers satisfying $r = s_0 + s_1 + 2s_2 + 3s_3$.

(iii) This readily follows from the part (ii). ∎

### 5.3 Twisted relative de Rham complex

The weight $\lambda := (1-2n)\omega_m$ is $\mathfrak{g}$-integral and $\mathfrak{r}$-dominant. Hence, there is an irreducible R-module $\mathbb{W}_\lambda$ with lowest weight $-\lambda$. Since $\mathfrak{r}$ is associated to $\{\alpha_m\}$, it follows that $\dim \mathbb{W}_\lambda = 1$ and so $\mathbb{W}_\lambda$ is also an irreducible Q-module. We will denote by $\mathcal{E}_{\mathfrak{q}}(\lambda)$ and $\mathcal{O}_{\mathfrak{q}}(\lambda)$ the sheaves of smooth and holomorphic sections of $W^{CS}_\lambda := \mathrm{G} \times_{\mathrm{Q}} \mathbb{W}_\lambda$, respectively. If $W$ is a vector bundle over $CS$, then we denote $W(\lambda) := W \otimes W^{CS}_\lambda$, i.e., we twist $W$ by tensoring with the line bundle $W^{CS}_\lambda$. It is not hard to see that $\eta^*\mathcal{E}_{\mathfrak{r}}(\lambda) \cong \mathcal{E}_{\mathfrak{q}}(\lambda)$ and $\eta^*\mathcal{O}_{\mathfrak{r}}(\lambda) = \mathcal{O}_{\mathfrak{q}}(\lambda)$ where we denote by the subscript $\mathfrak{r}$ the corresponding sheaves over $TS$.

We call $\mathcal{E}^{p,q}_\eta(\lambda) := \mathcal{E}^{p,q}_\eta \otimes_{\eta^{-1}\mathcal{E}_{\mathfrak{r}}} \eta^{-1}\mathcal{E}_{\mathfrak{r}}(\lambda)$ the *sheaf of twisted relative $(p,q)$-forms*. Consider the following sequence of canonical isomorphisms:

$$\begin{aligned}\mathcal{E}^{p,q}_\eta(\lambda) &\to \mathcal{E}^{p,q}_\eta \otimes_{\mathcal{E}_{\mathfrak{q}}} \mathcal{E}_{\mathfrak{q}} \otimes_{\eta^{-1}\mathcal{E}_{\mathfrak{r}}} \eta^{-1}\mathcal{E}_{\mathfrak{r}}(\lambda) \to \mathcal{E}^{p,q}_\eta \otimes_{\mathcal{E}_{\mathfrak{q}}} \eta^*\mathcal{E}_{\mathfrak{r}}(\lambda)\\ &\to \mathcal{E}^{p,q}_\eta \otimes_{\mathcal{E}_{\mathfrak{q}}} \mathcal{E}_{\mathfrak{q}}(\lambda) \to \mathcal{E}^{0,q}_{\mathfrak{q}} \otimes_{\mathcal{E}_{\mathfrak{q}}} \left(\mathcal{E}^{p,0}_\eta \otimes_{\mathcal{E}_{\mathfrak{q}}} \mathcal{E}_{\mathfrak{q}}(\lambda)\right) \to \mathcal{E}^{0,q}_{\mathfrak{q}}\left(\Lambda^p E^{CS*}(\lambda)\right).\end{aligned} \tag{5.5}$$

We see that $\mathcal{E}^{p,q}_\eta(\lambda)$ is isomorphic to the sheaf of smooth $(0,q)$-forms with values in $\Lambda^p E^{CS*}(\lambda)$. Hence, the Dolbeault differential induces a differential $\bar\partial\colon \mathcal{E}^{p,q}_\eta(\lambda) \to \mathcal{E}^{p,q+1}_\eta(\lambda)$ and a complex $(\mathcal{E}^{p,*}_\eta(\lambda), \bar\partial)$.

A section of $\mathcal{E}^{p,q}_\eta(\lambda)$ is by definition a finite sum of decomposable elements $\omega \otimes v$ where $\omega$ and $v$ are sections of $\mathcal{E}^{p,q}_\eta$ and $\eta^{-1}\mathcal{E}_{\mathfrak{r}}(\lambda)$, respectively. As any section of $\eta^{-1}\mathcal{E}_{\mathfrak{r}}$ as well as transition functions between sections of $\eta^{-1}\mathcal{E}_{\mathfrak{r}}(\lambda)$ belong to $\ker(\partial_\eta)$, it follows that there is a unique linear differential operator

$$\mathcal{E}^{p,q}_\eta(\lambda) \to \mathcal{E}^{p+1,q}_\eta(\lambda),$$

which satisfies $\omega \otimes v \mapsto \partial_\eta\omega \otimes v$. We denote the operator also by $\partial_\eta$ as there is no risk of confusion. It is clear that $\partial_\eta$ is a linear G-invariant differential operator of weighted order one.

**Proposition 5.3.** *Let $p, q \geq 0$ be integers.*

*(i) The sequence of sheaves $(\mathcal{E}^{(p,*)}_\eta(\lambda), \bar\partial)$ is exact.*

*(ii) The sequence of sheaves $(\mathcal{E}^{(*,q)}_\eta(\lambda), \partial_\eta)$ is exact.*

*(iii) There is a double complex $(\mathcal{E}^{p,q}_\eta(\lambda), \partial_\eta, (-1)^p\bar\partial)$ of fine sheaves with exact rows and columns.*

**Proof.** (i) By construction, the sequence is a Dolbeault complex and the claim follows.

(ii) The exactness follows immediately from Proposition 5.1(ii).

(iii) We need to verify that $[\bar\partial, \partial_\eta] = 0$. To see this, notice that a section of $\mathcal{E}^{p,q}_\eta(\lambda)$ can be locally written as a finite sum of elements as above with $v$ holomorphic. The claim then easily follows from Proposition 5.1(iii). ∎

Put $\Omega^*_\eta(\lambda) := \mathcal{E}^{*,0}_\eta(\lambda) \cap \ker(\bar\partial)$. The complex $(\mathcal{E}^{*,0}_\eta(\lambda), \partial_\eta)$ contains a sub-complex $(\Omega^*_\eta(\lambda), \partial_\eta)$ which we call the *twisted relative de Rham complex*. As in Proposition 5.2, one can easily see that $(\Omega^*_\eta(\lambda), \partial_\eta)$ is an exact sequence of sheaves of holomorhic sections. Following the proof of Proposition 5.2, we obtain the following:

**Proposition 5.4.** *The relative de Rham complex $(\Omega^*_\eta(\lambda), \partial_\eta)$ induces for each $r \geq 0$ a long exact sequence of vector bundles*

$$\big(\Lambda^\bullet E^{CS*} \otimes \mathfrak{gr}^{r-\bullet}(\lambda), \mathfrak{gr}\partial_\eta\big). \tag{5.6}$$

*Let $s_0 > 0$, $s_1, s_2, s_3 \geq 0$ be integers such that $s_0 + s_1 + 2s_2 + 3s_3 = r$. Then the sequence (5.6) contains a long exact subsequence*

$$\big(\Lambda^\bullet E^{CS*} \otimes S^{s_0-\bullet} E^{CS*} \otimes S^{s_1,s_2,s_3}(\lambda), \mathfrak{gr}\partial_\eta\big), \tag{5.7}$$

*where $S^{s_1,s_2,s_3}$ is defined in Proposition 5.2. The kernel of the first map in (5.6) is the bundle $\bigoplus_{s_1+2s_2+3s_3=r} S^{s_1,s_2,s_3}(\lambda)$.*

## 5.4 Relative BGG sequence

We know that $\Omega^p_\eta(\lambda)$ is isomorphic to the sheaf of holomorphic sections of $\Lambda^p E^{CS*}(\lambda) = \mathrm{G} \times_{\mathrm{Q}} (\Lambda^p \mathbb{E}^* \otimes \mathbb{W}_\lambda)$. The Q-module $\Lambda^p \mathbb{E}^*$ is not irreducible. Decomposing this module into irreducible Q-modules, we obtain from the relative twisted de Rham complex a relative BGG sequence and this will be crucial in the construction of the $k$-Dirac complexes. We will use notation from Section 3.2.

**Proposition 5.5.** *Let $a \in \mathbb{N}^{k,n}_{++}$ and $w_a \in W^{\mathfrak{q}}_{\mathfrak{r}}$ be as in Section 3.2. Then*

$$\Lambda^p \mathbb{E}^* \otimes \mathbb{W}_\lambda = \bigoplus_{a \in \mathbb{N}^{k,n}_{++} : |a| = p} \mathbb{W}_{\lambda_a} \qquad \text{and thus} \qquad \Omega^p_\eta(\lambda) = \bigoplus_{a \in \mathbb{N}^{n,k}_{++} : |a| = p} \mathcal{O}_{\mathfrak{q}}(\lambda_a), \tag{5.8}$$

*where $\mathbb{W}_{\lambda_a}$ is an irreducible Q-module with lowest weight $-\lambda_a := -w_a.\lambda$.*

*There is a linear G-invariant differential operator*

$$\partial^a_{a'} \colon \ \mathcal{O}_{\mathfrak{q}}(\lambda_a) \to \Omega^p_\eta(\lambda) \xrightarrow{\partial_\eta} \Omega^p_\eta(\lambda) \to \mathcal{O}_{\mathfrak{q}}(\lambda_{a'}) \tag{5.9}$$

*where the first map is the canonical inclusion and the last map is the canonical projection. If $a \not< a'$, then $\partial^a_{a'} = 0$.*

**Proof.** Recall from Section 3.1 that the semi-simple part $\mathfrak{r}^{ss}_0$ of $\mathfrak{r}_0$ is isomorphic to $\mathfrak{sl}(m, \mathbb{C})$ and that $\mathfrak{r}^{ss}_0 \cap \mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{r}^{ss}_0$. The direct sum decomposition from (5.8) then follows at once from the Kostant's version of the Bott–Borel–Weyl theorem (see [8, Theorem 3.3.5]) applied to $\mathbb{W}_\lambda$ and $(\mathfrak{r}^{ss}_0, \mathfrak{r}^{ss}_0 \cap \mathfrak{q})$ and the identity $\ell(w_a) = |a|$ from Lemma 3.6. Recall from [2, Section 8.7] that the graph of the relative BGG sequence coincides with the relative Hasse graph $W^{\mathfrak{q}}_{\mathfrak{r}}$. The last claim then follows from Lemma 3.4. ■

**Remark 5.6.** Let $a = (a_1, \ldots, a_k) \in \mathbb{N}^{k,n}_{++}$ and $b = (b_1, \ldots, b_n) \in \mathbb{N}^{n,k}_{++}$ be the conjugated partition. In order to compute $\lambda_a$ from Proposition 5.5, notice that

$$\lambda + \rho = \left(\frac{2k-1}{2}, \ldots, \frac{3}{2}, \frac{1}{2} \,\middle|\, -\frac{1}{2}, -\frac{3}{2}, \ldots, \frac{-2n+1}{2}\right). \tag{5.10}$$

Since $w_a(\omega_m) = \omega_m$, we have $w_a(\lambda) = \lambda$ and thus $\lambda_a = w_a(\lambda + \rho) - \rho = \lambda + w_a\rho - \rho$. By (3.10), it follows that

$$\lambda_a = \lambda + (-a_k, \ldots, -a_1 \,|\, b_1, b_2, \ldots, b_n).$$

### 5.5 Direct image of the relative BGG sequence

Recall from [2, Section 5.3] that given a $\mathfrak{g}$-integral and $\mathfrak{q}$-dominant weight $\nu$, there is at most one $\mathfrak{p}$-dominant weight in the $W^{\mathfrak{q}}_{\mathfrak{p}}$-orbit of $\nu$. If there is no $\mathfrak{p}$-dominant weight, then all direct images of $\mathcal{O}_{\mathfrak{q}}(\nu)$ vanish. If there is a $\mathfrak{p}$-dominant weight, say $\mu = w.\nu$ where $w \in W^{\mathfrak{q}}_{\mathfrak{p}}$, then $\tau^{\ell(w)}_{*}\mathcal{O}_{\mathfrak{q}}(\nu) \cong \mathcal{O}_{\mathfrak{p}}(\mu)$ is the unique non-zero direct image of $\mathcal{O}_{\mathfrak{q}}(\nu)$.

**Proposition 5.7.** *Let $n \geq k \geq 2$ and $a = (a_1, \dots, a_k) \in \mathbb{N}^{k,n}_{++}$. Put $\mu^{\pm} := \frac{1}{2}(-2n+1, \dots, -2n+1\,|\,1, \dots, 1, \pm 1)$ and $\mu_a := \mu^{*} - (a_k, \dots, a_1\,|\,0, \dots, 0)$ where $* = +$ if $d(a) \equiv n \mod 2$ and $* = -$ otherwise. Then*

$$\tau^{q}_{*}(\mathcal{O}_{\mathfrak{q}}(\lambda_a)) = \begin{cases} \mathcal{O}_{\mathfrak{p}}(\mu_a), & \text{if } a \in S^k,\ q = \ell(a) := \binom{n}{2} - q(a), \\ \{0\}, & \text{otherwise.} \end{cases}$$

**Proof.** By definition, each $w \in W^{\mathfrak{q}}_{\mathfrak{p}}$ fixes the first $k$ coefficients of $\lambda_a$ and so it is enough to look at the last $n$ coefficients. By Remark 5.6, it follows

$$w_a(\lambda + \rho) = \lambda_a + \rho = \left( \dots, i - a_i - \frac{1}{2}, \dots, \frac{1}{2} - a_1 \,\middle|\, b_1 - \frac{1}{2}, \dots, b_j - j + \frac{1}{2}, \dots \right),$$

where $b = (b_1, \dots, b_n)$ is the conjugated partition. Put $c := (c_1, \dots, c_n)$ where $c_j := \left|b_j - j + \frac{1}{2}\right|$. By (3.2) and Table 1, if $c_i = c_j$ for some $i \neq j$, then there cannot be a $\mathfrak{p}$-dominant weight in the $W^{\mathfrak{q}}_{\mathfrak{p}}$-orbit of $\lambda_a$. If $a \notin S^k$, then by Lemma 3.5 there is $s \in \{0, \dots, k-1\}$ such that[8] $w_a(k-s) = k+i \geq k$ and $w_a(k+s+1) = k+j \geq k$ for some distinct positive integers $i$ and $j$. By (5.10), it follows that $b_i - i + \frac{1}{2} = \frac{1}{2} + s$, $b_j - j - \frac{1}{2} = -\frac{1}{2} - s$ and thus $c_i = c_j$. Hence, all direct images of $\mathcal{O}_{\mathfrak{q}}(\lambda_a)$ are zero.

We may now suppose that $a \in S^k$. By the definition of $d(a)$, we have $b_{d(a)} - d(a) + \frac{1}{2} > 0 > b_{d(a)+1} - d(a) - \frac{1}{2}$. By Lemma 3.5 and (5.10), it follows that $(c_1, \dots, c_n)$ is a permutation of $\left(\frac{2n-1}{2}, \dots, \frac{3}{2}, \frac{1}{2}\right)$. As $\lambda_a$ is $\mathfrak{q}$-dominant, we know that the sequence $\left(b_1 - \frac{1}{2}, \cdots, b_j - j + \frac{1}{2}, \dots\right)$ is decreasing. Thus for each integer $i = 1, \dots, d(a)$, the set $\{j \colon i < j \leq n,\ c_i > c_j\}$ contains precisely $b_i - i$ distinct elements. Altogether, there are precisely $\sum_{\ell=1}^{d(a)} (b_\ell - \ell) = q(a)$ pairs $i < j$ such that $c_i > c_j$. Equivalently, there are $\ell(a)$ pairs $i < j$ such that $c_i < c_j$. It follows that the length of the permutation that maps $(c_1, \dots, c_n)$ to $\left(\frac{2n-1}{2}, \dots, \frac{3}{2}, \frac{1}{2}\right)$ is precisely $\ell(a)$. Now it is easy to see (recall (3.2)) that there is $w \in W^{\mathfrak{q}}_{\mathfrak{p}}$ such that $w.\lambda_a$ is $\mathfrak{p}$-dominant and $\ell(w) = \ell(a)$. As there are $n - d(a)$ negative numbers in the sequence $\left(b_1 - \frac{1}{2}, \dots, b_j - j + \frac{1}{2}, \dots\right)$, the last claim about the sign of the last coefficient of $w.\lambda_a$ also follows. This completes the proof. ∎

**Remark 5.8.** In Proposition 5.7 we recovered the $W^{\mathfrak{p}}$-orbit of the singular weight $\mu^+$ if $n$ is even and of $\mu^-$ if $n$ is odd which was computed in [14]. There is an automorphism of $\mathfrak{g}$ which swaps $\alpha_m$ and $\alpha_{m-1}$ and hence, it swaps also the associated parabolic subalgebras. If we cross in (3.3) the simple root $\alpha_{m-1}$ instead of $\alpha_m$, take $(1-2n)\omega_{m-1}$ as $\lambda$ and follow the computations given above, we will get the $W^{\mathfrak{p}}$-orbit of $\mu^+$ if $n$ is odd and of $\mu^-$ if $n$ is even. As also all other arguments presented in this paper work for the other case, we will obtain the other "half" of the $k$-Dirac complex from [22] as mentioned in Introduction.

### 5.6 Double complex of relative forms II

The direct sum decomposition from Proposition 5.5 together with the isomorphism in (5.5) gives a direct sum decomposition $\mathcal{E}^{p,q}_{\eta}(\lambda) = \bigoplus_{a \in \mathbb{N}^{k,n}_{++} : |a| = p} \mathcal{E}^{0,q}_{\mathfrak{q}}(\lambda_a)$. Let $a, a' \in \mathbb{N}^{k,n}_{++}$ be such that

[8] Notice that at this point we need that $n \geq k$.

$p = |a| = |a'| - 1$. Then there is a linear differential operator

$$\partial^a_{a'} \colon\ \mathcal{E}^{0,q}_{\mathfrak{q}}(\lambda_a) \to \mathcal{E}^{p,q}_{\eta}(\lambda) \xrightarrow{\partial_\eta} \mathcal{E}^{p+1,q}_{\eta}(\lambda) \to \mathcal{E}^{0,q}_{\mathfrak{q}}(\lambda_{a'}) \tag{5.11}$$

where the first map is the canonical inclusion and the last map is the canonical projection as in (5.9). We denote the differential operator by $\partial^a_{a'}$ as in (5.9) as there is no risk of confusion. Recall from Proposition 5.5 that $\partial^a_{a'} = 0$ if $a \not< a'$.

Suppose that $\mathcal{U}$ is an open, contractible and Stein subset of $M$. We put $\mathcal{E}^{p,q}_{\eta}(\tau^{-1}(\mathcal{U}), \lambda) := \Gamma(\tau^{-1}(\mathcal{U}), \mathcal{E}^{p,q}_{\eta}(\lambda))$, i.e., this is the space of sections of the sheaf $\mathcal{E}^{p,q}_{\eta}(\lambda)$ over $\tau^{-1}(\mathcal{U})$. Then there is a double complex

$$\begin{tikzcd}
 & {} & {} & \\
\cdots \arrow[r] & \mathcal{E}^{p,q+1}_{\eta}(\tau^{-1}(\mathcal{U}), \lambda) \arrow[r, "D''"] \arrow[u] & \mathcal{E}^{p+1,q+1}_{\eta}(\tau^{-1}(\mathcal{U}), \lambda) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathcal{E}^{p,q}_{\eta}(\tau^{-1}(\mathcal{U}), \lambda) \arrow[r, "D''"] \arrow[u, "D'"] & \mathcal{E}^{p+1,q}_{\eta}(\tau^{-1}(\mathcal{U}), \lambda) \arrow[r] \arrow[u, "D'"] & \cdots, \\
 & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd} \tag{5.12}$$

where $D' = (-1)^p \bar{\partial}$ and $D'' = \partial_\eta$. Put $T^i(\mathcal{U}) := \bigoplus_{p+q=i} \mathcal{E}^{p,q}_{\eta}(\tau^{-1}(\mathcal{U}), \lambda)$. We obtain three complexes $(T^*(\mathcal{U}), D')$, $(T^*(\mathcal{U}), D'')$ and $(T^*(\mathcal{U}), D' + D'')$.

**Lemma 5.9.** *Let $a \in \mathbb{N}^{k,n}_{++}$ and $\mathcal{U}$ be the open, contractible and Stein subset of $M$ as above. Then*

$$H^q\big(\tau^{-1}(\mathcal{U}), \mathcal{O}_{\mathfrak{q}}(\lambda_a)\big) = \begin{cases} \Gamma(\mathcal{U}, \mathcal{O}_{\mathfrak{p}}(\mu_a)), & a \in S^k,\ q = \ell(a), \\ \{0\}, & \text{otherwise}, \end{cases} \tag{5.13}$$

*and thus also*

$$H^{\binom{n}{2}+j}(T^*(\mathcal{U}), D') = \bigoplus_{a \in S^k_j} H^{\ell(a)}\big(\tau^{-1}(\mathcal{U}), \mathcal{O}_{\mathfrak{q}}(\lambda_a)\big).$$

**Proof.** The first claim follows from Proposition 5.7 and application of the Leray spectral sequence as explained in [2]. For the second claim, recall from [27, Theorem 3.20] that the sheaf cohomology is equal to the Dolbeault cohomology, i.e., there is an isomorphism

$$H^q\big(\tau^{-1}(\mathcal{U}), \mathcal{O}_{\mathfrak{q}}(\lambda_a)\big) \cong H^q\big(\mathcal{E}^{0,*}_{\mathfrak{q}}(\tau^{-1}(\mathcal{U}), \lambda_a), \bar{\partial}\big). \tag{5.14}$$

The cohomology group appears on the $(|a| + \ell(a)) = (d(a) + 2q(a) + \binom{n}{2} - q(a)) = (\binom{n}{2} + r(a))$-th diagonal of the double complex. Here, see Proposition 5.7, we use that $\ell(a) = \binom{n}{2} - q(a)$, the notation from (3.6) and $S^k_j = \{a \in S^k \colon r(a) = j\}$. ■

## 6 $k$-Dirac complexes

In Section 6 we will give the definition of differential operators in the $k$-Dirac complexes. It will be clear from the construction that the operators are linear, local and G-invariant. Later in Lemma 7.12 we will show that each operator is indeed a differential operator and we give an upper bound on its weighted order. The operators naturally form a sequence and we will prove in Theorem 6.2 that they form a complex which we call the $k$-Dirac complex.

Recall from Section 5.6 that $\mathcal{E}^{0,q}_{\mathfrak{q}}(\tau^{-1}(\mathcal{U}), \lambda_a)$ is the space of sections of the sheaf $\mathcal{E}^{0,q}_{\mathfrak{q}}(\lambda_a)$ over $\tau^{-1}(\mathcal{U})$. If $\alpha \in \mathcal{E}^{0,q}_{\mathfrak{q}}(\tau^{-1}(\mathcal{U}), \lambda_a)$ is $\bar{\partial}$-closed, then we will denote by $[\alpha] \in H^q(\tau^{-1}(\mathcal{U}), \mathcal{O}_{\mathfrak{q}}(\lambda_a))$ the corresponding cohomology class.

**Lemma 6.1.** *Let $j \geq 0$, $a \in S^k_j$, $a' \in S^k_{j+1}$ be such that $a < a'$ and $\mathcal{U}$ be the Stein set as above. Then there is a linear, local and* G*-invariant operator*

$$D^a_{a'}\colon\ \Gamma(\mathcal{U}, \mathcal{O}_{\mathfrak{p}}(\mu_a)) \to \Gamma(\mathcal{U}, \mathcal{O}_{\mathfrak{p}}(\mu_{a'})).$$

**Proof.** Let us for a moment put $\mathcal{V} := \tau^{-1}(\mathcal{U})$. Using the isomorphisms from (5.13), it is enough to define a map $H^{\ell(a)}(\mathcal{V}, \mathcal{O}_{\mathfrak{q}}(\lambda_a)) \to H^{\ell(a')}(\mathcal{V}, \mathcal{O}_{\mathfrak{q}}(\lambda_{a'}))$ which has the right properties. By assumption, we have $|a'|-|a| \in \{1, 2\}$. Let us first consider $|a'|-|a| = 1$. Then $q := \ell(a') = \ell(a)$ and by (5.11), we have the map $\partial^a_{a'}\colon \mathcal{E}^{0,q}_{\mathfrak{q}}(\mathcal{V}, \lambda_a) \to \mathcal{E}^{0,q}_{\mathfrak{q}}(\mathcal{V}, \lambda_{a'})$ in the double complex (5.12). The induced map on cohomology is $D^a_{a'}$.

If $|a'| - |a| = 2$, then $q := \ell(a) = \ell(a') + 1$ and we find that there are precisely two non-symmetric partitions $b, c \in \mathbb{N}^{k,n}_{++}$ such that $a < b < a'$ and $a < c < a'$. Then there is a diagram

$$\begin{array}{ccc} \mathcal{E}^{0,q}_{\mathfrak{q}}(\mathcal{V}, \lambda_a) \xrightarrow{(\partial^a_b, \partial^a_c)} & \mathcal{E}^{0,q}_{\mathfrak{q}}(\mathcal{V}, \lambda_b) \oplus \mathcal{E}^{0,q}_{\mathfrak{q}}(\mathcal{V}, \lambda_c) & \\ & \uparrow {\scriptstyle (-1)^p \bar{\partial}} & \\ & \mathcal{E}^{0,q-1}_{\mathfrak{q}}(\mathcal{V}, \lambda_b) \oplus \mathcal{E}^{0,q-1}_{\mathfrak{q}}(\mathcal{V}, \lambda_c) & \xrightarrow{\partial^b_{a'} + \partial^c_{a'}} \mathcal{E}^{0,q-1}_{\mathfrak{q}}(\mathcal{V}, \lambda_{a'}), \end{array} \tag{6.1}$$

which lives in the double complex (5.12). Let $\alpha \in \mathcal{E}^{0,q}_{\mathfrak{q}}(\mathcal{V}, \lambda_a)$ be $\bar{\partial}$-closed. Then $\partial^a_b \alpha$ and $\partial^a_c \alpha$ are also $\bar{\partial}$-closed and thus by Lemma 5.9 and the isomorphism (5.14), we can find $\beta$ and $\gamma$ such that $\partial^a_b \alpha = (-1)^p \bar{\partial}\beta$ and $\partial^a_c \alpha = (-1)^p \bar{\partial}\gamma$ where $p = |a| + 1$. Since the relative BGG sequence is a complex, we have

$$\bar{\partial}\big(\partial^b_{a'}\beta + \partial^c_{a'}\gamma\big) = (-1)^p(\partial^b_{a'}\partial^a_b + \partial^c_{a'}\partial^a_c)\alpha = 0,$$

which shows that $\partial^b_{a'}\beta + \partial^c_{a'}\gamma$ is a cocycle. Of course this elements depends on choices but we claim that $[\partial^b_{a'}\beta + \partial^c_{a'}\gamma]$ depends only on $[\alpha]$. It is easy to see that $[\partial^b_{a'}\beta + \partial^c_{a'}\gamma]$ does not depend on the choices of $\beta$ and $\gamma$. If $[\alpha] = 0$, say $\alpha = (-1)^{p-1}\bar{\partial}\varrho$, then we may put $\beta = -\partial^a_b \varrho$ and $\gamma = -\partial^a_c \varrho$ and thus $\partial^b_{a'}\beta + \partial^c_{a'}\gamma = -(\partial^b_{a'}\partial^a_b + \partial^c_{a'}\partial^a_c)\varrho = 0$. Hence, we can put $D^a_{a'}[\alpha] := [\partial^b_{a'}\beta + \partial^c_{a'}\gamma]$.

From the construction is clear that $D^a_{a'}$ is linear. The locality follows from the fact that $D^a_{a'}$ is compatible with restrictions to smaller Stein subsets of $\mathcal{U}$. As the operators in the double complex (5.12) are G-invariant, it is easy to verify that each operator $D^a_{a'}$ is G-invariant. ■

Put $\mathcal{O}_j := \bigoplus_{a \in S^k_j} \mathcal{O}_{\mathfrak{p}}(\mu_a)$ and $\mathcal{O}_j(\mathcal{U}) := \Gamma(\mathcal{U}, \mathcal{O}_j)$. If $a \in S^k_j$ and $s \in \mathcal{O}_j(\mathcal{U})$, then we denote by $s_a$ the $a$-th component of $s$ so that we may write $s = (s_a)_{a \in S^k_j}$. We call the following complex (6.2) the *k-Dirac complex.*

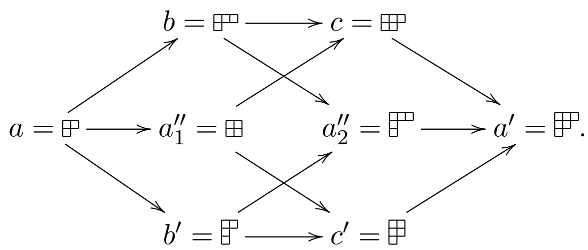
**Theorem 6.2.** *With the notation set above, there is a complex*

$$\mathcal{O}_0(\mathcal{U}) \xrightarrow{D_0} \mathcal{O}_1(\mathcal{U}) \to \cdots \to \mathcal{O}_j(\mathcal{U}) \xrightarrow{D_j} \mathcal{O}_{j+1}(\mathcal{U}) \to \cdots \tag{6.2}$$

*of linear* G*-invariant operators where*

$$(D_j s)_{a'} = \sum_{a < a'} D^a_{a'} s_a.$$

**Proof.** Let $a, a' \in S^k$ be such that $a < a'$, $r(a) = r(a') - 2$. We need to verify that $\sum\limits_{a'' \in S^k\colon a < a'' < a'} D^{a''}_{a'} D^a_{a''} = 0$. Observe that $|a'|-|a| \in \{3, 4\}$. Let us first assume that $|a'|-3 = |a|$. Then there are at most two symmetric partitions $a''$ such that $a < a'' < a'$. If there is only one such symmetric partition $a''$, then, since the relative BGG sequence is a complex, it follows easily that $D^{a''}_{a'}D^a_{a''}=0$. So we can assume that there are two symmetric partitions, say $a''_1$, $a''_2$. Consider for example



Then we can find $\beta$ and $\beta'$ so that $\partial^a_b\alpha=(-1)^p\bar\partial\beta$ and $\partial^a_{b'}\alpha=(-1)^p\bar\partial\beta'$ where $p=|b|=|b'|$. Then $[D^a_{a''_2}\alpha]=[\partial^b_{a''_2}\beta+\partial^{b'}_{a''_2}\beta']$ which implies

$$(-1)^{p+1}\bar\partial\big(\partial^b_c\beta\big)=(-1)^{p+1}\partial^b_c\bar\partial\beta=-\partial^b_c\partial^a_b\alpha=\partial^{a''_1}_c\partial^a_{a''_1}\alpha$$

and similarly $(-1)^{p+1}\bar\partial(\partial^{b'}_{c'}\beta')=\partial^{a''_1}_{c'}\partial^a_{a''_1}\alpha$. Hence, we conclude that

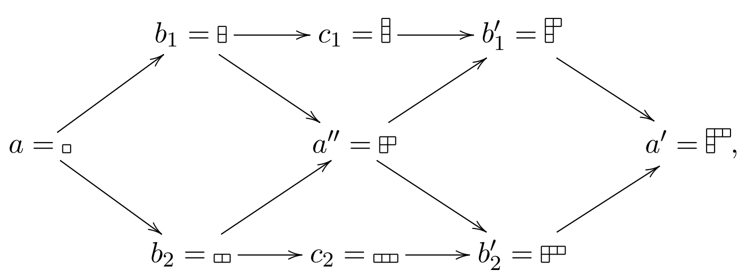
$$D^{a''_1}_{a'}D^a_{a''_1}[\alpha]=\big[\partial^c_{a'}\partial^b_c\beta+\partial^{c'}_{a'}\partial^{b'}_{c'}\beta'\big]$$

and thus

$$D^{a''_2}_{a'}D^a_{a''_2}[\alpha]=\big[\partial^{a''_2}_{a'}\big(\partial^b_{a''_2}\beta+\partial^{b'}_{a''_2}\beta'\big)\big]=-\big[\partial^c_{a'}\partial^b_c\beta+\partial^{c'}_{a'}\partial^{b'}_{c'}\beta'\big]=-D^{a''_1}_{a'}D^a_{a''_1}[\alpha].$$

This completes the proof when $|a'|=|a|+3$ and now we may assume $|a'|=|a|+4$.

We put $A'':=\{a''\in S^k|\ a<a''<a'\}$, $B:=\{b\in\mathbb{N}^{k,n}_{++}\,|\,\exists\,a''\in A''\colon a<b<a''\}$, $B':=\{b'\in\mathbb{N}^{k,n}_{++}\,|\,\exists\,a''\in A''\colon a''<b'<a'\}$ and finally $C:=\{c\in\mathbb{N}^{k,n}_{++}\setminus S^k\,|\,\exists\,b\in B,\ \exists\,b'\in B'\colon b'<c<b''\}$. Consider for example the diagram



where $A''=\{a''\}$, $B=\{b_1,b_2\}$, $B'=\{b'_1,b'_2\}$ and $C=\{c_1,c_2\}$. As above, the set $A''$ contains at most two elements but we will not need that.

Now we can proceed as above. There are $\beta_i$ such that $(-1)^p\bar\partial\beta_i=\partial^a_{b_i}\alpha$ where $p=|b_i|=|a|+1$ and so $[D^a_{a''_j}\alpha]=\big[\sum\limits_{b_i\in B}\partial^{b_i}_{a''_j}\beta_i\big]$ for every $a''_j\in A''$. As the relative BGG sequence is a complex, we have for each $c_\ell\in C$:

$$\bar\partial\bigg(\sum_{b_i\in B}\partial^{b_i}_{c_\ell}\beta_i\bigg)=\sum_{b_i\in B}\partial^{b_i}_{c_\ell}\bar\partial\beta_i=(-1)^p\sum_{b_i\in B}\partial^{b_i}_{c_\ell}\partial^a_{b_i}\alpha=0.$$

As above, there is $\gamma_\ell$ such that $(-1)^{p+1}\bar\partial\gamma_\ell=\sum\limits_{b_i\in B}\partial^{b_i}_{c_\ell}\beta_i$. Then for $b'_s\in B'$:

$$
\begin{aligned}
(-1)^{p+2}\bar\partial\Bigg(\sum_{c_\ell\in C}\partial^{c_\ell}_{b'_s}\gamma_\ell\Bigg) &= -\sum_{c_\ell\in C}\partial^{c_\ell}_{b'_s}((-1)^{p+1}\bar\partial\gamma_\ell) = -\sum_{c_\ell\in C,\ b_i\in B}\partial^{c_\ell}_{b'_s}\partial^{b_i}_{c_\ell}\beta_i\\
&= \sum_{b_i\in B,\ a''_j\in A''_j:\ b_i<a''_j<b'_s}\partial^{a''_j}_{b'_s}\partial^{b_i}_{a''_j}\beta_i = \sum_{a''_j\in A:\ a''_j<b'_s}\partial^{a''_j}_{b'_s}\Bigg(\sum_{b_i\in B}\partial^{b_i}_{a''_j}\beta_i\Bigg)\\
&= \sum_{a''_j\in A:\ a''_j<b'_s}\partial^{a''_j}_{b'_s}(D^a_{a''_j}\alpha).
\end{aligned}
$$

This implies that $\sum\limits_{a''_j\in A''}D^{a''}_{a'}D^a_{a''}([\alpha])$ is the cohomology class of

$$
\begin{aligned}
\sum_{a''_j\in A''}\Bigg(\sum_{b'_s\in B':\ a''_j<b'_s}\partial^{b'_s}_{a'}\Bigg(\sum_{c_\ell\in C}\partial^{c_\ell}_{b'_s}\gamma_\ell\Bigg)\Bigg) &= \sum_{b'_s\in B'}\sum_{c_\ell\in C}\partial^{b'_s}_{a'}\partial^{c_\ell}_{b'_s}\gamma_\ell\\
&= \sum_{c_\ell\in C}\sum_{b'_s\in B'}\partial^{b'_s}_{a'}\partial^{c_\ell}_{b'_s}\gamma_\ell = \sum_{c_\ell\in C}0=0.
\end{aligned}
$$

In the first equality we use the fact that given $b'_s\in B'$, there is only one $a''_j\in A''$ such that $a''_j<b'_s$ and in the third equality we use that the relative BGG sequence is a complex once more. ■

## 7 Formal exactness of $k$-Dirac complexes

We will proceed in Section 7 as follows. In Section 7.1 we will recall the definition of the normal bundle of the analytic subvariety $X_0:=\tau^{-1}(x_0)$ and give the definition of the weighted formal neighborhood of $X_0$. In Section 7.2 we will consider the double complex of twisted relative forms from Section 5 and restrict it to the weighted formal neighborhood of $X_0$. In Section 7.3 we will prove that the operators defined in Section 6 are differential operators and finally, in Theorem 7.14 we will prove that the $k$-Dirac complexes are formally exact.

### 7.1 Formal neighborhood of $\tau^{-1}(x_0)$

Let us first recall notation from Section 4.2. There is the 2-step filtration $\{0\}=F^M_0\subset F^M_{-1}\subset F^M_{-2}=TM$ and the 3-step filtration $\{0\}=F^{CS}_0\subset F^{CS}_{-1}\subset F^{CS}_{-2}\subset F^{CS}_{-3}=TM$. Moreover, $F^{CS}_{-1}$ decomposes as $E^{CS}\oplus F^{CS}$ where $E^{CS}=\ker(T\eta)$ and $F^{CS}=\ker(T\tau)$. From this it follows that $E^{CS}$ and $F^{CS}$ are integrable distributions. Dually, there are filtrations $T^*M=F^M_1\supset F^M_2\supset F^M_3=\{0\}$ and $T^*CS=F^{CS}_1\supset F^{CS}_2\supset F^{CS}_3\supset F^{CS}_4=\{0\}$ where $F^M_i$ is the annihilator of $F^M_{-i+1}$ and similarly for $F^{CS}_i$. We put $G^M_i:=F^M_i/F^M_{i+1}$ and $G^{CS}_i:=F^{CS}_i/F^{CS}_{i+1}$ so that

$$
\begin{aligned}
&gr(TM)=G^M_{-2}\oplus G^M_{-1},\qquad gr(T^*M)=G^M_1\oplus G^M_2,\\
&gr(TCS)=G^{CS}_{-3}\oplus G^{CS}_{-2}\oplus G^{CS}_{-1},\qquad gr(T^*CS)=G^{CS}_1\oplus G^{CS}_2\oplus G^{CS}_3,\\
&G^M_i\cong\big(G^M_{-i}\big)^*,\quad i=1,2\qquad\text{and}\qquad G^{CS}_i\cong\big(G^{CS}_{-i}\big)^*,\quad i=1,2,3.
\end{aligned}
$$

Let us now briefly recall Section 2.3. If $X$ is an analytic subvariety of a complex manifold $Y$, then the normal bundle $N_X$ of $X$ in $Y$ is the quotient $(TY|_X)/TX$ and the co-normal bundle $N^*_X$ is the annihilator of $TX$ inside $T^*X$. In particular, the origin $x_0$ can be viewed as an analytic subvariety of $M$ with local defining equation $X_1=0$, $X_2=0$ and $Y=0$ where the matrices are those as in (4.4). For each $i \geq 1$ there is the associated ($i$-th power of the) ideal sheaf $\mathcal{I}^i_{x_0}$. This is a sheaf of $\mathcal{O}_M$-modules such that

$$(\mathcal{I}_{x_0})^i_x = \begin{cases} (\mathcal{O}_M)_x, & x \neq x_0, \\ \mathcal{F}^i_{x_0}, & x = x_0, \end{cases}$$

where $\mathcal{O}_M$ is the structure sheaf on $M$, $\mathcal{F}^i_x = \{f \in (\mathcal{O}_M)_x \colon j^i_x f = 0\}$ and the subscript $x$ stands for the stalk at $x \in M$ of the corresponding sheaf.

Also recall from Section 2.2 the definition of weighted jets. For each $i \geq 0$, there is a short exact sequence of vector spaces

$$0 \to \mathfrak{F}^{i+1}_{x_0} \to \mathfrak{F}^i_{x_0} \to \mathfrak{gr}^{i+1}_{x_0} \to 0, \tag{7.1}$$

where $\mathfrak{F}^i_{x_0} := \{f \in \mathcal{O}_{x_0} \colon \mathrm{j}^i_{x_0} f = 0\}$. We will view (7.1) also as a short exact sequence of sheaves over $\{x_0\}$.

Put $X_0 := \tau^{-1}(x_0)$. Recall from Lemma 4.1 that $X_0$ is complex manifold which is biholomorphic to the connected component $\mathrm{Gr}^+_h(n,n)$ of $\mathbb{C}^n$ in the Grassmannian of maximal totally isotropic subspaces in $\mathbb{C}^{2n}$.

**Remark 7.1.** If $V^{CS}$ is a holomorphic vector bundle over $CS$, we will for brevity put $V := V^{CS}|_{X_0}$. We also put $\tau_0 := \tau|_{X_0}$.

**Lemma 7.2.**

*(i)* *$X_0$ is a closed analytic subvariety of $CS$ and there is an isomorphism of sheaves $\mathcal{I}_{X_0} \cong \tau^*\mathcal{I}_{x_0}$.*

*(ii)* *There is an isomorphism of vector bundles[9] $TX_0 \cong F$.*

*(iii)* *The normal bundle $N$ of $X_0$ in $CS$ is isomorphic to $\tau_0^* T_{x_0}M$. In particular, $N$ is a trivial holomorphic vector bundle.*

**Proof.** (i) By Lemma 4.2, $\tau^{-1}(\mathcal{X}) = \mathcal{X} \times X_0$ where $\mathcal{X} = (\mathbf{p} \circ \exp)(\mathfrak{g}_-)$. From this the claim easily follows.

(ii) As $X_0 = \tau^{-1}(x_0)$, it is clear that $TX_0 = \ker(T\tau)|_{X_0}$. But we know that $\ker(T\tau) = F$ and the claim follows.

(iii) By definition, $\tau_0^* T_{x_0}M = \{(x,v) \,|\, x \in X_0,\ v \in T_{x_0}M\}$. Hence, there is an obvious projection $TCS|_{X_0} \to \tau_0^* T_{x_0}M$, $(x,v) \mapsto (x, T_x\tau(v))$ which descends to an isomorphism $N \to \tau_0^* T_{x_0}M$. ■

Recall now the linear isomorphisms $\mathfrak{g}_i \cong (G^M_i)_{x_0}$, $i = -2,-1,1,2$, from (4.1). In particular, we can view $\mathfrak{g}_i$ as the fiber of $G^M_i$ over $\{x_0\}$ and thus also as a vector bundle over $\{x_0\}$. We use this point of view in the following definition.

**Definition 7.3.** Put $N^*_i := \tau_0^*\mathfrak{g}_i$, $i = 1,2$ and $\mathfrak{S}^\ell N^* := \tau_0^*\mathfrak{gr}^\ell$, $\ell = 0,1,2,\dots$.

Notice that $N^*_i$, $i = 1,2$ and $\mathfrak{S}^\ell N^*$, $\ell \geq 0$ are by definition trivial holomorphic vector bundles over $X_0$. Recall from the end of Section 2.2 that $gr^\ell_{x_0}$ is the subspace of $\mathfrak{gr}^\ell_{x_0}$ that is isomorphic to $S^\ell\mathfrak{g}_1$.

**Lemma 7.4.** *The co-normal bundle $N^*$ of $X_0$ in $CS$ is isomorphic to $\tau_0^* T^*_{x_0}M$ and the bundle $N^*_2$ is isomorphic to $G_3$. There are short exact sequences of vector bundles*

$$0 \to N^*_2 \to N^* \to N^*_1 \to 0 \qquad \textit{and} \qquad 0 \to G_2 \to N^*_1 \to E^* \to 0 \tag{7.2}$$

[9] Here we use notation set in Remark 7.1.

*over $X_0$. Moreover, for each $\ell \geq 0$ there are isomorphisms of vector bundles*

$$\mathfrak{S}^\ell N^* = \bigoplus_{\ell_1+2\ell_2=\ell} S^{\ell_1}N_1^* \otimes S^{\ell_2}N_2^* \qquad \textit{and} \qquad S^\ell N_1^* = \tau^* gr^\ell_{x_0}. \tag{7.3}$$

**Proof.** There is a canonical injective vector bundle map $\tau_0^* T^*_{x_0}M \to T^*CS$ and a moment of thought shows that its image is contained in $N^*$. By comparing dimensions of both vector bundles, we have $\tau_0^* T^*_{x_0}M \cong N^*$ and thus the first claim. It is clear that $N_2^* = \tau_0^*\mathfrak{g}_2$ is the annihilator of $F_{-2} = (T\tau)^{-1}(\mathfrak{g}_{-1})$ and since $G_3 = F^\perp_{-2}$, the second claim follows.

The first sequence in (7.2) is the pullback of the short exact sequence $0 \to \mathfrak{g}_2 \to T^*_{x_0}M \to \mathfrak{g}_1 \to 0$ and thus, it is short exact. The exactness of the latter sequence follows from the exactness of $0 \to G^{CS}_2 \to F^\perp/G^{CS}_3 \to E^{CS*} \to 0$ and the isomorphisms $N^* \cong F^\perp$, $G_3 \cong N_2^*$ and $N_1^* \cong N^*/N_2^*$.

The isomorphisms in (7.3) follow immediately from definitions and the isomorphism (4.2). ■

We know that $\mathfrak{S}^\ell N^*$ is a trivial holomorphic vector bundle over the compact base $X_0$. It follows that any global holomorphic section of $\mathfrak{S}^\ell N^*$ is a constant $\mathfrak{gr}^\ell$-valued function on $X_0$ and that $\mathfrak{S}^\ell N^*$ is trivialized by such sections. The same is obviously true also for $S^\ell N_1^*$. Let us formulate this as lemma.

**Lemma 7.5.** *The holomorphic vector bundles $\mathfrak{S}^\ell N^*$ and $S^\ell N_1^*$ are trivial and there are canonical isomorphisms $\mathfrak{gr}^\ell_{x_0} \to \Gamma(\mathcal{O}(\mathfrak{S}^\ell N^*))$ and $gr^\ell_{x_0} \to \Gamma(\mathcal{O}(S^\ell N_1^*))$ of finite-dimensional vector spaces.*

Let us finish this section by recalling the concept of formal neighborhoods (see [1, 26]). Let $\iota_0 \colon X_0 \hookrightarrow CS$ be the inclusion. Then $\mathcal{F}_{X_0} := \iota_0^{-1}\mathcal{I}_{X_0}$ is a sheaf of $\mathcal{O}_{X_0}$-modules whose stalk at $x \in X_0$ is the space of germs of holomorphic functions which are defined on some open neighborhood $\mathcal{V}$ of $x$ in $CS$ and which vanish on $\mathcal{V} \cap X_0$. Let us now view the vector space $\mathcal{F}_{x_0} = \{f \in (\mathcal{O}_M)_{x_0} \colon f(x_0) = 0\}$ also as a sheaf over $\{x_0\}$. Then (recall from Lemma 7.2) it is easy to see that $\mathcal{F}_{X_0} = \tau_0^*\mathcal{F}_{x_0}$. Observe that $\Gamma(\mathcal{F}_{X_0})$ is the space of equivalence classes of holomorphic functions which are defined on an open neighborhood of $X_0$ in $CS$ where two such functions belong to the same equivalence class if they agree on some possibly smaller open neighborhood of $X_0$.

The infinite-dimensional vector spaces from (7.1) form a decreasing filtration $\cdots \subset \mathfrak{F}^{i+1}_{x_0} \subset \mathfrak{F}^i_{x_0} \subset \cdots$ of $\mathcal{F}_{x_0} = \mathfrak{F}^0_{x_0}$. Then $\mathfrak{F}^i_{X_0} := \tau^*\mathfrak{F}^i_{x_0}$ is a sheaf of $\mathcal{O}_{X_0}$-modules which is naturally a sub-sheaf of $\mathcal{F}_{X_0}$. This induces a filtration $\cdots \subset \mathfrak{F}^{i+1}_{X_0} \subset \mathfrak{F}^i_{X_0} \subset \cdots$ of $\mathcal{F}_{X_0} = \mathfrak{F}^0_{X_0}$. Arguing as in Section 2.3, one can show that for each $i \geq 0$ there is a short exact sequence of sheaves

$$0 \to \mathfrak{F}^{i+1}_{X_0} \to \mathfrak{F}^i_{X_0} \to \mathcal{O}\big(\mathfrak{S}^{i+1}N^*\big) \to 0$$

and thus, the graded sheaf associated to the filtration $\mathcal{F}_{X_0}$ is isomorphic to $\bigoplus_{i\geq 1}\mathcal{O}(\mathfrak{S}^i N^*)$. Using the analogy with the classical formal neighborhood, we will call the pair $(X, \mathcal{O}^{(i)}_X)$ where $\mathcal{O}^{(i)}_X := \iota_0^{-1}\mathcal{O}_{CS}/\mathfrak{F}^{i+1}_{X_0}$ the *$i$-th weighted formal neighborhood of* $X_0$. Notice that the filtration $\{\mathfrak{F}^i_{X_0} \colon i = 0, 1, 2, \dots\}$ descends to a filtration of $\mathcal{O}^{(i)}_X$ and that the associated graded sheaf is isomorphic to $\bigoplus_{j\geq 0}^{i+1}\mathcal{O}(\mathfrak{S}^j N^*)$.

## 7.2 The double complex on the formal neighborhood of $\tau^{-1}(x_0)$

Recall from Section 5.4 that for each $a \in \mathbb{N}^{k,n}_{++}$ there is a Q-dominant and integral weight $\lambda_a$, an irreducible Q-module $\mathbb{W}_{\lambda_a}$ with lowest weight $-\lambda_a$ and an associated vector bundle $W^{CS}_{\lambda_a} = \mathrm{G} \times_{\mathrm{Q}} \mathbb{W}_{\lambda_a}$. We will denote by $W_{\lambda_a}$ the restriction of $W^{CS}_{\lambda_a}$ to $X_0$, by $\mathcal{O}(\lambda_a)$ the sheaf of holomorphic sections of $W_{\lambda_a}$, by $\mathcal{E}^{p,q}$ the sheaf of smooth $(p,q)$-forms over $X_0$ and by $\mathcal{E}^{p,q}(\lambda_a)$

the sheaf of $(p,q)$-forms with values in $W_{\lambda_a}$. If $V$ is another vector bundle over $X_0$, then we denote by $V(\lambda_a)$ the tensor product of $V$ with $W_{\lambda_a}$. We will use the notation set in (2.2) and (2.3).

**Lemma 7.6.** *There is for each $r := \ell + j \geq 0$ a long exact sequence a vector bundles over $X_0$:*

$$\mathfrak{S}^r N^*(\lambda) \xrightarrow{\mathfrak{d}_0} E^* \otimes \mathfrak{S}^{r-1} N^*(\lambda) \xrightarrow{\mathfrak{d}_1} \Lambda^2 E^* \otimes \mathfrak{S}^{r-2} N^*(\lambda) \xrightarrow{\mathfrak{d}_2} \cdots . \tag{7.4}$$

*This sequence contains a long exact subsequence*

$$S^r N_1^*(\lambda) \xrightarrow{\delta_0} E^* \otimes S^{r-1} N_1^*(\lambda) \xrightarrow{\delta_1} \Lambda^2 E^* \otimes S^{r-2} N_1^*(\lambda) \xrightarrow{\delta_2} \cdots . \tag{7.5}$$

**Proof.** In order to obtain the sequence (7.4), take the direct sum of all long exact sequences from (5.7) indexed by $s_0$, $s_1$, $s_2$ and $s_3$ where $s_0 + s_2 + 2s_3 = \ell + j$, $s_1 = 0$ and restrict it to $X_0$. The subsequence (7.5) is obtained similarly, we only add one more condition $s_3 = 0$. ■

Recall that each long exact sequence from (5.7) is induced by the relative twisted de Rham complex by restricting to weighted jets. Hence, also (7.4) and (7.5) are naturally induced by this complex.

**Remark 7.7.** Let $\mathcal{E}^{0,q}(\Lambda^j E^* \otimes \mathfrak{S}^\ell N^*(\lambda))$ be the sheaf of smooth $(0,q)$-forms with values in the corresponding vector bundle over $X_0$. The vector bundle map $\mathfrak{d}_j$ induces a map of sheaves

$$\mathcal{E}^{0,q}\big(\Lambda^j E^* \otimes \mathfrak{S}^\ell N^*(\lambda)\big) \to \mathcal{E}^{0,q}\big(\Lambda^{j+1} E^* \otimes \mathfrak{S}^{\ell-1} N^*(\lambda)\big), \tag{7.6}$$

which we also denote by $\mathfrak{d}_j$ as there is no risk of confusion.

Recall from (5.8) that $\Lambda^j \mathbb{E}^* \otimes \mathbb{W}_\lambda = \bigoplus_{a \in \mathbb{N}^{k,n}_{++}:\, |a|=j} \mathbb{W}_{\lambda_a}$ which gives direct sum decomposition $\mathcal{E}^{0,q}(\Lambda^j E^* \otimes \mathfrak{S}^\ell N^*(\lambda)) = \bigoplus_{a \in \mathbb{N}^{k,n}_{++}:\, |a|=j} \mathcal{E}^{0,q}(\mathfrak{S}^\ell N^*(\lambda_a))$. We see that if $a, a' \in \mathbb{N}^{k,n}_{++}$ are such that $|a| = |a'| - 1 = j$, then $\mathfrak{d}_j$ induces

$$\mathfrak{d}^a_{a'}\colon\ \mathcal{E}^{0,q}\big(\mathfrak{S}^\ell N^*(\lambda_a)\big) \to \mathcal{E}^{0,q}\big(\mathfrak{S}^{\ell-1} N^*(\lambda_{a'})\big) \tag{7.7}$$

in the same way $\partial_\eta$ induces in (5.9) the operator $\partial^a_{a'}$ in the relative BGG sequence. By Proposition 5.5, $\mathfrak{d}^a_{a'} = 0$ if $a \not< a'$.

**Remark 7.8.** Replacing (7.4) by (7.5) in Remark 7.7, we get a map of sheaves

$$\delta_j\colon\ \mathcal{E}^{0,q}\big(\Lambda^j E^* \otimes S^\ell N_1^*(\lambda)\big) \to \mathcal{E}^{0,q}\big(\Lambda^{j+1} E^* \otimes S^{\ell-1} N_1^*(\lambda)\big). \tag{7.8}$$

If $a, a'$ are as above, then there is a map

$$\delta^a_{a'}\colon\ \mathcal{E}^{0,q}\big(S^\ell N_1^*(\lambda_a)\big) \to \mathcal{E}^{0,q}\big(S^{\ell-1} N_1^*(\lambda_{a'})\big),$$

which is induced in the same way $\mathfrak{d}_j$ induces $\mathfrak{d}^a_{a'}$.

Even though the proof of Lemma 7.9 is trivial, it will be crucial later on.

**Lemma 7.9.** *Let $a \in \mathbb{N}^{k,n}_{++}$. Then*

$$(\tau_0)^q_*\big(\mathcal{O}\big(\mathfrak{S}^\ell N^*(\lambda_a)\big)\big) = H^q\big(X_0, \mathcal{O}\big(\mathfrak{S}^\ell N^*(\lambda_a)\big)\big) = \begin{cases} \mathfrak{gr}^\ell \mathbb{V}_{\mu_a}, \\ \{0\} \end{cases} \tag{7.9}$$

*and*

$$(\tau_0)^q_*\big(\mathcal{O}(S^\ell N_1^*(\lambda_a))\big) = H^q\big(X_0, \mathcal{O}(S^\ell N_1^*(\lambda_a))\big) = \begin{cases} gr^\ell \mathbb{V}_{\mu_a}, \\ \{0\}, \end{cases} \tag{7.10}$$

*where*[10] *in (7.9) and (7.10) the first possibility holds if and only if $a \in S^k$ and $q = \ell(a)$.*

[10] As above, we identify a sheaf over $\{x_0\}$ with its stalk.

**Proof.** The first equality in (7.9) is just the definition of $(\tau_0)^q_*$. The sheaf cohomology group in the middle is equal to the cohomology of the Dolbeault complex. In view of Lemma 7.5, $\Gamma(\mathcal{E}^{0,q}(\mathfrak{S}^\ell N^*(\lambda_a))) \cong \mathfrak{gr}^\ell_{x_0} \otimes \Gamma(\mathcal{E}^{0,q}(\lambda_a))$ and thus, the sheaf cohomology group is isomorphic to $\mathfrak{gr}^\ell_{x_0} \otimes H^q(X_0, \mathcal{O}(\lambda_a))$. By the Bott–Borel–Weil theorem, $H^q(X_0, \mathcal{O}(\lambda_a)) \cong \mathbb{V}_{\mu_a}$ if $a \in S^k$, $q = \ell(a)$ and vanishes otherwise. The second equality in (7.9) then follows from the isomorphism $\mathfrak{gr}^\ell_{x_0} \otimes \mathbb{V}_{\mu_a} \to \mathfrak{gr}^\ell \mathbb{V}_{\mu_a}$ from (2.3).

The isomorphism in (7.10) is proved similarly. We only use the other isomorphism

$$\Gamma\big(\mathcal{O}\big(S^\ell N_1^*\big)\big) \to gr^\ell_{x_0}$$

from Lemma 7.5 and the isomorphism $gr^\ell_{x_0} \otimes \mathbb{V}_{\mu_a} \to gr^\ell \mathbb{V}_{\mu_a}$. ∎

There is for each non-negative integer a certain double complex whose horizontal differential is (7.6) and the vertical differential is (up to sign) the Dolbeault differential. This is the double complex from Proposition 5.3 restricted to the weighted formal neighborhood of $X_0$.

**Proposition 7.10.** *Let $r \geq 0$ be an integer. Then there is a double complex $(\mathfrak{E}^{p,q}(r), d', d'')$ where:*

- $\mathfrak{E}^{p,q}(r) = \Gamma(\mathcal{E}^{0,q}(\Lambda^p E^* \otimes \mathfrak{S}^{r-p} N^*(\lambda)))$,
- *the vertical differential $d'$ is $(-1)^p \bar\partial$ where $\bar\partial$ is the standard Dolbeault differential and*
- *the horizontal differential $d''$ is $\mathfrak{d}_p$ from (7.6).*

*Moreover, we claim that:*

*(i)* $H^j(T^*(r), d' + d'') = 0$ *if* $j > \binom{n}{2}$ *where* $T^i(r) := \bigoplus_{p+q=i} \mathfrak{E}^{p,q}(r)$*;*

*(ii) the first page of the spectral sequence associated to the filtration by columns is*

$$\mathfrak{E}^{p,q}_1(r) = \bigoplus_{a \in S^k:\ |a|=p,\ \ell(a)=q} \mathfrak{gr}^{r-p} \mathbb{V}_{\mu_a};$$

*(iii) the spectral sequence degenerates on the second page.*

**Proof.** Recall from the proof of Proposition 5.2 that $\mathfrak{gr}\partial_\eta$ is induced from $\partial_\eta$ by passing to weighted jets (as explained at the end of Section 2.2) and, see Lemma 7.6, that $\mathfrak{d} = \mathfrak{d}_p$ is the restriction of the map $\mathfrak{gr}\partial_\eta$ to the sub-complex (7.4). Since $[\partial_\eta, \bar\partial] = 0$, we have that $[\mathfrak{d}, \bar\partial] = 0$ and thus also $d'd'' = -d''d'$. This shows the first claim.

(i) The rows of the double complex are exact as the sequence (7.4) is exact. Since $\dim X_0 = \binom{n}{2}$, it follows that $\mathfrak{E}^{p,q}_1(r) = 0$ whenever $q > \binom{n}{2}$. This proves the claim.

(ii) By definition, $\mathfrak{E}^{p,q}_1(r)$ is the $d'$-cohomology group in the $p$-th row and $q$-th column. The claim then follows from the direct sum decomposition from Remark 7.7 and Lemma 7.9.

(iii) The space $\mathfrak{gr}^{r-p}\mathbb{V}_{\mu_a}$ lives on the $|a|$-th vertical line and $\ell(a) = (\binom{n}{2} - q(a))$-th horizontal line of the first page of the spectral sequence and thus, on the $(|a| + \binom{n}{2} - q(a)) = (2q(a) + d(a) + \binom{n}{2} - q(a)) = (r(a) + \binom{n}{2})$-th diagonal. Choose $a' \in S^k$ such that $\mathfrak{gr}^{r-|a'|}\mathbb{V}_{\mu_{a'}}$ lives on the next diagonal and $a < a'$. This means that $r(a') = r(a) + 1$ and so $q(a) = q(a')$ or $q(a') = q(a) + 1$. In the first case, $\mathfrak{gr}^{r-|a'|}\mathbb{V}_{\mu_{a'}}$ lives on the $\ell(a)$-th row. In the second case, it lives on the $(\ell(a) - 1)$-th row. As $\mathfrak{d}^a_{a'} = 0$ if $a \not< a'$, it follows from definition that the differential on the $i$-th page is zero if $i > 2$. ∎

If we use the exactness of (7.5) instead of (7.4) and use the isomorphism (7.10) instead of (7.9), the proof of Proposition 7.10 gives the following.

**Proposition 7.11.** *The double complex from Proposition 7.10 contains a double complex $(F^{p,q}(r), d', d'')$ where $F^{p,q}(r) := \Gamma(\mathcal{E}^{0,q}(\Lambda^p E^* \otimes S^{r-p} N_1^*(\lambda)))$. Moreover we claim that:*

*(i)* $H^j(T^*(r), d' + d'') = 0$ *if* $j > \binom{n}{2}$ *where* $T^i(r) := \bigoplus_{p+q=i} F^{p,q}(r)$*;*

*(ii) the first page of the spectral sequence associated to the filtration by columns is*

$$F_1^{p,q}(r) := \bigoplus_{a \in S^k:\ |a|=p,\ \ell(a)=q} gr^{r-p}\mathbb{V}_{\mu_a};$$

*(iii) the spectral sequence degenerates on the second page.*

### 7.3 Long exact sequence of weighted jets

Let $a \in S^k$ and $\mathbb{V}_{\mu_a}$ be an irreducible P-module with lowest weight $-\mu_a$, see Proposition 5.7. Now we are ready to show that the linear operators defined in Lemma 6.1 are differential operators and we give an upper bound on their weighted order.

**Lemma 7.12.** *Let $a, a' \in S^k$ be such that $a < a'$ and $r(a') = r(a) + 1$. Then the operator $D^a_{a'}$ from Lemma 6.1 is a differential operator of weighted order at most $s := |a'| - |a|$.*

*Hence, $D^a_{a'}$ induces for each $i \geq 0$ a linear map*

$$\mathfrak{gr}D^a_{a'}:\ \mathfrak{gr}^i\mathbb{V}_{\mu_a} \to \mathfrak{gr}^{i-s}\mathbb{V}_{\mu_{a'}}, \tag{7.11}$$

*which restricts to a linear map*

$$grD^a_{a'}:\ gr^i\mathbb{V}_{\mu_a} \to gr^{i-s}\mathbb{V}_{\mu_{a'}}. \tag{7.12}$$

**Proof.** Let us make a few preliminary observations. Let $v \in \mathcal{O}_{\mathfrak{p}}(\mu_a)_{x_0}$. By the G-invariance of $D^a_{a'}$, it is obviously enough to show that $(D^{a'}_a v)(x_0)$ depends only on $\mathfrak{j}^s_{x_0} v$. We may assume that $v$ is defined on the Stein set $\mathcal{U}$ from Section 6 and so we can view $v$ as a cohomology class $[\alpha] = H^{\ell(a)}(\tau^{-1}(\mathcal{U}), \mathcal{O}_{\mathfrak{q}}(\lambda_a))$. A choice of Weyl structure (see [8]) and the isomorphisms (7.9) give for each integer $i \geq 0$ isomorphisms

$$\mathfrak{J}^i\mathbb{V}_{\mu_a} \to \bigoplus_{j=0}^{i} \mathfrak{gr}^j\mathbb{V}_{\mu_a} \to \bigoplus_{j=0}^{i} H^{\ell(a)}\big(X_0, \mathfrak{S}^j N^*(\lambda_a)\big).$$

Hence, the Taylor series of $v$ at $x_0$ determines an infinite[11] sum $\sum_{j=0}^{\infty}[v_j]$ where each $[v_j]$ belong to $H^{\ell(a)}(X_0, \mathfrak{S}^j(\lambda_a))$.

Now we can proceed with the proof. By assumption, $s \in \{1, 2\}$. If $s = 1$, then $\ell(a) = \ell(a')$. By definition, $D^a_{a'}v$ corresponds to $[\partial^a_{a'}\alpha] \in H^{\ell(a')}(\tau^{-1}(U), \mathcal{O}_{\mathfrak{q}}(\lambda_{a'}))$ and $\mathfrak{j}^i_{x_0}(D^a_{a'}v)$ can be viewed as $\sum_{j=0}^{i}[(\mathfrak{d}^a_{a'})v_{j+1}]$. But since $[\mathfrak{d}^a_{a'}(v_j)] \in H^{\ell(a)}(X_0, \mathfrak{S}^{j-1}(\lambda_a))$, it is clear that $D^a_{a'}(v)(x_0) = 0$ if $\mathfrak{j}^1_{x_0}v = 0$. This completes the proof when $s = 1$.

Notice that the linear map $\mathfrak{gr}D^a_{a'}$ fits into a commutative diagram

$$\begin{tikzcd}
\Gamma\big(\mathcal{E}^{0,\ell(a)}\big(\mathfrak{S}^i N^*(\lambda_a)\big)\big) \cap \ker\bar\partial \arrow[r, "\mathfrak{d}^a_{a'}"] \arrow[d] & \Gamma\big(\mathcal{E}^{0,\ell(a')}\big(\mathfrak{S}^{i-1} N^*(\lambda_{a'})\big)\big) \cap \ker\bar\partial \arrow[d] \\
H^{\ell(a)}\big(X_0, \mathfrak{S}^i N^*(\lambda_a)\big) \arrow[r] & H^{\ell(a')}\big(X_0, \mathfrak{S}^{i-1} N^*(\lambda_{a'})\big) \\
\mathfrak{gr}^i\mathbb{V}_{\mu_a} \arrow[r, "\mathfrak{gr}D^a_{a'}"] \arrow[u] & \mathfrak{gr}^{i-1}\mathbb{V}_{\mu_{a'}}, \arrow[u]
\end{tikzcd} \tag{7.13}$$

[11] We will at this point avoid discussion about the convergence of the sum as we will not need it.

where the lower vertical arrows are the isomorphisms from Lemma 7.9, the upper vertical arrows are the canonical projections and the map $\mathfrak{d}^a_{a'}$ is the one from (7.7).

Let us now assume $s=2$. In view of the diagram (6.1), we have to replace in (7.13) the map $\mathfrak{d}^a_{a'}$ by the diagram

$$\begin{tikzcd}
\Gamma(\mathcal{E}^{0,q}(\mathfrak{S}^i N^*(\lambda_a)))\cap \mathrm{Ker}(\bar\partial) \arrow[r, "(\mathfrak{d}^a_b)\oplus(\mathfrak{d}^a_c)"] & \Gamma(\mathcal{E}^{0,q}(\mathfrak{S}^{i-1}N^*(\lambda_b\oplus\lambda_c))) \\
\Gamma(\mathcal{E}^{0,q-1}(\mathfrak{S}^{i-1}N^*(\lambda_b\oplus\lambda_c))) \arrow[ur, "\bar\partial"] \arrow[r, "(\mathfrak{d}^b_{a'})+(\mathfrak{d}^c_{a'})"] & \Gamma(\mathcal{E}^{0,q-1}(\mathfrak{S}^{i-2}N^*(\lambda_{a'})))\cap \mathrm{Ker}(\bar\partial),
\end{tikzcd}$$

where we for brevity put $\mathfrak{S}^\bullet N^*(\lambda_b\oplus\lambda_c):=\mathfrak{S}^\bullet N^*(\lambda_b)\oplus\mathfrak{S}^\bullet N^*(\lambda_c)$. Following the same line of arguments as in the case $s=1$, we easily find that $D^a_{a'}(v)(x_0)=0$ whenever $\mathfrak{j}^2_{x_0}v=0$.

In order to prove the claim about $grD^a_{a'}$, we need to replace everywhere $\mathfrak{d}^a_{a'}$ by its restriction $\delta^a_{a'}$ and use (7.10) instead of (7.9). ■

In order to get rid of the factor $s$ in (7.11) and (7.12), we shift the gradings by introducing $\mathfrak{gr}^i\mathbb{V}_{\mu_a}[\uparrow]:=\mathfrak{gr}^{i-q(a)}\mathbb{V}_{\mu_a}$ and $gr^i\mathbb{V}_{\mu_a}[\uparrow]:=gr^{i-q(a)}\mathbb{V}_{\mu_a}$. We can now rewrite the maps from (7.11) and (7.12) as

$$\mathfrak{gr}D^a_{a'}\colon\ \mathfrak{gr}^\ell\mathbb{V}_{\mu_a}[\uparrow]\to\mathfrak{gr}^{\ell-1}\mathbb{V}_{\mu_{a'}}[\uparrow]\qquad\text{and}\qquad grD^a_{a'}\colon\ gr^\ell\mathbb{V}_{\mu_a}[\uparrow]\to gr^{\ell-1}\mathbb{V}_{\mu_{a'}}[\uparrow],$$

respectively, where $\ell\geq 0$ is the corresponding integer. We also put

$$\mathfrak{gr}^\ell\mathbb{V}_{j,i}[\uparrow]=\bigoplus_{a\in S^k_j\colon q(a)=i}\mathfrak{gr}^\ell\mathbb{V}_{\mu_a}[\uparrow],\qquad \mathfrak{gr}^\ell\mathbb{V}_j[\uparrow]=\bigoplus_{i=0}^{j}\mathfrak{gr}^\ell\mathbb{V}_{j,i}[\uparrow]$$

and

$$gr^\ell\mathbb{V}_{j,i}[\uparrow]=\bigoplus_{a\in S^k_j\colon q(a)=i}gr^\ell\mathbb{V}_{\mu_a}[\uparrow],\qquad gr^\ell\mathbb{V}_j[\uparrow]=\bigoplus_{i=0}^{j}gr^\ell\mathbb{V}_{j,i}[\uparrow].$$

We view $\mathfrak{gr}D^a_{a'}$ also as a map $\mathfrak{gr}^\ell\mathbb{V}_j[\uparrow]\to\mathfrak{gr}^{\ell-1}\mathbb{V}_{j+1}[\uparrow]$ by extending it from $\mathfrak{gr}^\ell\mathbb{V}_{\mu_a}[\uparrow]$ by zero to all the other summands. We put

$$\mathfrak{gr}D_j:=\sum_{a\in S^k_j,\ a'\in S^k_{j+1}\colon a<a'}\mathfrak{gr}D^a_{a'}\colon\ \mathfrak{gr}^\ell\mathbb{V}_j[\uparrow]\to\mathfrak{gr}^{\ell-1}\mathbb{V}_{j+1}[\uparrow]$$

and

$$\mathfrak{gr}(D_j)^i_{i'}\colon\ \mathfrak{gr}^\ell\mathbb{V}_{j,i}[\uparrow]\to\mathfrak{gr}^\ell\mathbb{V}_j[\uparrow]\xrightarrow{\mathfrak{gr}D_j}\mathfrak{gr}^{\ell-1}\mathbb{V}_{j+1}[\uparrow]\to\mathfrak{gr}^{\ell-1}\mathbb{V}_{j+1,i'}[\uparrow],$$

where the first map is the canonical inclusion and the last map is the canonical projection.

Recall from Section 3.2 that if $a<a'$, $a\in S^k_j$, $a'\in S^k_{j+1}$, then $q(a')\leq q(a)+1$. This implies that $\mathfrak{gr}(D_j)^i_{i'}=0$ if $i\neq i'$ or $i'\neq i+1$. Then $\mathfrak{gr}D_j$ is

$$\begin{pmatrix}\mathfrak{gr}^\ell\mathbb{V}_{j,0}[\uparrow]\\ \oplus\\ \mathfrak{gr}^\ell\mathbb{V}_{j,1}[\uparrow]\\ \oplus\\ \dots\end{pmatrix}\begin{matrix}\longrightarrow\\ \searrow\\ \longrightarrow\\ \searrow\\ \dots\end{matrix}\begin{pmatrix}\mathfrak{gr}^{\ell-1}\mathbb{V}_{j+1,0}[\uparrow]\\ \oplus\\ \mathfrak{gr}^{\ell-1}\mathbb{V}_{j+1,1}[\uparrow]\\ \oplus\\ \dots\end{pmatrix},$$

where the horizontal arrows and the diagonal arrows are $\mathfrak{gr}(D_j)^i_i$ and $\mathfrak{gr}(D_j)^i_{i+1}$, respectively.

We similarly define linear maps $grD_j\colon gr^\ell\mathbb{V}_j[\uparrow]\to gr^{\ell-1}\mathbb{V}_{j+1}[\uparrow]$ and $gr(D_j)^i_{i'}\colon gr^\ell\mathbb{V}_{j,i}[\uparrow]\to gr^{\ell-1}\mathbb{V}_{j+1,i'}[\uparrow]$.

**Remark 7.13.** Notice that

$$\begin{aligned}\mathfrak{gr}^\ell\mathbb{V}_{j,i}[\uparrow] &= \bigoplus_{a\in S^k_j:\ q(a)=i} \mathfrak{gr}^\ell\mathbb{V}_{\mu_a}[\uparrow] = \bigoplus_{a\in S^k:\ r(a)=j,\ q(a)=i} \mathfrak{gr}^{\ell-q(a)}\mathbb{V}_{\mu_a}\\ &= \bigoplus_{a\in S^k:\ |a|=i+j,\ \ell(a)=\binom{n}{2}-i} \mathfrak{gr}^{\ell-i}\mathbb{V}_{\mu_a} = \mathfrak{E}_1^{i+j,\binom{n}{2}-i}(\ell+j).\end{aligned}$$

Put $p := i+j$, $q := \binom{n}{2}-i$ and $r := \ell+j$. Then we can view $\mathfrak{gr}(D_j)^i_i$ and $\mathfrak{gr}(D_j)^i_{i+1}$ as maps

$$\mathfrak{E}_1^{p,q}(r) \to \mathfrak{E}_1^{p+1,q}(r) \qquad \text{and} \qquad \mathfrak{E}_1^{p,q}(r) \to \mathfrak{E}_1^{p+2,q-1}(r),$$

respectively. By the definition of $\mathfrak{gr}(D_j)^i_i$ from Lemma 7.12, it follows that we can view it as the differential $d_1$ on the first page of the spectral sequence from Proposition 7.10.

Suppose that $v \in \mathfrak{E}_1^{p,q}(s)$ satisfies $d_1(v) = 0$. Then we can apply the differential $d_2$ living on the second page to $v + \mathrm{im}(d_1)$ and, comparing this with the definition of $\mathfrak{gr}(D_j)^i_{i+1}$ from Lemma 7.12, we find that

$$d_2(v + \mathrm{im}(d_1)) = \mathfrak{gr}(D_j)^i_{i+1}(v) + \mathrm{im}(d_1). \tag{7.14}$$

Similarly we find that $gr^\ell\mathbb{V}_{j,i} = F_1^{p,q}(s)$ where $p$, $q$ and $s$ are as above. Moreover we can view $gr(D_j)^i_i$ and $gr(D_j)^i_{i+1}$ as maps

$$F_1^{p,q}(s) \to F_1^{p+1,q}(s) \qquad \text{and} \qquad F_1^{p,q}(s) \to F_1^{p+2,q-1}(s),$$

respectively. As the double complex from Proposition 7.11 is a sub-complex of the double complex from Proposition 7.10 and $gr(D_j)^i_{i'}$ is the restriction of $\mathfrak{gr}(D_j)^i_{i'}$ to the corresponding subspace, we see that $gr(D_j)^i_i$ coincides with the differential on the first page of the spectral sequence from Proposition 7.11 and that $gr(D_j)^i_{i+1}$ is related to the differential on the second page just as $\mathfrak{gr}(D_j)^i_{i+1}$ is related to $d_2$.

The exactness of the complex (7.15) for each $\ell + j \geq 0$ implies (see [24]) the exactness of the $k$-Dirac complex at the level of infinite weighted jets at any fixed point. Following [25], we say that the $k$-Dirac complex is formally exact. Notice that for application in [24], the exactness of the sub-complex (7.16) for each $\ell + j \geq 0$ is a crucial point in the proof of the local exactness of the descended complex and thus, in constructing the resolution of the $k$-Dirac operator.

**Theorem 7.14.** *The $k$-Dirac complex induces for each $\ell + j \geq 0$ a long exact sequence*

$$\mathfrak{gr}^{\ell+j}\mathbb{V}_0[\uparrow] \xrightarrow{\mathfrak{gr}D_0} \mathfrak{gr}^{\ell+j-1}\mathbb{V}_1[\uparrow] \to \dots \to \mathfrak{gr}^\ell\mathbb{V}_j[\uparrow] \xrightarrow{\mathfrak{gr}D_j} \mathfrak{gr}^{\ell-1}\mathbb{V}_{j+1}[\uparrow] \to \cdots \tag{7.15}$$

*of finite-dimensional vector spaces. The complex contains a sub-complex*

$$gr^{\ell+j}\mathbb{V}_0[\uparrow] \xrightarrow{grD_0} gr^{\ell+j-1}\mathbb{V}_1[\uparrow] \to \dots \to gr^\ell\mathbb{V}_j[\uparrow] \xrightarrow{grD_j} gr^{\ell-1}\mathbb{V}_{j+1}[\uparrow] \to \cdots, \tag{7.16}$$

*which is also exact.*

**Proof.** Let $v \in \mathfrak{gr}^\ell\mathbb{V}_j[\uparrow]$, $j \geq 1$ be such that $\mathfrak{gr}D_j(v) = 0$. Write $v = (v_0, \dots, v_j)$ with respect to the decomposition given above, i.e., $v_i \in \mathfrak{gr}^\ell\mathbb{V}_{j,i}[\uparrow]$. Assume that $v_0 = v_1 = \dots = v_{i-1} = 0$ and that $v_i \neq 0$. We have that $\mathfrak{gr}(D^i_i)(v_i) = 0$ and $\mathfrak{gr}(D^i_{i+1})(v_i) + \mathfrak{gr}(D^{i+1}_{i+1})(v_{i+1}) = 0$. If we view $v_i$ as an element of $\mathfrak{E}_1^{p,q}(s)$ as in Remark 7.13, we see that $d_1(v_i) = 0$ and by (7.14), we find that $d_2(v_i) = 0$. By Proposition 7.10, the spectral sequence $\mathfrak{E}^{p,q}(r)$ collapses on the second page and by part (i), we have that $\ker(d_2) = \mathrm{im}(d_2)$ beyond the $\binom{n}{2}$-th diagonal. By Remark 7.13 again,

$\mathfrak{gr}^\ell\mathbb{V}_{j,i}[\uparrow]$ lives on the $(\binom{n}{2}+j)$-th diagonal. We see that there are $t_{i-1}\in\mathfrak{gr}^{\ell+1}\mathbb{V}_{j-1,i-1}[\uparrow]$ and $t_i\in\mathfrak{gr}^{\ell+1}\mathbb{V}_{j-1,i}[\uparrow]$ such that $\mathfrak{gr}(D^{i-1}_{i-1})(t_{i-1})=0$ and $\mathfrak{gr}(D^{i-1}_{i})(t_{i-1})+\mathfrak{gr}(D^{i}_{i})(t_i)=v_i$. Hence, we can kill the lowest non-zero component of $v$ and repeating this argument finitely many times, we see that there is $t\in\mathfrak{gr}^{\ell+1}\mathbb{V}_{j-1}[\uparrow]$ such that $v=\mathfrak{gr}D_{j-1}(t)$.

The proof of the exactness of the second sequence (7.16) proceeds similarly. We only replace $\mathfrak{gr}^\ell\mathbb{V}_j[\uparrow]$ by $gr^\ell\mathbb{V}_j[\uparrow]$, $\mathfrak{gr}^\ell\mathbb{V}_{j,i}[\uparrow]$ by $gr^\ell\mathbb{V}_{j,i}[\uparrow]$, use that the second spectral sequence from Proposition 7.11 has the same key properties as the spectral sequence from Proposition 7.10 and the end of Remark 7.13. ■

### Acknowledgements

The author is grateful to Vladimír Souček for his support and many useful conversations. The author would also like to thank to Lukáš Krump for the possibility of using his package for the Young diagrams. The author wishes to thank to the unknown referees for many helpful suggestions which considerably improved this article. The research was partially supported by the grant 17-01171S of the Grant Agency of the Czech Republic.

## References

[1] Baston R.J., Quaternionic complexes, *J. Geom. Phys.* **8** (1992), 29–52.

[2] Baston R.J., Eastwood M.G., The Penrose transform. Its interaction with representation theory, *Oxford Mathematical Monographs, Oxford Science Publications*, The Clarendon Press, Oxford University Press, New York, 1989.

[3] Bureš J., Damiano A., Sabadini I., Explicit resolutions for the complex of several Fueter operators, *J. Geom. Phys.* **57** (2007), 765–775.

[4] Bureš J., Souček V., Complexes of invariant operators in several quaternionic variables, *Complex Var. Elliptic Equ.* **51** (2006), 463–485.

[5] Čap A., Salač T., Parabolic conformally symplectic structures I: definition and distinguished connections, *Forum Math.*, to appear, arXiv:1605.01161.

[6] Čap A., Salač T., Parabolic conformally symplectic structures II: parabolic contactification, *Ann. Mat. Pura Appl.*, to appear, arXiv:1605.01897.

[7] Čap A., Salač T., Parabolic conformally symplectic structures III: invariant differential operators and complexes, arXiv:1701.01306.

[8] Čap A., Slovák J., Parabolic geometries. I. Background and general theory, *Mathematical Surveys and Monographs*, Vol. 154, Amer. Math. Soc., Providence, RI, 2009.

[9] Čap A., Slovák J., Souček V., Bernstein–Gelfand–Gelfand sequences, *Ann. of Math.* **154** (2001), 97–113, math.DG/0001164.

[10] Čap A., Souček V., Relative BGG sequences: I. Algebra, *J. Algebra* **463** (2016), 188–210, arXiv:1510.03331.

[11] Čap A., Souček V., Relative BGG sequences: II. BGG machinery and invariant operators, *Adv. Math.* **320** (2017), 1009–1062, arXiv:1510.03986.

[12] Colombo F., Sabadini I., Sommen F., Struppa D.C., Analysis of Dirac systems and computational algebra, *Progress in Mathematical Physics*, Vol. 39, Birkhäuser Boston, Inc., Boston, MA, 2004.

[13] Colombo F., Souček V., Struppa D.C., Invariant resolutions for several Fueter operators, *J. Geom. Phys.* **56** (2006), 1175–1191.

[14] Franek P., Generalized Dolbeault sequences in parabolic geometry, *J. Lie Theory* **18** (2008), 757–774, arXiv:0710.0093.

[15] Goodman R., Wallach N.R., Symmetry, representations, and invariants, *Graduate Texts in Mathematics*, Vol. 255, Springer, Dordrecht, 2009.

[16] Hörmander L., An introduction to complex analysis in several variables, D. Van Nostrand Co., Inc., Princeton, N.J. – Toronto, Ont. – London, 1966.

[17] Krump L., A resolution for the Dirac operator in four variables in dimension 6, *Adv. Appl. Clifford Algebr.* **19** (2009), 365–374.

[18] Morimoto T., Lie algebras, geometric structures and differential equations on filtered manifolds, in Lie Groups, Geometric Structures and Differential Equations – One Hundred Years After Sophus Lie (Kyoto/Nara, 1999), *Adv. Stud. Pure Math.*, Vol. 37, Math. Soc. Japan, Tokyo, 2002, 205–252.

[19] Nacinovich M., Complex analysis and complexes of differential operators, in Complex Analysis (Trieste, 1980), *Lecture Notes in Math.*, Vol. 950, Springer, Berlin – New York, 1982, 105–195.

[20] Sabadini I., Struppa D.C., Sommen F., Van Lancker P., Complexes of Dirac operators in Clifford algebras, *Math. Z.* **239** (2002), 293–320.

[21] Salač T., $k$-Dirac operator and the Cartan–Kähler theorem, *Arch. Math. (Brno)* **49** (2013), 333–346, arXiv:1304.0956.

[22] Salač T., $k$-Dirac operator and parabolic geometries, *Complex Anal. Oper. Theory* **8** (2014), 383–408, arXiv:1201.0355.

[23] Salač T., $k$-Dirac operator and the Cartan–Kähler theorem for weighted differential operators, *Differential Geom. Appl.* **49** (2016), 351–371, arXiv:1601.08077.

[24] Salač T., Resolution of the $k$-Dirac operator, *Adv. Appl. Clifford Algebr.* **28** (2018), 28:3, 19 pages, arXiv:1705.10168.

[25] Spencer D.C., Overdetermined systems of linear partial differential equations, *Bull. Amer. Math. Soc.* **75** (1969), 179–239.

[26] Ward R.S., Wells Jr. R.O., Twistor geometry and field theory, *Cambridge Monographs on Mathematical Physics*, Cambridge University Press, Cambridge, 1990.

[27] Wells Jr. R.O., Differential analysis on complex manifolds, *Graduate Texts in Mathematics*, Vol. 65, 2nd ed., Springer-Verlag, New York – Berlin, 1980.