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NNSC-Cobordism of Bartnik Data in High Dimensions

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Abstract. In this short note, we formulate three problems relating to nonnegative scalar curvature (NNSC) fill-ins. Loosely speaking, the first two problems focus on: *When are* $(n-1)$*-dimensional Bartnik data* $\left(\Sigma_i^{n-1}, \gamma_i, H_i\right)$*,* $i = 1, 2$*, NNSC-cobordant?* (i.e., there is an n-dimensional compact Riemannian manifold $\left(\Omega^n, g\right)$ with scalar curvature $R(g) \geq 0$ and the boundary $\partial\Omega = \Sigma_1 \cup \Sigma_2$ such that γ_i is the metric on Σ_i^{n-1} induced by g, and H_i is the mean curvature of Σ_i in $\left(\Omega^n, g\right)$). If $\left(\mathbb{S}^{n-1}, \gamma_{\rm std}, 0\right)$ is positive scalar curvature (PSC) cobordant to $\left(\Sigma_1^{n-1}, \gamma_1, H_1\right)$, where $\left(\mathbb{S}^{n-1}, \gamma_{\rm std}\right)$ denotes the standard round unit sphere then $\left(\Sigma_1^{n-1}, \gamma_1, H_1\right)$ admits an NNSC fill-in. Just as Gromov's conjecture is connected with positive mass theorem, our problems are connected with Penrose inequality, at least in the case of $n = 3$. Our third problem is on $\Lambda\left(\Sigma^{n-1}, \gamma\right)$ defined below.

Key words: scalar curvature; NNSC-cobordism; quasi-local mass; fill-ins

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Dedicate this paper to Professor Misha Gromov
on the occasion of his 75th birthday.

Bartnik data $\left(\Sigma^{n-1}, \gamma, H\right)$ consists of an $(n-1)$-dimensional orientable Riemannian manifold $\left(\Sigma^{n-1}, \gamma\right)$ and a smooth function H defined on Σ^{n-1} which serves as the mean curvature of Σ^{n-1}. One basic problem in Riemannian geometry is to study: *under what conditions is it that* γ *is induced by a Riemannian metric* g *with nonnegative scalar curvature, for example, defined on* Ω^n*, and* H *is the mean curvature of* Σ *in* $\left(\Omega^n, g\right)$ *with respect to the outward unit normal vector?* Indeed, this problem was proposed by M. Gromov recently (see [8, Problem A] and [9, Sections 3.3 and 3.6]).

On the other hand, when $n = 3$, for each Bartnik data $\left(\Sigma^2, \gamma, H\right)$ may be associated with certain quasi-local masses, for instance, when the Gaussian curvature K of γ is positive, $(\mathbb{S}^2, \gamma)$ can be isometrically embedded into $\mathbb{R}^3$ with mean curvature H_0 (with respect to the outward unit normal vector of the embedded image in $\mathbb{R}^3$), with this embedding we may define Brown–York mass for $\left(\mathbb{S}^2, \gamma, H\right)$ [4, 5] as

$$\mathfrak{m}_{\rm BY}\left(\mathbb{S}^2; \gamma, H\right) = \frac{1}{8\pi}\int_{\mathbb{S}^2} (H_0 - H)\, {\rm d}\sigma_\gamma.$$

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If $(\mathbb{S}^2, \gamma, H)$ admits an NNSC fill-in and $H > 0$, it was shown that $\mathfrak{m}_{\mathrm{BY}}(\mathbb{S}^2; \gamma, H) \geq 0$ [22]. There are several pieces of interesting work on NNSC fill-ins relating to positivity of Brown–York mass (for instance see [13, 14]). Obviously, positivity of Brown–York mass is one necessary condition for the existence of such a fill-in, but it is far from sufficient. *It was shown that for Bartnik data $(\mathbb{S}^2, \gamma, H)$ with positive Gaussian curvature and $H > 0$, let H_0 be the mean curvature of isometric embedding of $(\mathbb{S}^2, \gamma)$ in $\mathbb{R}^3$, if $\mathfrak{m}_{\mathrm{BY}}(\mathbb{S}^2; \gamma, H) = 0$ and $H \neq H_0$ then there is a constant ϵ depending only on $(\mathbb{S}^2, \gamma, H)$ such that for any $\tilde{H} > H - \epsilon$, $(\mathbb{S}^2, \gamma, \tilde{H})$ admits no NNSC fill-ins* [14, Theorem 3].

If $K > -\kappa^2$ where κ is a constant, then $(\mathbb{S}^2, \gamma)$ can be isometrically embedded into the hyperbolic space with constant sectional curvature $-\kappa^2$, and we can make use of such embedding to define a generalized Brown–York mass, moreover if $H > 0$ we were able to prove its positivity [25]. Clearly, this positivity of generalized Brown–York mass is also a kind of necessary condition for the Bartnik data with $K > -\kappa^2$ and $H > 0$ to admit NNSC fill-ins.

For Bartnik data (Σ^2, γ, H), we can define its Hawking mass as following:

$$\mathfrak{m}_{\mathrm{H}}(\Sigma, \gamma, H) = \sqrt{\frac{\mathrm{Area}(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \, \mathrm{d}\sigma_\gamma\right).$$

It should be interesting to explore similar relation between Hawking mass or other quasi-local masses of the Bartnik data with its NNSC fill-ins. Unfortunately, it is not easy to obtain a lower bound of the Hawking mass which depends only on (Σ^2, γ).

In the investigation of above Gromov's NNSC fill-in problem, we often need to deal with NNSC-cobordisms of Bartnik data which may have its own interests. More specifically, *given Bartnik data $(\Sigma_i^{n-1}, \gamma_i, H_i)$, $i = 1, 2$, we say $(\Sigma_1^{n-1}, \gamma_1, H_1)$ is NNSC-cobordant to $(\Sigma_2^{n-1}, \gamma_2, H_2)$ if there is an orientable n-dimensional manifold (Ω^n, g) with $\partial\Omega^n = \Sigma_1^{n-1} \cup \Sigma_2^{n-1}$, $R(g) \geq 0$, $\gamma_i = g|_{\Sigma_i}$, $i = 1, 2$, H_1 is the mean curvature of Σ_1^{n-1} in (Ω^n, g) with respect to inward unit normal vector, and H_2 is the mean curvature of Σ_2^{n-1} in (Ω^n, g) with respect to outward unit normal vector.* Our first problem is:

Problem 1. *Given Bartnik data $(\Sigma_i^{n-1}, \gamma_i, H_i)$, $i = 1, 2$, when are they NNSC-cobordant?*

By using surgery arguments (see [10, 21]), it is not difficult to show that if Bartnik data $(\Sigma_i^{n-1}, \gamma_i, H_i)$, $i = 1, 2$ can be filled in with positive scalar curvature metrics, then $(\Sigma_1^{n-1}, \gamma_1, -H_1)$ is NNSC-cobordant to $(\Sigma_2^{n-1}, \gamma_2, H_2)$. Another possible relevant notion to this is so called "PSC-concordant". Namely, *two PSC-metrics γ_0 and γ_1 on Σ^{n-1} are said to be PSC-concordant if there is a PSC-metric g on the cylinder $\Sigma \times I$ which are the product $\gamma_0 + \mathrm{d}t^2$ near $\Sigma \times \{0\}$ and $\gamma_1 + \mathrm{d}t^2$ near $\Sigma \times \{1\}$* (see [28]), in that case, $(\Sigma^{n-1}, \gamma_0, 0)$ is NNSC-cobordant to $(\Sigma^{n-1}, \gamma_1, 0)$. By index theory, it is known that there are countable infinity distinct PSC-concordant classes for $\mathbb{S}^{4k-1}$, for any positive integer $k \geq 2$. When two PSC-metrics γ_0 and γ_1 are isotopic, i.e., they can be connected by a continuous path γ_t, $t \in [0, 1]$, and for each $t \in [0, 1]$, γ_t is a PSC-metric. Then we may use quasi-spherical metric to show that *if H_1 is not too large then $(\mathbb{S}^2, \gamma_0, H_0)$ is NNSC-cobordant to $(\mathbb{S}^2, \gamma_1, H_1)$, here H_0 can be any given smooth positive function* (see [1, 22, 23]). On the other hand, *when H_1 is large enough we are able to show $(\mathbb{S}^2, \gamma_i, H_i)$, $i = 0, 1$, cannot be NNSC-cobordant* [2].

Let γ_0 be a Riemannian metric on $\mathbb{S}^2$ with its first eigenvalue $\lambda_1(-\Delta_0 + K) > 0$, here Δ_0 is the Laplacian operator of γ_0, then it was shown in [18] *that $(\mathbb{S}^2, \gamma_0, 0)$ is NNSC-cobordant to $(\mathbb{S}^2, \gamma_{\mathrm{rou}}, H)$ provided $\mathfrak{m}_{\mathrm{H}}(\mathbb{S}^2, \gamma_{\mathrm{rou}}, H) > \sqrt{\frac{\mathrm{Area}(\mathbb{S}^2, \gamma_0)}{16\pi}}$, here γ_{rou} denotes the round metric on $\mathbb{S}^2$*. For a generalization to the case of Bartnik data with constant mean curvature surfaces see [6, Theorem 1.1], and higher-dimensional analogues see [7, Theorems 1.1 and 1.2], and [19, Proposition 2.1]. An NNSC fill-in by a conformal blow-down argument which may have deep relation to Problem 1 please see the proof of Theorem 1.2 in [11]. For deep discussion on PSC-concordant relation for two PSC-metrics on a manifold from topological point of view, please see [29, 30] and references therein.

As we mentioned above, one obstruction of the above NNSC fill-in problem is from positivity of certain quasi-local mass (for instance, Brown–York mass, see [22, 26]). It may be reasonable to think that there may be a potential obstruction of NNSC-cobordism problem which is from Penrose-type inequality (for Penrose inequality, see [3, 12], for local Penrose inequality, see [15, 20, 24, 27]). For instance, we observed that *if* $\left(\mathbb{S}^2, \gamma_2, H_2\right)$ *is with positive Gaussian curvature and* $H_2 > 0$, *and* $\left(\Sigma_1^2, \gamma_1, H_1\right)$ *is NNSC-cobordant to* $\left(\mathbb{S}^2, \gamma_2, H_2\right)$, *then* $\mathfrak{m}_{\mathrm{BY}}\left(\mathbb{S}^2; \gamma_2, H_2\right) \geq \mathfrak{m}_{\mathrm{H}}\left(\Sigma_1^2, \gamma_1, H_1\right)$ *provided* $\mathfrak{m}_{\mathrm{H}}\left(\Sigma_1^2, \gamma_1, H_1\right) \leq 0$ [2].

To our knowledge, even the following simple case is still unknown:

Problem 2. *Given Bartnik data* $\left(\mathbb{S}^{n-1}, g_1, H\right)$ *and* $\left(\mathbb{S}^{n-1}, g_0, 0\right)$, *both are with positive scalar curvature, what is the largest* $\inf\limits_{\mathbb{S}^{n-1}} H$ *so that* $\left(\mathbb{S}^{n-1}, g_0, 0\right)$ *is NNSC-cobordant to* $\left(\mathbb{S}^{n-1}, g_1, H\right)$*?*

Remark 1.

- By the arguments of [26, Theorem 1.4] and some gluing technique, we are able to show that for any PSC-metric g_1 on $\mathbb{S}^{n-1}$, no matter whether g_1 is PSC-concordant to g_0 or not, there is a constant H so that $\left(\mathbb{S}^{n-1}, g_0, 0\right)$ is NNSC-cobordant to $\left(\mathbb{S}^{n-1}, g_1, H\right)$ and the ambient manifold bounded by these Bartnik data is diffeomorphic to $\mathbb{S}^{n-1} \times [0,1]$ provided g_0 is the standard round metric on $\mathbb{S}^{n-1}$ [2].
- If g_0 is the standard round metric on $\mathbb{S}^{n-1}$, then by gluing arguments, the largest $\inf\limits_{\mathbb{S}^{n-1}} H$ in Problem 2 is the corresponding number for $\left(\mathbb{S}^{n-1}, g_1, H\right)$ to admit NNSC fill-ins.[1]
- As we know, $\int_{\Sigma^2} H \, \mathrm{d}\mu_1$ and $\int_{\Sigma^2} H^2 \, \mathrm{d}\mu_1$ are closely related to Brown–York mass and Hawking mass respectively, they are also involved in classical Minkowski's inequality for a convex surface and Willmore functional for a surface in $\mathbb{R}^3$, so, it may also be interesting to ask what the possible largest values of $\int_{\mathbb{S}^{n-1}} H \, \mathrm{d}\mu_1$ and $\int_{\mathbb{S}^{n-1}} H^2 \, \mathrm{d}\mu_1$ are, especially for $n = 3$.

For an orientable closed null-cobordant Riemannian manifold $\left(\Sigma^{n-1}, \gamma\right)$, define $\Lambda\left(\Sigma^{n-1}, \gamma\right)$ by

$$\Lambda\left(\Sigma^{n-1}, \gamma\right) = \sup \left\{ \int_{\Sigma} H \, \mathrm{d}\mu_\gamma \,\middle|\, \left(\Sigma^{n-1}, \gamma, H\right) \text{ admits an NNSC fill-in} \right\}.$$

In the case of $n = 3$ and $H > 0$, the above Λ was introduced in [16, 17], and also some interesting properties were discussed therein. An open problem on an estimate of $\Lambda\left(\Sigma^{n-1}, \gamma\right)$ was proposed in [9, p. 31], and a partial result in the case of $H > 0$ was obtained in [26, Theorem 1.3].

Suppose $\left(\mathbb{S}^2, \gamma\right)$ is a 2-dimensional surface with positive Gaussian curvature, then it can be isometrically embedded into $\mathbb{R}^3$, let H_0 be the mean curvature of the embedding image with respect to the outward unit normal vector, then we have:

Problem 3. *Is* $\Lambda\left(\mathbb{S}^2, \gamma\right) = \int_{\mathbb{S}^2} H_0 \, \mathrm{d}\mu_\gamma$*?*

The affirmative answer implies the positivity of Brown–York mass without assumption of positivity of the mean curvature.

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