Symmetry, Integrability and Geometry: Methods and Applications SIGMA **16** (2020), 055, 15 pages

# A Note on Disk Counting in Toric Orbifolds

*Kwokwai CHAN* †[1], *Cheol-Hyun CHO* †[2], *Siu-Cheong LAU* †[3], *Naichung Conan LEUNG* †[4] *and Hsian-Hua TSENG* †[5]

†[1] *Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong*
E-mail: *kwchan@math.cuhk.edu.hk*

†[2] *Department of Mathematical Sciences, Research Institute in Mathematics, Seoul National University, Gwanak-Gu, Seoul, South Korea*
E-mail: *chocheol@snu.ac.kr*

†[3] *Department of Mathematics and Statistics, Boston University, Boston, MA, USA*
E-mail: *lau@math.bu.edu*

†[4] *The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong*
E-mail: *leung@math.cuhk.edu.hk*

†[5] *Department of Mathematics, Ohio State University, 100 Math Tower, 231 West 18th Ave., Columbus, OH 43210, USA*
E-mail: *hhtseng@math.ohio-state.edu*

Received January 24, 2020, in final form June 11, 2020; Published online June 17, 2020
https://doi.org/10.3842/SIGMA.2020.055

**Abstract.** We compute orbi-disk invariants of compact Gorenstein semi-Fano toric orbifolds by extending the method used for toric Calabi–Yau orbifolds. As a consequence the orbi-disc potential is analytic over complex numbers.

*Key words:* orbifold; toric; open Gromov–Witten invariants; mirror symmetry; SYZ

*2020 Mathematics Subject Classification:* 53D37; 14J33

## 1 Introduction

The mirror map plays a central role in the study of mirror symmetry. It provides a canonical local isomorphism between the Kähler moduli and the complex moduli of the mirror near a large complex structure limit. Such an isomorphism is crucial to counting of rational curves using mirror symmetry.

The mirror map is a transformation from the complex coordinates of the Hori–Vafa mirror moduli to the canonical coordinates obtained from period integrals. In [4] and [5], we derived an enumerative meaning of the inverse mirror maps for toric Calabi–Yau orbifolds and compact semi-Fano toric manifolds in terms of genus 0 open (orbifold) Gromov–Witten invariants (or (orbi-)disk invariants). Namely, we showed that coefficients of the inverse mirror map are equal to generating functions of virtual counts of stable (orbi-)disks bounded by a regular Lagrangian moment map fiber. In particular it gives a way to effectively compute all such invariants.

It is interesting to compare this with the mirror family constructed by Gross–Siebert [19], which is written in canonical coordinates [24]. In [19, Conjecture 0.2], it was conjectured that the wall-crossing functions in their construction are generating functions of open Gromov–Witten invariants. Our results verify this conjecture in the toric setting, namely, we showed that the SYZ mirror family [25], constructed using open Gromov–Witten invariants, is written in canonical coordinates.

In this short note we extend our method in [4] to derive an explicit formula for the orbi-disk invariants in the case of compact Gorenstein semi-Fano toric orbifolds; see Theorem 3.6 for the explicit formulas. This proves [3, Conjecture] for such orbifolds, generalizing [5, Theorem 1.2]:

**Theorem 1.1** (open mirror theorem)**.** *For a compact Gorenstein semi-Fano toric orbifold, the orbi-disk potential is equal to the* (*extended*) *Hori–Vafa superpotential via the mirror map.*

See (3.6) for the definition of the orbi-disc potential. We remark that the open crepant resolution conjecture [3, Conjecture 1] may be studied using this computation and techniques of analytical continuation in [4, Appendix A].

**Corollary 1.2.** *There exists an open neighborhood around the large volume limit where the orbi-disk potential converges.*

This generalizes [5, Theorem 7.6] to the orbifold case.

# 2 Preparation

## 2.1 Toric orbifolds

### 2.1.1 Construction

Following [2], a *stacky fan* is the combinatorial data $(\Sigma, \boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1})$, where $\Sigma$ is a simplicial fan contained in the $\mathbb{R}$-vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, $N$ is a lattice of rank $n$, and $\{\boldsymbol{b}_i \,|\, 0 \le i \le m-1\} \subset N$ are generators of 1-dimensional cones of $\Sigma$. $\boldsymbol{b}_i$ are called the *stacky vectors*.

Choose $\boldsymbol{b}_m, \ldots, \boldsymbol{b}_{m'-1} \in N$ so that they are contained in the support of the fan $\Sigma$ and they generate $N$ over $\mathbb{Z}$. An *extended stacky fan* in the sense of [22] is the data

$$\left(\Sigma, \{\boldsymbol{b}_i\}_{i=0}^{m-1} \cup \{\boldsymbol{b}_j\}_{j=m}^{m'-1}\right). \tag{2.1}$$

The vectors $\{\boldsymbol{b}_j\}_{j=m}^{m'-1}$ are called *extra vectors*.

The *fan map* associated to an extended stacky fan (2.1) is defined by

$$\phi\colon\ \widetilde{N} := \bigoplus_{i=0}^{m'-1} \mathbb{Z} e_i \to N, \qquad \phi(e_i) := \boldsymbol{b}_i \quad \text{for } i = 0, \ldots, m'-1.$$

$\phi$ is surjective and yields an exact sequence of groups called the *fan sequence*:

$$0 \longrightarrow \mathbb{L} := \mathrm{Ker}(\phi) \xrightarrow{\psi} \widetilde{N} \xrightarrow{\phi} N \longrightarrow 0. \tag{2.2}$$

Clearly $\mathbb{L} \simeq \mathbb{Z}^{m'-n}$. Tensoring (2.2) with $\mathbb{C}^\times$ yields the following sequence:

$$0 \longrightarrow G := \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{C}^\times \longrightarrow \widetilde{N} \otimes_{\mathbb{Z}} \mathbb{C}^\times \simeq (\mathbb{C}^\times)^{m'} \xrightarrow{\phi_{\mathbb{C}^\times}} \mathbb{T} := N \otimes_{\mathbb{Z}} \mathbb{C}^\times \to 0, \tag{2.3}$$

which is exact. Note that $G$ is an algebraic torus.

By definition, the set of *anti-cones* is

$$\mathcal{A} := \left\{ I \subset \{0, 1, \ldots, m'-1\} \,\middle|\, \sum_{i \notin I} \mathbb{R}_{\ge 0} \boldsymbol{b}_i \text{ is a cone in } \Sigma \right\}.$$

This terminology is justified because for $I \in \mathcal{A}$, the complement of $I$ in $\{0, 1, \ldots, m'-1\}$ indexes generators of a cone in $\Sigma$. For $I \in \mathcal{A}$, the collection $\{Z_i \,|\, i \in I\}$ generates an ideal in $\mathbb{C}[Z_0, \ldots, Z_{m'-1}]$, which in turn determines a subvariety $\mathbb{C}^I \subset \mathbb{C}^{m'}$. Set

$$U_{\mathcal{A}} := \mathbb{C}^{m'} \setminus \bigcup_{I \notin \mathcal{A}} \mathbb{C}^I.$$

The map $G \to (\mathbb{C}^\times)^{m'}$ in (2.3) defines a $G$-action on $\mathbb{C}^{m'}$ and hence a $G$-action on $U_{\mathcal{A}}$. This action is effective and has finite stabilizers, because $N$ is torsion-free (see [22, Section 2]). The *toric orbifold* associated to $\left(\Sigma, \{\boldsymbol{b}_i\}_{i=0}^{m-1} \cup \{\boldsymbol{b}_j\}_{j=m}^{m'-1}\right)$ is defined to be the following quotient stack:

$$\mathcal{X}_\Sigma := [U_{\mathcal{A}}/G].$$

The standard $(\mathbb{C}^\times)^{m'}$-action on $U_{\mathcal{A}}$ induces a $\mathbb{T}$-action on $\mathcal{X}_\Sigma$ via (2.3).

The coarse moduli space of the toric orbifold $\mathcal{X}_\Sigma$ is the toric variety $X_\Sigma$ associated to the fan $\Sigma$. In this paper we assume that $X_\Sigma$ is *semi-projective*, i.e., $X_\Sigma$ has a $\mathbb{T}$-fixed point and the natural map $X_\Sigma \to \operatorname{Spec} H^0(X_\Sigma, \mathcal{O}_{X_\Sigma})$ is projective, or equivalently, $X_\Sigma$ arises as a GIT quotient of a complex vector space by an abelian group (see [20, Section 2]). This assumption is required for the toric mirror theorem of [11] to hold. More detailed discussions on semi-projective toric varieties can be found in [12, Section 7.2].

### 2.1.2 Twisted sectors

Consider a $d$-dimensional cone $\sigma \in \Sigma$ generated by $\boldsymbol{b}_\sigma = (\boldsymbol{b}_{i_1}, \dots, \boldsymbol{b}_{i_d})$. Define

$$\mathrm{Box}_{\boldsymbol{b}_\sigma} := \left\{ \nu \in N \,\middle|\, \nu = \sum_{k=1}^{d} t_k \boldsymbol{b}_{i_k},\ t_k \in [0,1) \cap \mathbb{Q} \right\}.$$

$\{\boldsymbol{b}_{i_1}, \dots, \boldsymbol{b}_{i_d}\}$ generates a submodule $N_{\boldsymbol{b}_\sigma} \subset N$. One can check that there is a bijection between $\mathrm{Box}_{\boldsymbol{b}_\sigma}$ and the finite group $G_{\boldsymbol{b}_\sigma} := (N \cap \mathrm{Span}_{\mathbb{R}} \boldsymbol{b}_\sigma)/N_{\boldsymbol{b}_\sigma}$. Furthermore, if $\tau$ is a subcone of $\sigma$, then $\mathrm{Box}_{\boldsymbol{b}_\tau} \subset \mathrm{Box}_{\boldsymbol{b}_\sigma}$. Define

$$\mathrm{Box}^\circ_{\boldsymbol{b}_\sigma} \colon\ \mathrm{Box}_{\boldsymbol{b}_\sigma} \setminus \bigcup_{\tau \precneqq \sigma} \mathrm{Box}_{\boldsymbol{b}_\tau}, \qquad \mathrm{Box}(\Sigma) := \bigcup_{\sigma \in \Sigma^{(n)}} \mathrm{Box}_{\boldsymbol{b}_\sigma} = \bigsqcup_{\sigma \in \Sigma} \mathrm{Box}^\circ_{\boldsymbol{b}_\sigma},$$
$$\mathrm{Box}'(\Sigma) = \mathrm{Box}(\Sigma) \setminus \{0\},$$

where $\Sigma^{(n)}$ is the set of $n$-dimensional cones in $\Sigma$.

Following the description of the inertia orbifold of $\mathcal{X}_\Sigma$ in [2], for $\nu \in \mathrm{Box}(\Sigma)$, we denote by $\mathcal{X}_\nu$ the corresponding component of the inertia orbifold of $\mathcal{X} := \mathcal{X}_\Sigma$. Note that $\mathcal{X}_0 = \mathcal{X}_\Sigma$ as orbifolds. Elements $\nu \in \mathrm{Box}'(\Sigma)$ correspond to *twisted sectors* of $\mathcal{X}$, namely non-trivial connected components of the inertia orbifold of $\mathcal{X}$.

Following [7], the direct sum of singular cohomology groups of components of the inertia orbifold of $\mathcal{X}$, subject to a degree shift, is called the *Chen–Ruan orbifold cohomology* $H^*_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q})$ of $\mathcal{X}$. More precisely,

$$H^d_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q}) = \bigoplus_{\nu \in \mathrm{Box}} H^{d - 2\,\mathrm{age}(\nu)}(\mathcal{X}_\nu; \mathbb{Q}),$$

where $\mathrm{age}(\nu)$ is called the *degree shifting number*[1] in [7] of the twisted sector $\mathcal{X}_\nu$. In case of toric orbifolds, age has a combinatorial description [2]: if $\nu = \sum\limits_{k=1}^{d} t_k \boldsymbol{b}_{i_k} \in \mathrm{Box}(\Sigma)$ where $\{\boldsymbol{b}_{i_1}, \dots, \boldsymbol{b}_{i_d}\}$ generates a cone in $\Sigma$, then

$$\mathrm{age}(\nu) = \sum_{k=1}^{d} t_k \in \mathbb{Q}_{\geq 0}.$$

[1] Following Miles Reid, it is now more commonly called *age*.

Using $\mathbb{T}$-actions on twisted sectors induced from that on $\mathcal{X}$, we can define $\mathbb{T}$-*equivariant Chen–Ruan orbifold cohomology* $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X};\mathbb{Q})$ by replacing singular cohomology with $\mathbb{T}$-equivariant cohomology $H^*_{\mathbb{T}}(-)$. Namely

$$H^d_{\mathrm{CR},\mathbb{T}}(\mathcal{X};\mathbb{Q}) = \bigoplus_{\nu\in\mathrm{Box}} H^{d-2\,\mathrm{age}(\nu)}_{\mathbb{T}}(\mathcal{X}_\nu;\mathbb{Q}).$$

By general properties of equivariant cohomology, $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X};\mathbb{Q})$ is a module over $H^*_{\mathbb{T}}(\mathrm{pt},\mathbb{Q})$ and admits a map $H^*_{\mathrm{CR},\mathbb{T}}(\mathcal{X};\mathbb{Q}) \to H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$ called *non-equivariant limit*.

### 2.1.3 Toric divisors, Kähler cones, and Mori cones

We continue using the notations in Sections 2.1.1 and 2.1.2. Applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ to the fan sequence (2.2), we obtain the following exact sequence:[2]

$$0 \longrightarrow M := N^\vee = \mathrm{Hom}(N,\mathbb{Z}) \xrightarrow{\phi^\vee} \widetilde{M} := \widetilde{N}^\vee = \mathrm{Hom}\big(\widetilde{N},\mathbb{Z}\big) \xrightarrow{\psi^\vee} \mathbb{L}^\vee = \mathrm{Hom}(\mathbb{L},\mathbb{Z}) \longrightarrow 0,$$

which is called the *divisor sequence*. Line bundles on $\mathcal{X} = [\mathcal{U}_{\mathcal{A}}/G]$ correspond to $G$-equivariant line bundles on $\mathcal{U}_{\mathcal{A}}$. In view of (2.3), $\mathbb{T}$-equivariant line bundles on $\mathcal{X}$ correspond to $(\mathbb{C}^\times)^{m'}$-equivariant line bundles on $\mathcal{U}_{\mathcal{A}}$. Because the codimension of $\cup_{I\notin\mathcal{A}}\mathbb{C}^I \subset \mathbb{C}^{m'}$ is at least 2, the Picard groups satisfy:

$$\mathrm{Pic}(\mathcal{X}) \simeq \mathrm{Hom}(G,\mathbb{C}^\times) \simeq \mathbb{L}^\vee, \qquad \mathrm{Pic}_{\mathbb{T}}(\mathcal{X}) \simeq \mathrm{Hom}\big((\mathbb{C}^\times)^{m'},\mathbb{C}^\times\big) \simeq \widetilde{N}^\vee = \widetilde{M}.$$

The natural map $\mathrm{Pic}_{\mathbb{T}}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X})$ is identified with the map $\psi^\vee\colon \widetilde{M} \to \mathbb{L}^\vee$ appearing in the divisor sequence.

The elements $\{e_i^\vee \,|\, i = 0,1,\dots,m'-1\} \subset \widetilde{M} \simeq \mathrm{Pic}_{\mathbb{T}}(\mathcal{X})$ dual to $\{e_i \,|\, i = 0,1,\dots,m'-1\} \subset \widetilde{N}$ correspond to $\mathbb{T}$-equivariant line bundle on $\mathcal{X}$ which we denote by $D_i^{\mathbb{T}}$, $i = 0,1,\dots,m'-1$. The collection

$$\{D_i := \psi^\vee(e_i^\vee) \,|\, 0 \le i \le m-1\} \subset \mathbb{L}^\vee \simeq \mathrm{Pic}(\mathcal{X})$$

consists of toric prime divisors corresponding to the generators $\{\boldsymbol{b}_i \,|\, 0 \le i \le m-1\}$ of 1-dimensional cones in $\Sigma$. Elements $D_i^{\mathbb{T}}$, $0 \le i \le m-1$ are $\mathbb{T}$-equivariant lifts of these divisors. There are natural maps

$$\widetilde{M}\otimes\mathbb{Q} \overset{\psi^\vee\otimes\mathbb{Q}}{\to} \mathbb{L}^\vee\otimes\mathbb{Q},$$

$$\big(\widetilde{M}\otimes\mathbb{Q}\big)\Big/\left(\sum_{j=m}^{m'-1}\mathbb{Q}D_j^{\mathbb{T}}\right) \simeq H^2_{\mathbb{T}}(\mathcal{X},\mathbb{Q}) \to H^2(\mathcal{X},\mathbb{Q}) \simeq \big(\mathbb{L}^\vee\otimes\mathbb{Q}\big)\Big/\left(\sum_{j=m}^{m'-1}\mathbb{Q}D_j\right).$$

Together with the natural quotient maps, they fit into a commutative diagram.

As explained in [21, Section 3.1.2], there is a canonical splitting of the quotient map $\mathbb{L}^\vee\otimes\mathbb{Q} \to H^2(\mathcal{X};\mathbb{Q})$. For $m \le j \le m'-1$, let $I_j \in \mathcal{A}$ be the anticone of the cone containing $\boldsymbol{b}_j$. This allows us to write $\boldsymbol{b}_j = \sum_{i\notin I_j} c_{ji}\boldsymbol{b}_i$ for $c_{ji} \in \mathbb{Q}_{\ge 0}$.

Tensoring the fan sequence (2.2) with $\mathbb{Q}$, we may find a unique $D_j^\vee \in \mathbb{L}\otimes\mathbb{Q}$ such that values of the natural pairing $\langle -,-\rangle$ between $\mathbb{L}^\vee$ and $\mathbb{L}$ satisfy

$$\langle D_i, D_j^\vee\rangle = \begin{cases} 1 & \text{if } i = j,\\ -c_{ji} & \text{if } i \notin I_j,\\ 0 & \text{if } i \in I_j\setminus\{j\}.\end{cases} \tag{2.4}$$

[2] The map $\psi^\vee : \widetilde{M} \to \mathbb{L}^\vee$ is surjective since $N$ is torsion-free.

Using $D_j^\vee$ we get a decomposition

$$\mathbb{L}^\vee \otimes \mathbb{Q} = \operatorname{Ker}\left( (D_m^\vee, \dots, D_{m'-1}^\vee) \colon \mathbb{L}^\vee \otimes \mathbb{Q} \to \mathbb{Q}^{m'-m} \right) \oplus \bigoplus_{j=m}^{m'-1} \mathbb{Q} D_j. \tag{2.5}$$

We can view $H^2(\mathcal{X};\mathbb{Q})$ as a subspace of $\mathbb{L}^\vee \otimes \mathbb{Q}$ because $\operatorname{Ker}\left((D_m^\vee, \dots, D_{m'-1}^\vee)\right)$ can be identified with $H^2(\mathcal{X};\mathbb{Q})$ via the map $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X};\mathbb{Q})$.

Define *extended Kähler cone* of $\mathcal{X}$ to be

$$\widetilde{C}_{\mathcal{X}} := \bigcap_{I \in \mathcal{A}} \left( \sum_{i \in I} \mathbb{R}_{>0} D_i \right) \subset \mathbb{L}^\vee \otimes \mathbb{R}.$$

The Kähler cone $C_{\mathcal{X}}$ is the image of $\widetilde{C}_{\mathcal{X}}$ under $\mathbb{L}^\vee \otimes \mathbb{R} \to H^2(\mathcal{X};\mathbb{R})$. The splitting (2.5) of $\mathbb{L}^\vee \otimes \mathbb{Q}$ yields a splitting $\widetilde{C}_{\mathcal{X}} = C_{\mathcal{X}} + \sum_{j=m}^{m'-1} \mathbb{R}_{>0} D_j$.

By (2.2), $\mathbb{L}^\vee$ has rank equal to $r := m' - n$. The rank of $H_2(\mathcal{X};\mathbb{Z})$ is $r' := r - (m' - m) = m - n$. We choose an integral basis

$$\{p_1, \dots, p_r\} \subset \mathbb{L}^\vee,$$

such that $p_a$ is in the closure of $\widetilde{C}_{\mathcal{X}}$ for all $a$ and $p_{r'+1}, \dots, p_r \in \sum_{i=m}^{m'-1} \mathbb{R}_{\geq 0} D_i$. We get a nef basis $\{\bar{p}_1, \dots, \bar{p}_{r'}\}$ for $H^2(\mathcal{X};\mathbb{Q})$ as images of $\{p_1, \dots, p_{r'}\}$ under the quotient map $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X};\mathbb{Q})$. For $r' + 1 \leq a \leq r$, the images satisfies $\bar{p}_a = 0$.

We choose equivariant lifts of $p_a$'s, namely $\{p_1^{\mathbb{T}}, \dots, p_r^{\mathbb{T}}\} \subset \widetilde{M} \otimes \mathbb{Q}$ such that $\psi^\vee(p_a^{\mathbb{T}}) = p_a$ for all $a$. We also require that for $a = r'+1, \dots, r$ the images $\bar{p}_a^{\mathbb{T}}$ of $p_a^{\mathbb{T}}$ under the natural map $\widetilde{M} \otimes \mathbb{Q} \to H^2_{\mathbb{T}}(\mathcal{X}, \mathbb{Q})$ satisfies $\bar{p}_a^{\mathbb{T}} = 0$.

The coefficients $Q_{ia} \in \mathbb{Z}$ in the equations $D_i = \sum_{a=1}^{r} Q_{ia} p_a$ assemble to a matrix $(Q_{ia})$. The images[3] $\bar{D}_i$ of $D_i$ under the map $\mathbb{L}^\vee \otimes \mathbb{Q} \to H^2(\mathcal{X};\mathbb{Q})$ can be expressed as

$$\bar{D}_i = \sum_{a=1}^{r'} Q_{ia} \bar{p}_a, \qquad i = 0, \dots, m-1.$$

Their equivariant lifts $\bar{D}_i^{\mathbb{T}}$ can be expressed as

$$\bar{D}_i^{\mathbb{T}} = \sum_{a=1}^{r'} Q_{ia} \bar{p}_a^{\mathbb{T}} + \lambda_i, \qquad \text{where} \quad \lambda_i \in H^2(B\mathbb{T};\mathbb{Q}).$$

For $i = m, \dots, m'-1$, we have $\bar{D}_i = 0$ in $H^2(\mathcal{X};\mathbb{R})$ and $\bar{D}_i^{\mathbb{T}} = 0$.

Localization gives the following description of $H^{\leq 2}_{\mathrm{CR},\mathbb{T}}$:

$$H^0_{\mathrm{CR},\mathbb{T}}(\mathcal{X}, K_{\mathbb{T}}) = K_{\mathbb{T}} \mathbf{1}, \qquad H^2_{\mathrm{CR},\mathbb{T}}(\mathcal{X}, K_{\mathbb{T}}) = \bigoplus_{a=1}^{r'} K_{\mathbb{T}} \bar{p}_a^{\mathbb{T}} \oplus \bigoplus_{\nu \in \mathrm{Box}, \mathrm{age}(\nu)=1} K_{\mathbb{T}} \mathbf{1}_\nu.$$

Here $K_{\mathbb{T}}$ is the field of fractions of $H^*_{\mathbb{T}}(\mathrm{pt}, \mathbb{Q})$, $\mathbf{1} \in H^0(\mathcal{X}, \mathbb{Q})$ and $\mathbf{1}_\nu \in H^0(\mathcal{X}_\nu, \mathbb{Q})$ are fundamental classes.

[3] $\bar{D}_i$ is the class of the toric prime divisor $D_i$.

Let

$$\{\gamma_1, \dots, \gamma_r\} \subset \mathbb{L}, \qquad \gamma_a = \sum_{i=0}^{m'-1} Q_{ia} e_i \in \widetilde{N},$$

be the basis dual to $\{p_1, \dots, p_r\} \subset \mathbb{L}^\vee$. $H_2^{\mathrm{eff}}(\mathcal{X};\mathbb{Q})$ admits a basis $\{\gamma_1, \dots, \gamma_{r'}\}$, and we have $Q_{ia} = 0$ when $m \le i \le m'-1$ and $1 \le a \le r'$.

Set

$$\mathbb{K} := \{d \in \mathbb{L} \otimes \mathbb{Q} \,|\, \{j \in \{0,1,\dots,m'-1\} \,|\, \langle D_j, d\rangle \in \mathbb{Z}\} \in \mathcal{A}\},$$
$$\mathbb{K}_{\mathrm{eff}} := \{d \in \mathbb{L} \otimes \mathbb{Q} \,|\, \{j \in \{0,1,\dots,m'-1\} \,|\, \langle D_j, d\rangle \in \mathbb{Z}_{\ge 0}\} \in \mathcal{A}\}.$$

Elements of $\mathbb{K}_{\mathrm{eff}}$ should be interpreted as effective curve classes. Elements of $\mathbb{K}_{\mathrm{eff}} \cap H_2(\mathcal{X};\mathbb{R})$ should be viewed as classes of stable maps $\mathbb{P}(1,m) \to \mathcal{X}$ for some $m \in \mathbb{Z}_{\ge 0}$. See, e.g., [21, Section 3.1] for more details.

**Definition 2.1.** A toric orbifold $\mathcal{X}$ is called *semi-Fano* if $c_1(\mathcal{X}) \cdot \alpha > 0$ for every effective curve class $\alpha$, in other words, $-K_{\mathcal{X}}$ is nef.

For $d \in \mathbb{K}$, put[4]

$$\nu(d) := \sum_{i=0}^{m'-1} \lceil \langle D_i, d\rangle \rceil \boldsymbol{b}_i \in N,$$

and let $I_d := \{j \in \{0,1,\dots,m'-1\} \,|\, \langle D_j, d\rangle \in \mathbb{Z}\} \in \mathcal{A}$. Then $\nu(d) \in \mathrm{Box}$ because

$$\nu(d) = \sum_{i=0}^{m'-1} (\{-\langle D_i, d\rangle\} + \langle D_i, d\rangle) \boldsymbol{b}_i = \sum_{i=0}^{m'-1} \{-\langle D_i, d\rangle\} \boldsymbol{b}_i = \sum_{i \notin I_d} \{-\langle D_i, d\rangle\} \boldsymbol{b}_i.$$

## 2.2 Genus 0 open orbifold GW invariants according to [9]

Let $(\mathcal{X}, \omega)$ be a toric Kähler orbifold of complex dimension $n$, equipped with the standard toric complex structure $J_0$ and a toric Kähler structure $\omega$. Denote by $(\Sigma, \boldsymbol{b})$ the stacky fan that defines $\mathcal{X}$, where $\boldsymbol{b} = (\boldsymbol{b}_0, \dots, \boldsymbol{b}_{m-1})$ and $\boldsymbol{b}_i = c_i v_i$.

Let $L \subset \mathcal{X}$ be a Lagrangian torus fiber of the moment map $\mu_0\colon \mathcal{X} \to M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$, and let $\beta \in \pi_2(\mathcal{X}, L) = H_2(\mathcal{X}, L; \mathbb{Z})$ be a relative homotopy class.

### 2.2.1 Holomorphic orbi-disks and their moduli spaces

A *holomorphic orbi-disk* in $\mathcal{X}$ with boundary in $L$ is a continuous map

$$w\colon\ (\mathbb{D}, \partial\mathbb{D}) \to (\mathcal{X}, L)$$

satisfying the following conditions:

1. $(\mathbb{D}, z_1^+, \dots, z_l^+)$ is an *orbi-disk* with interior marked points $z_1^+, \dots, z_l^+$. More precisely $\mathbb{D}$ is analytically the disk $D^2 \subset \mathbb{C}$ so that for $j = 1, \dots, l$, the orbifold structure at $z_j^+$ is given by a disk neighborhood of $z_j^+$ uniformized by the branched covering map $\mathrm{br}\colon z \to z^{m_j}$ for some $m_j \in \mathbb{Z}_{>0}$. (If $m_j = 1$, $z_j^+$ is not an orbifold point.)

[4] For a real number $\lambda \in \mathbb{R}$, let $\lceil \lambda \rceil$, $\lfloor \lambda \rfloor$ and $\{\lambda\}$ denote the ceiling, floor and fractional part of $\lambda$ respectively.

2. For any $z_0 \in \mathbb{D}$, there is a disk neighborhood of $z_0$ with a branched covering map $\mathrm{br}\colon z \to z^m$, and there is a local chart $\left(V_{w(z_0)}, G_{w(z_0)}, \pi_{w(z_0)}\right)$ of $\mathcal{X}$ at $w(z_0)$ and a local holomorphic lifting $\widetilde{w}_{z_0}$ of $w$ satisfying $w \circ \mathrm{br} = \pi_{w(z_0)} \circ \widetilde{w}_{z_0}$.
3. The map $w$ is *good* (in the sense of Chen–Ruan [6]) and *representable*. In particular, for each $z_j^+$, the associated group homomorphism

$$h_p\colon\ \mathbb{Z}_{m_j} \to G_{w(z_j^+)}$$

between local groups which makes $\widetilde{w}_{z_j^+}$ equivariant, is *injective*.

The *type* of a map $w$ as above is defined to be $\boldsymbol{x} := (\mathcal{X}_{\nu_1}, \dots, \mathcal{X}_{\nu_l})$. Here $\nu_j \in \mathrm{Box}(\Sigma)$ is the image of the generator $1 \in \mathbb{Z}_{m_j}$ under $h_j$.

There are two notions of Maslov index for an orbi-disk. The *desingularized Maslov index* $\mu^{\mathrm{de}}$ is defined by desingularizing the interior singularities of the pull-back bundle $w^*T\mathcal{X}$. Namely, the bundle $w^*T\mathcal{X}$ over an orbi-disk $\left(\mathbb{D}, z_1^+, \dots, z_l^+\right)$ cannot be trivialized due to the orbifold structure, but we can obtain another bundle $|w^*T\mathcal{X}|$ by modifying the bundle near orbifold points (see Chen–Ruan [6] for more details). This is called a desingularization of $w^*T\mathcal{X}$ and it is a smooth bundle over the orbi-disk, hence is a trivial bundle. We can compute the Maslov index of the boundary Lagrangian loop relative to this trivialization, and it is called the desingularized Maslov index. See [9, Section 3] for more details and [9, Section 5] for an explicit formula in the toric case.

The *Chern–Weil* (*CW*) *Maslov index* $\mu_{\mathrm{CW}}$ is defined as the integral of the curvature of a unitary connection on $w^*T\mathcal{X}$ which preserves the Lagrangian boundary condition, see [10] (and also [9, Section 3.3] for a relation with $\mu^{\mathrm{de}}$). The following lemma, which appeared as [4, Lemma 3.1], computes the CW Maslov indices of disks. This is an orbifold version of the formula in [1, Lemma 3.1].

**Lemma 2.2.** *Let $(\mathcal{X}, \omega, J)$ be a Kähler orbifold of complex dimension $n$. Let $\Omega$ be a non-zero meromorphic $n$-form on $\mathcal{X}$ which has at worst simple poles. Let $D \subset \mathcal{X}$ be the pole divisor of $\Omega$. Suppose also that the generic points of $D$ are smooth. Then for a special Lagrangian submanifold $L \subset \mathcal{X} \setminus D$, the CW Maslov index of a class $\beta \in \pi_2(\mathcal{X}, L)$ is given by*

$$\mu_{\mathrm{CW}}(\beta) = 2\beta \cdot D.$$

*Here, $\beta \cdot D$ is defined by writing $\beta$ as a fractional linear combination of homotopy classes of smooth disks.*

The classification of orbi-disks in a symplectic toric orbifold has been worked out in [9, Theorem 6.2]. It is similar to the classification of holomorphic discs in toric manifolds [8]. In the classification, the *basic disks* corresponding to the stacky vectors (and twisted sectors) play a basic role.

**Theorem 2.3** ([9, Corollaries 6.3 and 6.4])**.** *Let $\mathcal{X}$ be a toric Kähler orbifold and let $L$ be a fiber of the toric moment map.*

1. *The* smooth *holomorphic disks of Maslov index* 2 (*modulo $\mathbb{T}^n$-action and reparametrizations of the domain*) *are in bijective correspondence with the stacky vectors $\{\boldsymbol{b}_0, \dots, \boldsymbol{b}_{m-1}\}$. Denote the homotopy classes of these disks by $\beta_0, \dots, \beta_{m-1}$.*
2. *The holomorphic orbi-disks with one interior orbifold marked point and desingularized Maslov index* 0 (*modulo $\mathbb{T}^n$-action and reparametrizations of the domain*) *are in bijective correspondence with the twisted sectors $\nu \in \mathrm{Box}'(\Sigma)$ of the toric orbifold $\mathcal{X}$. Denote the homotopy classes of these orbi-disks by $\beta_\nu$.*

**Lemma 2.4** ([9, Lemma 9.1])**.** *For $\mathcal{X}$ and $L$ as above, the relative homotopy group $\pi_2(\mathcal{X}, L)$ is generated by the classes $\beta_i$ for $i = 0, \dots, m-1$ together with $\beta_\nu$ for $\nu \in \mathrm{Box}'(\Sigma)$.*

As in [9], these generators of $\pi_2(\mathcal{X}, L)$ are called *basic disk classes*. They are the analogue of Maslov index 2 disk classes in toric manifolds.

Let

$$\mathcal{M}^{\mathrm{op,main}}_{k+1,l}(\mathcal{X}, L, \beta, \boldsymbol{x})$$

be the moduli space of good representable stable maps from bordered orbifold Riemann surfaces of genus zero with $k+1$ boundary marked points $z_0, z_1, \dots, z_k$ and $l$ interior (orbifold) marked points $z_1^+, \dots, z_l^+$ in the homotopy class $\beta$ of type $\boldsymbol{x} = (\mathcal{X}_{\nu_1}, \dots, \mathcal{X}_{\nu_l})$. The superscript "*main*" is meant to indicate the connected component on which the boundary marked points respect the cyclic order of $S^1 = \partial D^2$. According to [9, Lemma 2.5], $\mathcal{M}^{\mathrm{op,main}}_{k+1,l}(\mathcal{X}, L, \beta, \boldsymbol{x})$ has real virtual dimension

$$n + \mu_{\mathrm{CW}}(\beta) + k + 1 + 2l - 3 - 2\sum_{j=1}^{l} \mathrm{age}(\nu_j).$$

By [9, Proposition 9.4], if $\mathcal{M}^{\mathrm{op,main}}_{1,1}(\mathcal{X}, L, \beta)$ is non-empty and if $\partial\beta$ is not in the sublattice generated by $\boldsymbol{b}_0, \dots, \boldsymbol{b}_{m-1}$, then there exist $\nu \in \mathrm{Box}'(\Sigma)$, $k_0, \dots, k_{m-1} \in \mathbb{N}$ and $\alpha \in H_2^{\mathrm{eff}}(\mathcal{X})$ such that $\beta = \beta_\nu + \sum_{i=0}^{m-1} k_i\beta_i + \alpha$, where $\alpha$ is realized by a union of holomorphic (orbi-)spheres. The CW Maslov index of $\beta$ written in this way is given by $\mu_{\mathrm{CW}}(\beta) = 2\,\mathrm{age}(\nu) + 2\sum_{i=0}^{m-1} k_i + 2c_1(\mathcal{X}) \cdot \alpha$.

### 2.2.2 Orbi-disk invariants

Pick twisted sectors $\mathcal{X}_{\nu_1}, \dots, \mathcal{X}_{\nu_l}$ of the toric orbifold $\mathcal{X}$. Consider the moduli space

$$\mathcal{M}^{\mathrm{op,main}}_{1,l}(\mathcal{X}, L, \beta, \boldsymbol{x})$$

of good representable stable maps from bordered orbifold Riemann surfaces of genus zero with one boundary marked point and $l$ interior orbifold marked points of type $\boldsymbol{x} = (\mathcal{X}_{\nu_1}, \dots, \mathcal{X}_{\nu_l})$ representing the class $\beta \in \pi_2(\mathcal{X}, L)$. According to [9], $\mathcal{M}^{\mathrm{op,main}}_{1,l}(\mathcal{X}, L, \beta, \boldsymbol{x})$ can be equipped with a virtual fundamental chain, which has an expected dimension $n$ if the following equality holds:

$$\mu_{\mathrm{CW}}(\beta) = 2 + \sum_{j=1}^{l} (2 \cdot \mathrm{age}(\nu_j) - 2). \tag{2.6}$$

Throughout the paper, we make the following assumptions.

**Assumption 2.5.** *We assume that the toric orbifold $\mathcal{X}$ is semi-Fano* (*see Definition* 2.1) *and* Gorenstein.[5] *Moreover, we assume that the type $\boldsymbol{x}$ consists of twisted sectors with* age $\leq 1$.[6]

Then the age of every twisted sector of $\mathcal{X}$ is a non-negative integer. Since a basic orbi-disk class $\beta_\nu$ has Maslov index $2\,\mathrm{age}(\nu)$, we see that every non-constant stable disk class has at least Maslov index 2.

[5] This means that $K_{\mathcal{X}}$ is Cartier.

[6] This assumption does not impose any restriction in the construction of the SYZ mirror over $H^{\leq 2}_{\mathrm{CR}}(\mathcal{X})$. We do not discuss mirror construction in this paper.

Moreover, the virtual fundamental chain $\left[\mathcal{M}_{1,l}^{\mathrm{op,main}}(\mathcal{X}, L, \beta, \boldsymbol{x})\right]^{\mathrm{vir}}$ has expected dimension $n$ when $\mu_{\mathrm{CW}}(\beta) = 2$, and in fact we get a virtual fundamental *cycle* because $\beta$ attains the minimal Maslov index, thus preventing disk bubbling to occur. Therefore the following definition of *genus* 0 *open orbifold GW invariants* (also known as *orbi-disk invariants*) is independent of the choice of perturbations of the Kuranishi structures:[7]

**Definition 2.6** (orbi-disk invariants)**.** Let $\beta \in \pi_2(\mathcal{X}, L)$ be a relative homotopy class with Maslov index given by (2.6). Suppose that the moduli space $\mathcal{M}_{1,l}^{\mathrm{op,main}}(\mathcal{X}, L, \beta, \boldsymbol{x})$ has a virtual fundamental cycle of dimension $n$. Then we define

$$n_{1,l,\beta}^{\mathcal{X}}([\mathrm{pt}]_L; \mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l}) := \mathrm{ev}_{0*}\left(\left[\mathcal{M}_{1,l}^{\mathrm{op,main}}(\mathcal{X}, L, \beta, \boldsymbol{x})\right]^{\mathrm{vir}}\right) \in H_n(L; \mathbb{Q}) \cong \mathbb{Q},$$

where $\mathrm{ev}_0 \colon \mathcal{M}_{1,l}^{\mathrm{op,main}}(\mathcal{X}, L, \beta, \boldsymbol{x}) \to L$ is the evaluation map at the boundary marked point, $[\mathrm{pt}]_L \in H^n(L; \mathbb{Q})$ is the point class of $L$, and $\mathbf{1}_{\nu_j} \in H^0(\mathcal{X}_{\nu_j}; \mathbb{Q}) \subset H_{\mathrm{CR}}^{2\,\mathrm{age}(\nu_j)}(\mathcal{X}; \mathbb{Q})$ is the fundamental class of the twisted sector $\mathcal{X}_{\nu_j}$.

**Remark 2.7.** The Kuranishi structures in this paper are the same as those defined in [14, 15], incorporating the works [6, 7] for the interior orbifold marked points. This has been explained in [9, Section 10]. We also refer the readers to [13, Appendix] and [16] for the detailed construction, and to [23] (and its forthcoming sequels) for a different approach.

The moduli spaces considered here are in fact much simpler than those in [14, 15] (and [13]) because we only need to consider stable disks with just one disk component which is minimal, and hence disk bubbling does not occur. In particular, we do not have codimension-one boundary components, and hence the above definition is independent of choices of Kuranishi perturbations.

For a basic (orbi-)disk with at most one interior orbifold marked point, the corresponding moduli space $\mathcal{M}_{1,0}^{\mathrm{op,main}}(\mathcal{X}, L, \beta_i)$ (or $\mathcal{M}_{1,1}^{\mathrm{op,main}}(\mathcal{X}, L, \beta_\nu, \nu)$ when $\beta_\nu$ is a basic orbi-disk class) is regular and can be identified with $L$. Thus the associated invariants are evaluated as follows [9]:

1. For $\nu \in \mathrm{Box}'$, we have $n_{1,1,\beta_\nu}^{\mathcal{X}}([\mathrm{pt}]_L; \mathbf{1}_\nu) = 1$.
2. For $i \in \{0, \ldots, m-1\}$, we have $n_{1,0,\beta_i}^{\mathcal{X}}([\mathrm{pt}]_L) = 1$.

When there are more interior orbifold marked points or when the disk class is not basic, the corresponding moduli space is in general non-regular and virtual theory is involved in the definition, making the invariant much more difficult to compute.

# 3 Geometric constructions

Let $\beta \in \pi_2(\mathcal{X}, L)$ be a disk class with $\mu_{\mathrm{CW}}(\beta) = 2$. By the discussion in Section 2.2, we can write

$$\beta = \beta_{\mathbf{d}} + \alpha$$

with $\alpha \in H_2(\mathcal{X}, \mathbb{Z})$, $c_1(\mathcal{X}) \cdot \alpha = 0$ and either $\beta_{\mathbf{d}} \in \{\beta_0, \ldots, \beta_{m-1}\}$ or $\beta_{\mathbf{d}} \in \mathrm{Box}'(\mathcal{X})$ is of age 1. Denote by $\boldsymbol{b}_d \in N$ the element corresponding to $\beta_{\mathbf{d}}$.

Recall that the fan polytope $\mathcal{P} \subset N_{\mathbb{R}}$ is the convex hull of the vectors $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}$. Note that $\boldsymbol{b}_d \in \mathcal{P}$. Denote by $F(\boldsymbol{b}_d)$ the minimal face of the fan polytope $\mathcal{P}$ that contains the vector $\boldsymbol{b}_d$. Let $F$ be a facet of $\mathcal{P}$ that contains $F(\boldsymbol{b}_d)$. Let $\Sigma_{\beta_{\mathbf{d}}} \subset \Sigma$ be the minimal convex subfan containing all $\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}\} \cap F$. The vectors

$$\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m'-1}\} \cap \sum_{\boldsymbol{b}_j \in \{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}\} \cap F} \mathbb{Q}_{\geq 0} \boldsymbol{b}_j \tag{3.1}$$

[7] In the general case one may restrict to torus-equivariant perturbations, as did in [14, 15, 17].

determine a fan map $\mathbb{Z}^p \to N$ (where $p$ is the number of vectors above). Let

$$\mathcal{X}_{\beta_{\mathbf{d}}} \subset \mathcal{X}$$

be the associated toric suborbifold (of the same dimension $n$).

**Lemma 3.1.** *$\mathcal{X}_{\beta_{\mathbf{d}}}$ is a toric Calabi–Yau orbifold.*

**Proof.** All the generators in (3.1) lie in the hyperplane containing $F$. Since $\mathcal{X}$ is Gorenstein, this hyperplane has a defining equation $\nu = 0$ for some primitive vector $\nu \in M$. Hence $\mathcal{X}_{\beta_{\mathbf{d}}}$ is toric Calabi–Yau. ■

**Example 3.2.** Consider $\mathbb{P}^2/\mathbb{Z}_3$, whose fan is shown in the left of Fig. 1. If $\beta_{\mathbf{d}}$ corresponds to the vector $(1,0)$ (which is marked as '113' in the figure), then $\Sigma_{\beta_{\mathbf{d}}}$ is the cone spanned by $v_2 = (2,-1)$ and $v_3 = (-1,2)$. If $\beta_{\mathbf{d}}$ corresponds to the vector $v_3$, then $\Sigma_{\beta_{\mathbf{d}}}$ can be taken to be the cone spanned by $v_2$, $v_3$, or the cone spanned by $v_1$, $v_3$. In both cases, the corresponding toric Calabi–Yau orbifold is $\mathbb{C}^2/\mathbb{Z}_3$.

Note that $\mathcal{X}_{\beta_{\mathbf{d}}}$ depends on the choice of the face $F$, not just $\beta_{\mathbf{d}}$. We use $\mathcal{X}_{\beta_{\mathbf{d}}}$ to compute open Gromov–Witten invariants of $\mathcal{X}$ in class $\beta = \beta_{\mathbf{d}} + \alpha$.

In what follows we show that $\mathcal{X}_{\beta_{\mathbf{d}}} \subset \mathcal{X}$ contains all stable orbi-disks of $\mathcal{X}$ of class $\beta$. First, we have the following analogue of [5, Proposition 5.6].

**Lemma 3.3.** *Let $f\colon \mathcal{D} \cup \mathcal{C} \to \mathcal{X}$ be a stable orbi-disk map in the class $\beta = \beta_{\mathbf{d}} + \alpha$, where $\mathcal{D}$ is a (possibly orbifold) disk and $\mathcal{C}$ is a (possibly orbifold) rational curve such that $f_*[\mathcal{D}] = \beta_{\mathbf{d}}$ and $f_*[\mathcal{C}] = \alpha$ with $c_1(\alpha) = 0$. Then we have*

$$f(\mathcal{C}) \subset \bigcup_{\boldsymbol{b}_j \in F(\boldsymbol{b}_d)} D_j,$$

*and $[f(\mathcal{C})] \cdot D_j = 0$ whenever $\boldsymbol{b}_j \notin F(\boldsymbol{b}_d)$.*

**Proof.** Since $c_1(\alpha) = 0$, $f(\mathcal{C})$ should lie in toric divisors of $\mathcal{X}$. Recall that $\beta_{\mathbf{d}}$ achieves the minimal Maslov index 2, and hence there is no disc bubbling.

Suppose $\beta_{\mathbf{d}}$ is a smooth disk class. Then each sphere component $\mathcal{C}_0$ meeting the disk component $\mathcal{D}$ maps into the divisor $D_{\mathbf{d}}$ and it should have non-negative intersection with other toric divisors. By [18, Lemma 4.5] which easily extends to the simplicial setting, we have the desired statement for $f(\mathcal{C}_0)$.

If $\beta_{\mathbf{d}}$ is an orbi-disk class, then we can write the corresponding $\boldsymbol{b}_d \in N$ as $\boldsymbol{b}_d = \sum_{\boldsymbol{b}_i \in \sigma} c_i \boldsymbol{b}_i$, with $\sum_i c_i = 1, c_i \in [0,1) \cap \mathbb{Q}$. For a sphere component $\mathcal{C}_0$ meeting the disk component $\mathcal{D}$, we have $f(\mathcal{C}_0) \subset \bigcup_{\boldsymbol{b}_i \in \sigma} D_i$ and each $\boldsymbol{b}_i \in \sigma$ satisfies $\boldsymbol{b}_i \in F(\boldsymbol{b}_d)$. Hence $f(\mathcal{C}_0) \subset \bigcup_{\boldsymbol{b}_i \in F(\boldsymbol{b}_d)} D_i$ and $f(\mathcal{C}_0) \cdot D_j = 0$ for $\boldsymbol{b}_j \notin F(\boldsymbol{b}_d)$.

Let $\mathcal{C}_1 \subset \mathcal{C}$ be a sphere component meeting $\mathcal{C}_0$, then we have $f(\mathcal{C}_1) \subset F(\boldsymbol{b}_j)$ for some $\boldsymbol{b}_j \in F(\boldsymbol{b}_d)$ by the intersection condition. Now, we can follow the proof of [5, Proposition 5.6] shows that $f(\mathcal{C}_1) \subset \bigcup_{\boldsymbol{b}_i \in F(\boldsymbol{b}_d)} D_i$. The result follows by repeating this argument for one sphere component at a time. ■

Partition $\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}\} \cap F(\boldsymbol{b}_d)$ into the disjoint union of two subsets,

$$\{\boldsymbol{b}_0, \ldots, \boldsymbol{b}_{m-1}\} \cap F(\boldsymbol{b}_d) = F(\boldsymbol{b}_d)^c \coprod F(\boldsymbol{b}_d)^{nc},$$

where $\boldsymbol{b}_i \in F(\boldsymbol{b}_d)^c$ if $D_i \subset \mathcal{X}_{\beta_{\mathbf{d}}}$ and $\boldsymbol{b}_i \in F(\boldsymbol{b}_d)^{nc}$ if $D_i \not\subset \mathcal{X}_{\beta_{\mathbf{d}}}$.

**Lemma 3.4.** *Let $f\colon \mathcal{D}\cup\mathcal{C}\to\mathcal{X}$ be as in Lemma 3.3. Then we have $f(\mathcal{D}\cup\mathcal{C})\subset\mathcal{X}_{\beta_{\mathbf{d}}}$.*

**Proof.** Certainly $f(\mathcal{D})\subset\mathcal{X}_{\beta_{\mathbf{d}}}$. We claim that

$$f(\mathcal{C})\subset\bigcup_{\boldsymbol{b}_j\in F(\boldsymbol{b}_d)^c} D_j, \tag{3.2}$$

from which the lemma follows.

To see (3.2), we write $\mathcal{C}=\mathcal{C}_c\cup\mathcal{C}_{nc}$ where $\mathcal{C}_c$ consists of components of $\mathcal{C}$ which lie in $\bigcup_{\boldsymbol{b}_j\in F(\boldsymbol{b}_d)^c} D_j$, and $\mathcal{C}_{nc}$ consists of the remaining components. Set $A:=f_*[\mathcal{C}_c]$ and $B:=f_*[\mathcal{C}_{nc}]$. Then $\alpha=A+B$. Since $-K_{\mathcal{X}}$ is nef and $-K_{\mathcal{X}}\cdot\alpha=0$, we have $-K_{\mathcal{X}}\cdot A=0=-K_{\mathcal{X}}\cdot B$. Write $B=\sum_k c_kB_k$ as an effective linear combination of the classes $B_k$ of irreducible 1-dimensional torus-invariant orbits in $\mathcal{X}$. Again because $-K_{\mathcal{X}}$ is nef, we have $-K_{\mathcal{X}}\cdot B_k=0$ for all $k$. Each $B_k$ corresponds to an $(n-1)$-dimensional cone $\sigma_k\in\Sigma$. In the expression $B=\sum_k c_kB_k$, there is at least one (non-zero) $B_k$ which is not contained in $\bigcup_{\boldsymbol{b}_j\in F(\boldsymbol{b}_d)^c} D_j$. As a consequence, either $\sigma_k$ contains a ray $\mathbb{R}_{\geq 0}\boldsymbol{b}_j$ with $\boldsymbol{b}_j\notin F(\boldsymbol{b}_d)$, or there exists a $\boldsymbol{b}_j\notin F(\boldsymbol{b}_d)$ such that $\sigma_k$ and $\boldsymbol{b}_j$ span an $n$-dimensional cone in $\Sigma$.

Since $B_k$ is not contained in $\bigcup_{\boldsymbol{b}_j\in F(\boldsymbol{b}_d)^c} D_j$, we see that if $\boldsymbol{b}_i\in F(\boldsymbol{b}_d)^c$ then $\boldsymbol{b}_i\notin\sigma_k$. Also, $D\cdot B_k\geq 0$ for every toric prime divisor $D\subset\mathcal{X}$ not corresponding to a ray in $\sigma_k$.

By [18, Lemma 4.5] (which easily extends to the simplical setting), we have $D\cdot B_k=0$ for every toric prime divisor $D\subset\mathcal{X}$ corresponding to an element in $\{\boldsymbol{b}_1,\dots,\boldsymbol{b}_m\}\setminus F(\sigma_k)$, where $F(\sigma_k)\subset\mathcal{P}$ is the minimal face of $\mathcal{P}$ containing rays in $\sigma_k$. Since the divisors $D\subset\mathcal{X}$ corresponding to $\{\boldsymbol{b}_1,\dots,\boldsymbol{b}_m\}\setminus F(\sigma_k)$ span $H^2(\mathcal{X})$, we have $B_k=0$, a contradiction. ∎

Let $\boldsymbol{x}=(\mathcal{X}_{\nu_1},\dots,\mathcal{X}_{\nu_l})$ be an $l$-tuple of twisted sectors of $\mathcal{X}_{\beta_{\mathbf{d}}}$. Then Lemma 3.4 implies that the natural inclusion $\mathcal{M}^{\mathrm{op,main}}_{1,l}(\mathcal{X}_{\beta_{\mathbf{d}}},L,\beta,\boldsymbol{x})\hookrightarrow\mathcal{M}^{\mathrm{op,main}}_{1,l}(\mathcal{X},L,\beta,\boldsymbol{x})$ is a bijection. Since $\mathcal{X}_{\beta_{\mathbf{d}}}\subset\mathcal{X}$ is open, the local deformations and obstructions of stable discs in $\mathcal{X}_{\beta_{\mathbf{d}}}$ and their inclusion in $\mathcal{X}$ are isomorphic. It follows that

**Proposition 3.5.** *The moduli spaces $\mathcal{M}^{\mathrm{op,main}}_{1,l}(\mathcal{X},L,\beta,\boldsymbol{x})$ of disks in $\mathcal{X}$ is isomorphic as Kuranishi spaces to the moduli spaces $\mathcal{M}^{\mathrm{op,main}}_{1,l}(\mathcal{X}_{\beta_{\mathbf{d}}},L,\beta,\boldsymbol{x})$ of disks in $\mathcal{X}_{\beta_{\mathbf{d}}}$. Consequently*

$$n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_L;\mathbf{1}_{\nu_1},\dots,\mathbf{1}_{\nu_l})=n^{\mathcal{X}_{\beta_{\mathbf{d}}}}_{1,l,\beta}([\mathrm{pt}]_L;\mathbf{1}_{\nu_1},\dots,\mathbf{1}_{\nu_l}).$$

Since $\mathcal{X}_{\beta_{\mathbf{d}}}$ is a toric Calabi–Yau orbifold, the open Gromov–Witten invariants $n^{\mathcal{X}_{\beta_{\mathbf{d}}}}_{1,l,\beta}([\mathrm{pt}]_L;\mathbf{1}_{\nu_1},\dots,\mathbf{1}_{\nu_l})$ have been computed in [4]. By Proposition 3.5, this gives open Gromov–Witten invariants of $\mathcal{X}$. Explicitly they are given as follows.

Using the toric data of $\mathcal{X}_{\beta_{\mathbf{d}}}$, we define

$$\begin{aligned}
\Omega_j^{\mathcal{X}_{\beta_{\mathbf{d}}}} &:= \{d\in\mathbb{K}_{\mathrm{eff}}\mid \nu(d)=0,\langle D_j,d\rangle\in\mathbb{Z}_{<0}\text{ and}\\
&\qquad \langle D_i,d\rangle\in\mathbb{Z}_{\geq 0}\ \forall\, i\neq j\},\qquad j=0,1,\dots,m-1,\\
\Omega_j^{\mathcal{X}_{\beta_{\mathbf{d}}}} &:= \{d\in\mathbb{K}_{\mathrm{eff}}\mid \nu(d)=\boldsymbol{b}_j\text{ and }\langle D_i,d\rangle\notin\mathbb{Z}_{<0}\ \forall\, i\},\qquad j=m,m+1,\dots,m'-1,\\
A_j^{\mathcal{X}_{\beta_{\mathbf{d}}}}(y) &:= \sum_{d\in\Omega_j^{\mathcal{X}_{\beta_{\mathbf{d}}}}} y^d\frac{(-1)^{-\langle D_j,d\rangle-1}(-\langle D_j,d\rangle-1)!}{\prod_{i\neq j}\langle D_i,d\rangle!},\qquad j=0,1,\dots,m-1,\\
A_j^{\mathcal{X}_{\beta_{\mathbf{d}}}}(y) &:= \sum_{d\in\Omega_j^{\mathcal{X}_{\beta_{\mathbf{d}}}}} y^d\prod_{i=0}^{m'-1}\frac{\prod_{k=\lceil\langle D_i,d\rangle\rceil}^{\infty}(\langle D_i,d\rangle-k)}{\prod_{k=0}^{\infty}(\langle D_i,d\rangle-k)},\qquad j=m,m+1,\dots,m'-1,
\end{aligned}$$

$$
\begin{aligned}
\log q_a &= \log y_a + \sum_{j=0}^{m-1} Q_{ja} A_j^{\mathcal{X}_{\beta_{\mathbf{d}}}}(y), \qquad a = 1, \ldots, r', \\
\tau_{\boldsymbol{b}_j} &= A_j^{\mathcal{X}_{\beta_{\mathbf{d}}}}(y), \qquad j = m, \ldots, m'-1,
\end{aligned} \tag{3.3}
$$

**Theorem 3.6.** *If $\beta_{\mathbf{d}} = \beta_{i_0}$ is a basic smooth disk class corresponding to the ray generated by $\boldsymbol{b}_{i_0}$ for some $i_0 \in \{0, 1, \ldots, m-1\}$, then we have*

$$
\begin{aligned}
&\sum_{\alpha \in H_2^{\mathrm{eff}}(\mathcal{X})} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \mathrm{Box}'(\Sigma_{\beta_{\mathbf{d}}})^{\mathrm{age}=1}} \frac{\prod_{i=1}^{l} \tau_{\nu_i}}{l!} n^{\mathcal{X}}_{1,l,\beta_{i_0}+\alpha}\left([\mathrm{pt}]_L; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}\right) q^{\alpha} \\
&\qquad = \exp\left(-A_{i_0}^{\mathcal{X}_{\beta_{\mathbf{d}}}}(y(q,\tau))\right)
\end{aligned} \tag{3.4}
$$

*via the inverse $y = y(q,\tau)$ of the toric mirror map (3.3).*

*If $\beta_{\mathbf{d}} = \beta_{\nu_{j_0}}$ is a basic orbi-disk class corresponding to $\nu_{j_0} \in \mathrm{Box}'(\Sigma)^{\mathrm{age}=1}$ for some $j_0 \in \{m, m+1, \ldots, m'-1\}$, then we have*

$$
\begin{aligned}
&\sum_{\alpha \in H_2^{\mathrm{eff}}(\mathcal{X})} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \mathrm{Box}'(\Sigma_{\beta_{\mathbf{d}}})^{\mathrm{age}=1}} \frac{\prod_{i=1}^{l} \tau_{\nu_i}}{l!} n^{\mathcal{X}}_{1,l,\beta_{\nu_{j_0}}+\alpha}([\mathrm{pt}]_L; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}) q^{\alpha} \\
&\qquad = y^{D_{j_0}^{\vee}} \exp\left(-\sum_{i \notin I_{j_0}} c_{j_0 i} A_i^{\mathcal{X}_{\beta_{\mathbf{d}}}}(y(q,\tau))\right),
\end{aligned} \tag{3.5}
$$

*via the inverse $y = y(q,\tau)$ of the toric mirror map (3.3), where $D_{j_0}^{\vee} \in \mathbb{K}_{\mathrm{eff}}$ is the class defined in (2.4), $I_{j_0} \in \mathcal{A}$ is the anticone of the minimal cone containing $\boldsymbol{b}_{j_0} = \nu_{j_0}$ and $c_{j_0 i} \in \mathbb{Q} \cap [0,1)$ are rational numbers such that $\boldsymbol{b}_{j_0} = \sum_{i \notin I_{j_0}} c_{j_0 i} \boldsymbol{b}_i$.*

**Proof.** By Proposition 3.5, $n^{\mathcal{X}}_{1,l,\beta}([\mathrm{pt}]_L; \mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l}) = n^{\mathcal{X}_{\beta_{\mathbf{d}}}}_{1,l,\beta}([\mathrm{pt}]_L; \mathbf{1}_{\nu_1}, \ldots, \mathbf{1}_{\nu_l})$, and so the l.h.s. of (3.4) is equal to

$$
\sum_{\alpha \in H_2^{\mathrm{eff}}(\mathcal{X}_{\beta_{\mathbf{d}}})} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \mathrm{Box}'(\Sigma_{\beta_{\mathbf{d}}})^{\mathrm{age}=1}} \frac{\prod_{i=1}^{l} \tau_{\nu_i}}{l!} n^{\mathcal{X}_{\beta_{\mathbf{d}}}}_{1,l,\beta_{i_0}+\alpha}\left([\mathrm{pt}]_L; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}\right) q^{\alpha},
$$

which in turn is equal to $\exp\left(-A_{i_0}^{\mathcal{X}_{\beta_{\mathbf{d}}}}(y(q,\tau))\right)$ by [4, Theorem 1.4]. The deduction for (3.5) is similar. ∎

To combine all the invariants into a single expression, one defines the orbi-disc potential

$$
W = \sum_{\beta_{\mathbf{d}}} \sum_{\alpha \in H_2^{\mathrm{eff}}(\mathcal{X})} \sum_{l \geq 0} \sum_{\nu_1, \ldots, \nu_l \in \mathrm{Box}'(\Sigma_{\beta_{\mathbf{d}}})^{\mathrm{age}=1}} \frac{\prod_{i=1}^{l} \tau_{\nu_i}}{l!} q^{\alpha} n^{\mathcal{X}}_{1,l,\beta_{\mathbf{d}}+\alpha}\left([\mathrm{pt}]_L; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}\right) Z^{\beta_{\mathbf{d}}}, \tag{3.6}
$$

where $\beta_{\mathbf{d}}$ runs over all the basic smooth or orbi-disc classes, and $Z^{\beta_{\mathbf{d}}}$ are monomials associated to $\beta_{\mathbf{d}}$. See [5, Definition 19] for more detail. The above theorem gives an explicit expression of $W$ via the mirror map.

**Example 3.7.** $\mathbb{P}^2/\mathbb{Z}_3$ is a Gorenstein Fano toric orbifold. Its fan and polytope pictures are shown in Fig. 1. It has three toric divisors $D_1$, $D_2$, $D_3$ corresponding to the rays generated by $v_1 = (-1,-1)$, $v_2 = (2,-1)$, $v_3 = (-1,2)$. By pairing with the dual vectors $(1,0)$ and $(0,1)$, the linear equivalence relations are $2D_2 - D_3 - D_1 \sim 0$ and $2D_3 - D_2 - D_1 \sim 0$, and so $D_1 \sim D_2 \sim D_3$. It has three orbifold points corresponding to the three vertices in the polytope picture. Locally it is $\mathbb{C}^2/\mathbb{Z}_3$ around each orbifold point.

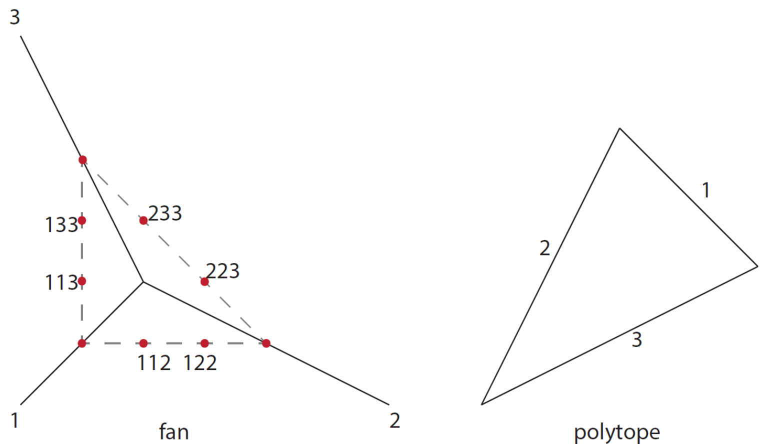


**Figure 1.** The fan and polytope picture for $\mathbb{P}^2/\mathbb{Z}_3$.

Fix a Lagrangian torus fiber. $\mathbb{P}^2/\mathbb{Z}_3$ has nine basic orbi-disk classes corresponding to the nine lattice points on the boundary of the fan polytope. Three of them are smooth disk classes and denote them by $\beta_1$, $\beta_2$, $\beta_3$. The basic orbi-disk classes corresponding to the two lattice points $(2v_1 + v_2)/3$ and $(v_1 + 2v_2)/3$ are denoted by $\beta_{112}$ and $\beta_{122}$, which pass through the twisted sectors $\nu_{112}$ and $\nu_{122}$ respectively. Then $2\beta_1 + \beta_2 - 3\beta_{112}$ (or $2\beta_2 + \beta_1 - 3\beta_{122}$) is the class of a constant orbi-sphere passing through the twisted sector $\nu_{112}$ (or $\nu_{122}$ resp.). In particular the area of $\beta_{112}$ equals to $(2\beta_1 + \beta_2)/3$. Other basic orbi-disk classes have similar notations.

Theorem 3.6 provides a formula for the open GW invariants $n^{\mathcal{X}}_{1,l,\beta_{112}}\left([\mathrm{pt}]_L; \prod_{i=1}^{l} \mathbf{1}_{\nu_i}\right)$ where $\nu_i$ is either $\nu_{112}$ or $\nu_{122}$ for each $i$. To write down the invariants more systematically, we consider the open GW potential as follows.

Let $q$ be the Kähler parameter of the smooth sphere class $\beta_1 + \beta_2 + \beta_3 \in H_2(\mathbb{P}^2/\mathbb{Z}_3)$. The basic orbi-disk classes correspond to monomials in the disk potential $q^{\beta} z^{\partial\beta}$, where $q^{\beta_1} = q^{\beta_2} = q^{\beta_{112}} = q^{\beta_{122}} = 1$, $q^{\beta_3} = q^{\beta_1+\beta_2+\beta_3} = q$, $q^{\beta_{223}} = q^{(2\beta_2+\beta_3)/3} = q^{\beta_3/3} = q^{1/3}$, and similar for other basic orbi-disk classes. The Kähler parameters corresponding to the twisted sectors $\nu_{112}$, $\nu_{122}$ are denoted as $\tau_{112}, \tau_{122}$ (and similar for other twisted sectors).

By [4, Example 1, Section 6.5], the open GW potential for $\mathbb{C}^2/\mathbb{Z}_3$ is given by

$$w(z - \kappa_0(\tau_{112}, \tau_{122}))(z - \kappa_1(\tau_{112}, \tau_{122}))(z - \kappa_2(\tau_{112}, \tau_{122}))$$

where

$$\kappa_k(\tau_1, \tau_2) = \zeta^{2k+1} \prod_{r=1}^{2} \exp\left(\frac{1}{3}\zeta^{(2k+1)r}\tau_r\right), \qquad \zeta := \exp\left(\pi\sqrt{-1}/3\right).$$

By Proposition 3.5, the disk invariants of $\mathbb{P}^2/\mathbb{Z}_3$ equal to those of $\mathbb{C}^2/\mathbb{Z}_3$. Thus the open GW potential of $\mathbb{P}^2/\mathbb{Z}_3$ is given by

$$W = z^{-1}w^{-1}(z - \kappa_0(\tau_{112}, \tau_{122}))(z - \kappa_1(\tau_{112}, \tau_{122}))(z - \kappa_2(\tau_{112}, \tau_{122}))$$

$$
\begin{aligned}
&+ z^{-1}w^{-1}\big(q^{1/3}w - \kappa_0(\tau_{113},\tau_{133})\big)\big(q^{1/3}w - \kappa_1(\tau_{113},\tau_{133})\big)\big(q^{1/3}w - \kappa_2(\tau_{113},\tau_{133})\big)\\
&+ z^2w^{-1}\big(q^{1/3}z^{-1}w - \kappa_0(\tau_{223},\tau_{233})\big)\big(q^{1/3}z^{-1}w - \kappa_1(\tau_{223},\tau_{233})\big)\\
&\times \big(q^{1/3}z^{-1}w - \kappa_2(\tau_{223},\tau_{233})\big) - z^{-1}w^{-1} - z^2w^{-1} - qz^{-1}w^2.
\end{aligned}
$$

Then the generating functions of open orbifold GW for $\beta_{112}$ and $\beta_{122}$ are given by the coefficients of $w^{-1}$ and $zw^{-1}$ in $W$ respectively. The first few terms are given by the following table.

| $n_{(a,b)}$ | $a=0$ | $a=1$ | $a=2$ | $a=3$ | $a=4$ | $a=5$ | $a=6$ |
|---|---|---|---|---|---|---|---|
| $b=0$ | 0 | 1 | 0 | 0 | 1/648 | 0 | 0 |
| $b=1$ | 0 | 0 | −1/18 | 0 | 0 | −1/29160 | 0 |
| $b=2$ | 1/6 | 0 | 0 | 1/972 | 0 | 0 | 1/3149280 |
| $b=3$ | 0 | −1/162 | 0 | 0 | −1/104976 | 0 | 0 |
| $b=4$ | 0 | 0 | 1/11664 | 0 | 0 | 1/18895680 | 0 |
| $b=5$ | −1/9720 | 0 | 0 | −1/1574640 | 0 | 0 | −1/5101833600 |
| $b=6$ | 0 | 1/524880 | 0 | 0 | 1/340122240 | 0 | 0 |

In the above table,

$$
n_{(a,b)} = n_{1,a+b,\beta_{112}}\big([\mathrm{pt}]_L; \mathbf{1}_{\nu_{112}}^{\otimes a}, \mathbf{1}_{\nu_{122}}^{\otimes b}\big) = n_{1,a+b,\beta_{122}}\big([\mathrm{pt}]_L; \mathbf{1}_{\nu_{112}}^{\otimes b}, \mathbf{1}_{\nu_{122}}^{\otimes a}\big).
$$

We observe that all invariants satisfy ‘reciprocal integrality’, namely their reciprocals are integers. Moreover, all these integers are divisible by 6. $n_{(k,k)} = 0$. Furthermore, the sign is alternating with respect to $b$.

## Acknowledgments

K. Chan was supported by a Hong Kong RGC grant CUHK14314516 and direct grants from CUHK. C.-H. Cho was supported by the NRF grant funded by the Korea government(MSIT) (No. 2017R1A22B4009488). S.-C. Lau was partially supported by the Simons collaboration grant #580648. N.C. Leung was supported by Hong Kong RGC grants CUHK14302215 & CUHK14303516 and direct grants from CUHK. H.-H. Tseng was supported in part by NSF grant DMS-1506551.

# References

[1] Auroux D., Mirror symmetry and $T$-duality in the complement of an anticanonical divisor, *J. Gökova Geom. Topol.* **1** (2007), 51–91, arXiv:0706.3207.

[2] Borisov L.A., Chen L., Smith G.G., The orbifold Chow ring of toric Deligne–Mumford stacks, *J. Amer. Math. Soc.* **18** (2005), 193–215, arXiv:math.AG/0309229.

[3] Chan K., Cho C.-H., Lau S.-C., Tseng H.-H., Lagrangian Floer superpotentials and crepant resolutions for toric orbifolds, *Comm. Math. Phys.* **328** (2014), 83–130, arXiv:1208.5282.

[4] Chan K., Cho C.-H., Lau S.-C., Tseng H.-H., Gross fibrations, SYZ mirror symmetry, and open Gromov–Witten invariants for toric Calabi–Yau orbifolds, *J. Differential Geom.* **103** (2016), 207–288, arXiv:1306.0437.

[5] Chan K., Lau S.-C., Leung N.C., Tseng H.-H., Open Gromov–Witten invariants, mirror maps, and Seidel representations for toric manifolds, *Duke Math. J.* **166** (2017), 1405–1462, arXiv:1209.6119.

[6] Chen W., Ruan Y., Orbifold Gromov–Witten theory, in Orbifolds in Mathematics and Physics (Madison, WI, 2001), *Contemp. Math.*, Vol. 310, Amer. Math. Soc., Providence, RI, 2002, 25–85, arXiv:math.AG/0103156.

[7] Chen W., Ruan Y., A new cohomology theory of orbifold, *Comm. Math. Phys.* **248** (2004), 1–31, arXiv:math.AG/0004129.

[8] Cho C.-H., Oh Y.-G., Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds, *Asian J. Math.* **10** (2006), 773–814, arXiv:math.SG/0308225.

[9] Cho C.-H., Poddar M., Holomorphic orbi-discs and Lagrangian Floer cohomology of symplectic toric orbifolds, *J. Differential Geom.* **98** (2014), 21–116, arXiv:1206.3994.

[10] Cho C.-H., Shin H.-S., Chern–Weil Maslov index and its orbifold analogue, *Asian J. Math.* **20** (2016), 1–19, arXiv:1202.0556.

[11] Coates T., Corti A., Iritani H., Tseng H.-H., A mirror theorem for toric stacks, *Compos. Math.* **151** (2015), 1878–1912, arXiv:1310.4163.

[12] Cox D.A., Little J.B., Schenck H.K., Toric varieties, *Graduate Studies in Mathematics*, Vol. 124, Amer. Math. Soc., Providence, RI, 2011.

[13] Fukaya K., Oh Y.-G., Ohta H., Ono K., Lagrangian intersection Floer theory: anomaly and obstruction, *AMS/IP Studies in Advanced Mathematics*, Vol. 46, Amer. Math. Soc., Providence, RI, 2009.

[14] Fukaya K., Oh Y.-G., Ohta H., Ono K., Lagrangian Floer theory on compact toric manifolds. I, *Duke Math. J.* **151** (2010), 23–174, arXiv:0802.1703.

[15] Fukaya K., Oh Y.-G., Ohta H., Ono K., Lagrangian Floer theory on compact toric manifolds II: bulk deformations, *Selecta Math. (N.S.)* **17** (2011), 609–711, arXiv:0810.5654.

[16] Fukaya K., Oh Y.-G., Ohta H., Ono K., Technical details on Kuranishi structure and virtual fundamental chain, arXiv:1209.4410.

[17] Fukaya K., Oh Y.-G., Ohta H., Ono K., Lagrangian Floer theory and mirror symmetry on compact toric manifolds, *Astérisque* **376** (2016), vi+340 pages, arXiv:1009.1648.

[18] González E., Iritani H., Seidel elements and mirror transformations, *Selecta Math. (N.S.)* **18** (2012), 557–590, arXiv:1103.4171.

[19] Gross M., Siebert B., From real affine geometry to complex geometry, *Ann. of Math.* **174** (2011), 1301–1428, arXiv:math.AG/0703822.

[20] Hausel T., Sturmfels B., Toric hyperKähler varieties, *Doc. Math.* **7** (2002), 495–534, arXiv:math.AG/0203096.

[21] Iritani H., An integral structure in quantum cohomology and mirror symmetry for toric orbifolds, *Adv. Math.* **222** (2009), 1016–1079, arXiv:0903.1463.

[22] Jiang Y., The orbifold cohomology ring of simplicial toric stack bundles, *Illinois J. Math.* **52** (2008), 493–514, arXiv:math.AG/0504563.

[23] McDuff D., Wehrheim K., Smooth Kuranishi atlases with isotropy, *Geom. Topol.* **21** (2017), 2725–2809, arXiv:1508.01556.

[24] Ruddat H., Siebert B., Period integrals from wall structures via tropical cycles, canonical coordinates in mirror symmetry and analyticity of toric degenerations, *Publ. Math. Inst. Hautes Études Sci.*, to appear, arXiv:1907.03794.

[25] Strominger A., Yau S.-T., Zaslow E., Mirror symmetry is $T$-duality, *Nuclear Phys. B* **479** (1996), 243–259, arXiv:hep-th/9606040.