

Representations and Conjugacy Classes of Semisimple Quasitriangular Hopf Algebras

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Abstract. In this paper we give two general formulae for the Müger centralizers in the category of representations of a semisimple quasitriangular Hopf algebra. The first formula is given in the terms of the Drinfeld map associated to the quasitriangular Hopf algebra. The second formula for the Müger centralizer is given in the terms of the conjugacy classes introduced by Cohen and Westreich in [*J. Algebra* **283** (2005), 42–62]. In the case of a factorizable Hopf algebra these formulae extend some particular cases obtained by the author in [*Math. Z.* **279** (2015), 227–240].

Key words: quasi-triangular Hopf algebras; centralizers; braided fusion categories; normal coideal subalgebras

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1 Introduction

The notion of centralizer in a braided fusion category was introduced by Müger in [25]. It was shown in [12, Theorem 3.13] that the centralizer of a nondegenerate fusion subcategory of a braided category is a categorical complement of the nondegenerate subcategory. This principle is the basis of many classification results of braided fusion categories, see for example papers [11, 12, 15] and references therein.

Despite its importance, in the current literature there is no concrete known formula for the Müger centralizer of all fusion subcategories of a given fusion category. Only few particular cases are completely known in the literature. For instance, in the same paper [25], Müger described the centralizer of all fusion subcategories of the category of finite-dimensional representations of a Drinfeld double of a finite abelian group. More generally, for the category of representations of a (twisted) Drinfeld double of an arbitrary finite group, not necessarily abelian, a similar formula was then given in [26]. For the braided center of Tambara–Yamagami categories, in [16], the centralizer can be described by computing completely the S -matrix of the modular category. In [5] a different approach gave a partial formula for the centralizer of fusion subcategories of a braided equivariantized fusion category.

Given a fusion subcategory \mathcal{D} of a braided fusion category \mathcal{C} , the notion of *Müger centralizer of \mathcal{D}* was introduced in [12]. The centralizer \mathcal{D}' is defined as the fusion subcategory \mathcal{D}' of \mathcal{C} generated by all simple objects X of \mathcal{C} satisfying

$$c_{X,Y}c_{Y,X} = \text{id}_{X \otimes Y}$$

for all objects $Y \in \mathcal{O}(\mathcal{D})$ (see also [25]). For a fusion category \mathcal{C} as usually, we denote by $\mathcal{O}(\mathcal{C})$ the set of isomorphism classes of simple objects of \mathcal{C} .

If (A, R) is a quasitriangular Hopf algebra then the category $\text{Rep}(A)$ of finite-dimensional A -modules is a braided category with the braiding given by

$$c_{M,N}: M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto R_{21}(n \otimes m) = R^{(2)}n \otimes R^{(1)}m,$$

for any two objects $M, N \in \text{Rep}(A)$.

Given a quasitriangular Hopf algebra (A, R) one can also define the Drinfeld map

$$\phi_R: A^* \rightarrow A, \quad f \mapsto (f \otimes \text{id})(R_{21}R) = f(Q_1)Q_2,$$

where $Q = R_{21}R$ is the monodromy matrix.

We prove the following theorem which gives a general description for the centralizer of any fusion subcategory of the category of representations of a quasitriangular Hopf algebra:

Theorem 1.1. *Let (A, R) be a semisimple quasitriangular Hopf algebra and L be a left normal coideal subalgebras of A . Then*

$$\text{Rep}(A//L)' = \text{Rep}(A//M), \quad \text{where } M = \phi_R((A//L)^*).$$

We denote by F_0, F_1, \dots, F_r the central primitive idempotents of the character ring $C(A)$ where $F_0 = t$ is the idempotent integral of A^* . Following [7] one can define the conjugacy classes \mathcal{C}^j of A as $\mathcal{C}^j := \Lambda \leftarrow F_j A^*$, where Λ is an idempotent integral of A and $a \leftarrow f = \langle f, a_1 \rangle a_2$ for all $a \in A$ and $f \in A^*$. It is well known that these conjugacy classes are the simple $D(A)$ -submodules of the induced $D(A)$ -module $\mathbb{k} \uparrow_A^{D(A)} \simeq A$, see [31].

Let (A, R) be a semisimple quasitriangular Hopf algebra and $V_0 = \mathbb{k}, \dots, V_r$ be a complete set of isomorphism classes of irreducible A -modules. Let also $\text{Irr}(A) = \{\chi_0 = \epsilon, \chi_1, \dots, \chi_r\}$ be the set of irreducible characters afforded by these modules and $E_i \in \mathcal{Z}(A)$ be the associated central primitive idempotent of the irreducible character χ_i . Since the Drinfeld map $\phi_R: C(A) \rightarrow \mathcal{Z}(A)$ is an algebra map we may suppose that $\phi_R(F_j) = \sum_{i \in \mathcal{A}_j} E_i$ for some subset $\mathcal{A}_j \subseteq \{0, \dots, r\}$.

Since $\phi_R(1) = 1$ we obtain a partition for the set of indices of all irreducible representations $\{0, 1, \dots, r\} = \bigsqcup_{j \in \mathcal{J}} \mathcal{A}_j$. For any $0 \leq i \leq r$ we denoted by $m(i)$ the unique index $j \in \mathcal{J}$ such that $i \in \mathcal{A}_j$. Therefore in this way we obtain a unique function

$$m: \{0, 1, \dots, r\} \rightarrow \mathcal{J}$$

with the property that $E_i \phi_R(F_{m(i)}) \neq 0$ for all $i \in \{0, 1, \dots, r\}$.

Our second main result is the following:

Theorem 1.2. *Suppose that (A, R) is a semisimple quasitriangular Hopf algebra and L is a left normal coideal subalgebra of A . With the above notations one has*

$$\mathcal{O}(\text{Rep}(A//L)') = \{\chi_i \mid \mathcal{C}^{m(i)} \subseteq L\}.$$

Recall that the quasitriangular Hopf algebra (A, R) is called *factorizable* if the Drinfeld map $\phi_R: A^* \rightarrow A$ is an isomorphism of algebras. In this case, its restriction $\phi_R|_{C(A)}: C(A) \rightarrow \mathcal{Z}(A)$ is an isomorphism of algebras. For a factorizable semisimple Hopf algebra we can record the primitive central idempotents F_j of $C(A)$ such that $F_j := \phi_R^{-1}(E_j)$ any $1 \leq j \leq r$. With these notations, $m(i) = i$ for any $0 \leq i \leq r$ and Theorem 1.2 implies the following:

Corollary 1.3. *Let (A, R) be a semisimple factorizable Hopf algebra and L be a left normal coideal subalgebra of A . Then with the above notations one has that*

$$\mathcal{O}(\text{Rep}(A//L)') = \{\chi_i \mid \mathcal{C}^i \subseteq L\}.$$

Shortly, this paper is organized as follows. In Section 2 we recall the basic notions of Hopf algebras and fusion categories that are used throughout this paper. In this section we also prove a canonical decomposition of a left normal coideal subalgebra in terms of the decomposition of its integral, see equation (2.2). In Section 2.5 we recall the main properties of quasitriangular Hopf algebras and their associated Drinfeld maps. In Section 3 we prove Theorem 1.1 and some consequences of it. In particular we apply Theorem 1.1 to the adjoint subcategory of the category of representations of a factorizable Hopf algebra. In this way we obtain a relation, via the Drinfeld map, between the Hopf center and the first commutator of a factorizable Hopf algebra.

In Section 4 we prove Theorem 1.2. Some consequences of this result are also described. In Section 5 we give an example, by considering the semisimple quasitriangular Hopf algebra H_8 of dimension 8. Based on our results we are able to compute the function m in this case and therefore the centralizer of any fusion subcategory of $\text{Rep}(H_8)$.

We work over an algebraically closed field \mathbb{k} of characteristic zero. The comultiplication and antipode of a Hopf algebra are denoted by Δ and S respectively. We use Sweedlers notation for comultiplication with the sigma symbol dropped. All the other Hopf algebra notations are those used in [24].

2 Preliminaries

Let A be a finite-dimensional semisimple Hopf algebra over an algebraically closed field \mathbb{k} of characteristic zero. Then A is also cosemisimple and $S^2 = \text{id}$ [21]. The character ring $C(A) := \mathcal{G}_0(A) \otimes_{\mathbb{Z}} \mathbb{k}$ is a semisimple subalgebra of A^* and it has a vector space basis given by the set $\text{Irr}(A)$ of irreducible characters of A , see [30]. Moreover, $C(A) = \text{Cocom}(A^*)$, the space of cocommutative elements of A^* . By duality, the character ring of A^* is a semisimple subalgebra of A and $C(A^*) = \text{Cocom}(A)$. If M is an A -representation with character χ then M^* is also an A -representation with character $\chi^* = \chi \circ S$. This induces an involution “ $*$ ”: $C(A) \rightarrow C(A)$ on $C(A)$.

Throughout of this paper we denote by Λ an idempotent integral of A and by t an idempotent integral of A^* . Moreover one has that $t(\Lambda) = \frac{1}{\dim_{\mathbb{k}}(A)}$. Recall also [20] that

$$\dim_{\mathbb{k}}(A)\Lambda = \sum_{d \in \text{Irr}(A^*)} \epsilon(d)d$$

is the regular character of A^* . Dually

$$\dim_{\mathbb{k}}(A)t = \sum_{\chi \in \text{Irr}(A)} \chi(1)\chi$$

is the regular character of A .

Recall that a *left coideal subalgebra* of A is a subalgebra $L \subseteq A$ with $\Delta(L) \subset A \otimes L$. Then L is called *left normal coideal subalgebra* if L is closed under the left adjoint action of A , i.e., $a_1 l S(a_2) \in L$ for any $l \in L$ and any $a \in A$. Recall also from [1] that given a left coideal subalgebra L of A there is a unique element $\Lambda_L \in L$ (called *integral*) such that $l\Lambda_L = \epsilon(l)\Lambda_L$ for all $l \in L$, see also [19]. Then the coideal subalgebra L is normal if and only if Λ_L is central, i.e., $\Lambda_L \in Z(A)$.

For any left normal coideal subalgebra L of A the augmentation ideal AL^+ is a Hopf ideal and it has the following form $AL^+ = A(1 - \Lambda_L) = \text{Ann}_A(\Lambda_L)$. Thus one can define the Hopf quotient $A//L := A/AL^+$. It is well known that any fusion subcategories of $\text{Rep}(A)$ can be written as $\text{Rep}(A//L)$ for some Hopf quotient of A .

Remark 2.1. Since A is free as left L -module [29] it follows that the map

$$A \otimes_L \mathbb{k} \simeq A\Lambda_L, \quad a \otimes_L 1 \mapsto a\Lambda_L$$

is an isomorphism of A -modules. Moreover, by [2, Proposition 3.11] it follows that the regular module of the quotient Hopf algebra $A//L$ is isomorphic to the induced module $A \otimes_L \mathbb{k}$.

2.1 Duality between the character ring and the center

Let A be a semisimple Hopf algebra over the ground field \mathbb{k} . Let us denote by $\text{Irr}(A)$ the set of irreducible characters of A . We suppose that $\text{Irr}(A) = \{\chi_0, \chi_1, \dots, \chi_r\}$. Without loss of generality we may suppose that $\chi_0 = \epsilon$. Let also E_0, E_1, \dots, E_r be the corresponding central primitive central idempotents in A . The evaluation form

$$C(A) \otimes \mathcal{Z}(A) \rightarrow \mathbb{k}, \quad \chi \otimes a \mapsto \chi(a)$$

is nondegenerate. A pair of dual bases for this form is given by $\{\chi_i, \frac{1}{n_i} E_i\}$ since $\langle \chi_i, \frac{1}{n_j} E_j \rangle = \delta_{i,j}$ for any $1 \leq i, j \leq r$.

2.2 Another pair of dual basis in the commutative case

Let A be a semisimple Hopf algebra with a commutative character ring. According to [7] in the case of a commutative ring $C(A)$ there is another pair of dual bases corresponding to this nondegenerate form. This pair of dual bases is given in terms of the conjugacy class sums as defined in [7].

The *conjugacy classes* \mathcal{C}^j of A are defined as $\mathcal{C}^j = \Lambda \leftarrow F_j A^*$, where $\Lambda = \Lambda_A$ is a two-sided idempotent integral of A and $\{F_j\}_j$ is the (complete) set of central primitive idempotents of the semisimple algebra $C(A)$. This notion of conjugacy classes generalizes the usual notion of conjugacy classes in finite groups.

Example 2.2. Let G be a finite group and $A = \mathbb{k}G$ be the associated group algebra. It is easy to see that the conjugacy classes as defined above coincide with the usual notion of conjugacy class in a group. Indeed, let $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_r$ be the usual conjugacy classes of G . Then the set of central primitive idempotents of $C(\mathbb{k}G)$ can be described as $p_j = \sum_{h \in \mathcal{C}^j} p_h$ where $p_h \in \mathbb{k}^G$ is defined as $p_h(g) = \delta_{g,h}$. Since $\Lambda = \frac{1}{|G|} \sum_{g \in G} g$ it follows that

$$\mathcal{C}^j = \Lambda \leftarrow p_j \mathbb{k}^G = \mathbb{k}[\mathcal{C}_j]$$

is the vector sub-space of $\mathbb{k}G$ generated by all group elements of \mathcal{C}_j .

Recall that the Fourier transform $\mathcal{F}: A^* \rightarrow A$ defined by $f \mapsto \Lambda \leftarrow f$ for any $f \in A^*$ is a \mathbb{k} -linear isomorphism. Since $A^* = \bigoplus_{j=0}^r F_j A^*$ and $\mathcal{C}^j = \mathcal{F}(F_j A^*)$ one has

$$A = \bigoplus_{j=0}^r \mathcal{C}^j.$$

One can also define the corresponding *conjugacy class sums*

$$\mathbf{C}_j = \Lambda \leftarrow (\dim A) F_j.$$

Note that $\mathbf{C}_j \in \mathcal{Z}(A)$ and since $\dim_{\mathbb{k}} \mathcal{Z}(A) = \dim_{\mathbb{k}} C(A)$ it follows that $\mathcal{C}^j \cap \mathcal{Z}(A) = \mathbb{k} \mathbf{C}_j$. Note also that $\dim_{\mathbb{k}}(A)\Lambda = \sum_{j=0}^r \mathbf{C}_j$. Indeed, $\Lambda = \Lambda \leftarrow \epsilon = \Lambda \leftarrow \sum_{j=0}^r F_j = \frac{1}{\dim_{\mathbb{k}}(A)} \sum_{j=0}^r \mathbf{C}_j$.

Example 2.3. If $A = \mathbb{k}G$ then

$$\mathbf{C}_j = \left(\frac{1}{|G|} \sum_{g \in G} g \right) \leftarrow |G| p_j = \sum_{g \in G} p_j(g) g = \sum_{h \in \mathcal{C}_j} h$$

is the usual class sum of a conjugacy class \mathcal{C}_j .

Remark 2.4. By the class equation for semisimple Hopf algebras, see [22], one has that the value $n_j := \frac{\dim_{\mathbb{k}} A^*}{\dim_{\mathbb{k}} (A^* F_j)}$ is an integer. Moreover as in [7, equation (11)] one can deduce that $F_j(\Lambda) = \frac{1}{n_j}$.

Example 2.5. If $A = \mathbb{k}G$ then $n_j = \frac{|G|}{|\mathcal{C}_j|}$ is the order of the centralizer of any group element $g_j \in \mathcal{C}_j$.

This implies that a second pair of dual bases for the form of equation (2.1) can be given by $\{F_i, \frac{n_i}{\dim_{\mathbb{k}}(A)} \mathbf{C}_i\}$, see also [7, equation (17)]. This can be written as $\langle F_i, \frac{n_j}{\dim_{\mathbb{k}}(A)} \mathbf{C}_j \rangle = \delta_{i,j}$.

Indeed, one has

$$\left\langle F_i, \frac{n_j}{\dim_{\mathbb{k}}(A)} \mathbf{C}_j \right\rangle = \frac{n_j}{\dim_{\mathbb{k}}(A)} \langle F_i, \Lambda \leftarrow (\dim_{\mathbb{k}}(A)) F_j \rangle = n_j \langle F_j F_i, \Lambda \rangle = \delta_{i,j} n_j F_j(\Lambda) = \delta_{i,j}.$$

2.3 Decomposition of the integral

Let L be a left normal coideal subalgebra of a semisimple Hopf algebra A with a commutative character ring $\mathcal{C}(A)$. We shall use the notation $\lambda_L \in (A//L)^*$ for the idempotent integral of the Hopf algebra $(A//L)^*$. Clearly $\lambda_L \in \mathcal{C}((A//L)^*) \subset \mathcal{C}(A^*)$ and we may suppose that

$$\lambda_L = \sum_{j \in \mathcal{I}_L} F_j \tag{2.1}$$

for some subset of indices $\mathcal{I}_L \subseteq \{0, 1, \dots, r\}$. Note that by [10, Lemma 1.1] $\widetilde{\Lambda}_L := \Lambda \leftarrow \lambda_L$ is a left integral for L . It follows from above that

$$\widetilde{\Lambda}_L = \frac{1}{\dim_{\mathbb{k}}(A)} \sum_{j \in \mathcal{I}_L} \mathbf{C}_j.$$

Then one has

$$L = \Lambda_L \leftarrow A^* = \bigoplus_{j \in \mathcal{I}_L} \mathbf{C}_j \leftarrow A^* = \bigoplus_{j \in \mathcal{I}_L} \mathcal{C}^j.$$

Note also that

$$\epsilon(\mathbf{C}_j) = \dim_{\mathbb{k}}(A) F_j(\Lambda) = \frac{\dim_{\mathbb{k}}(A)}{n_j} = \dim_{\mathbb{k}}(F_j A^*) = \dim_{\mathbb{k}}(\mathcal{C}^j).$$

Thus

$$\epsilon(\widetilde{\Lambda}_L) = \frac{1}{\dim_{\mathbb{k}}(A)} \left(\sum_{j \in \mathcal{I}_L} \epsilon(\mathbf{C}_j) \right) = \frac{1}{\dim_{\mathbb{k}}(A)} \left(\sum_{j \in \mathcal{I}_L} \dim_{\mathbb{k}}(\mathcal{C}^j) \right) = \frac{\dim_{\mathbb{k}}(L)}{\dim_{\mathbb{k}}(A)}.$$

It follows then that

$$\Lambda_L = \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(L)} \widetilde{\Lambda}_L = \frac{1}{\dim_{\mathbb{k}}(L)} \sum_{j \in \mathcal{I}_L} \mathbf{C}_j \tag{2.2}$$

is a formula for the idempotent integral of L .

Define the functional $p_{\mathcal{C}^j} \in A^*$ as the unique linear functional that coincides to ϵ on \mathcal{C}^j and it is equal to zero on the other conjugacy classes \mathcal{C}_l with $l \neq j$. The following lemma was proven in [3, Theorem 5.13].

Lemma 2.6. *Suppose that A is a semisimple Hopf algebra with a commutative character ring $C(A)$. Let $\{F_j\}_{0 \leq j \leq r}$ be a complete set of central primitive idempotents of $C(A)$. Then $F_j = p_{C_j}$ for all $0 \leq j \leq r$.*

Equation (2.2) and the above lemma implies the following:

Lemma 2.7. *Let L be a left normal coideal subalgebra of a semisimple Hopf algebra A . With the above notations one has $j \in \mathcal{I}_L \iff F_j(\Lambda_L) \neq 0$.*

Let A be a semisimple Hopf algebra with a commutative character ring $C(A)$. Then $\{F_j\}$ form a \mathbb{k} -linear basis for $C(A)$ and for any character $\chi \in C(A)$ one can write $\chi = \sum_{j=0}^r \alpha_{\chi,j} F_j$ with $\alpha_{\chi,j} \in \mathbb{k}$. Previous lemma implies the following result, see also [8, Theorem 1.12].

Proposition 2.8. *Let A be a semisimple Hopf algebra with a commutative character ring and $\chi \in C(A)$. Then one has $\chi \in C(A//L) \iff \chi F_j = \chi(1)F_j$ for all $j \in \mathcal{I}_L$.*

Proof. We may suppose that $\chi = \chi_M$ is the character of an A -module M . If $\chi \in C(A//L)$ then $\chi \lambda_L = \chi(1) \lambda_L$ and equation (2.1) implies that $\chi F_j = \chi(1)F_j$, for any $j \in \mathcal{J}_L$. Conversely if $\chi F_j = \chi(1)F_j$ for all $j \in \mathcal{J}_L$ then $\chi \lambda_L = \lambda_L \chi(1)$. Thus

$$\chi(\Lambda_L) = \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(L)} \chi(\widetilde{\Lambda}_L) = \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(L)} \chi(\lambda_L \rightarrow \Lambda) = \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(L)} \chi \lambda_L(\Lambda) = \frac{\dim_{\mathbb{k}}(A)}{\dim_{\mathbb{k}}(L)} \chi(1) \lambda_L(\Lambda).$$

On the other hand note that $\lambda_L(\Lambda) = \epsilon(\widetilde{\Lambda}_L) = \frac{\dim_{\mathbb{k}}(L)}{\dim_{\mathbb{k}}(A)}$ by equation (2.2). Thus $\chi(\Lambda_L) = \chi(1)$ which shows that the restriction of the A -module M to L is trivial. It follows that $M \in \text{Rep}(A//L)$. \blacksquare

Example 2.9. If $A = \mathbb{k}G$ then a left normal coideal subalgebra of A is of the form $L = \mathbb{k}N$ for some normal subgroup $N \trianglelefteq G$. Then

$$\lambda_L = \sum_{C_j \subseteq N} p_j \quad \text{and} \quad \mathcal{I}_L = \{j \mid C_j \subseteq N\}.$$

2.4 Left kernels and Burnside formula

Let M be an A -module and let $\text{LKer}_A(M)$ be the *left kernel* of M . Recall [1] that $\text{LKer}_A(M)$ is defined by

$$\text{LKer}_A(M) = \{a \in A \mid a_1 \otimes a_2 m = a \otimes m, \text{ for all } m \in M\}.$$

Then by [1] it follows that $\text{LKer}_A(M)$ is the largest left coideal subalgebra of A that acts trivially on M . It is also a left normal coideal subalgebra. For example, if $A = \mathbb{k}G$ is the group algebra of a finite group G and M is a $\mathbb{k}G$ -module then $\text{LKer}_A(M) = \mathbb{k} \ker_G(M)$, where $\ker_G(M) = \{g \in G \mid g.m = m, \text{ for all } m \in M\}$ is the usual kernel of M .

Next theorem generalizes a well known result of Brauer in the representation theory of finite groups.

Theorem 2.10 ([1, Theorem 4.2.1]). *Suppose that M is a finite-dimensional module over a semisimple Hopf algebra A . Then*

$$\langle M \rangle = \text{Rep}(A/\text{LKer}_A(M)),$$

where $\langle M \rangle$ is the fusion subcategory of $\text{Rep}(A)$ generated by M .

This implies that for any left normal coideal subalgebra L of A one has that

$$\bigcap_{M \in \text{Irr}(A//L)} \text{LKer}_A(M) = L. \quad (2.3)$$

The previous theorem also implies that any fusion subcategory of $\text{Rep}(A)$ is of the type $\text{Rep}(A//L)$ for some left normal coideal subalgebra L of A . Moreover, for any $V \in \text{Rep}(A)$ one has

$$M \in \text{Rep}(A//L) \iff \text{LKer}_A(M) \supseteq L. \quad (2.4)$$

2.5 Quasitriangular and factorizable Hopf algebras

Recall that a Hopf algebra A is called *quasitriangular* if A admits an R -matrix, i.e., an element $R \in A \otimes A$ satisfying the following properties:

- 1) $R\Delta(x) = \Delta^{\text{cop}}(x)R$ for all $x \in A$,
- 2) $(\Delta \otimes \text{id})(R) = R^{(1)} \otimes r^{(1)} \otimes R^{(2)}r^{(2)}$,
- 3) $(\text{id} \otimes \Delta)(R) = R^{(1)}r^{(1)} \otimes r^{(2)} \otimes R^{(2)}$,
- 4) $(\text{id} \otimes \epsilon)(R) = 1 = (\epsilon \otimes \text{id})(R)$.

Here $R = r = R^{(1)} \otimes R^{(2)} = r^{(1)} \otimes r^{(2)}$. If (A, R) is a quasitriangular Hopf algebra then the category of representations is a braided fusion category with the braiding given by

$$c_{M,N}: M \otimes N \rightarrow N \otimes M, \quad m \otimes n \mapsto R_{21}(n \otimes m) = R^{(2)}n \otimes R^{(1)}m$$

for any two left A -modules $M, N \in \text{Rep}(A)$ (see [18]). Recall that $R_{21} := R^{(2)} \otimes R^{(1)}$. Denote $Q := R_{21}R$. Then the monodromy of two objects is defined as

$$c_{M,N}c_{N,M}: M \otimes N \rightarrow N \otimes M, \quad (m \otimes n) \mapsto R^{(2)}R^{(1)}m \otimes R^{(2)}R^{(1)}n = Q(m \otimes n).$$

A quasitriangular Hopf algebra (A, R) is called *factorizable* if and only if the Drinfeld map

$$\phi_R: A^* \rightarrow A, \quad f \mapsto (f \otimes \text{id})(R_{21}R)$$

is an isomorphism of vector spaces. In this situation, following [28, Theorem 2.3] ϕ_R maps the character ring $C(A)$ onto the center $\mathcal{Z}(A)$ of A and the restriction $\phi_R|_{C(A)}$ is an isomorphism of algebras.

Remark 2.11. By [27, Lemma 4.1] one has that $\phi_R(C)$ is a left normal coideal for any subcoalgebra C of A^* .

One can also define the map ${}_R\phi: A^* \rightarrow A$ by ${}_R\phi(f) = (\text{id} \otimes f)(Q)$ for all $f \in A^*$. Moreover by [28, Theorem 2.1] one has that ${}_R\phi(f\chi) = {}_R\phi(f) {}_R\phi(\chi)$ for all $f \in A^*$ and $\chi \in C(A)$. Thus ${}_R\phi|_{C(A)}: C(A) \rightarrow \mathcal{Z}(A)$ is an isomorphism of \mathbb{k} -algebras.

Remark 2.12. By [27, Lemma 2.3] one has that $S \circ {}_R\phi = \phi_R \circ s$ where S and s are the antipodes of A and A^* respectively.

In the case of a factorizable Hopf algebra ${}_R\phi$ is also bijective and moreover by [9] the two maps ${}_R\phi$ and ϕ_R coincide on the character ring $C(A)$.

3 Proof of Theorem 1.1 on Müger centralizer

In this section we prove the first main theorem mentioned in the introduction. Given a fusion subcategory \mathcal{D} of a braided fusion category \mathcal{C} , recall that the Müger centralizer \mathcal{D}' is defined as the fusion subcategory of \mathcal{C} generated by all simple objects X of \mathcal{C} satisfying

$$c_{X,Y}c_{Y,X} = \text{id}_{X \otimes Y}$$

for all objects $Y \in \mathcal{O}(\mathcal{D})$ (see also [25]). Recall that $\mathcal{O}(\mathcal{D})$ denotes the set of isomorphism classes of simple objects of \mathcal{D} .

Let A be a semisimple quasitriangular Hopf algebra over \mathbb{k} and $\mathcal{D} = \text{Rep}(A//L)$ be a fusion subcategory of $\text{Rep}(A)$ where L is a left normal coideal subalgebra of A .

Lemma 3.1. *Let (A, R) be a semisimple quasitriangular Hopf algebra and L, N be two left normal coideal subalgebras. Then the following assertions are equivalent:*

- 1) $\text{Rep}(A//N) \subseteq \text{Rep}(A//L)'$,
- 2) the following equation holds in $A \otimes A$:

$$Q(\Lambda_L \otimes \Lambda_N) = \Lambda_L \otimes \Lambda_N, \tag{3.1}$$

- 3) $N \supseteq \phi_R((A//L)^*)$.

Proof. (1) \iff (2) It is well known that two fusion subcategories of $\text{Rep}(A)$ centralize each other if and only if their regular representations centralize. Thus one needs to show that the two regular representations of $A//L$ and $A//N$ centralize each other if and only if equation (3.1) holds. On the other hand from the definition of the braiding in $\text{Rep}(A)$ the two representations centralize each other if and only if $Q = R^{(2)}r^{(1)} \otimes R^{(1)}r^{(2)}$ acts as identity on their tensor product $A//L \otimes A//N$. By Remark 2.1 one has $\mathbb{k} \uparrow_L^A \otimes \mathbb{k} \uparrow_N^A = A\Lambda_L \otimes A\Lambda_N$. Since Λ_L and Λ_N are central elements of A it is clear that Q acts as identity on this subspace of $A \otimes A$ if and only if equation (3.1) holds.

(2) \implies (3) By [10, Lemma 1.1] one has $(A//L)^* = \Lambda_L \rightharpoonup A^*$. Therefore $\phi_R(\Lambda_L \rightharpoonup f) = f(Q^1\Lambda_L)Q^2$ for any $f \in A^*$. From here, applying equation (3.1) it follows

$$\phi_R(\Lambda_L \rightharpoonup f)\Lambda_N = f(Q^1\Lambda_L)Q^2\Lambda_N = f(\Lambda_L)\Lambda_N. \tag{3.2}$$

On the other hand note that

$$\epsilon_A(\phi_R(\Lambda_L \rightharpoonup f)) = f(\Lambda_L). \tag{3.3}$$

If $L' := \phi_R((A//L)^*)$ then equation (3.3) gives that $A(L')^+ \subseteq AN^+$ and by [2, Lemma 6.2] one has that $L' \subseteq N$.

(3) \implies (2) In this situation equation (3.2) is satisfied for any $f \in A^*$ which shows that equation (3.1) also holds. \blacksquare

Remark 3.2. If $\mathcal{B} \subseteq \mathcal{D}$ are fusion subcategories of a braided fusion category \mathcal{C} then clearly $\mathcal{D}' \subseteq \mathcal{B}'$. In particular if $\text{Rep}(A//N) \subseteq \text{Rep}(A//L)'$ then by centralizing once more one has that $\text{Rep}(A//L) \subseteq \text{Rep}(A//L)'' \subseteq \text{Rep}(A//N)'$. Thus the three above conditions from the previous lemma are also equivalent to:

- 1) $\text{Rep}(A//L) \subseteq \text{Rep}(A//N)'$,
- 2) $Q(\Lambda_N \otimes \Lambda_L) = (\Lambda_N \otimes \Lambda_L)$,
- 3) $L \supseteq \phi_R((A//N)^*)$.

3.1 Proof of Theorem 1.1

Proof. Let $M := \phi_R((A//L)^*)$ as in the statement of the theorem. By Remark 2.11 it is well-known that M is also a left normal coideal subalgebra of A . Suppose also that $\text{Rep}(A//L)' = \text{Rep}(A//L^\circ)$ for some other left normal coideal subalgebra L° of A .

We need to prove that $L^\circ = M$. Note that for any left normal coideal subalgebra N of A one has $\text{Rep}(A//N) \subseteq \text{Rep}(A//L)'$ if and only if $\text{Rep}(A//N) \subseteq \text{Rep}(A//L^\circ)$, i.e. $L^\circ \subseteq N$. Then the previous Lemma 3.1 shows that for any left normal coideal subalgebra N of A one has

$$L^\circ \subseteq N \iff M \subseteq N.$$

In particular, for $N = L^\circ$ one obtains that $M \subseteq L^\circ$. For $N = M$ one obtains the other inclusion $L^\circ \subseteq M$. Thus $L^\circ = M$ and the proof is complete. \blacksquare

Theorem 1.1 can be rewritten as following by using the notion of left kernel of an A -module, see Section 2.4:

Theorem 3.3. *Let (A, R) be a quasitriangular semisimple Hopf algebra and L be a left normal coideal subalgebra of A . If $M \in \text{Rep}(A)$ then the following assertions are equivalent:*

- 1) $M \in \text{Rep}(A//L)'$,
- 2) $\phi_R((A//L)^*) \subseteq \text{LKer}_A(M)$.

Proof. One has that $\text{Rep}(A//L)' = \text{Rep}(A//L^\circ)$ where $L^\circ = \phi_R((A//L)^*)$. Then by equation (2.4) one has $M \in \text{Rep}(A//L)' \iff \text{LKer}_A(M) \supseteq L^\circ$. \blacksquare

3.2 On the commutators and Hopf centre

Given a fusion category \mathcal{C} we denote by \mathcal{C}_{pt} the fusion subcategory generated by the invertible objects of \mathcal{C} . In the case $\mathcal{C} = \text{Rep}(A)$ for a semisimple Hopf algebra we have that \mathcal{C}_{pt} is the full abelian subcategory generated by one-dimensional modules. It was shown in [2] that $\text{Rep}(A)_{\text{pt}} = \text{Rep}(A/I)$ where $I := \{ab - ba \mid a, b \in A\}$ is the first commutator of the \mathbb{k} -algebra A . Moreover the commutator ideal $[A, A]$ is a Hopf ideal and by Takeuchi's correspondence it corresponds to a left normal coideal subalgebra A' . Thus $A(A')^+ = I$ and

$$\text{Rep}(A)_{\text{pt}} = \text{Rep}(A//A').$$

Moreover by [2] A' is the smallest left normal coideal subalgebra L with the property that $A//L$ is a commutative Hopf algebra. A' is called the *the commutator* of A .

Example 3.4. If $A = \mathbb{k}G$ then $A' = \mathbb{k}G'$ where G' is the first commutator of G , i.e., $G' = [G, G]$.

For a fusion category \mathcal{C} , recall that the adjoint subcategory \mathcal{C}_{ad} is defined as the smallest fusion subcategory generated by all objects of the type $X \otimes X^*$ with X a simple object of \mathcal{C} . If $\mathcal{C} = \text{Rep}(A)$ is the category of representations of a semisimple Hopf algebra A then it is well-known that

$$\text{Rep}(A)_{\text{ad}} = \text{Rep}(A//K(A)),$$

where $K(A)$ is the Hopf centre of A , i.e., largest central Hopf subalgebra of A .

Recall that a braided fusion subcategory \mathcal{C} is called *nondegenerate* if its Müger center is trivial, i.e., $\mathcal{C}' = \text{Vec}$. If \mathcal{C} is a nondegenerate braided fusion category then by [11, Corollary 3.11] one has

$$(\mathcal{C}_{\text{ad}})' = \mathcal{C}_{\text{pt}}. \tag{3.4}$$

Since $\mathcal{D}' = \mathcal{D}$ for any fusion subcategory \mathcal{D} of a nondegenerate braided category \mathcal{C} we can also write that

$$(\mathcal{C}_{\text{pt}})' = \mathcal{C}_{\text{ad}}. \quad (3.5)$$

Proposition 3.5. *Let A be a factorizable semisimple Hopf algebra. With the above notations, one has that*

$$\phi_R((A//K(A))^*) = A' \quad \text{and} \quad \phi_R((A/I)^*) = K(A).$$

Proof. For a quasitriangular Hopf algebra (A, R) it is well known that $\text{Rep}(A)$ is nondegenerate if and only if A is a factorizable Hopf algebra. In this case equation (3.4) gives that

$$\text{Rep}(A//K(A))' = \text{Rep}(A//A').$$

Then Theorem 1.1 gives $\phi_R((A//K(A))^*) = A'$.

Similarly, equation (3.5) gives that

$$\text{Rep}(A//A')' = \text{Rep}(A//K(A))$$

and Theorem 1.1 implies $K(A) = \phi_R((A//A')^*)$. ■

4 Conjugacy classes and Müger centralizer

In this section we will prove Theorem 1.2. Suppose that (A, R) is a semisimple quasitriangular Hopf algebra. Let as above V_0, V_1, \dots, V_r be a complete set of isomorphism classes of irreducible A -modules. Let also $\text{Irr}(A) = \{\chi_0, \chi_1, \dots, \chi_r\}$ be the set of irreducible characters afforded by these modules and $\{E_0, \dots, E_r\}$ be their associated central primitive idempotents of A . Without loss of generality we may suppose that $V_0 = \mathbb{k}$ is the trivial A -module and therefore $\chi_0 = \epsilon$ and $E_0 = \Lambda$.

Recall by [14], that $\text{Rep}(A)$ is a ribbon category with the canonical ribbon element $v = u^{-1}$, where $u := S(R^1)R^2$ is the Drinfeld element of (A, R) . With respect to the canonical ribbon structure given by this ribbon element, the S -matrix of (A, R) has entries

$$s_{ii'} := \text{tr}_{V_i \otimes V_{i'}}(Q) = (\chi_i \otimes \chi_{i'})(Q) = \langle \chi_{i'}, \phi_R(\chi_i) \rangle.$$

It follows from [12] that one has $|s_{ii'}| \leq \chi_i(1)\chi_{i'}(1)$ and $V_i, V_{i'}$ centralize each other if and only if $s_{ii'} = \chi_i(1)\chi_{i'}(1)$.

The Drinfeld map $\phi_R: C(A) \rightarrow \mathcal{Z}(A)$ is an algebra map and we may suppose as in the introduction that

$$\phi_R(F_j) = \sum_{i \in \mathcal{A}_j} E_i$$

for some subset $\mathcal{A}_j \subseteq \{0, \dots, r\}$.

Without loss of generality we may also suppose that $F_0 = t$, the idempotent integral of A^* . Then $\phi_R(F_0)$ is the idempotent integral of $\Phi(A) := \phi_R(A^*)$ since $\phi_R(f)\phi_R(F_0) = \phi_R(fF_0) = f(1)\phi_R(F_0) = \epsilon(\phi_R(f))\phi_R(F_0)$ for any $f \in A^*$ and also $\epsilon(\phi_R(F_0)) = F_0(1) = 1$.

Note that the set \mathcal{A}_j is empty if and only if $\phi_R(F_j) = 0$. Denote by $\mathcal{J} \subseteq \{0, 1, \dots, r\}$ the set of all indices j with \mathcal{A}_j not a empty set. Since $\phi_R(1) = 1$ we obtain in this way a partition for the set of indices of all irreducible representations $\{0, 1, \dots, r\} = \bigsqcup_{j \in \mathcal{J}} \mathcal{A}_j$.

For any index $0 \leq i \leq r$ we denote by $m(i)$ the unique index $j \in \mathcal{J}$ such that $i \in \mathcal{A}_j$. Therefore in this way we obtain a unique function

$$m: \{0, 1, \dots, r\} \rightarrow \mathcal{J}$$

with the property that $E_i \phi_R(F_{m(i)}) \neq 0$ for all $i \in \{0, 1, \dots, r\}$.

Recall from Section 2.4 the definition of the left kernel of an A -module.

Lemma 4.1. *Let (A, R) be a quasitriangular Hopf algebra and $V_i, V_{i'}$ be two irreducible A -representations. Then, with the above notations the following assertions are equivalent:*

- 1) V_i and $V_{i'}$ centralize each other in $\text{Rep}(A)$,
- 2) $\chi_{i'} F_{m(i)} = \chi_{i'}(1) F_{m(i)}$,
- 3) $\chi_i F_{m(i')} = \chi_i(1) F_{m(i')}$,
- 4) $\mathcal{C}^{m(i)} \subseteq \text{LKer}_A(V_{i'})$,
- 5) $\mathcal{C}^{m(i')} \subseteq \text{LKer}_A(V_i)$.

Proof. For any character $\chi \in C(A)$ write as above $\chi = \sum_{j=0}^r \alpha_{\chi,j} F_j$. Then one has that

$$\phi_R(\chi) = \sum_{j=0}^r \alpha_{\chi,j} \phi_R(F_j) = \sum_{j=0}^r \alpha_{\chi,j} \left(\sum_{s \in \mathcal{A}_j} E_s \right).$$

With these formulae note that

$$s_{ii'} = \langle \chi_i, \phi_R(\chi_{i'}) \rangle = \left\langle \chi_i, \sum_{j \in \mathcal{J}} \sum_{s \in \mathcal{A}_j} \alpha_{\chi_{i'},j} E_s \right\rangle = \chi_i(1) \alpha_{\chi_{i'}, m(i)},$$

where $m(i)$ as above, is the unique index $j \in \mathcal{J}$ with $i \in \mathcal{A}_j$. Therefore we see that V_i centralize $V_{i'}$ if and only if

$$\alpha_{\chi_{i'}, m(i)} = \chi_{i'}(1) \iff \chi_{i'} F_{m(i)} = \chi_{i'}(1) F_{m(i)} \iff \mathcal{C}^{m(i)} \subseteq \text{LKer}_A(V_{i'}). \quad (4.1)$$

The equivalence of assertions (2) and (4) follows from [9, Theorem 3.6]. The rest of the equivalences follow from the symmetry property of the centralizer. \blacksquare

Remark 4.2. The above lemma also shows that if V_i centralizes $V_{i'}$ then V_i centralize all $V_{i''}$ with $i'' \in \mathcal{A}_{m(i')}$.

Next theorem is a generalization of Theorem 1.2.

4.1 Proof of Theorem 1.2

Proof. Using the previous lemma we have the following equalities:

$$\begin{aligned} \mathcal{O}(\text{Rep}(A//L)') &= \bigcap_{M \in \text{Irr}(A//L)} \mathcal{O}(\langle M \rangle') = \bigcap_{M \in \text{Irr}(A//L)} \{V_i \mid \mathcal{C}^{m(i)} \subseteq \text{LKer}_A(M)\} \\ &= \left\{ V_i \mid \mathcal{C}^{m(i)} \subseteq \bigcap_{M \in \text{Irr}(A//L)} \text{LKer}_A(M) \right\}. \end{aligned}$$

On the other hand by equation (2.3) one has that

$$\bigcap_{M \in \text{Irr}(A//L)} \text{LKer}_A(M) = L$$

and therefore $\mathcal{O}(\text{Rep}(A//L)') = \{\chi_i \mid \mathcal{C}^{m(i)} \subseteq L\}$. \blacksquare

Another description of the simple objects of $\text{Rep}(A//L)'$ is given in the following:

Proposition 4.3. *If (A, R) is a semisimple quasitriangular Hopf algebra then*

$$\mathcal{O}(\text{Rep}(A//L)') = \{\chi_i \mid m(i) \in \mathcal{I}_L\} = \{\chi_i \mid F_{m(i)}(\Lambda_L) \neq 0\}$$

for any left normal coideal subalgebra L of A .

Proof. By Theorem 1.2 one has that $\mathcal{O}(\text{Rep}(A//L)') = \{\chi_i \mid \mathcal{C}^{m(i)} \subseteq L\}$, i.e., $\mathcal{O}(\text{Rep}(A//L)') = \{\chi_i \mid m(i) \in \mathcal{I}_L\}$. On the other hand by Lemma 2.7 one has that $\mathcal{C}^{m(i)} \subseteq L$ if and only if $F_{m(i)}(\Lambda_L) \neq 0$. ■

Corollary 3.3 together with Theorem 1.2 gives the following:

Theorem 4.4. *Let (A, R) be a quasitriangular semisimple Hopf algebra and L be a left normal coideal subalgebra of A . For an irreducible representation V_i of A we have that the following assertions are equivalent:*

- 1) $V_i \in \mathcal{O}(\text{Rep}(A//L)')$,
- 2) $\mathcal{C}^{m(i)} \subseteq L$,
- 3) $\phi_R((A//L)^*) \subseteq \text{LKer}_A(V_i)$.

4.2 Description of $\Phi(A)$

As above denote by $\Phi(A) := \phi_R(A^*)$ the image of the Drinfeld map.

Proposition 4.5. *Suppose that (A, R) is a quasitriangular Hopf algebra and $\phi_R(F_0) = \sum_{i \in \mathcal{A}_0} E_i$ where $F_0 = t$ is the idempotent integral of A^* . Then*

- 1) $\text{Rep}(A)' = \text{Rep}(A//\Phi(A))$,
- 2) $\text{Irr}(A//\Phi(A)) = \{\chi_i \mid i \in \mathcal{A}_0\}$,
- 3) $\Phi(A) = \bigoplus_{j \in \mathcal{J}} \mathcal{C}^j$.

Proof. By Theorem 1.1 one has that

$$\text{Rep}(A)' = \text{Rep}(A//\mathbb{k})' = \text{Rep}(A//\phi_R(A^*)) = \text{Rep}(A//\Phi(A)).$$

On the other hand, Theorem 1.2 gives the following equality

$$\mathcal{O}(\text{Rep}(A)') = \mathcal{O}(\text{Rep}(A//\mathbb{k})') = \{\chi_i \mid \mathcal{C}^{m(i)} \subseteq \mathbb{k}\} = \{\chi_i \mid \mathcal{C}^{m(i)} = \mathbb{k}\}.$$

It is easy to see that $\mathcal{C}^{m(i)} = \mathbb{k}$ if and only if $m(i) = 0$. Indeed, $\mathcal{C}^{m(i)} = \mathcal{F}(F_{m(i)}A^*) = \Lambda \leftarrow F_{m(i)}A^*$ and $\mathbb{k} = \mathcal{F}(F_0A^*)$. Since \mathcal{F} is bijective the statement follows. Therefore $\mathcal{O}(\text{Rep}(A//\Phi(A))) = \mathcal{O}(\text{Rep}(A)') = \{\chi_i \mid m(i) = 0\} = \{\chi_i \mid i \in \mathcal{A}_0\}$.

For the last item note that $\mathcal{O}(\text{Rep}(A)) = \mathcal{O}(\text{Rep}(A//\Phi(A)))' = \{\chi_i \mid \mathcal{C}^{m(i)} \subseteq \Phi(A)\}$. This implies that $\Phi(A) = \bigoplus_{j \in \mathcal{J}} \mathcal{C}^j$. ■

4.3 Proof of Corollary 1.3

This is now a particular case of Theorem 1.2.

Proof. Note that if A is factorizable then ϕ_R is bijective and every set \mathcal{A}_j is a singleton. Moreover $\phi_R(F_0) = E_0$, the integral of A in this case. Then, without loss of generality, after a permutation of the indices, we may suppose $\phi_R(F_i) = E_i$ and therefore $m(i) = i$ for all $0 \leq i \leq r$. Then the statement of Theorem 1.2 becomes Theorem 1.3. \blacksquare

For the rest of this subsection we suppose that A is a semisimple factorizable Hopf algebra. As explained above without loss of generality we may also assume $\phi_R(F_i) = E_i$ and therefore that the function $m: \{0, 1, \dots, r\} \rightarrow \{0, 1, \dots, r\}$ is the identity map.

Proposition 4.6. *Suppose that (A, R) is a semisimple factorizable Hopf algebra. Then for any irreducible A -module V_i one has that*

$$\text{LKer}_A(V_i) = \bigoplus_{\{i' \mid V_{i'} \text{ centralizes } V_i\}} \mathcal{C}^{i'}.$$

Proof. Since $\mathcal{J} = \{0, 1, \dots, r\}$ in this case, by equation (4.1) and [9, Theorem 3.6] one has that

$$\chi_i F_{i'} = \chi_i(1) F_{i'} \iff \mathcal{C}^{i'} \subseteq \text{LKer}_A(V_i) \iff V_i \text{ centralizes } V_{i'}.$$

It follows that in this case one has

$$\text{LKer}_A(V_i) = \bigoplus_{\mathcal{C}^{i'} \subseteq \text{LKer}_A(V_i)} \mathcal{C}^{i'} = \bigoplus_{\{i' \mid V_i \text{ centralizes } V_{i'}\}} \mathcal{C}^{i'}. \quad \blacksquare$$

Remark 4.7.

1. From the previous proposition, in the case of a semisimple factorizable Hopf algebra one can deduce that for any two irreducible characters χ_i and $\chi_{i'}$ one has $\mathcal{C}^{i'} \subseteq \text{LKer}_A(V_i) \iff \mathcal{C}^i \subseteq \text{LKer}_A(V_{i'})$
2. Recall that in [4, Theorem 1.4] it is shown that

$$\mathcal{O}(\text{Rep}(A//K)') = \{\chi_i \mid F_i(\Lambda_K) \neq 0\}$$

for any normal Hopf subalgebra K of a factorizable Hopf algebra A . Note that Proposition 4.3 generalizes the above result from normal Hopf subalgebras K to left normal coideal subalgebras L of A . It also drops the factorizability assumption on A .

Define $C_{V_i} := C_{\chi_i} \subset A^*$ as the subcoalgebra of A^* generated by χ_i . By [9, Lemma 4.2(i)], in the factorizable case one has that $\phi_R(C_{V_i}) = \mathcal{C}_i$. for all $0 \leq i \leq r$.

Remark 4.8. Let A be a semisimple factorizable Hopf algebra and L a left normal coideal subalgebra of A . If $\mathcal{D} := \text{Rep}(A//L)$ then $(A//L)^* = \bigoplus_{\chi_j \in \mathcal{O}(\mathcal{D})} C_{V_j}$ and $L^\circ = \phi_R((A//L)^*) =$

$\bigoplus_{\chi_j \in \mathcal{O}(\mathcal{D})} \phi_R(C_{V_j})$, i.e., $L^\circ = \bigoplus_{\chi_j \in \mathcal{O}(\mathcal{D})} \mathcal{C}^j$. This gives another proof for Theorem 1.3 in the case of a factorizable Hopf algebra since in this case $\mathcal{D}'' = \mathcal{D}$.

5 Example H_8

In this section we compute the centralizer of any fusion subcategory of the quasi-triangular Hopf algebra H_8 , the unique semisimple non-trivial Hopf algebra of dimension 8. We note that the category of representations $\text{Rep}(H_8)$ is a braided Tambara–Yamagami category and therefore $\text{Rep}(H_8) \subset \text{Rep}(D(H_8))$. The S -matrix of the center of a Tambara–Yamagami was computed in [16]. Using this one can describe completely the centralizer of any fusion subcategory of $\text{Rep}(H_8)$. However, we decided to include this example here to illustrate how Theorem 1.2 can be applied in a concrete example.

The eight-dimensional semisimple Hopf algebra (see [17, 23]) is generated by $\{x, y, z\}$ subject to the relations

$$x^2 = y^2 = z^2 = 1, \quad xz = zx, \quad zy = yz, \quad xyz = yx.$$

The comultiplication is given by

$$\Delta(x) = xe_0 \otimes x + xe_1 \otimes y, \quad \Delta(y) = ye_1 \otimes x + ye_0 \otimes y, \quad \Delta(z) = z \otimes z, \quad (5.1)$$

and the counit by $\epsilon(x) = \epsilon(y) = \epsilon(z) = 1$. The antipode has the formulae

$$S(x) = xe_0 + ye_1, \quad S(y) = xe_1 + ye_0, \quad S(z) = z.$$

Based on equation (5.1) one can compute that

$$\Delta(xy) = xye_0 \otimes xy + xye_1 \otimes yx, \quad \Delta(yx) = yxe_0 \otimes yx + yxe_1 \otimes xy.$$

It can also be checked that

$$\Delta(xz) = xe_0 \otimes xz - xe_1 \otimes yz, \quad \Delta(yz) = ye_0 \otimes yz - ye_1 \otimes xz.$$

Since z is a central element in H_8 , there are two central orthogonal idempotents:

$$e_0 = \frac{1}{2}(1 + z), \quad e_1 = \frac{1}{2}(1 - z).$$

5.1 Dual basis of H^*

H_8 has a \mathbb{k} -linear basis given by the set of elements $\{1, x, y, z, xy, yx, xz, yz\}$. We consider its linear dual basis on H_8^* given by $\{p_1, p_x, p_y, p_z, p_{xy}, p_{yx}, p_{xz}, p_{yz}\}$.

It is easy to see that the idempotent integrals of H_8 and H_8^* are given by

$$\Lambda = \frac{e_0}{4}(1 + x + y + xy) = \frac{1}{8}(1 + x + y + xy + z + zx + zy + yx), \quad \lambda = p_1.$$

5.2 H_8^* representations

Since H_8 is a self dual Hopf algebra [13] it has also four 1-dimensional representations given by the group like elements of H_8 and a 2-dimensional representation. One has that

$$G(H_8) = \{1, g_1, g_2, z\},$$

with $g_1 = xy(e_0 + ie_1)$, $g_2 = xy(e_0 - ie_1)$. It can be easily checked that $g_1g_2 = z$, $zg_i = g_i$ and $g_i^2 = 1$. Moreover, the set of central grouplike elements of H_8 is given by $\tilde{G}(H_8) = \{1, z\}$.

5.2.1 On the 2-dimensional comodule

From equation (5.1) one can compute that

$$\begin{aligned}\Delta(xe_0) &= xe_0 \otimes xe_0 + xe_1 \otimes ye_1, & \Delta(xe_1) &= xe_0 \otimes xe_1 + xe_1 \otimes ye_0, \\ \Delta(ye_0) &= ye_0 \otimes ye_0 + ye_1 \otimes xe_1, & \Delta(ye_1) &= ye_0 \otimes ye_1 + ye_1 \otimes xe_0.\end{aligned}$$

This shows that $W = \mathbb{k}w_1 \oplus \mathbb{k}w_2$ is a left H_8 -comodule with the comodule structure given by

$$\rho(w_1) = xe_0 \otimes w_1 + xe_1 \otimes w_2, \quad \rho(w_2) = ye_1 \otimes w_1 + ye_0 \otimes w_2.$$

Thus

$$H_8 = \mathbb{k}1 \oplus \mathbb{k}g_1 \oplus \mathbb{k}g_2 \oplus \mathbb{k}z \oplus (\mathbb{k}xe_0 \oplus \mathbb{k}ye_1) \oplus (\mathbb{k}ye_0 \oplus \mathbb{k}xe_1) \quad (5.2)$$

is a decomposition of H_8 into simple left H_8 -comodules. We denote $M_0 = \mathbb{k}1$, $M_1 = \mathbb{k}g_1$, $M_2 = \mathbb{k}g_2$, $M_3 = \mathbb{k}z$ the four one-dimensional H_8^* -modules. Moreover, $M_4 := (\mathbb{k}xe_0 \oplus \mathbb{k}ye_1)$ is an irreducible H_8^* -module and $M_4 \simeq M_5 := (\mathbb{k}ye_0 \oplus \mathbb{k}xe_1)$.

5.3 Fourier transform

Based on the comultiplication formulae one can compute the Fourier transform

$$\mathcal{F}: H_8^* \rightarrow H_8, \quad f \mapsto \dim_{\mathbb{k}}(H_8)f \rightarrow \Lambda.$$

After some computations it follows that under \mathcal{F} one has

$$p_1 \mapsto 1, \quad p_z \mapsto z, \quad p_x \mapsto xe_0 + ye_1, \quad p_{xz} \mapsto xe_0 - ye_1$$

and

$$p_y \mapsto ye_0 + xe_1, \quad p_{yz} \mapsto ye_0 - xe_1, \quad p_{xy} \mapsto yx, \quad p_{yx} \mapsto xy.$$

5.4 Irreducible H_8 -modules and their characters

H_8 has four 1-dimensional modules V_0, V_1, V_2, V_3 and a 2-dimensional irreducible module V . The action of the generators on these modules is given as follows.

For V_0 , the trivial H -module the action is given by $xv = yv = zv = v$ and therefore the character is given by

$$\chi_0 = p_1 + p_z + p_x + p_y + p_{xy} + p_{yx} + p_{xz} + p_{yz}.$$

For V_1 the action is given by $xv = -v$, $yv = v$, $zv = v$. The character χ_1 of V_1 has $\chi_1(1) = \chi_1(z) = \chi_1(y) = \chi_1(yz) = 1$, $\chi_1(x) = \chi_1(yx) = \chi_1(xy) = \chi_1(xz) = -1$. Thus

$$\chi_1 = p_1 + p_z - p_x + p_y - p_{xy} - p_{yx} - p_{xz} + p_{yz}.$$

For V_2 the action is given by $xv = -v$, $yv = -v$, $zv = v$ and the character is given by

$$\chi_2 = p_1 + p_z - p_x - p_y + p_{xy} + p_{yx} - p_{xz} - p_{yz}.$$

For V_3 one has $xv = v$, $yv = -v$, $zv = v$ and the character is given by

$$\chi_3 = p_1 + p_z + p_x - p_y - p_{xy} - p_{yx} + p_{xz} - p_{yz}.$$

For $V_4 = V$, the 2-dimensional simple module, if $V = \mathbb{k}v_1 \oplus \mathbb{k}v_2$ then the action of generators is given by

$$xv_1 = v_1, \quad yv_1 = v_2, \quad zv_1 = -v_1, \quad xv_2 = -v_2, \quad yv_2 = v_1, \quad zv_2 = -v_2.$$

The character χ_4 of V has the values $\chi_4(1) = 2\chi_4(z) = -2$, $\chi_4(x) = \chi_4(y) = \chi_4(yx) = \chi_4(xy) = \chi_4(xz) = \chi_4(zx) = 0$. This gives that

$$\chi_4 = 2p_1 - 2p_z.$$

5.4.1 Multiplication of the characters

It can be easily checked that $\chi_4^2 = \sum_{i=0}^3 \chi_i$, and $\chi_i \chi_4 = \chi_4 \chi_i = \chi_i$ for all $0 \leq i \leq 3$. Moreover $\chi_i \chi_j = \chi_k$ if $\{i, j, k\} = \{1, 2, 3\}$.

5.5 Central primitive idempotent of the character ring

Based on the above multiplication, the central primitive idempotents of $C(H_8)$ can be computed as follows

$$\begin{aligned} F_0 &= \frac{1}{8}(\epsilon + \chi_1 + \chi_2 + \chi_3 + 2\chi_4) = p_1 && \text{is the integral of } H_8^*, \\ F_1 &= \frac{1}{4}(\epsilon + \chi_1 - \chi_2 - \chi_3) = p_y + p_{yz}, && F_2 = \frac{1}{4}(\epsilon - \chi_1 + \chi_2 - \chi_3) = p_{xy} + p_{yx}, \\ F_3 &= \frac{1}{4}(\epsilon - \chi_1 - \chi_2 + \chi_3) = p_x + p_{xz}, && F_4 = \frac{1}{8}(\epsilon + \chi_1 + \chi_2 + \chi_3 - 2\chi_4) = p_z. \end{aligned}$$

Note that F_4 is the central primitive idempotent of H_8^* attached to the central grouplike element $z \in \bar{G}(H_8) := G(H_8) \cap \mathcal{Z}(H_8)$.

5.6 Conjugacy class sums of H_8

Using the above formulae for the central primitive idempotents since $C_i = 8(\Lambda \leftarrow F_i)$ it follows that

$$\begin{aligned} C_0 &= 8(\Lambda \leftarrow F_0) = 1, && C_1 = 8(\Lambda \leftarrow F_1) = 2ye_0, \\ C_2 &= 8(\Lambda \leftarrow F_2) = xy + yx = g_1 + g_2, \end{aligned}$$

and

$$C_3 = 8(\Lambda \leftarrow F_3) = 2xe_0, \quad C_4 = 8(\Lambda \leftarrow F_4) = z.$$

5.7 Description of the adjoint action

Using the antipode formulae one can see that the adjoint action of H_8 on itself can be given by

$$x.a = ax, \quad y.a = yay, \quad z.a = a, \quad \text{for all } a \in H_8.$$

5.8 Conjugacy classes of H_8

Recall that the conjugacy classes are simple $D(H_8)$ -modules [31]. We rewrite the decomposition of H_8 into left H_8 -comodules from equation (5.2) as follows

$$H_8 = \mathbb{k}1 \oplus \mathbb{k}z \oplus (\mathbb{k}g_1 \oplus \mathbb{k}g_2) \oplus (\mathbb{k}xe_0 \oplus \mathbb{k}ye_1) \oplus (\mathbb{k}xe_1 \oplus \mathbb{k}ye_0).$$

Moreover, by the above description of the left adjoint action it can be checked each of the above five subspaces is closed under the adjoint action of H_8 .

Clearly, the first two subspaces are irreducible $D(H_8)$ -modules being one-dimensional. Since $C_0 \in \mathbb{k}1$ and $C_4 = z \in \mathbb{k}z$ we deduce that

$$\mathcal{C}^0 = \mathbb{k}1, \quad \mathcal{C}^4 = \mathbb{k}z.$$

It can be checked directly that the third $D(H_8)$ -module is an irreducible $D(H_8)$ -module since

$$x.g_1 = g_2, \quad x.g_2 = g_1, \quad y.g_1 = g_2, \quad y.g_2 = g_1.$$

Since $C_2 = g_1 + g_2 = xy + yx$ it follows that

$$\mathcal{C}^2 = \mathbb{k}g_1 \oplus \mathbb{k}g_2.$$

Similarly one can check that the simple H_8 -comodule $M_4 = \mathbb{k}xe_0 \oplus \mathbb{k}ye_1$ is an irreducible $D(H_8)$ -module and since $C_3 = 2xe_0 = x + xz \in M_4$ we can say that

$$\mathcal{C}^3 = \mathbb{k}xe_0 \oplus \mathbb{k}ye_1.$$

By the same argument

$$\mathcal{C}^1 = \mathbb{k}xe_1 \oplus \mathbb{k}ye_0.$$

5.9 Presentation of the central primitive idempotents of H_8

The associated central primitive idempotents of V_0, V_1, V_2, V_3, V can be computed as

$$E_0 = \frac{e_0}{4}(1 + x + y + xy) = \Lambda$$

is the central primitive idempotent of χ_0 , i.e., the idempotent integral Λ_{H_8} of H_8 ,

$$E_1 = \frac{e_0}{4}(1 - x + y - xy)$$

is the central primitive idempotent of χ_1 ,

$$E_2 = \frac{e_0}{4}(1 - x - y + xy)$$

is the central primitive idempotent of χ_2 ,

$$E_3 = \frac{e_0}{4}(1 + x - y - xy)$$

is the central primitive idempotent of χ_3 ,

$$E_4 = e_1$$

is the central primitive idempotent of χ_4 .

5.10 The R -matrix and the Drinfeld map

It is well-known that H_8 is a semisimple quasitriangular Hopf algebra [23] with the R -matrix given by

$$R = \frac{1}{2}(1 \otimes 1 + g_2 \otimes 1 + 1 \otimes g_1 - g_2 \otimes g_1).$$

It follows that

$$\begin{aligned} Q = R_{21}R &= \frac{1}{4}(1 \otimes (1 + z + g_1 + g_2) + g_1 \otimes (1 + g_1 - g_2 - z) \\ &\quad + g_2 \otimes (1 - g_1 + g_2 - z) + z \otimes (1 - g_1 - g_2 + z)). \end{aligned}$$

5.10.1 The Drinfeld map on \mathbb{k} -linear basis

One can compute that the Drinfeld map $\phi_R: H_8^* \rightarrow H_8$, $f \mapsto f(Q^1)Q^2$ is given by

$$\begin{aligned} p_1 &\mapsto \Lambda_{\mathbb{k}G} = \frac{1}{4}(1 + g_1)(1 + g_2) = e_{0,0}, & p_z &\mapsto \frac{1}{4}(1 - g_1)(1 - g_2) = e_{11}, \\ p_{xy} &\mapsto \frac{1}{4}(1 + ig_1 - ig_2 - z), & p_{yx} &\mapsto \frac{1}{4}(1 - ig_1 + ig_2 - z), & p_x, p_y, p_{xz}, p_{yz} &\mapsto 0. \end{aligned}$$

5.10.2 The Drinfeld map given on the central primitive idempotents of the character ring

The following formulae hold for the Drinfeld map:

$$\phi_R(F_0) = E_0 + E_2, \quad \phi_R(F_1) = \phi_R(F_3) = 0, \quad \phi_R(F_2) = E_4, \quad \phi_R(F_4) = E_1 + E_3.$$

5.10.3 The map m and the subset \mathcal{J}

It follows that in this case the subset $\mathcal{J} = \{0, 2, 4\}$ and the function $m: \{0, 1, 2, 3, 4\} \rightarrow \mathcal{J}$ is given by

$$m(0) = m(2) = 0, \quad m(1) = m(3) = 4, \quad m(4) = 2. \quad (5.3)$$

Therefore $\phi_R(H_8^*) = \mathbb{k}G(H_8)$ and by Proposition 4.5 one has

$$\mathcal{O}(\text{Rep}(H_8)') = \{\chi_i \mid m(i) = 0\} = \{\chi_0, \chi_2\}.$$

5.11 Description of the centralizer based on the function m

In this subsection we describe the centralizer of any fusion subcategory of $\text{Rep}(H_8)$. For the entire category, $\text{Rep}(H_8)$ this was done above.

There are two normal coideal subalgebras of H_8 which are not Hopf subalgebras,

$$L_1 = \mathbb{k}\langle 1, z, ye_0, xe_1 \rangle = \text{LKer}_H(V_1)$$

and

$$L_3 = \mathbb{k}\langle 1, z, xe_0, ye_1 \rangle = \text{LKer}_H(V_3).$$

One has

$$L_1 = \mathcal{C}^0 \oplus \mathcal{C}^1 \oplus \mathcal{C}^4, \quad L_2 = \mathcal{C}^0 \oplus \mathcal{C}^3 \oplus \mathcal{C}^4.$$

Theorem 2.10 gives that $\text{Rep}(H_8/\text{LKer}_{H_8}(V_i)) \simeq \langle V_i \rangle$, the fusion subcategory of $\text{Rep}(H_8)$ generated by V_i . Using Theorem 1.2 it follows

$$\text{Rep}(A//L_1)' = \{\chi_i \mid \mathcal{C}^{m(i)} \subseteq L_1\} = \{\chi_i \mid m(i) \in \{0, 1, 4\}\}.$$

By description of the function m from equation (5.3) it follows that

$$\mathcal{O}(\langle V_1 \rangle') = \{\chi_0, \chi_1, \chi_2, \chi_3\}.$$

Similarly, one has that

$$\mathcal{O}(\langle V_3 \rangle') = \text{Rep}(A//L_3)' = \{\chi_i \mid \mathcal{C}^{m(i)} \subseteq L_3\} = \{\chi_i \mid m(i) \in \{0, 1, 3\}\}.$$

By description of the function m from equation (5.3) it follows that

$$\mathcal{O}(\langle V_3 \rangle') = \{\chi_0, \chi_1, \chi_2, \chi_3\}.$$

Note that for the central linear character $\chi_2 \in \mathcal{Z}(H_8^*)$ one has

$$L_2 := \text{LKer}_H(V_2) = \mathbb{k}G(H_8) = \mathcal{C}^0 \oplus \mathcal{C}^2 \oplus \mathcal{C}^4.$$

As above, using Theorem 1.2 it follows

$$\mathcal{O}(\langle V_2 \rangle') = \{\chi_i \mid m(i) \in \{0, 2, 4\}\} = \mathcal{O}(\text{Rep}(H_8)).$$

On the other hand since $\langle V_4 \rangle = \text{Rep}(H_8)$ Theorem 2.10 gives that $L_4 = \text{LKer}_{H_8}(V) = \mathbb{k} = \mathcal{C}^0$. Thus, using again Theorem 1.2 it follows that

$$\mathcal{O}(\langle V_4 \rangle') = \{\chi_i \mid m(i) = 0\} = \{\chi_0, \chi_2\} = \mathcal{O}(\text{Rep}(H_8)').$$

All the four one-dimensional centralize each other. V centralizes only χ_2 .

5.11.1 On the first commutator and adjoint subcategory

One has that the first commutator of H_8 is given by $H'_8 = \mathbb{k}\langle 1, z \rangle$ and moreover, for this Hopf algebra $\mathcal{C}_{\text{ad}} = \mathcal{C}_{\text{pt}}$. Thus in this case

$$\mathcal{C}'_{\text{ad}} = \langle V_1, V_2, V_3 \rangle' = \bigcap_{i=0}^3 \langle V_i \rangle' = \mathcal{C}_{\text{pt}}.$$

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