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Curvature-Dimension Condition Meets Gromov's n-Volumic Scalar Curvature

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Abstract. We study the properties of the n-volumic scalar curvature in this note. Lott–Sturm–Villani's curvature-dimension condition $\mathrm{CD}(\kappa, n)$ was showed to imply Gromov's n-volumic scalar curvature $\geq n\kappa$ under an additional n-dimensional condition and we show the stability of n-volumic scalar curvature $\geq \kappa$ with respect to smGH-convergence. Then we propose a new weighted scalar curvature on the weighted Riemannian manifold and show its properties.

Key words: curvature-dimension condition; n-volumic scalar curvature; stability; weighted scalar curvature $\mathrm{Sc}_{\alpha,\beta}$

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1 Introduction

The concept of lower bounded curvature on the metric space or the metric measure space has evolved to a rich theory due to Alexandrov's insight. The stability of Riemannian manifolds with curvature bounded below is another deriving force to extend the definition of the curvature bounded below to a broader space. However, the scalar curvature (of Riemannian metrics) bounded below was yet absent from this picture. Gromov proposed a synthetic treatment of scalar curvature bounded below, which was called the n-volumic scalar curvature bounded below, and offered some pertinent conjectures in [18, Section 26]. Motivated by the $\mathrm{CD}(\kappa, n)$ condition, we add an n-dimension condition to the Gromov's definition and introduce the definition of $\mathrm{Sc}_{\alpha,\beta}$ on the smooth metric measure space. Details will be given later.

Theorem 1.1. *Assume that the metric measure space (X^n, d, μ) satisfies n-dimensional condition and the curvature-dimension condition $\mathrm{CD}(\kappa, n)$ for $\kappa \geq 0$ and $n \geq 2$, then (X^n, d, μ) satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq n\kappa$.*

Theorem 1.2. *If compact metric measure spaces (X_i^n, d_i, μ_i) with $\mathrm{Sc}^{\mathrm{vol}_n}(X_i^n) \geq \kappa \geq 0$ and SC-radius $r_{x_i^n} \geq R > 0$ and (X_i^n, d_i, μ_i) strongly measured Gromov–Hausdorff converge to the compact metric measure space (X^n, d, μ) with n-dimensional condition, then X^n also satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq \kappa$ and the SC-radius $r_{X^n} \geq R$.*

Theorem 1.3. *Let $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ be the closed smooth metric measure space with $\mathrm{Sc}_{\alpha,\beta} > 0$, then we have the following conclusions:*

1. *If M^n is a spin manifold, $\alpha \in \mathbb{R}$ and $\beta \geq \frac{|\alpha|^2}{4}$, then the harmonic spinors of M^n vanish.*
2. *If the dimension $n \geq 3$, $\alpha \in \mathbb{R}$ and $\beta \geq \frac{(n-2)|\alpha|^2}{4(n-1)}$, then there is a metric $\tilde{g}$ conformal to g with positive scalar curvature.*

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3. *If the dimension $n \geq 3$, $\alpha = 2$, $\beta \geq \frac{n-2}{n-1}$ and $\left(N^{n-1}, \bar{g}\right)$ is the compact L_f-stable minimal hypersurface of $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$, then there exists a PSC-metric conformal to $\bar{g}$ on N^{n-1}, where $\bar{g}$ is the induced metric of g on N^{n-1}.*
4. *Assume M^n is a spin manifold and there exists a smooth 1-contracting map $h\colon (M^n, g) \to (S^n, g_{\mathrm{st}})$ of non-zero degree. If $\alpha \in \mathbb{R}$, $\beta \geq \frac{|\alpha|^2}{4}$ and $\mathrm{Sc}_{\alpha,\beta} \geq n(n-1)$, then h is an isometry between the metrics g and g_{st}.*

The paper is organized as follows. In Section 2, we introduce the notions and show that $\mathrm{CD}(\kappa, n)$ implies $\mathrm{Sc}^{\mathrm{vol}_n} \geq (n-1)\kappa$. In Section 3, we show the stability of spaces with $\mathrm{Sc}^{\mathrm{vol}_n} \geq \kappa$. In Section 4, we present the properties of the smooth metric measure space with $\mathrm{Sc}_{\alpha,\beta} > 0$.

2 CD meets n-volumic scalar curvature

The n-dimensional Aleksandrov space with curvature $\geq \kappa$ equipped with the volume-measure satisfies Lott–Villani–Sturm's weak curvature-dimension condition for dimension n and curvature $(n-1)\kappa$, i.e., $\mathrm{CD}((n-1)\kappa, n)$, was shown by Petrunin for $\kappa = 0$ (and said that for general curvature $\geq \kappa$ the result followed in a similar way) [32] and then Zhang–Zhu investigated the general case [43]. We will modify Gromov's definition of n-volumic scalar curvature bounded below in [18, Section 26] to fill the picture, which means Lott–Sturm–Villani's Ricci curvature ≥ 0 implies Gromov's scalar curvature ≥ 0.

The metric measure space (mm-space) $X = (X, d, \mu)$ means that d is the complete separable length metric on X and μ is the locally finite full support Borel measure on X equipped with its Borel σ-algebra. Say that an mm-space $X = (X, d, \mu)$ is locally volume-wise smaller (or not greater) than another such space $X' = (X', d', \mu')$ and write $X <_{\mathrm{vol}} X'$ ($X \leq_{\mathrm{vol}} X'$), if all ϵ-balls in X are smaller (or not greater) than the ϵ-balls in X', $\mu(B_\epsilon(x)) < \mu'(B_\epsilon(x'))(\mu(B_\epsilon(x)) \leq \mu'(B_\epsilon(x'))$, for all $x \in X, x' \in X'$ and the uniformly small ϵ which depends on X and X'.

From now on, the Riemannian 2-sphere $\left(S^2(\gamma), d_S, \mathrm{vol}_S\right)$ is endowed with round metric such that the scalar curvature equal to $2\gamma^{-2}$, $\left(\mathbf{R}^{n-2}, d_E, \mathrm{vol}_E\right)$ is endowed with Euclidean metric with flat scalar curvature and the product manifold $S^2(\gamma) \times \mathbf{R}^{n-2}$ is endowed with the Pythagorean product metrics $d_{S\times E} := \sqrt{d_S^2 + d_E^2}$ and the volume $\mathrm{vol}_{S\times E} := \mathrm{vol}_S \otimes \mathrm{vol}_E$.

Thus, we have $S^2(\gamma) <_{\mathrm{vol}} \mathbf{R}^2$. If $0 < \gamma_1 < \gamma_2$, then $S^2(\gamma_1) <_{\mathrm{vol}} S^2(\gamma_2)$. Furthermore, $S^2(\gamma) \times \mathbf{R}^{n-2} <_{\mathrm{vol}} \mathbf{R}^n$. If $0 < \gamma_1 < \gamma_2$, then $S^2(\gamma_1) \times \mathbf{R}^{n-2} <_{\mathrm{vol}} S^2(\gamma_2) \times \mathbf{R}^{n-2}$.

Definition 2.1 (Gromov's n-volumic scalar curvature)**.** Gromov's n-volumic scalar curvature of X is bounded below by 0 for $X = (X, d, \mu)$ if X is locally volume-wise not greater than $\mathbf{R}^n$.

Gromov's n-volumic scalar curvature of X bounds from below by $\kappa > 0$ for $X = (X, d, \mu)$ if X is locally volume-wise smaller than $S^2(\gamma) \times \mathbf{R}^{n-2}$ for all $\gamma > \sqrt{\frac{2}{\kappa}}$, i.e., $X <_{\mathrm{vol}} S^2(\gamma) \times \mathbf{R}^{n-2}$ and $\gamma > \sqrt{\frac{2}{\kappa}}$, where $S^2(\gamma) \times \mathbf{R}^{n-2} = \left(S^2(\gamma) \times \mathbf{R}^{n-2}, d_{S\times E}, \mathrm{vol}_{S\times E}\right)$.

The n-volumic scalar curvature is sensitive to the scaling of the measure, but the curvature condition $\mathrm{CD}(\kappa, n)$ of Lott–Villani–Sturm [38, Definition 1.3] is invariant up to scalars of the measure only [38, Proposition 1.4(ii)]. Therefore, the n-dimensional condition needs to be put into the definition of Gromov's n-volumic scalar curvature. In fact, the n-dimensional condition is the special case of Young's point-wise dimension in dynamical systems [41, Theorem 4.4].

Definition 2.2 (n-dimensional condition)**.** For given positive natural number n, the mm-space $X = (X, d, \mu)$ satisfies the n-dimensional condition if

$$\lim_{r\to 0} \frac{\mu(B_r(x))}{\mathrm{vol}_E(B_r(\mathbf{R}^n))} = 1$$

for every $x \in X$, where $B_r(\mathbf{R}^n)$ is the closed r-ball in the Euclidean space $\mathbf{R}^n$ and the $B_r(x)$ is the closed r-ball with the center $x \in X$.

From now on, the superscript of n in the space X^n means the mm-space (X^n, d, μ) satisfies n-dimensional condition.

Note that a closed smooth n-manifold M^n ($n \geq 3$) admits a Riemannian metric with constant negative scalar curvature and a Riemannian metric of non-negative scalar curvature which is not identically zero, then by a conformal change of the metric we get a metric of positive scalar curvature according to Kazdan–Warner theorem [25]. Furthermore, if there is a scalar-flat Riemannian metric g on M^n, but g is not Ricci-flat metric, then g can be deformed to a metric with positive scalar curvature according to Kazdan theorem [24, Theorem B] or by using Ricci-flow with an easy argument. Hence we will focus more on promoting the positive scalar curvature to positive n-volumic scalar curvature.

Definition 2.3 (n-volumic scalar curvature)**.** Assume $X^n = (X^n, d, \mu)$ is the compact mm-space and satisfies the n-dimensional condition, we call

1. the n-volumic scalar curvature of X^n is positive, i.e., $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) > 0$, if there exists $r_{X^n} > 0$ such that the measures of ϵ-balls in X^n are smaller than the volumes of ϵ-balls in $\mathbf{R}^n$ for $0 < \epsilon \leq r_{X^n}$.
2. the n-volumic scalar curvature of X^n is bounded below by 0, i.e., $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq 0$, if there exists $r_{X^n} > 0$ such that the measures of ϵ-balls in X^n are not greater than the volumes of ϵ-balls in $\mathbf{R}^n$ for $0 < \epsilon \leq r_{X^n}$.

   The r_{X^n} is called scalar curvature radius (SC-radius) of X^n for $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq 0$.
3. the n-volumic scalar curvature of X^n is bounded below by $\kappa > 0$, i.e., $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq \kappa > 0$, if, for any γ with $\gamma > \sqrt{\frac{2}{\kappa}}$, there exists $r_{X^n,\gamma} > 0$ such that the measures of ϵ-balls in X^n are smaller than the volumes of ϵ-balls in $S^2(\gamma) \times \mathbf{R}^{n-2}$ for $0 < \epsilon \leq r_{X^n,\gamma}$.

   We call $r_{X^n} := \inf\limits_{\gamma > \sqrt{\frac{2}{\kappa}}} r_{X^n,\gamma}$ is the SC-radius of X^n for $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq \kappa > 0$.

In particular, we will focus on the case of $\inf\limits_{\gamma > \sqrt{\frac{2}{\kappa}}} r_{X^n,\gamma} \neq 0$ for stability in Section 3.

If the mm-space X^n is locally compact, then the definition of the n-volumic scalar curvature bounded below only modifies the definition of the $r_{X^n,\gamma} > 0$ to a positive continuous function of X^n.

Two mm-spaces (X^n, d, μ) and (X_1^n, d_1, μ_1) are isometric if there exists a one-to-one map $f\colon X^n \to X_1^n$ such that $d_1(f(a), f(b)) = d(a, b)$ for a and b are in X^n and $f_*\mu = \mu_1$, where $f_*\mu$ is the push-forward measure, i.e., $f_*\mu(U) = \mu\big(f^{-1}(U)\big)$ for a measureable subset $U \subset X_1^n$. If X^n satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq \kappa \geq 0$, then each mm-space (X_1^n, d_1, μ_1) that is isometric to (X^n, d, μ) also satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(X_1^n) \geq \kappa \geq 0$.

Proposition 2.4. *Let g be a C^2-smooth Riemannian metric on a closed oriented n-manifold M^n with induced metric measure space $(M^n, d_g, \mathrm{dVol}_g)$, then the scalar curvature of g is positive, $\mathrm{Sc}_g > 0$, if and only if $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) > 0$, and $\mathrm{Sc}_g \geq \kappa > 0$ if and only if $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq \kappa > 0$.*

Proof. For a C^2-smooth Riemannian metric g, one has

$$\mathrm{dVol}_g(B_r(x)) = \mathrm{vol}_E(B_r(\mathbf{R}^n)) \left[1 - \frac{\mathrm{Sc}_g(x)}{6(n+2)} r^2 + O(r^4)\right]$$

for $B_r(x) \subset M^n$ as $r \to 0$. Hence $(M^n, d_g, \mathrm{dVol}_g)$ satisfies the n-dimensional condition.

If we have $\mathrm{Sc}_g > 0$, then, since M^n is compact, there exists $r_{M^n} > 0$, so that $\mathrm{dVol}_g(B_r(x)) < \mathrm{vol}_E(B_r(\mathbf{R}^n))$ for all $0 < r \leq r_{M^n}$. On the other hand, if there exists $r_{M^n} > 0$ such that $\mathrm{dVol}_g(B_r(x)) < \mathrm{vol}_E(B_r(\mathbf{R}^n))$ for all $0 < r \leq r_{M^n}$, then Sc_g must be greater than 0.

If $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq \kappa > 0$, then $\mathrm{Sc}_g \geq \kappa > 0$. Otherwise, assume there exist small $\epsilon > 0$ such that $\mathrm{Sc}_g \geq \kappa - \epsilon > 0$. That means that there exists a point x_0 in M^n such that $\mathrm{Sc}_g(x_0) = \kappa - \epsilon$, as M^n is compact and the scalar curvature is a continuous function on M^n. Thus, we can find a small r-ball $B_r(x_0)$ such that the volume of $B_r(x_0)$ is greater than the volume of the r-ball in the $S^2(\gamma) \times \mathbf{R}^{n-2}$ for $\gamma = \sqrt{\frac{2}{\kappa - \frac{\epsilon}{2}}}$, which is a contradiction.

On the other hand, $\mathrm{Sc}_g \geq \kappa > 0$ implies $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq \kappa > 0$. Assume $\mathrm{Sc}_g(x_1) = \kappa$ for some $x_1 \in M^n$, then there exists r_1 such that $\mathrm{dVol}_g(B_{r_1}(x)) \leq \mathrm{dVol}_g(B_{r_1}(x_1))$ for r_1-balls in M^n and

$$\mathrm{dVol}_g(B_r(x_1)) = \mathrm{vol}_E(B_r(\mathbf{R}^n)) \left[1 - \frac{\kappa}{6(n+2)} r^2 + O(r^4)\right]$$

as $r \to 0$. Thus, for any γ with $\gamma > \sqrt{\frac{2}{\kappa}}$, there exists $r_{M^n,\gamma} > 0$ such that the measures of ϵ-balls in M^n are smaller than the volumes of ϵ-balls in $S^2(\gamma) \times \mathbf{R}^{n-2}$ for $0 < \epsilon \leq r_{M^n,\gamma}$, i.e., $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq \kappa > 0$. ■

Therefore, we have $S^n_{\frac{\kappa}{n-1}} <_{\mathrm{vol}} S^2(\gamma) \times \mathbf{R}^{n-2}$ for all $\gamma > \sqrt{\frac{2}{n\kappa}}$. Here $S^n_{\frac{\kappa}{n-1}}$ is the Riemannian manifold S^n with constant sectional curvature $\frac{\kappa}{n-1}$.

Remark 2.5. For a closed smooth Riemannian manifold (M^n, g), $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq 0$ implies $\mathrm{Sc}_g \geq 0$. Otherwise, there exists a point in M^n such that the scalar curvature is negative, then the volume of small ball will be greater than the volume of the small ball in Euclidean space, which is a contradiction.

On the other hand, one can consider the case of the scalar-flat metric, i.e., $\mathrm{Sc}_g \equiv 0$. If g is a strongly scalar-flat metric, meaning a metric with scalar curvature zero such that M^n has no metric with positive scalar curvature, then g is also Ricci flat according to Kazdan theorem above. Thus, we have

$$\mathrm{dVol}_g(B_r(x)) = \mathrm{vol}_E(B_r(\mathbf{R}^n)) \left[1 - \frac{\|\mathrm{Rie}(x)\|_g^2}{120(n+2)(n+4)} r^4 + O(r^6)\right]$$

for $B_r(x) \subset M^n$ as $r \to 0$ [14, Theorem 3.3]. Here Rie is the Riemannian tensor. Therefore, if g is a not flat metric, then $M^n <_{\mathrm{vol}} \mathbf{R}^n$. If g is a flat metric, then $M^n \leq_{\mathrm{vol}} \mathbf{R}^n$. Thus $\mathrm{Sc}_g \geq 0$ implies $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq 0$ for a strongly scalar-flat metric g.

However, $\mathrm{Sc}_g \geq 0$ may not imply $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq 0$. There are a lot of scalar-flat metrics but not strongly scalar flat metrics, i.e., $\mathrm{Sc}_g \equiv 0$ but not $\mathrm{Ricc}_g \neq 0$. For instance, the product metric on $S^2(1) \times \Sigma$, where Σ is a closed hyperbolic surface, is the scalar-flat metric, but not the Ricci-flat metric. For those metrics, we have

$$\mathrm{dVol}_g(B_r(x)) = \mathrm{vol}_E(B_r(\mathbf{R}^n)) \left[1 + \frac{-3\|\mathrm{Rie}(x)\|_g^2 + 8\|\mathrm{Ricc}(x)\|_g^2}{360(n+2)(n+4)} r^4 + O(r^6)\right]$$

for $B_r(x) \subset M^n$ as $r \to 0$ [14, Theorem 3.3]. If $8\|\mathrm{Ricc}(x)\|_g^2 > -3\|\mathrm{Rie}(x)\|_g^2$ for some point, then $\mathrm{Sc}_g \geq 0$ does not imply $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq 0$.

Theorem 2.6. *Assume that the mm-space (X^n, d, μ) satisfies n-dimensional condition and the curvature-dimension condition $\mathrm{CD}(\kappa, n)$ for $\kappa \geq 0$ and $n \geq 2$, then (X^n, d, μ) satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq n\kappa$.*

Proof. In fact, one only needs the generalized Bishop–Gromov volume growth inequality, which is implied by the curvature-dimension of X^n [38, Theorem 2.3].

(i) If $\kappa = 0$, then

$$\frac{\mu(B_r(x))}{\mu(B_R(x))} \geq \Big(\frac{r}{R}\Big)^n$$

for all $0 < r < R$. That is

$$\frac{\mu(B_r(x))}{\mathrm{vol}_E(B_r(\mathbf{R}^n))} = \frac{\mu(B_r(x))}{\alpha(n) r^n} \geq \frac{\mu(B_R(x))}{\alpha(n) R^n} = \frac{\mu(B_R(x))}{\mathrm{vol}_E(B_R(\mathbf{R}^n))},$$

where $\alpha(n) = \frac{\mathrm{vol}_E(B_r(\mathbf{R}^n))}{r^n}$. Combining the n-dimensional condition,

$$\lim_{r\to 0} \frac{\mu(B_r(x))}{\mathrm{vol}_E(B_r(\mathbf{R}^n))} = 1,$$

that implies $\mathrm{Sc}^{\mathrm{vol}_n}(X) \geq 0$.

(ii) If $\kappa > 0$, then

$$\frac{\mu(B_r(x))}{\mu(B_R(x))} \geq \frac{\int_0^r \big[\sin\big(t\sqrt{\frac{\kappa}{(n-1)}}\big)\big]^{n-1} \mathrm{d}t}{\int_0^R \big[\sin\big(t\sqrt{\frac{\kappa}{(n-1)}}\big)\big]^{n-1} \mathrm{d}t}$$

for all $0 < r \leq R \leq \pi\sqrt{\frac{(n-1)}{\kappa}}$.

Since the scalar curvature of the product manifold $S^2(\gamma) \times \mathbf{R}^{n-2}$ is $n\kappa$, where $\gamma = \sqrt{\frac{2}{n\kappa}}$, then there exists $C_1, C_2 > 0$ such that

$$1 - \frac{n\kappa}{6(n+2)} r_1^2 - C_2 r_1^4 \leq \mathrm{vol}_{S\times E}\widetilde{(B_{r_1}(y))} := \frac{\mathrm{vol}_{S\times E}(B_{r_1}(y))}{\mathrm{vol}_E(B_{r_1}(\mathbf{R}^n))} \leq 1 - \frac{n\kappa}{6(n+2)} r_1^2 + C_2 r_1^4,$$

for $y \in S^2(\gamma) \times \mathbf{R}^{n-2}$ and $r_1 \leq C_1$, where C_1, C_2 are decided by the product manifold $S^2(\gamma) \times \mathbf{R}^{n-2}$.

Let

$$\mu\widetilde{(B_r(x))} := \frac{\mu(B_r(x))}{\mathrm{vol}_E(B_r(\mathbf{R}^n))}$$

and

$$f(r) := \frac{\int_0^r \big[\sin\big(t\sqrt{\frac{\kappa}{(n-1)}}\big)\big]^{n-1} \mathrm{d}t}{\mathrm{vol}_E(B_r(\mathbf{R}^n))},$$

then the generalized Bishop–Gromov inequality can be re-formulated as

$$\mu\widetilde{(B_R(x))} \leq \mu\widetilde{(B_r(x))} \frac{f(R)}{f(r)}$$

for all $0 < r < R \leq \pi\sqrt{\frac{(n-1)}{\kappa}}$. The asymptotic expansion of $f(r)$ is

$$f(r) = \frac{\frac{1}{n} r^n \big[\frac{\kappa}{(n-1)}\big]^{\frac{(n-1)}{2}} - \frac{(n-1)}{6(n+2)} r^{n+2} \big[\frac{\kappa}{(n-1)}\big]^{\frac{n+1}{2}} + O\big(r^{n+4}\big)}{\mathrm{vol}_E(B_r(\mathbf{R}^n))}$$

as $r \to 0$. Thus, the asymptotic expansion of $\frac{f(R)}{f(r)}$ is

$$\frac{f(R)}{f(r)} = \frac{1 - \frac{n\kappa}{6(n+2)}R^2 + O\big(R^4\big)}{1 - \frac{n\kappa}{6(n+2)}r^2 + O\big(r^4\big)}$$

as $R \to 0$, $r \to 0$. The n-dimensional condition, $\lim_{r\to 0} \mu(\widetilde{B_r(x)}) = 1$, implies that

$$\mu(\widetilde{B_R(x)}) \leq 1 - \frac{n\kappa}{6(n+2)}R^2 + O\big(R^4\big)$$

as $R \to 0$. Therefore, for any κ' with $0 < \kappa' < \kappa$, there exists $\epsilon_{\kappa'} > 0$ such that for any $0 < R \leq \epsilon_{\kappa'}$, we have

$$\mu(\widetilde{B_R(x)}) < \mathrm{vol}_{S\times E}(\widetilde{B_R(y)}),$$

where $\mathrm{vol}_{S\times E}(\widetilde{B_R(y)}) = \frac{\mathrm{vol}_{S\times E}(B_R(y))}{\mathrm{vol}_E(B_R(\mathbf{R}^n))}$ is defined as before, the balls $B_R(y)$ are in $S^2(\gamma) \times \mathbf{R}^{n-2}$ and $\gamma = \sqrt{\frac{2}{n\kappa'}}$. That is $X^n <_{\mathrm{vol}} S^2(\gamma) \times \mathbf{R}^{n-2}$, for all $\gamma > \sqrt{\frac{2}{n\kappa}}$, i.e., $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq n\kappa$.

In fact, one has the classical Bishop inequality by adding the n-dimensional condition to the generalized Bishop–Gromov volume growth inequality. It means that

- if $\kappa = 0$, $\mu(B_R(x)) \leq \mathrm{vol}_E(B_R(\mathbf{R}^n))$ for all $R > 0$,
- if $\kappa > 0$, $\mu(B_R(x)) \leq \mathrm{vol}_{S^n}\big(B_R(S^n_{\frac{\kappa}{n-1}})\big)$ for $0 < R \leq \pi\sqrt{\frac{(n-1)}{\kappa}}$.

In other words, if $\kappa = 0$, $X^n \leq_{\mathrm{vol}} \mathbf{R}^n$. If $\kappa > 0$, $X^n \leq_{\mathrm{vol}} S^n_{\frac{\kappa}{n-1}}$. We have $S^n_{\frac{\kappa}{n-1}} <_{\mathrm{vol}} S^2(\gamma) \times \mathbf{R}^{n-2}$ for all $\gamma > \sqrt{\frac{2}{n\kappa}}$. Then $X^n <_{\mathrm{vol}} S^2(\gamma) \times \mathbf{R}^{n-2}$ for all $\gamma > \sqrt{\frac{2}{n\kappa}}$.

Thus, we also get $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq n\kappa$. ■

Remark 2.7. Hence the mm-space (X^n, d, μ) with $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq n\kappa$ includes the mm-spaces that satisfies n-dimensional condition and the generalized Bishop–Gromov volume growth inequality as stated in the proof, e.g., the mm-spaces with the Riemannian curvature condition $\mathrm{RCD}(\kappa, n)$ [2] or with the measure concentration property $\mathrm{MCP}(\kappa, n)$ [29].

Question 2.8. *Let* $\mathrm{Al}^n(1)$ *be an orientable compact* n*-dimensional Aleksandrov space with curvature* ≥ 1*, then do all continuous maps* ϕ *from* $\mathrm{Al}^n(1)$ *to the sphere* S^n *with standard metric of non-zero degree satisfy* $\mathrm{Lip}(\phi) \geq C(n)$*? Here* $\mathrm{Lip}(\phi)$ *is the Lipschitz constant of* ϕ*,* ϕ *maps the boundary of* $\mathrm{Al}^n(1)$ *to a point in* S^n *and* $C(n)$ *is a constant depending only on the dimension* n*.*

Question 2.9. *Assume the compact mm-space* (X, d, μ) *satisfies the curvature-dimension condition* $\mathrm{CD}(n-1, n)$*,* n*-dimensional condition and the covering dimension is also* n*, then do all continuous maps* ϕ *from* (X, d, μ) *to the sphere* S^n *with standard metric, where* ϕ *is non-trivial in the homotopy class of maps, satisfy* $\mathrm{Lip}(\phi) \geq C_1(n)$*, where* $C_1(n)$ *is a constant depending only on* n*?*

Remark 2.10. The questions above are inspired by Gromov's spherical Lipschitz bounded theorem in [19, Section 3] and the results above. The finite covering dimension is equal to the cohomological dimension over integer ring $\mathbb{Z}$ for the compact metric space according to the Alexandrov theorem. The best constant of $C(n)$ and $C_1(n)$ would be 1 if both questions have positive answers.

Proposition 2.11 (quadratic scaling)**.** *Assume the compact mm-space (X^n, d, μ) satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq \kappa > 0$, then $\mathrm{Sc}^{\mathrm{vol}_n}(\lambda X^n) \geq \lambda^{-2}\kappa > 0$ and $r_{\lambda X^n} = \lambda r_{X^n}$ for all $\lambda > 0$, where $\lambda X^n := (X^n, \lambda \cdot d, \lambda^n \cdot \mu)$.*

Proof. First, we will show that the n-dimensional condition is stable under scaling. Let $d' := \lambda \cdot d$, $\mu' := \lambda^n \cdot \mu$, $B'_r(x)$ be an r-ball in the (X^n, d'), and $B_r(x)$ be an r-ball in the (X^n, d), then $B'_r(x) = B_{\frac{r}{\lambda}}(x)$ as the subset in the X^n. One has

$$\lim_{r\to 0} \frac{\mu'(B'_r(x))}{\mathrm{vol}_E(B_r(\mathbf{R}^n))} = \lim_{r\to 0} \frac{\mu'(B_{\frac{r}{\lambda}}(x))}{\mathrm{vol}_E(B_r(\mathbf{R}^n))} = \lim_{r\to 0} \frac{\lambda^n \cdot \mu(B_{\frac{r}{\lambda}}(x))}{\lambda^n \cdot \mathrm{vol}_E(B_{\frac{r}{\lambda}}(\mathbf{R}^n))} = 1,$$

then λX^n satisfies the n-dimensional condition.

Since $\lambda \cdot \big(S^2(\gamma) \times \mathbf{R}^{n-2}\big) = \lambda \cdot S^2(\gamma) \times \lambda \cdot \mathbf{R}^{n-2} = S^2(\lambda\gamma) \times \lambda \cdot \mathbf{R}^{n-2}$, we have

$$\lambda \cdot X^n <_{\mathrm{vol}} \lambda \cdot \big(S^2(\gamma) \times \mathbf{R}^{n-2}\big) = S^2(\lambda\gamma) \times \lambda \cdot \mathbf{R}^{n-2}$$

for all $\lambda\gamma > \sqrt{\frac{2}{\lambda\kappa}}$ and $0 < \epsilon \leq \lambda r_{X^n}$. That means $\mathrm{Sc}^{\mathrm{vol}_n}(\lambda X^n) \geq \lambda^{-2}\kappa > 0$ and $r_{\lambda \cdot X^n} = \lambda r_{X^n}$. ■

We also have $\mathrm{Sc}^{\mathrm{vol}_n}(\lambda X^n) \geq 0$ (> 0), if $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq 0$ (> 0).

Remark 2.12. Since the n-dimensional condition and definition of n-volumic scalar curvature is locally defined, we have the following construction.

- Global to local: Let the locally compact mm-space (X^n, d, μ) satisfy $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq \kappa \geq 0$ and $Y^n \subset X$ be an open subset. Then, if (Y^n, d_Y) is a complete length space, $(Y^n, d_Y, \mu\llcorner_Y)$ satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(Y^n) \geq \kappa \geq 0$ and $r_{Y^n} = r_{X^n}$. Where d_Y is the induced metric of d and $\mu\llcorner_Y$ is the restriction operator, namely, $\mu\llcorner_Y(A) := \mu(Y^n \cap A)$ for $A \subset X^n$.
- Local to global: Let $\{Y_i^n\}_{i\in I}$ be a finite open cover of a locally compact mm-space (X^n, d, μ). Assume that (Y_i^n, d_{Y_i}) is a complete length space and $(Y_i^n, d_{Y_i}, \mu\llcorner_{Y_i})$ satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(Y_i^n) \geq \kappa \geq 0$, then (X^n, d, μ) satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq \kappa \geq 0$ and r_{X^n} can be chosen as a partition of unity of the functions $\{r_{Y_i^n}\}_{i\in I}$.

Question 2.13. *Assume that $\mathrm{Sc}^{\mathrm{vol}_{n_1}}(X_1^{n_1}) \geq \kappa_1 (\geq 0)$ for the compact mm-space $(X_1^{n_1}, d_1, \mu_1)$ and $\mathrm{Sc}^{\mathrm{vol}_{n_2}}\big(X_2^{n_2}\big) \geq \kappa_2 (\geq 0)$ for the compact mm-space $(X_2^{n_2}, d_2, \mu_2)$, then do we have*

$$\mathrm{Sc}^{\mathrm{vol}_{n_1+n_2}}\big(X_1^{n_1} \times X_2^{n_2}\big) \geq \kappa_1 + \kappa_2, \qquad r_{X_1^n \times X_2^n} = \min\{r_{X_1^n}, r_{X_2^n}\}$$

for $\big(X_1^{n_1} \times X_2^{n_2}, d_3, \mu_3\big)$? Here $X_1^{n_1} \times X_2^{n_2}$ is endowed with the measure $\mu_3 := \mu_1 \otimes \mu_2$ and with the Pythagorean product metric $d_3 := \sqrt{d_1^2 + d_2^2}$.

3 smGH-convergence

Let $\{\mu_n\}_{n\in\mathbf{N}}$ and μ be Borel measures on the space X, then the sequence $\{\mu_n\}_{n\in\mathbf{N}}$ is said to converge *strongly* (also called setwise convergence in other literature) to a limit μ if $\lim\limits_{n\to\infty} \mu_n(\mathcal{A}) = \mu(\mathcal{A})$ for every $\mathcal{A}$ in the Borel σ-algebra.

A map $f\colon X \to Y$ is called an ϵ-isometry between compact metric spaces X and Y, if $|d_X(a, b) - d_Y(f(a), f(b))| \leq \epsilon$ for all $a, b \in X$ and it is almost surjective, i.e., for every $y \in Y$, there exists an $x \in X$ such that $d_Y(f(x), y) \leq \epsilon$.

In fact, if f is an ϵ-isometry $X \to Y$, then there is a (4ϵ)-isometry $f'\colon Y \to X$ such that for all $x \in X$, $y \in Y$, $d_X(f' \circ f(x), x) \leq 3\epsilon$, $d_Y(f \circ f'(y), y) \leq \epsilon$.

Definition 3.1 (smGH-convergence)**.** Let $(X_i, d_i, \mu_i)_{i\in\mathbf{N}}$ and (X, d, μ) be compact mm-spaces. X_i converges to X in the strongly measured Gromov–Hausdorff topology (smGH-convergence) if there are measurable ϵ_i-isometries $f_i\colon X_i \to X$ such that $\epsilon_i \to 0$ and $f_{i*}\mu_i \to \mu$ in the strong topology of measures as $i \to \infty$.

If the spaces $(X_i, d_n, \mu_i, p_i)_{i\in\mathbf{N}}$ and (X, d, μ, p) are locally compact pointed mm-spaces, it is said that X_i converges to X in the pointed strongly measured Gromov–Hausdorff topology (psmGH-convergence) if there are sequences $r_i \to \infty$, $\epsilon_i \to 0$, and measurable pointed ϵ_i-isometries $f_i\colon B_{r_i}(p_i) \to B_{r_i}(p)$, such that $f_{i*}\mu_i \to \mu$, where the convergence is strong convergence.

Remark 3.2. Let $(X_i, d_i, \mu_i)_{i\in\mathbf{N}}$ converge to (X, d, μ) in the measured Gromov–Hausdorff topology, then there are measurable ϵ_i-isometries $f_i\colon X_i \to X$ such that $f_{i*}\mu_i$ weakly converges to μ. If there is a Borel measure ν on X such that $\sup\limits_i f_{i*}\mu_i \le \nu$, i.e., $\sup\limits_i f_{i*}\mu_i(\mathcal{A}) \le \nu(\mathcal{A})$ for every $\mathcal{A}$ in the Borel σ-algebra on X, then X_i smGH-converges to X (see [26, Lemma 4.1]).

Remark 3.3. The n-dimensional condition is not preserved by the measured Gromov–Hausdorff convergence as the following example shows. Let $\{a_iS^2 := (S^2, a_id_S)\}$ ($a_i \in (0,1)$) be a sequence of space, then the limit of a_iS^2 under the measured Gromov–Hausdorff convergence is a point when a_i goes to 0. The limit exists as the Ricci curvature of a_iS^2 is bounded below by 1.

Remark 3.4. The n-dimensional condition is not preserved by the smGH-convergence since the limits of $\lim\limits_{r\to 0}\lim\limits_{i\to\infty} \frac{\mu_i(B_r(x))}{\mathrm{vol}_E(B_r(\mathbf{R}^n))}$ may not be commutative for some mm-spaces (X, d, μ_i). Assume the total variation distance of the measures goes to 0 as $i \to \infty$, i.e.,

$$d_{TV}(\mu_i, \mu) := \sup_{\mathcal{A}} |\mu_i(\mathcal{A}) - \mu(\mathcal{A})| \to 0,$$

where $\mathcal{A}$ runs over the Borel σ-algebra of X, then the limits are commutative.

One can also define *the total variation Gromov–Hausdorff convergence* (*tvGH-convergence*) for mm-spaces by replacing the strong topology with the topology induced by the total variation distance in definition of smGH-convergence. Then tvGH-convergence implies smGH-convergence and the n-dimensional condition is preserved by tvGH-convergence.

Theorem 3.5 (stability)**.** *If compact mm-spaces (X_i^n, d_i, μ_i) with $\mathrm{Sc}^{\mathrm{vol}_n}(X_i^n) \ge \kappa \ge 0$, SC-radius $r_{X_i^n} \ge R > 0$, and (X_i^n, d_i, μ_i) smGH-converge to the compact mm-space (X^n, d, μ) with n-dimensional condition, then X^n also satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \ge \kappa$ and the SC-radius $r_{X^n} \ge R$.*

Proof. Fix an $x \in X^n$ and let $B_r(x)$ be the small r-ball on X^n where $r < R$, then there exists $x_i \in X_i^n$ such that $f_i^{-1}(B_r(x)) \subset B_{r+4\epsilon_i}(x_i)$ where $B_{r+4\epsilon_i}(x_i) \subset X_i^n$ and $r + 4\epsilon_i \le R$. Thus, $f_{i*}\mu_i(B_r(x)) \le \mu_i(B_{r+4\epsilon_i}(x_i))$.

- For $\kappa = 0$, since $\mathrm{Sc}^{\mathrm{vol}_n}(X_i^n) \ge 0$ and SC-radius$\ge R > 0$, then $\mu_i(B_r(x_i)) \le \mathrm{vol}_E(B_r(\mathbf{R}^n))$ for all $0 < r \le R$ and all i. Therefore, $f_{i*}\mu_i(B_r(x)) < \mu_i(B_{r+4\epsilon_i}(x_i)) \le \mathrm{vol}_E(B_{r+4\epsilon_i}(\mathbf{R}^n))$ for $r + 4\epsilon_i \le R$. Since ϵ_i that is not related to r can be arbitrarily small, then $\mu(B_r(x)) \le \mathrm{vol}_E(B_r(\mathbf{R}^n))$.
- For $\kappa > 0$, we have $\mu_i(B_r(x_i)) < \mathrm{vol}_{S\times E}\big(B_r\big(S^2(\gamma) \times \mathbf{R}^{n-2}\big)\big)$ for all $0 < r \le R$, all i, and $\gamma > \sqrt{\frac{2}{\kappa}}$. Thus, $f_{i*}\mu_i(B_r(x)) < \mu_i(B_{r+4\epsilon_i}(x_i)) < \mathrm{vol}_{S\times E}\big(B_{r+4\epsilon_i}\big(S^2(\gamma) \times \mathbf{R}^{n-2}\big)\big)$ for $r + 4\epsilon_i \le R$. Since ϵ_i that is not related to r can be arbitrarily small, then $\mu(B_r(x)) \le \mathrm{vol}_{S\times E}\big(B_r\big(S^2(\gamma) \times \mathbf{R}^{n-2}\big)\big)$ for $\gamma > \sqrt{\frac{2}{\kappa}}$. Thus, $\mu(B_r(x)) < \mathrm{vol}_{S\times E}\Big(B_r\Big(S^2\Big(\sqrt{\frac{2}{\kappa+\epsilon'}}\Big) \times \mathbf{R}^{n-2}\Big)\Big)$, where $0 < \epsilon'$ is independence on r and ϵ' can as small as we want. Therefore, we have $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \ge \kappa$. ■

Definition 3.6 (tangent space)**.** The mm-space (Y, d_Y, μ_Y, o) is a tangent space of (X^n, d, μ) at $p \in X^n$ if there exists a sequence $\lambda_i \to \infty$ such that $(X^n, \lambda_i \cdot d, \lambda_i^n \cdot \mu, p)$ psmGH-converges to (Y, d_Y, μ_Y, o) as $\lambda_i \to \infty$.

Therefore, (Y, d_Y, μ_Y, o) also satisfies the n-dimensional condition and can be written as Y^n.

Corollary 3.7. *Assume the compact mm-space (X^n, d, μ) with $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq \kappa \geq 0$ and the tangent space (Y^n, d_Y, μ_Y, o) of X^n exists at the point p, then (Y^n, d_Y, μ_Y, o) satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(Y^n)$ ≥ 0 and the SC-radius$\geq r_{X^n}$.*

Proof. Since the n-volumic scalar curvature has the quadratic scaling property, i.e., $\mathrm{Sc}^{\mathrm{vol}_n}(\lambda X^n)$ $\geq \lambda^{-2}\kappa \geq 0$ and $r_{\lambda X^n} = \lambda r_{X^n}$ for all $\lambda > 0$, where $\lambda X^n := (X^n, \lambda \cdot d, \lambda^n \cdot \mu)$, then $\mathrm{Sc}^{\mathrm{vol}_n}(Y^n) \geq 0$ is implied by the stability theorem. ∎

The mm-spaces with $\mathrm{Sc}^{\mathrm{vol}_n} \geq 0$ includes some of the Finsler manifolds, for instance, $\mathbf{R}^n$ equipped with any norm and with the Lebesgue measure satisfies $\mathrm{Sc}^{\mathrm{vol}_n} \geq 0$ and any smooth compact Finsler manifold is a $\mathrm{CD}(\kappa, n)$ space for appropriate finite κ and n [30]. It is well-known that Gigli's infinitesimally Hilbertian [12] can be seen as the Riemannian condition in $\mathrm{RCD}(\kappa, n)$ space. Thus, infinitesimally Hilbertian can also be used as a Riemannian condition in the mm-spaces with $\mathrm{Sc}^{\mathrm{vol}_n} \geq 0$.

Definition 3.8 ($\mathrm{RSC}(\kappa, n)$ space)**.** The compact mm-space (X^n, d, μ) with the n-dimensional condition is a Riemannian n-volumic scalar curvature$\geq \kappa$ space ($\mathrm{RSC}(\kappa, n)$ space) if it is infinitesimally Hilbertian and satisfies the $\mathrm{Sc}^{\mathrm{vol}_n}(X^n) \geq \kappa \geq 0$.

Note that any finite-dimensional Alexandrov spaces with curvature bounded below are infinitesimally Hilbertian. Then

$$\mathrm{Al}^n(\kappa) \Rightarrow \mathrm{RCD}((n-1)\kappa, n) \Rightarrow \mathrm{RSC}((n(n-1)\kappa, n)$$

on $(X^n, d, \mathcal{H}^n)$, where the measure $\mathcal{H}^n$ is the n-dimensional Hausdorff measure that satisfies the n-dimensional condition.

Question 3.9. *Are $\mathrm{RSC}(\kappa, n)$ spaces stable under tvGH-convergence?*

Remark 3.10 (convergence of compact mm-spaces)**.** For the compact metric measure spaces with probability measures, one can consider mGH-convergence, Gromov–Prokhorov convergence, Gromov–Hausdorff–Prokhorov convergence, Gromov–Wasserstein convergence, Gromov–Hausdorff–Wasserstein convergence, Gromov's $\underline{\Box}$-convergence, Sturm's $\mathbb{D}$-convergence [39, Section 27], and Gromov–Hausdorff-vague convergence [3]. smGH-convergence implies those convergences for compact metric measure spaces with probability measures, since the measures converge strongly in smGH-convergence and converge weakly in other situations.

Note that mm-spaces with infinitesimally Hilbertian are not stable under mGH-convergence [12]. It is not clear if the infinitesimally Hilbertian are preserved under smGH-convergence or tvGH-convergence.

4 Smooth mm-space with $\mathrm{Sc}_{\alpha,\beta} > 0$

Let the smooth metric measure space $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ (also known as the weighted Riemannian manifold in some references), where f is a C^2-function on M^n, g is a C^2-Riemannian metric and $n \geq 2$, satisfy the curvature-dimension condition $\mathrm{CD}(\kappa, n)$ for $\kappa \geq 0$, then M^n also satisfies $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq n\kappa$.

Motivated by the importance of the Ricci Bakry-Emery curvature, i.e.,

$$\mathrm{Ric}_f^M = \mathrm{Ricc} + \mathrm{Hess}(f),$$

the weighted sectional curvature of smooth mm-space was proposed and discussed in [40]. On the other hand, Perelman defined and used the P-scalar curvature in his $\mathcal{F}$-functional in [31, Section 1]. Inspired by the P-scalar curvature, i.e., $\mathrm{Sc}_g + 2\bigtriangleup_g f - \|\bigtriangledown_g f\|_g^2$, we propose another scalar curvature on the smooth mm-space.

Definition 4.1 (weighted scalar curvature $\mathrm{Sc}_{\alpha,\beta}$)**.** The weighted scalar curvature $\mathrm{Sc}_{\alpha,\beta}$ on the smooth mm-space $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ is defined by

$$\mathrm{Sc}_{\alpha,\beta} := \mathrm{Sc}_g + \alpha\bigtriangleup_g f - \beta\|\bigtriangledown_g f\|_g^2.$$

Note that the Laplacian $\bigtriangleup_g$ here is the trace of the Hessian and $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq \kappa \geq 0$ is equivalent to $\mathrm{Sc}_{\alpha,\beta} \geq \kappa \geq 0$ for $\alpha = 3$ and $\beta = 3$ (see [33, Theorem 8] or the proof of Corollary 4.10 below).

Example 4.2.

1. For $\alpha = \frac{2(n-1)}{n}$ and $\beta = \frac{(n-1)(n-2)}{n^2}$, the $\mathrm{Sc}_{\frac{2(n-1)}{n},\frac{(n-1)(n-2)}{n^2}}$ is the Chang–Gursky–Yang's conformally invariant scalar curvature for the smooth mm-space [7]. That means for a C^2-smooth function w on M^n, one has
$$\mathrm{Sc}_{\frac{2(n-1)}{n},\frac{(n-1)(n-2)}{n^2}}\left(\mathrm{e}^{2w}g\right) = \mathrm{e}^{-2w}\mathrm{Sc}_{\frac{2(n-1)}{n},\frac{(n-1)(n-2)}{n^2}}(g).$$

2. For $\alpha = 2$ and $\beta = \frac{m+1}{m}$, where $m \in \mathbb{N}\cup\{0,\infty\}$, the $\mathrm{Sc}_{2,\frac{m+1}{m}}$ is Case's weighted scalar curvature and Case also defined and studied the weighted Yamabe constants in [6]. Case's weighted scalar curvature is the classical scalar curvature if $m = 0$. If $m = \infty$, then it is Perelman's P-scalar curvature.

Note that the results in this paper are new for those examples.

4.1 Spin manifold and $\mathrm{Sc}_{\alpha,\beta} > 0$

For an orientable closed surface with density $\left(\Sigma, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ with $\mathrm{Sc}_{\alpha,\beta} > 0$ and $\beta \geq 0$, then the inequality,

$$0 < \int_\Sigma \mathrm{Sc}_{\alpha,\beta}\,\mathrm{dVol}_g = \int_\Sigma \left(\mathrm{Sc}_g + \alpha\bigtriangleup_g f - \beta\|\bigtriangledown_g f\|_g^2\right)\mathrm{dVol}_g = 4\pi\chi(\Sigma) - \beta\int_\Sigma \|\bigtriangledown_g f\|_g^2\,\mathrm{dVol}_g,$$

implies that $\chi(\Sigma) > 0$. Thus, Σ is a 2-sphere.

The following proposition of vanishing harmonic spinors is owed to Perelman essentially and the proof is borrowed from [1, Proposition 1].

Proposition 4.3 (vanishing harmonic spinors)**.** *Assume the smooth mm-space* $\left(M^n,g,\mathrm{e}^{-f}\mathrm{dVol}_g\right)$ *is closed and spin. If* $\alpha \in \mathbb{R}$, $\beta \geq \frac{|\alpha|^2}{4}$ *and* $\mathrm{Sc}_{\alpha,\beta} > 0$, *then the harmonic spinor of* M^n *vanishes.*

Proof. Let ψ be a harmonic spinor, though the Schrödinger–Lichnerowicz–Weitzenboeck formula

$$\mathbb{D}^2 = \bigtriangledown^*\bigtriangledown + \frac{1}{4}\mathrm{Sc}_g,$$

one has

$$\begin{aligned}0 &= \int_M \left[\| \triangledown_g \psi\|_g^2 + \frac{1}{4}\big(\mathrm{Sc}_{\alpha,\beta} - \alpha \triangle_g f + \beta \| \triangledown_g f\|_g^2\big)\|\psi\|_g^2\right] \mathrm{dVol}_g \\ &= \int_M \left[\| \triangledown_g \psi\|_g^2 + \left(\frac{1}{4}\mathrm{Sc}_{\alpha,\beta} + \frac{\beta}{4}\| \triangledown_g f\|_g^2\right)\|\psi\|_g^2 + \frac{\alpha}{4}\big\langle \triangledown_g f, \triangledown_g \|\psi\|_g^2\big\rangle_g\right] \mathrm{dVol}_g.\end{aligned}$$

Then one gets

$$\begin{aligned}\frac{|\alpha|}{4}\big|\big\langle \triangledown_g f, \triangledown_g \|\psi\|_g^2\big\rangle_g\big| &\leq \frac{|\alpha|}{4}\big(c\| \triangledown_g f\|_g \|\psi\|_g \times 2c^{-1}\| \triangledown_g \psi\|_g\big) \\ &\leq \frac{|\alpha|c^2}{8}\| \triangledown_g f\|_g^2\|\psi\|_g^2 + \frac{c^{-2}|\alpha|}{2}\| \triangledown_g \psi\|_g^2.\end{aligned}$$

Therefore,

$$0 \geq \int_M \left[\left(1 - \frac{c^{-2}|\alpha|}{2}\right)\| \triangledown_g \psi\|_g^2 + \frac{2\beta - c^2|\alpha|}{8}\| \triangledown_g f\|_g^2\|\psi\|_g^2 + \frac{1}{4}\mathrm{Sc}_{\alpha,\beta}\|\psi\|_g^2\right] \mathrm{dVol}_g,$$

where $c \neq 0$. If $c^{-2}|\alpha| \leq 2$, $\beta \geq \frac{c^2|\alpha|}{2}$ and $\mathrm{Sc}_{\alpha,\beta} > 0$, then $\psi = 0$. So the conditions $\alpha \in \mathbb{R}$ and $\beta \geq \frac{|\alpha|^2}{4}$ are needed

$$\begin{aligned}\frac{|\alpha|}{4}\big|\big\langle \triangledown_g f, \triangledown_g \|\psi\|_g^2\big\rangle_g\big| &\leq \frac{|\alpha|}{4}\big(\| \triangledown_g f\|_g \|\psi\|_g \times 2\| \triangledown_g \psi\|_g\big) \\ &= \frac{|\alpha|}{2}\big(c_1\| \triangledown_g f\|_g \|\psi\|_g \times c_1^{-1}\| \triangledown_g \psi\|_g\big) \\ &\leq \frac{|\alpha|}{4}\big(c_1^2\| \triangledown_g f\|_g^2\|\psi\|_g^2 + c_1^{-2}\| \triangledown_g \psi\|_g^2\big).\end{aligned}$$

Thus,

$$0 \geq \int_M \left[\left(1 - \frac{c_1^{-2}|\alpha|}{4}\right)\| \triangledown_g \psi\|_g^2 + \frac{\beta - c_1^2|\alpha|}{4}\| \triangledown_g f\|_g^2\|\psi\|_g^2 + \frac{1}{4}\mathrm{Sc}_{\alpha,\beta}\|\psi\|_g^2\right] \mathrm{dVol}_g,$$

where $c_1 \neq 0$. If $c_1^{-2}|\alpha| \leq 4$, $\beta \geq c_1^2|\alpha|$ and $\mathrm{Sc}_{\alpha,\beta} > 0$, then $\psi = 0$. Also the conditions $\alpha \in \mathbb{R}$ and $\beta \geq \frac{|\alpha|^2}{4}$ are needed. ∎

The following 3 corollaries come from the proposition of vanishing of harmonic spinors.

Corollary 4.4. *Assume the smooth mm-space* $\big(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\big)$ *is closed and spin. If* $\alpha \in \mathbb{R}$, $\beta \geq \frac{|\alpha|^2}{4}$ *and* $\mathrm{Sc}_{\alpha,\beta} > 0$, *then the* $\widehat{A}$*-genus and the Rosenberg index of* M^n *vanish.*

Proof. Since the $C^*(\pi_1(M^n))$-bundle in the construction of the Rosenberg index [35] is flat, there are no correction terms due to curvature of the bundle. Then the Schrödinger–Lichnerowicz–Weitzenboeck formula and the argument in the proof of vanishing harmonic spinors can be applied without change. ∎

Corollary 4.5. *Assume that* M^n *is a closed spin* n*-manifold and* f *is a smooth function on* M^n*. If one of the following conditions is met,*

(1) $N \subset M^n$ *is a codimension one closed connected submanifold with trivial normal bundle, the inclusion of fundamental groups* $\pi_1\big(N^{n-1}\big) \to \pi_1(M^n)$ *is injective and the Rosenberg index of* N *does not vanish, or*

(2) *$N \subset M^n$ is a codimension two closed connected submanifold with trivial normal bundle, $\pi_2(M^n) = 0$, the inclusion of fundamental groups $\pi_1(N^{n-1}) \to \pi_1(M^n)$ is injective and the Rosenberg index of N does not vanish, or*

(3) *$N = N_1 \cap \cdots \cap N_k$, where $N_1 \cdots N_k \subset M$ are closed submanifolds that intersect mutually transversely and have trivial normal bundles. Suppose that the codimension of N_i is at most two for all $i \in \{1\dot{k}\}$ and $\pi_2(N) \to \pi_2(M)$ is surjective and $\hat{A}(N) \neq 0$,*

then M^n does not admit a Riemnannian metric g such that the smooth mm-space $\big(M^n, g, \mathrm{e}^{-f}\,\mathrm{dvol}_g\big)$ satisfies $\mathrm{Sc}_{\alpha,\beta} > 0$ for the dimension $n \geq 3$, $\alpha \in \mathbb{R}$ and $\beta \geq \frac{|\alpha|^2}{4}$.

Proof. The results in the [22, Theorem 1.1] and [42, Theorem 1.9] can be applied to show that the Rosenberg index of M^n does not vanish and Corollary 4.4 implies the theorem. ■

Let $\mathcal{R}_f(M^n) := \{(g, f)\}$ be the space of densities, where g is a smooth Riemannian metric on M^n and f is a smooth function on M^n and $\mathcal{R}^+_f(M^n) \subset \mathcal{R}_f(M^n)$ is the subspace of densities such that the smooth mm-space $\big(M^n, g, \mathrm{e}^{-f}\,\mathrm{dvol}_g\big)$ satisfies $\mathrm{Sc}_{\alpha,\beta} > 0$. Furthermore, let $\mathcal{R}^+_f(M^n)$ be endowed with the smooth topology.

Corollary 4.6. *Assume M^n is a closed spin n-manifold, $n \geq 3$, $\alpha \in \mathbb{R}$ and $\beta \geq \frac{|\alpha|^2}{4}$ and $\mathcal{R}^+_f(M^n) \neq \varnothing$, then there exists a homomorphism*

$$A_{m-1}\colon\ \pi_{m-1}(\mathcal{R}^+_f(M^n)) \to KO_{n+m}$$

such that

- $A_0 \neq 0$, *if* $n \equiv 0, 1 \pmod 8$,
- $A_1 \neq 0$, *if* $n \equiv -1, 0 \pmod 8$,
- $A_{8j+1-n} \neq 0$, *if* $n \geq 7$ *and* $8j - n \geq 0$.

Proof. Since the results in the [23, Section 4.4] and [9] depend on the existence of exotic spheres with non-vanishing α-invariant. Let $\phi\colon M^n \to M^n$ be a diffeomorphism of M^n and $(g, f) \in \mathcal{R}^+_f(M^n)$, then $(\phi^* g, f \circ \phi)$ is also in $\mathcal{R}^+_f(M^n)$. Combining it with Proposition 4.3 shows that Hitchin's construction of the map A [23, Proposition 4.6] can be applied to the case of $\mathcal{R}^+_f(M^n)$ and then we can finish the proof with the arguments in [23, Section 4.4] and [9, Section 2.5]. ■

4.2 Conformal to PSC-metrics

Proposition 4.7 (conformal to PSC-metrics)**.** *Let $\big(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\big)$ be a closed smooth mm-space with $\mathrm{Sc}_{\alpha,\beta} > 0$. If the dimension $n \geq 3$, $\alpha \in \mathbb{R}$ and $\beta \geq \frac{(n-2)|\alpha|^2}{4(n-1)}$, then there is a metric $\tilde{g}$ conformal to g with positive scalar curvature (PSC-metric).*

Proof. One only needs to show for all nontrivial u, $\int_M -u L_g u\,\mathrm{dVol}_g > 0$ as in the Yamabe problem [36], where

$$L_g := \triangle_g - \frac{n-2}{4(n-1)}\mathrm{Sc}_g$$

is conformal Laplacian operator. To see this,

$$\begin{aligned}\int_M -uL_g u\,\mathrm{dVol}_g &= \int_M \left[\|\bigtriangledown_g u\|_g^2 + \frac{n-2}{4(n-1)}\mathrm{Sc}_g u^2\right]\mathrm{dVol}_g\\ &= \int_M \left[\|\bigtriangledown_g u\|_g^2 + \frac{n-2}{4(n-1)}\left(\mathrm{Sc}_{\alpha,\beta} - \alpha \bigtriangleup_g f + \beta\|\bigtriangledown_g f\|_g^2\right)u^2\right]\mathrm{dVol}_g\\ &= \int_M \left[\|\bigtriangledown_g u\|_g^2 + \frac{n-2}{4(n-1)}\left(\mathrm{Sc}_{\alpha,\beta} + \beta\|\bigtriangledown_g f\|_g^2\right)u^2\right.\\ &\quad \left. + \frac{\alpha(n-2)}{2(n-1)}\langle\bigtriangledown_g f, \bigtriangledown_g u\rangle_g u\right]\mathrm{dVol}_g.\end{aligned}$$

Through the inequality

$$\langle\bigtriangledown_g f, \bigtriangledown_g u\rangle_g u \le c_2\|\bigtriangledown_g f\|_g u \times c_2^{-1}\|\bigtriangledown_g u\|_g \le \frac{c_2^2\|\bigtriangledown_g f\|_g^2 u^2 + c_2^{-2}\|\bigtriangledown_g u\|_g^2}{2},$$

one gets

$$\begin{aligned}\int_M -uL_g u\,\mathrm{dVol}_g \ge \int_M &\left[\left(1 - \frac{|\alpha|c_2^{-2}(n-2)}{4(n-1)}\right)\|\bigtriangledown_g u\|_g^2\right.\\ &\left. + \frac{\left(\beta - |\alpha|c_2^{-2}\right)(n-2)}{4(n-1)}\|\bigtriangledown_g f\|_g^2 u^2 + \frac{n-2}{4(n-1)}\mathrm{Sc}_{\alpha,\beta}u^2\right]\mathrm{dVol}_g,\end{aligned}$$

where $c_2 \neq 0$.

If $|\alpha|c_2^{-2} \le \frac{4(n-1)}{n-2}$, $\beta \ge c_2^2|\alpha|$ and $\mathrm{Sc}_{\alpha,\beta} > 0$, then

$$\int_M -uL_g u\,\mathrm{dVol}_g > 0.$$

So the conditions $n > 2$, $\alpha \in \mathbb{R}$ and $\beta \ge \frac{(n-2)\alpha^2}{4(n-1)}$ are needed. ■

Remark 4.8. The proof was borrowed from [1, Proposition 2]. The two propositions above offer a geometric reason why the condition of the vanishing of $\widehat{A}$-genus (without simply connected condition) does not imply that M^n can admit a PSC-metric for the closed spin manifold M^n.

The proposition of conformal to PSC-metrics has following 3 corollaries.

Corollary 4.9 (weighted spherical Lipschitz bounded)**.** *Let $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ be a closed orientable smooth mm-space with $\mathrm{Sc}_{\alpha,\beta} \ge \kappa > 0$, $3 \le n \le 8$, $\alpha \in \mathbb{R}$ and $\beta \ge \frac{(n-2)|\alpha|^2}{4(n-1)}$, then the Lipschitz constant of the continuous map ϕ from $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ to the sphere S^n with standard metric of non-zero degrees has uniformly non-zero lower bounded.*

Proof. There is a metric $\tilde{g}$ conformal to g with scalar curvature $\ge n(n-1)$ by the proposition of conformal PSC-metrics. For the continuous map ϕ from $(M^n, \tilde{g})$ to S^n of non-zero degrees, the Lipschitz constant of ϕ is greater than a constant that depends only on the dimensions n by Gromov's spherical Lipschitz bounded theorem [19, Section 3]. Since the conformal function has the positive upper bound by the compactness of the manifold, then the Lipschitz constant has uniformly non-zero lower bounded. ■

Corollary 4.10. *For the closed smooth mm-space $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ $(n \ge 3)$ with $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) > 0$, there is a metric $\hat{g}$ conformal to g with PSC-metric. In particular, the $\widehat{A}$-genus and Rosenberg index vanish with additional spin condition.*

For the closed orientable smooth mm-space $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ $(3 \le n \le 8)$ with $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \ge \kappa > 0$, then the Lipschitz constant of the continuous map ϕ from $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ to the sphere S^n with standard metric of non-zero degrees has uniformly non-zero lower bounded.

Proof. The volume of the small disk of $\left(M^n, g, \mathrm{e}^{-f}\, \mathrm{dVol}_g\right)$ was computed in [33, Theorem 8],

$$\mu(B_r(x)) = \mathrm{vol}_E(B_r(\mathbf{R}^n)) \left[1 - \frac{\mathrm{Sc}_g + 3 \triangle_g f - 3\| \bigtriangledown_g f\|_g^2}{6(n+2)} r^2 + O\left(r^4\right)\right]$$

as $r \to 0$. Since $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) > 0$, i.e., $\mu(B_r(x)) < \mathrm{vol}_E(B_r)$ as $r \to 0$, then

$$\mathrm{Sc}_g + 3 \triangle_g f - 3\| \bigtriangledown_g f\|_g^2 > 0.$$

Therefore, the propositions of vanishing harmonic spinors and of conformal PSC-metrics and Corollary 4.4 imply it. ■

Remark 4.11. Since any weighted Riemannian manifold (with non-trivial Borel measure) is infinitesimally Hilbertian (see [28]), Corollary 4.10 also works for $\left(M^n, g, \mathrm{e}^{-f}\, \mathrm{dVol}_g\right)$ with $\mathrm{RSC}(\kappa, n)$ condition.

Enlargeability as an obstruction to the existence of a PSC-metric on a closed manifold was introduced by Gromov–Lawson. We call a manifold enlargeable as Gromov–Lawson's definition in [21, Definition 5.5]

Corollary 4.12. *Assume M^n $(n \geq 3)$ is a closed spin smooth enlargeable manifold, then $\mathcal{R}_f^+(M^n)$ is an empty set for $\alpha \in \mathbb{R}$ and $\beta \geq \frac{(n-2)|\alpha|^2}{4(n-1)}$.*

In particular, $\left(\mathbb{T}^n, g, \mathrm{e}^{-f}\, \mathrm{dVol}_g\right)$ does not satisfy $\mathrm{Sc}^{\mathrm{vol}_n}(\mathbb{T}^n) > 0$ for any C^2-smooth Riemannian metrics g and C^2-smooth functions f on the torus $\mathbb{T}^n$.

Proof. Since a closed enlargeable manifold cannot carry a PSC-metric [21, Theorem 5.8], Proposition 4.7 implies $\mathcal{R}_f^+(M^n) = \varnothing$ for $\alpha \in \mathbb{R}$ and $\beta \geq \frac{(n-2)|\alpha|^2}{4(n-1)}$.

$\mathbb{T}^n$ is an important example of enlargeable manifolds and then Corollary 4.10 implies that $\left(\mathbb{T}^n, g, \mathrm{e}^{-f}\, \mathrm{dVol}_g\right)$ does not satisfy $\mathrm{Sc}^{\mathrm{vol}_n}(\mathbb{T}^n) > 0$ for $n \geq 3$. For dimension 2, the conditions of $\mathrm{Sc}_{\alpha,\beta} > 0$ and $\beta \geq 0$ imply that the oriented surface is 2-sphere. ■

4.3 f-minimal hypersurface and $\mathrm{Sc}_{\alpha,\beta} > 0$

In addition to using the Dirac operator method, Schoen–Yau's minimal hypersurface method [37] is another main idea. For an immersed orientable hypersurface $N^{n-1} \subset M^n$, the weighted mean curvature vector H_f of N^{n-1} is defined by Gromov in [16, Section 9.4.E],

$$H_f = H + (\bigtriangledown_g f)^{\perp},$$

where H is the mean curvature vector field of the immersion, $(\cdot)^{\perp}$ is the projection on the normal bundle of N^{n-1}. The first and second variational formulae for the weighted volume functional of N^{n-1} were derived in Bayle's thesis (also see [34]). We take the detailed presentation of such derivation for [8]. The $\left(N^{n-1}, \bar{g}\right)$ with the induced metric is called f-minimal hypersurface if the weighted mean curvature vector H_f vanishes identically.

In fact, the definition of f-minimal hypersurface can also be derived from the first variational formula. Furthermore, an f-minimal hypersurface is a minimal hypersurface of $(M^n, \tilde{g})$, where $\tilde{g}$ is the conformal metric of g, $\tilde{g} = \mathrm{e}^{-\frac{2f}{n-1}} g$.

The connection between the geometry of the ambient smooth mm-space and the f-minimal hypersurfaces occurs via the second variation of the weighted volume functional. For a hypersurface $\left(N^{n-1}, \bar{g}\right)$, the L_f operator is defined by

$$L_f := \triangle_f + |A|^2 + \mathrm{Ricc}_f^M(\nu, \nu),$$

where ν is the unit normal vector, $|A|^2$ denotes the square of the norm of the second fundamental form A of N^{n-1} and

$$\triangle_f := \triangle_{\bar{g}} - \langle \triangledown_{\bar{g}} f, \triangledown_{\bar{g}} \cdot \rangle$$

is the weighted Laplacian. Through the second variational formula, a two-sided f-minimal hypersurface N^{n-1} is stable (called L_f-stable) if for any compactly supported smooth function $u \in C_c^\infty(N^{n-1})$, it holds that

$$-\int_N u L_f u \mathrm{e}^{-f} \,\mathrm{dVol}_{\bar{g}} \geq 0.$$

Proposition 4.13. *Let $(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g)$ be a closed orientable smooth mm-space with $\mathrm{Sc}_{\alpha,\beta} > 0$ and $(N^{n-1}, \bar{g})$ be the compact L_f-stable minimal hypersurface of $(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g)$. If the dimension $n \geq 3$, $\alpha = 2$, and $\beta \geq \frac{n-2}{n-1}$, then there exists a PSC-metric conformal to $\bar{g}$ on N^{n-1}.*

Proof. The f-minimal hypersurface $(N^{n-1}, \bar{g})$ is L_f-stable if and only if $(N^{n-1}, \bar{\tilde{g}})$ is stable as a minimal hypersurface on $(M^n, \tilde{g})$, where $\tilde{g} := \mathrm{e}^{-\frac{2f}{n-1}} g$ and $\bar{\tilde{g}}$ is the induced metric of $\tilde{g}$ (see [8, Appendix]). On the other hand, the scalar curvature of $(M^n, \tilde{g})$ is

$$\mathrm{Sc}_{\tilde{g}} = \mathrm{e}^{\frac{f}{n-1}} \left(\mathrm{Sc}_g + 2 \triangle_g f - \frac{n-2}{n-1} \| \triangledown_g f \|_g^2 \right).$$

Thus, $\mathrm{Sc}_{\alpha,\beta} > 0$ with $n \geq 3$, $\alpha = 2$, and $\beta \geq \frac{n-2}{n-1}$ imply $\mathrm{Sc}_{\tilde{g}} > 0$. Then the standard Schoen–Yau's argument can be applied to show that $\bar{\tilde{g}}$ conformal to a PSC-metric on N^{n-1}. ■

Remark 4.14. The minimal hypersurface method poses a stricter condition to the valid range of α, β than that of the Dirac operator method.

Since the oriented closed manifolds with a PSC-metric in 2 and 3 dimensions are classified by Gauess–Bonnet theorem and Perelman–Thurston geometrization theorem, then Proposition 4.13 can give the following elementary applications:

Corollary 4.15. *Let $(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g)$ be a closed orientable smooth mm-space with $\mathrm{Sc}_{\alpha,\beta} > 0$.*

1. *If $n = 3$, $\alpha = 2$, and $\beta \geq \frac{1}{2}$, then there is no closed immersed L_f-stable minimal 2-dimensional surface with positive genus.*
2. *If $n = 4$, $\alpha = 2$, and $\beta \geq \frac{2}{3}$, then the closed immersed L_f-stable minimal 3-dimensional submanifold must be spherical 3-manifolds, $S^2 \times S^1$ or the connected sum of spherical 3-manifolds and copies of $S^2 \times S^1$.*

Remark 4.16 (historical remark)**.** The prototype of Corollary 4.15(1) is the Schoen–Yau's classic result, which said that the oriented closed 3-manifold with a PSC-metric has no compact immersed stable minimal surface of positive genus [37]. The Schoen–Yau result had been generalized to Perelman's P-scalar curvature > 0 by Fan [10]. Note that one can also consider the noncompact immersed L_f-stable minimal 2-dimensional surface under the condition of Corollary 4.15(1) since an oriented complete stable minimal surface in a complete oriented 3-manifold with a PSC-metric is conformally equivalent to the complex plane $\mathbb{C}$ showed by Fischer–Colbrie–Schoen [11].

The smooth mm-space with $\mathrm{Sc}_{\alpha,\beta} > 0$ under suit ranges of α and β implies the manifold admits PSC-metrics, but the manifold (itself) that can admit PSC-metrics does not necessarily imply there exists $\mathrm{Sc}_{\alpha,\beta} > 0$.

Question 4.17. *Does the smooth mm-space with* $\mathrm{Sc}_{\alpha,\beta} > 0$ *under suitable ranges of* α *and* β *give more topological restriction on the manifold than the PSC-metric on the manifold?*

Question 4.18. *Let* M *be a closed smooth manifold,* f *be a smooth function on* M *and* h *be a smooth function that is negative for some point* p *on* M*. What is the range of* α *and* β *such that there exists a smooth Riemannian metric* g *on* M *satisfying*

$$\mathrm{Sc}_g + \alpha \triangle_g f - \beta \| \nabla_g f \|_g^2 = h,$$

i.e., $\mathrm{Sc}_{\alpha,\beta}(g) = h$*?*

Let (M, g_i) be smooth Riemannian manifolds and $\{g_i\}_{i\in\mathbf{N}}$ C^0-converges to g, then $\{g_i\}_{i\in\mathbf{N}}$ also smGH-converges to g. Gromov showed that the scalar curvature$\geq \kappa$ is stable under C^0-convergence in [17, Section 1.8].

Question 4.19. *Assume smooth mm-spaces* $\left(M^n, g_i, \mathrm{e}^{-f}\,\mathrm{dVol}_{g_i}\right)$ *all satisfy* $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq 0$ *such that* $\{g_i\}_{i\in\mathbf{N}}$ C^2*-converges to the smooth Riemannian metric* g *on* M^n*, then does* $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ *also satisfy* $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq 0$*?*

Question 4.20. *Let mm-spaces* $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ *with* $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq \kappa > 0$*, where* M^n *is a closed smooth manifold,* g *and* f *are* C^0*-smooth, then does there exist a PSC-metric on* M^n*?*

Since the role of $\mathrm{Sc}_{\alpha,\beta} > 0$ on the smooth mm-space is similar to the role of $\mathrm{Sc} > 0$ on the manifold, one can try to extend the knowledge about $\mathrm{Sc} > 0$ to $\mathrm{Sc}_{\alpha,\beta} > 0$.

4.4 Weighted rigidity

Gromov's conjecture that said if a smooth Riemannian metric g satisfies $g \geq g_{\mathrm{st}}$ and $\mathrm{Sc}(g) \geq \mathrm{Sc}(g_{\mathrm{st}}) = n(n-1)$ on S^n then $g = g_{\mathrm{st}}$, was proved by Llarull [27] and called Llarull rigidity theorem. A map $h\colon (M^n, g_M) \to (N^n, g_N)$ is said to be ϵ-contracting if $\|h_* v\|_{g_N} \leq \epsilon \|v\|_{g_N}$ for all tangent vectors v on M^n

Proposition 4.21 (weighted rigidity)**.** *Assume the smooth mm-space* $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ *is closed and spin and there exists a smooth* 1*-contracting map* $h\colon (M^n, g) \to (S^n, g_{\mathrm{st}})$ *of non-zero degree. If* $\alpha \in \mathbb{R}$*,* $\beta \geq \frac{|\alpha|^2}{4}$ *and* $\mathrm{Sc}_{\alpha,\beta} \geq n(n-1)$*, then* h *is an isometry between the metrics* g *and* g_{st}*. Furthermore, if* $\alpha > 0$*, then* f *is a constant function.*

Proof. One just need to insert the tricks in the proof of Proposition 4.3 to the proof in [27, Theorem 4.1]. Following the setup of Llarull, we only prove the even-dimensional $(2n)$ case without loss of generality.

First, we will show that h is an isometry. Fix $p \in M^{2n}$. Let $\{e_1, \ldots, e_{2n}\}$ be a g-orthonormal tangent fame near p such that $(\nabla_g e_k)_p = 0$ for each k. Let $\{\epsilon_1, \ldots, \epsilon_{2n}\}$ be a g_{st}-orthonormal tangent frame near $h(p) \in S^{2n}$ such that $(\nabla_{g_{\mathrm{st}}} \epsilon_k)_{h(p)} = 0$ for each k. Moreover, the bases $\{e_1, \ldots, e_{2n}\}$ and $\{\epsilon_1, \ldots, \epsilon_{2n}\}$ can be chosen so that $\epsilon_j = \lambda_j h_* e_j$ for appropriate $\{\lambda_j\}_{j=1}^{2n}$. This is possible since h_* is symmetric. Since h is 1-contracting map, $\lambda_k \geq 1$ for each k.

Then one constructs the twisted vector bundles $S \bigotimes E$ over M^{2n} as Llarull did. Let R^E be the curvature tensor of E and ψ be a twisted spinor, then one gets

$$\left\langle R^E \psi, \psi \right\rangle_g \geq -\frac{1}{4} \sum_{i \neq j} \frac{1}{\lambda_i \lambda_j} \|\psi\|_g.$$

For the twisted Dirac operator $\mathbb{D}_E$, one has $\mathbb{D}_E^2 = \nabla^* \nabla + \frac{1}{4}\mathrm{Sc}_g + R^E$ and

$$\begin{aligned}\int_M \left\langle \mathbb{D}_E^2\psi, \psi\right\rangle_g \mathrm{dVol}_g &= \int_M \Big[\| \nabla_g \psi\|_g^2 + \frac{1}{4}\big(\mathrm{Sc}_{\alpha,\beta} - \alpha \triangle_g f + \beta\| \nabla_g f\|_g^2\big)\|\psi\|_g^2 \\ &\quad + \left\langle R^E\psi, \psi\right\rangle_g\Big] \mathrm{dVol}_g \\ &= \int_M \Big[\| \nabla_g \psi\|_g^2 + \left(\frac{1}{4}\mathrm{Sc}_{\alpha,\beta} + \frac{\beta}{4}\| \nabla_g f\|_g^2\right)\|\psi\|_g^2 \\ &\quad + \frac{\alpha}{4}\langle \nabla_g f, \nabla_g \|\psi\|_g^2\rangle_g + \left\langle R^E\psi, \psi\right\rangle_g\Big]\mathrm{dVol}_g.\end{aligned}$$

Because $\lambda_k \geq 1$ for each k, one gets

$$\left\langle R^E\psi, \psi\right\rangle_g \geq \frac{-2n(2n-1)}{4}\|\psi\|_g$$

and then

$$\begin{aligned}\frac{|\alpha|}{4}|\langle \nabla_g f, \nabla_g\|\psi\|_g^2\rangle_g| &\leq \frac{|\alpha|}{4}\big(\| \nabla_g f\|_g\|\psi\|_g \times 2\| \nabla_g \psi\|_g\big) \\ &= \frac{|\alpha|}{2}\big(c_1\| \nabla_g f\|_g\|\psi\|_g \times c_1^{-1}\| \nabla_g \psi\|_g\big) \\ &\leq \frac{|\alpha|}{4}\big(c_1^2\| \nabla_g f\|_g^2\|\psi\|_g^2 + c_1^{-2}\| \nabla_g \psi\|_g^2\big),\end{aligned}$$

where $c_1 \neq 0$. Therefore,

$$\begin{aligned}\int_M \left\langle \mathbb{D}_E^2\psi, \psi\right\rangle_g \mathrm{dVol}_g \geq \int_M &\Bigg[\left(1 - \frac{c_1^{-2}|\alpha|}{4}\right)\| \nabla_g \psi\|_g^2 + \frac{\beta - c_1^2|\alpha|}{4}\| \nabla_g f\|_g^2\|\psi\|_g^2 \\ &+ \frac{1}{4}(\mathrm{Sc}_{\alpha,\beta} - 2n(2n-1))\|\psi\|_g^2\Bigg]\mathrm{dVol}_g.\end{aligned}$$

Furthermore, since $\alpha \in \mathbb{R}$, $\beta \geq \frac{|\alpha|^2}{4}$ and $\mathrm{Sc}_{\alpha,\beta} \geq 2n(2n-1)$, one can choose c_1 such that $c_1^{-2}|\alpha| \leq 4$, then $\beta - c_1^2|\alpha| \geq 0$. Thus,

$$\int_M \left\langle \mathbb{D}_E^2\psi, \psi\right\rangle_g \mathrm{dVol}_g \geq \int_M \frac{1}{4}[\mathrm{Sc}_{\alpha,\beta} - 2n(2n-1)]\|\psi\|_g^2 \,\mathrm{dVol}_g \geq 0.$$

The fact $\mathrm{Index}(\mathbb{D}_{E^+}) \neq 0$ implies $\ker(\mathbb{D}_E) \neq 0$ and then $\mathrm{Sc}_{\alpha,\beta} = 2n(2n-1)$. Then using the inequality $\left\langle R^E\psi, \psi\right\rangle_g \geq -\frac{1}{4}\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}\|\psi\|_g$, one gets

$$\int_M \left\langle \mathbb{D}_E^2\psi, \psi\right\rangle_g \mathrm{dVol}_g \geq \int_M \frac{1}{4}\left[\sum_{i\neq j}\left(1 - \frac{1}{\lambda_i\lambda_j}\right)\right]\|\psi\|_g^2 \,\mathrm{dVol}_g \geq 0.$$

Choosing $\psi \neq 0$ such that $\mathbb{D}_E\psi = 0$, one has

$$0 \leq 1 - \frac{1}{\lambda_i\lambda_j} \leq 0$$

for $i \neq j$. Thus, $\lambda_k = 0$ for all $1 \leq k \leq 2n$ and h is an isometry.

Second, we will show that f is a constant function. Since $\mathrm{Sc}_{\alpha,\beta} = 2n(2n-1)$, $\mathrm{Sc}_g = 2n(2n-1)$, $\alpha > 0$ and $\beta \geq \frac{|\alpha|^2}{4}$, then $\triangle_g f \geq 0$. One has

$$\int_M \triangle_g f \,\mathrm{dVol}_g = 0$$

for a closed manifold M^n, so one gets $\triangle_g f = 0$. That implies $\nabla_g f = 0$ so that f is a constant function on M^n. ∎

Corollary 4.22. *Let the closed and spin smooth mm-space* $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ *with* $\mathrm{Sc}^{\mathrm{vol}_n}(M^n) \geq n(n-1)$ *and there exists a smooth* 1*-contracting map* $h\colon (M^n, g) \to (S^n, g_{\mathrm{st}})$ *of non-zero degree, then* h *is an isometry between the metrics* g *and* g_{st}*.*

Proof. Combining the weighted rigidity theorem and the proof of Corollary 4.10 can imply it. ■

As Llarull rigidity theorem (and the weighted rigidity theorem) still holds if the condition that h is 1-contracting is replaced by the condition that h is area-contracting, Gromov called such metrics area-extremal metrics, asked which manifolds possess area-extremal metrics, and conjectured that Riemannian symmetric spaces should have area-extremal metrics [15], [18, Section 17] and [20, Section 4.2]. Goette–Semmelmann showed that several classes of symmetric spaces with non-constant curvatures are area-extremal [13].

Question 4.23. *Can Goette–Semmelmann's results* [13] *be generalized to the smooth mm-space with* $\mathrm{Sc}_{\alpha,\beta} > 0$ *under other suitable conditions?*

Since Corollary 4.15(1) showed that the closed orientable immersed L_f-stable minimal 2-dimensional surface in the closed orientable smooth mm-space $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$ with $\mathrm{Sc}_{2,\beta} > 0$ $\left(\beta \geq \frac{1}{2}\right)$ is 2-sphere, then one can consider rigidity of area-minimizing 2-sphere in 3-dimensional smooth mm-space. Bray's volume comparison theorem [4, Chapter 3, Theorem 18] is another rigidity theorem that needs the conditions of Ricci curvature and scalar curvature bounded below. There are other rigidity phenomena involving scalar curvature, see [5].

Question 4.24. *Can Bray's volume comparison theorem be extended to the smooth mm-space?*

Question 4.25. *What is the correct Einstein field equation on the smooth mm-space?*

If one replaces the Ricci and scalar curvature on the left hand side of Einstein field question by Ricc_f^M *and* $\mathrm{Sc}_{\alpha,\beta}$ *for the smooth mm-space* $\left(M^n, g, \mathrm{e}^{-f}\,\mathrm{dVol}_g\right)$*, then what is the stress–energy tensor on the right hand side in this case?*

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